COk-H06464-27-P17561

BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

ABRAHAM ADRIAN ALBERT
TOMLINSON FORT
LAWRENCE MURRAY GRAVES
PAUL ALTHAUS SMITH

WITH THE ASSISTANCE OF

P. O. BELL
MORRIS MARDEN
VIRGIL SNYDER
GARRETT BIRKHOFF
J. A. SHOHAT
J. L. DOOB
O. F. [illegible]CHILLING
MO[illegible] WARD

VOLUME 48
JANUARY TO DECEMBER, 1942

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK

The Collegiate Press
GEORGE BANTA PUBLISHING COMPANY
MENASHA, WIS.

CALCUTTA UNIVERSITY
CENTRAL LIBRARY.

BULLETIN
OF THE
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27

LETTER FROM MARSTON MORSE

Institute for Advanced Study
Princeton, New Jersey
November 8, 1941.

The object of this letter is to give to interested mathematicians a report concerning developments in the problem of the proper use of mathematicians of the draft age.

At a July meeting of the Roster of Scientific and Specialized Personnel[1] in Washington these matters were discussed at length and a number of those present, including the writer, urged that the Roster is the body best equipped to take up the problem of the proper use of scientists. It should be understood at the beginning of this letter that the Roster has no power to defer scientists, but is concerned only with making recommendations as to whether or not a man may be regarded as "necessary" in the sense of the law.

In order to make proper recommendations it was necessary that the Roster obtain considerable additional information concerning men of draft age on its lists. The matter of procedure has now been systematized. Any man who is on the Roster and who finds himself likely to be called may write the Roster for a standard questionnaire. Among other things this questionnaire seeks to find out the man's present occupation and scientific status and the names of references. These references and the man's employer are then sent questionnaires calculated to assist in determining whether the man may be regarded as "necessary."

The Roster than calls in men in the various scientific fields to review the assembled data and advise the Roster on each individual case. The mathematicians were the first so treated, and Murnaghan, Hotelling and Morse recently served as consultants for the Roster. The Roster had assembled excellent data bearing on 300 cases, each of which was carefully reviewed by each of the three mathematicians. In general we tried to follow the resolutions concerning deferment which were voted at the recent Chicago meeting of the mathematicians and are published in the Bulletin.

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[1] The Roster of Scientific and Specialized Personnel, Washington, D. C., is a sufficient address.

The Roster makes its recommendations to the Selective Service Headquarters, which in turn *may* pass on the recommendations to the local boards. The local boards are free to follow the recommendations or not. We shall know more about the practice a little later.

In case a man is inducted, the Roster acts in an advisory capacity to the personnel department of the Adjutant General's office, offering information which will aid the personnel officers in properly using the scientists on the Roster.

In addition to making recommendations as to individual men, the Roster also seeks information concerning shortages in various scientific fields and passes on this information to the Selective Service Headquarters and the Labor Office. Our Committee on the Supply and Demand for Mathematicians will materially aid the Roster in this respect, and will help to clarify the situation.

In conclusion I wish to recommend that problems of deferment or use of scientists on the Roster be taken up with the Roster whenever they arise. With the advice of consultants in the various fields, the Roster is the group best organized and equipped to handle these questions.

Very truly yours,
MARSTON MORSE

THE OCTOBER MEETING IN NEW YORK

The three hundred eighty-first meeting of the American Mathematical Society was held at Columbia University on Saturday, October 25, 1941. The attendance included the following one hundred sixty-seven members of the Society:

A. A. Albert, G. K. Anderson, R. L. Anderson, Lawrence Annenberg, R. G. Archibald, L. A. Aroian, S. F. Barber, E. F. Beckenbach, M. F. Becker, Gertrude Blanch, H. F. Bohnenblust, S. G. Bourne, C. B. Boyer, A. T. Brauer, H. W. Brinkmann, H. C. Brodie, A. B. Brown, R. S. Burington, Jewell H. Bushey, S. S. Cairns, D. E. Christie, J. A. Clarkson, M. D. Clement, R. M. Cohn, T. F. Cope, Richard Courant, Ruth Crucet, J. H. Curtiss, M. D. Darkow, Norman Davids, D. R. Davis, Jesse Douglas, Arnold Dresden, Jacques Dutka, J. E. Eaton, M. L. Elbeback, Paul Erdös, F. H. Escott, J. M. Feld, Aaron Fialkow, W. B. Fite, Edward Fleisher, A. I. Forsythe, J. S. Frame, K. O. Friedrichs, H. P. Geiringer, B. P. Gill, Saul Gorn, L. M. Graves, M. C. Gray, C. C. Grove, E. J. Gumbel, Margaret Gurney, W. W. Gutzman, D. W. Hall, F. C. Hall, N. A. Hall, Hans-Karl Hammer, D. C. Harkin, M. H. Heins, Edward Helly, Olaf Helmer, E. H. C. Hildebrandt, L. S. Hill, Banesh Hoffmann, T. R. Hollcroft, M. W. Hopkins, Harold Hotelling, E. M. Hull, R. P. Isaacs, S. A. Joffe, R. A. Johnson, B. W. Jones, Shizuo Kakutani, G. K. Kalisch, Edward Kasner, L. S. Kennison, A. R. Kirby, J. R. Kline, E. R. Kolchin, B. O. Koopman, Arthur Korn, Jack Laderman, M. E. Ladue, Solomon Lefschetz, B. A. Lengyel, Howard Levi, Marie Litzinger, E. R. Lorch, L. A. Lorch, Jack Lorell, A. N. Lowan, Eugene Lukacs, L. A. MacColl, C. C. MacDuffee, H. F. MacNeish, Dorothy Maharam, P. T. Maker, A. J. Maria, D. H. Maria, Walther Mayer, H. L. Mintzer, A. K. Mitchell, Don Mittleman, Morris Monsky, Marston Morse, F. J. Murray, C. A. Nelson, O. E. Neugebauer, J. C. Oxtoby, Everett Pitcher, E. L. Post, Walter Prenowitz, M. H. Protter, R. G. Putnam, H. A. Rademacher, C. J. Rees, M. S. Rees, R. W. Rempfer, H. J. Riblet, Moses Richardson, M. S. Robertson, S. L. Robinson, Benjamin Rosenbaum, J. E. Rosenthal, Arthur Sard, L. J. Savage, Rubin Schatten, Henry Scheffé, I. J. Schoenberg, L. W. Sheridan, Seymour Sherman, Max Shiffman, L. P. Siceloff, James Singer, L. L. Smail, D. M. Smiley, M. F. Smiley, P. A. Smith, Ernst Snapper, Virgil Snyder, Abraham Spitzbart, Wolfgang Sternberg, J. J. Stoker, W. C. Strodt, Alvin Sugar, M. M. Sullivan, Fred Supnick, J. D. Tamarkin, R. M. Thrall, W. R. Transue, W. J. Trjitzinsky, J. W. Tukey, J. L. Vanderslice, Oswald Veblen, G. C. Webber, Louis Weisner, G. W. Whaples, O. L. Wheeler, H. S. White, Hassler Whitney, John Williamson, Audrey Wishard, Jacob Wolfowitz, M. A. Woodbury, Leo Zippin, Antoni Zygmund.

The meeting opened in the morning with a general session for contributed papers at which Professor W. J. Trjitzinsky presided. President Marston Morse presided at the afternoon session which began with a short business meeting for consideration of the amendment of Section 9 of Article VII of the By-Laws proposed by the Council. The amendment was adopted. The section as amended is given below:

Section 9. Any member not in arrears of dues may become a life member on the payment of a sum to be determined in accordance with actuarial principles. A life member shall have for life the status and privileges of an ordinary member, withou further payment of dues. After October 25, 1941, no additional applications for lif memberships will be accepted.

After the business meeting Professor I. J. Schoenberg gave an address entitled *Some analytical aspects of isometric imbedding.*

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 9 were read at the morning session; those numbered 10 to 27 were read by title. Dr. Bers was introduced by Dr. A. M. Gelbart.

1. Rubin Schatten: *On the direct product of Banach spaces.* (Abstract 47-11-476.)

2. Walter Prenowitz: *Descriptive geometries as multigroups.* (Abstract 47-11-481.)

3. E. F. Beckenbach: *On the analytic prolongation of a minimal surface.* (Abstract 47-9-418.)

4. S. S. Cairns: *The space of a variable geodesic complex on a sphere.* (Abstract 47-9-438.)

5. J. H. Curtiss: *On the distribution of the quotient of two chance variables.* (Abstract 47-9-434.)

6. E. J. Gumbel: *Simple tests for given statistical hypotheses.* (Abstract 47-11-485.)

7. Arthur Korn: *On the problem of pulsating spheres in a liquid.* (Abstract 47-11-478.)

8. Ernst Snapper: *Co-maximal linear sets and products of linear sets.* (Abstract 47-11-460.)

9. B. W. Jones: *The number of classes in related genera of quadratic forms.* (Abstract 48-1-13.)

10. Lipman Bers: *A convergence theorem for analytic functions of two variables.* (Abstract 47-11-467-*t*.)

11. D. G. Bourgin: *Linear topological spaces.* (Abstract 47-9-437-*t*.)

12. A. B. Brown: *Independent parameters for sets of functions.* (Abstract 47-11-468-*t*.)

13. S. S. Cairns: *Topological mapping of a Brouwer 4-manifold on an analytic Riemannian 4-manifold.* (Abstract 47-9-439-*t*.)

14. A. D. Campbell: *On the application of an algebra of sets to group theory.* (Abstract 47-11-455-*t*.)

15. George Comenetz: *Isotropic curves on a surface.* (Abstract 47-11-479-*t*.)

16. J. J. DeCicco: *Bi-isothermal systems of curves.* (Abstract 47-11-480-*t*.)

17. Howard Levi: *A characterization of polynomial rings by means of order relations*. (Abstract 47-11-458-*t*.)

18. A. N. Lowan, Norman Davids and Arthur Levinson: *Tables for Gauss' mechanical quadrature formula*. (Abstract 47-9-413-*t*.)

19. A. N. Lowan and William Horenstein: *On the function* $H(m, a, x) = \exp(-ix)F(m+1-ia, 2m+2, 2ix)$. (Abstract 47-11-470-*t*.)

20. R. E. O'Connor: *A theorem of Diophantine approximation and an application to the values of a linear form*. (Abstract 47-11-462-*t*.)

21. R. E. O'Connor: *Representation of integers by power-products of two integers*. (Abstract 47-11-463-*t*.)

22. R. E. O'Connor: *Representation of integers by power-products of two real numbers*. (Abstract 47-11-464-*t*.)

23. Harry Pollard: *Legendre polynomials and the Hausdorff moment problem*. (Abstract 47-9-400-*t*.)

24. Maxwell Reade and E. F. Beckenbach: *Square averages and a class of harmonic polynomials*. (Abstract 47-11-474-*t*.)

25. J. F. Ritt: *Complete difference ideals*. (Abstract 47-11-475-*t*.)

26. A. R. Schweitzer: *Concerning general abstract relational spaces*. III. (Abstract 47-9-430-*t*.)

27. H. S. Wall: *The behavior of certain Stieltjes continued fractions near the singular line*. (Abstract 47-11-477-*t*.)

T. R. HOLLCROFT,
Associate Secretary

THE NOVEMBER MEETING IN MANHATTAN

The three hundred eighty-second meeting of the American Mathematical Society was held at Kansas State College, Manhattan, Kansas, on Friday and Saturday, November 21–22, 1941. The attendance was about eighty including the following forty-seven members of the Society:

A. A. Albert, M. N. Arnoldy, W. L. Ayres, R. W. Babcock, Wealthy Babcock, F. L. Black, L. M. Blumenthal, W. C. Brenke, C. C. Camp, R. D. Daugherty, H. P. Doole, Paul Eberhart, W. E. Ferguson, Cornelius Gouwens, O. H. Hamilton, J. J. L. Hinrichsen, D. L. Holl, W. C. Janes, H. E. Jordan, D. E. Kibbey, E. H. Larguier, P. E. Lewis, A. T. Lonseth, J. V. McKelvey, Anna Marm, U. G. Mitchell, T. A. Mossman, G. C. Munro, E. N. Oberg, R. S. Pate, T. A. Pierce, P. S. Pretz, G. B. Price P. G. Robinson, D. H. Rock, L. L. Runge, G. E. Schweigert, C. E. Sealander, M. E Shanks, D. T. Sigley, E. R. Smith, G. W. Smith, E. B. Stouffer, W. T. Stratton C. B. Tucker, L. E. Ward, J. W. T. Youngs.

The meeting opened Friday afternoon in Mathematics Hall with a session for short papers, Professor W. T. Stratton presiding. Following this, Professor G. B. Price gave an address entitled *Linear extensions*. Professor L. M. Blumenthal presided at this lecture. On Saturday morning in Willard Hall with Professor C. C. Camp presiding, Professor A. A. Albert gave an address with the title *Some mathematical aspects of cryptography*. The meeting closed with a second session of contributed papers with Dean E. B. Stouffer presiding.

The dinner on Friday evening was held at the Country Club on a hill with a fine view of Manhattan and the surrounding country. The number attending the dinner was eighty-two. Professor W. T. Stratton presided at the dinner. After musical selections by Miss Hilda Grossman, President F. D. Farrell of Kansas State College welcomed the mathematicians and Associate Secretary W. L. Ayres responded for the Society. Professor U. G. Mitchell spoke on mathematics as an intellectual invention and urged the training of mathematicians for the social sciences. Dean R. W. Babcock talked on mathematical contemporaries urging that students be taught to look on mathematics as a living science and be told of living mathematicians.

Titles and cross references to the abstracts of the papers read at this meeting follow below. Papers 1 to 5 were read Friday afternoon, papers 6 to 10 on Saturday morning, papers 11 to 13, whose abstract numbers are followed by the letter *t*, were read by title. Mr. R. H. Tripp and Mr. A. E. Engelbrecht were introduced by Professor D. L. Holl.

1. D. L. Holl: *Consolidation of elastic earth layers*. (Abstract 48-1-68.)

2. A. A. Albert: *Quadratic forms permitting composition.* (Abstract 8-1-1.)

3. J. W. T. Youngs: *Concerning arc-curves and basic sets.* (Abstract 47-9-453.)

4. M. E. Shanks: *Irreducible transformations of the discontinuum into continua.* Preliminary report. (Abstract 48-1-97.)

5. J. J. L. Hinrichsen: *Bounds for the libration points in a restricted problem of n bodies.* (Abstract 48-1-67.)

6. L. M. Blumenthal: *Metric characterisation of n-dimensional elliptic space* $\mathcal{E}_{nr}$. Preliminary report. (Abstract 48-1-75.)

7. R. S. Pate: *Functional homomorphisms.* I. Preliminary report. (Abstract 48-1-17.)

8. G. C. Munro: *Systems of linear differential equations with constant coefficients.* (Abstract 48-1-45.)

9. R. H. Tripp: *Bending of a thin plate having the form of a parallelogram.* (Abstract 48-1-73.)

10. A. E. Engelbrecht: *Circular plates with large deflections.* (Abstract 48-1-63.)

11. C. V. Robinson: *Spherical theorems of Helly type and congruence indices of spherical caps.* (Abstract 47-11-482-*t*.)

12. R. H. Bruck and T. L. Wade: *The number of independent components of the tensor* ${}_{[a]}T_{i_1,\cdots,i_p}$. (Abstract 47-11-454-*t*.)

13. Isaac Opatowski: *On the theory of lethal irradiation of micro-organisms.* II. (Abstract 48-1-69-*t*.)

W. L. AYRES,
Associate Secretary

THE NOVEMBER MEETING IN PASADENA

The three hundred eighty-third meeting of the American Mathematical Society was held at the California Institute of Technology, Pasadena, California, on November 22, 1941. The attendance was about seventy, including the following thirty-eight members of the Society:

O. W. Albert, L. D. Ames, R. S. Arthur, H. M. Bacon, Harry Bateman, Clifford Bell, J. L. Brenner, F. A. Butter, P. H. Daus, G. C. Evans, G. E. Forsythe, W. H. Glenn, H. J. Hamilton, Einar Hille, P. G. Hoel, C. G. Jaeger, Glenn James, D. H. Lehmer, R. J. Levit, S. H. Levy, G. F. McEwen, G. W. Mackey, A. D. Michal, W. T. Puckett, W. C. Roberts, G. E. F. Sherwood, D. V. Steed, J. D. Swift, P. M. Swingle, Gabor Szegö, A. E. Taylor, T. Y. Thomas, S. E. Urner, F. A. Valentine, Morgan Ward, W. M. Whyburn, E. R. Worthington, Max Zorn.

General sessions were held in the morning and in the afternoon, the sessions opening at 10 A.M. and 2 P.M., respectively. The morning session was presided over at different times by Professors T. Y. Thomas, Gabor Szegö, and Harry Bateman, and the afternoon session by Professor G. C. Evans.

By invitation of the Program Committee, Professor W. M. Whyburn of the University of California, Los Angeles, delivered an hour address on *Differential systems with general boundary conditions*.

Following the meeting Professor and Mrs. A. D. Michal served tea at their home for members of the Society and their friends.

The titles and cross references to abstracts of papers read at the meeting are given below, abstracts whose numbers are followed by *t* having been read by title. Mr. R. C. James, Mr. Knox Millsaps, and Mr. Edmund Pinney were introduced by Professor A. D. Michal, Mr. S. P. Avann by Professor Morgan Ward, Mr. C. B. Barker by Professor C. B. Morrey, Jr., Dr. O. G. Owens by Professor Hans Lewy, and Mr. R. D. Gordon by Professor Max Zorn.

1. G. W. Mackey: *On the lattice of closed linear subspaces of a normed linear space*. (Abstract 47-11-472.)

2. D. H. Lehmer: *Properties of the coefficients of the modular invariant* $J(\tau)$. (Abstract 47-11-457.)

3. R. J. Levit: *Fields in terms of a single operation*. (Abstract 47-11-459.)

4. Morgan Ward: *Conditions for a lattice to be a Boolean algebra*. (Abstract 48-1-23.)

5. R. C. James: *Linearly arc-wise connected topological groups*. (Abstract 48-1-92.)

6. Knox Millsaps: *First order differentials of functions with arguments in topological groups and values in topological abelian groups.* (Abstract 48-1-96.)

7. Edmund Pinney: *Theory of functions on linear topological spaces to Banach spaces.* (Abstract 48-1-50.)

8. S. P. Avann: *The τ-function and its application to semi-modular lattices.* (Abstract 48-1-3.)

9. P. M. Swingle: *Topological foundations of an uncertain mathematics.* (Abstract 48-1-99.)

10. G. F. McEwen: *On the probability that a ratio of random numbers will depart from a harmonic ratio by less than a given amount.* (Abstract 48-1-89.)

11. C. B. Barker: *The Lagrange multiplier rule for two dependent and two independent variables.* (Abstract 48-1-28.)

12. R. D. Gordon: *An application of the Cauchy integral to a problem in probability theory.* (Abstract 48-1-88.)

13. Einar Hille: *On the oscillation of differential transforms.* II. (Abstract 48-1-37.)

14. R. M. Robinson: *Bounded univalent functions.* (Abstract 48-1-54-*t*.)

15. O. G. Owens: *An explicit formula for the solution of the ultrahyperbolic equation in four independent variables.* (Abstract 48-1-48-*t*.)

16. R. P. Dilworth: *A theorem on lattice homomorphisms.* (Abstract 48-1-10-*t*.)

17. R. P. Dilworth: *Dependence relations in a Birkhoff lattice.* (Abstract 48-1-11-*t*.)

18. Nathaniel Coburn: *Unitary curves in unitary space.* (Abstract 48-1-76-*t*.)

19. R. C. James: *Normable topological abelian groups.* (Abstract 48-1-93-*t*.)

20. František Wolf: *On majorants of analytic functions.* (Abstract 48-1-62-*t*.)

21. S. P. Avann: *Lattices with arbitrary automorphism groups.* (Abstract 48-1-2-*t*.)

22. Gabor Szegö: *On the oscillation of differential transforms.* I. (Abstract 48-1-57-*t*.)

T. M. Putnam,
Associate Secretary

BOOK REVIEWS

Diophantische Gleichungen. By T. Skolem. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 5, Heft 4. Berlin, Springer, 1938. 130 pp. R.M. 15.

Included in the author's preface is this statement, "Hier dagegen ist es versucht worden, eine zusammenfassende Darstellung der Theorie der diophantischen Gleichungen zu geben von den einfachsten bis zu den schwierigsten, die man bis jetzt hat bewaltigen können, aber nur soweit sie allgemeinen Methoden zugänglich sind. Es gibt bekanntlich auf diesem Gebiete sehr viele Untersuchungen über ganz spezielle Gleichungen; auf solche wird nicht eingegangen."

All the equations considered by Skolem, with the exception of several types considered in Chapter II, are of the form

$$f(x_1, x_2, \cdots, x_n) = 0,$$

the left-hand member of the equation being a polynomial in the unknowns $x_1, x_2, \cdots, x_n$ with given integral coefficients, the problem being to find all rational solutions or, in particular, integral solutions. Obviously, not all equations of this type which have been treated in the literature could have been considered in a pamphlet of this size. Hence he has elected, as he states above, to take up mainly what he regards as the most general types of equations concerning which definite results have been found. For example, congruences are special kinds of diophantine equations, but Skolem has wisely omitted any discussion of them, except for certain systems considered on two pages in the first chapter.

Many theorems are stated and proofs of some are given. When the proofs are not given references are usually indicated which enable us to find the demonstrations in the literature.

Diophantine analysis is noteworthy for the great interest which mathematicians have exhibited in certain methods which so far have been applied only to comparatively special equations. A number of these are not mentioned by Skolem, and we think it will be illuminating to refer to them in the course of this review.

In the first chapter of the book, the author considers linear equations of the form

$$\sum_{s=1}^{n} a_{rs}x_s = b_r,$$

$r=1, 2, \cdots, m$, and gives a number of the results of Heger, Smith and Frobenius. This work connects up with the theory of the elementary divisors of a matrix.

In Chapter II Skolem treats equations which are linear in each unknown, in particular the bilinear equation

$$\sum_{rs} a_{rs} x_r y_s = 0.$$

In this general theory it is convenient to employ equations in determinant form.

The third chapter treats quadratic equations in two or more unknowns. The homogeneous case is treated first. Smith's criterion for the integral solution of

$$ax^2 + a_1x_1^2 + a_2x_2^2 + 2bx_1x_2 + 2b_1xx_2 + 2b_2xx_1 = 0$$

is given. Proceeding to the equation

$$\sum_{i=1}^{n} a_i x_i^2 = 0, \tag{1}$$

the beautiful theory for the case $n=3$ is treated. Meyer's results on (1) are stated and also this theorem that (1) for $n=5$ is always satisfied in integers not all zero with integers $a_1, a_2, \cdots, a_5$, not zero, and not all of like sign. The first complete proof is by Dickson.[1] Skolem gives Mordell's proof of this after first giving his own proof of Hasse's results concerning the congruence

$$\sum_{i=1}^{n} a_i x_i^2 \equiv 0 \pmod{m},$$

on which the latter shows the solution of (1) depends, if f is indefinite.

Next the nonhomogeneous quadratic equations are treated. He considers the classic theory of representation of integers by ternary forms, but does not mention a number of theories concerning quadratic forms which have so far attracted great attention from many mathematicians, such as the representation of integers as the sum of squares and closely related topics developed by Liouville, Eisenstein, Minkowski, Smith, Bell, Nasimoff, Hardy, Uspensky, Mordell and others, using the methods of elliptic functions, analytic additive number theory, for example. Also, the application of quaternions and related linear algebras to such questions as developed by A. Hurwitz, Dickson and others is not taken up.

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[1] Studies in the Theory of Numbers, pp. 68–70.

In the fourth chapter entitled "Multiplikative Gleichungen" the author gives, with the proof, the results of Bell and Ward on the equations,

$$A_i x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_k^{a_{ik}} = B_i y_1^{b_{i1}} y_2^{b_{i2}} \cdots y_s^{b_{is}}.$$

He then takes up equations obtained by expanding

$$N(\alpha_1 x_1 + \cdots + \alpha_n x_n)$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are integers in the algebraic field K of degree n, N denotes the norm in K, and setting

$$N = a, \tag{2}$$

where a is a rational integer, and treats by the use of units in K the problem of finding other sets of solutions of (2), having given one set, as well as methods for determining if there exists any set.

He then examines equations of the form

$$f(x_1, x_2, \cdots, x_n) = k y_1^{e_1} y_2^{e_2} \cdots y_m^{e_m},$$

where f is decomposable in some algebraic field. The results of Ward[2] on the equation

$$x^2 - dy^2 = z^m, \qquad d > 0,$$

are given. Ward obtains the primitive solutions of this by use of the field defined by $d^{1/2}$.

Dickson's solution of

$$q = ax^2 + bxy + cy^2 = w_1 \cdots w_n,$$

is mentioned, then a generalization to the equation

$$q = k z_1^{a_1} \cdots z_n^{a_n}$$

is discussed by Skolem. He remarks that the methods he employs extend to the equation,

$$f(x_1, \cdots, x_m) = k z_1^{a_1} \cdots z_n^{a_n}$$

where f is decomposable into m linear factors in some algebraic field, and where the degree of $f > 2$. For the case $k = 1$, $a_1 = \cdots = a_n = 1$, a solution involving the use of ideals was given by Wahlin.

The fifth chapter is concerned with the application of geometry to various types of diophantine equations, when we are seeking rational solutions. It is in this chapter, in the sixth and in the last chapter, that it appears the most general results are to be found in the work. The curves

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[2] Transactions of this Society, vol. 38 (1935), pp. 447–457.

$$f(x, y) = 0, \quad \text{or} \quad f(x, y, z) = 0$$

in homogeneous coordinates are classified for arithmetical purposes according to their genus, the coefficients of the equations being integral. Two curves are called equivalent when they are connected with each other by a birational transformation with rational coefficients. Thus two curves

$$f_1(x, y) = 0, \qquad f_2(\xi, \eta) = 0$$

are equivalent if the coordinates x, y are rationally expressible with rational coefficients in terms of ξ, η, and conversely.

If the curve is of genus zero, it was shown by Poincaré that it is equivalent to a straight line or a conic, from which the theory of the rational points on the original curve is easily derived.

For a curve of genus unity Poincaré also proved that if it has a rational point, it is equivalent to a curve of the third degree, and, in particular, the cubic is equivalent to a curve whose equation can be written in the Weierstrass normal form

$$y^2 = 4x^3 - g_2x - g_3. \tag{3}$$

The latter statement was proved by Mordell in 1912.

There is a well known geometric method for finding rational points on a curve when certain rational points are known initially. If one of the latter points is P (not a point of inflexion), the tangent at P meets the curve in another point P_1 whose coordinates are determined by an equation of the first degree and hence are rational. If Q is another such point, the chord PQ will meet the cubic in a rational point R, in general different from P.

To interpret this process analytically Poincaré and Hurwitz noted that the equation (3) could be given in parametric form by the use of Weierstrass elliptic p functions so that

$$x = p(u), \qquad y = p'(u). \tag{3a}$$

It was proved by Mordell by the use of such ideas that all the rational points on (3) could be found from a finite number of rational points, so that, using (3a), if $u_1, u_2, \cdots, u_n$ satisfy (2) all points are given by $m_1u_1+m_2u_2+ \cdots +m_nu_n$, where the m's are integers, that is, the parametric values for all the rational points form an additive Abelian group with a finite basis.

Skolem mentions the work of Fueter and Brunner on the equation

$$x^3 - y^2 = D.$$

They were able to determine the number of solutions of this equation for various values of D, using the field defined by $(-D)^{1/2}$. The author then considers curves of genus $p > 1$ and notes that it has been conjectured that such curves contain only a finite number of rational points. A curve of this type is given by the Fermat equation

$$x^n + y^n = z^n.$$

In this connection he mentions only the work of Kummer, Wieferich, Mirimanoff, Furtwangler, and Kapferer, on Fermat's last theorem.

A rational system of points is defined as a finite system of points such that all the symmetric functions of their parametric values are rational. Using this idea Weil showed that Mordell's result concerning the parametric values associated with the rational points on a cubic curve forming an additive Abelian group could be extended to curves of genus $p > 1$.

In Chapter VI Skolem goes into detail concerning the theorems associated with the names of Thue and Siegel. These theorems enable us to state that certain equations of the type

$$f(x, y) = c,$$

have only a finite number of solutions. These ideas seem to have had their beginnings in the work of Runge (1887), who proved among other results the following one:

Let $f(x, y)$ be a polynomial with integral coefficients of degree n irreducible in the rational field, the homogeneous part of f of highest degree not being the power of an irreducible polynomial. Then $f(x, y) = c$ has only a finite number of solutions in rational integers.

Thue's theorem of the year 1909 has to do with a polynomial of a type a bit different from that treated in Runge's theorem. It is as follows:

If $f(x, y)$ is a homogeneous irreducible polynomial of degree greater than 2 with integral coefficients and c is a given integer not equal to 0, *the equation $f(x, y) = c$ has only a finite number of integral solutions.*

The author proves this theorem as well as a number of lemmas concerning the approximation to a real algebraic number by the use of a rational number, on which these investigations depend. Using these results it is possible to show that the equation

$$ay^2 + by + c = dx^n,$$
$$a(b^2 - 4ac)d \neq 0, \qquad n \geqq 3,$$

has only a finite number of solutions as well as the equation

$$y^2 = ax^n + bx^{n-1} + \cdots + k$$

where, in the latter, the right-hand member has at least three different zeros.

After considering a number of special equations of the form

$$x^n + dy^n = \pm 1$$

where $n=3$ or 4, Skolem applies the theory of p-adic numbers to the equation

$$N(\alpha x + \beta y + \gamma z) = k$$

where α, β and γ are integers in an algebraic field K of degree n. He finds equations of this type which have only a finite number of solutions for $n=5$.

We now signalize a problem which seems fundamental in this subject. If we consider the irreducible equation

$$f(x_1, x_2, \cdots, x_k) = c$$

where f is of degree n with integral coefficients and with c integral and also f homogeneous we know from the theory of units in an algebraic field that for $k=n$ and $c=1$, there exist equations of this type with an infinity of integral solutions. On the other hand, if $k=2$, $n>2$, Thue's theorem states that there cannot be more than a finite number of solutions. The question is, how far must k be increased to obtain equations of this type with an infinity of solutions? If $n=3$, we have $k=3$.

The arithmetical theory of Hermitian forms is not considered, likewise Waring's theorem. It is not exactly surprising that the latter topic has been omitted, as it would merit a volume in itself.

Skolem has written a very interesting book. It is surprising how much arithmetical meat he has packed into the space he employs.

H. S. VANDIVER

Sur les Fonctions Orthogonales de Plusieurs Variables Complexes, avec les Applications à la Théorie des Fonctions Analytiques. By Stefan Bergman. New York, Interscience Publishers, 1941. 62 pp. $1.50.

This book was to appear as one of the Mémorial des Sciences Mathématiques series, but circumstances were such that the edition reached only the galley proof stage. The book is a photostatic edition made from the galley proofs. As a result, the characters are sometimes a little blurred and in some places the lines are uneven.

At the outset the author states that many important results in the theory of one complex variable cannot be extended to functions of several complex variables. To find a technique analogous to that of one variable it is necessary to introduce a new approach. One such approach is the method of orthogonal functions for n variables in a domain B. In this book, however, the author limits himself to the properties of functions of two complex variables. The methods used and the results obtained can be extended to functions of several complex variables. The power of the method of orthogonal functions lies in the following properties of the kernel function $\sum\phi^{(\nu)}(z_1, z_2) \cdot\overline{\phi^{(\nu)}(t_1, t_2)} = K_B(z_1, z_2, \bar{t}_1, \bar{t}_2)$, established by the author, of a system of functions closed for the class, $F(B)$, of functions analytic and of summable square in B. First, the function $K_B(z, \bar{t})$, $z=\{z_1, z_2\}$, $\bar{t}=\{\bar{t}_1, \bar{t}_2\}$ is regular in the variables $z_1, z_2, \bar{t}_1, \bar{t}_2$. Second, $K_B(z, \bar{t})$ is independent of the particular choice of the closed system $\{\phi^{(\nu)}\}$. Third, the minimum value of the integral $\int_B |h|^2 dw$, $dw = dx_1 dy_1 dx_2 dy_2$, where h belongs to $F(B)$, and $h(t_1, t_2)=1$, that is, h is normalized at the point (t_1, t_2), is equal to $[K_B(t, \bar{t})]^{-1}=\lambda_B(t)$, and this minimum is attained for $h=f=K_B(z, \bar{t})/K_B(t, \bar{t})$. It is from these properties that he builds up a theory of orthogonal functions for two complex variables and uses this to deal with analytic functions of two complex variables. This method of the kernel function is very neatly applied with much success to the study of pseudo-conformal transformations. It is extremely interesting that these results can also be applied to the study of linear partial differential equations.

This theory of conformal transformations is essentially based on the Riemannian mapping theorem which states that every simply connected domain possessing at least two boundary points can be mapped conformally into a circle, and the Schwarz lemma. This lemma, as formulated by Pick, states that if a transformation takes the unit circle into its interior, then the hyperbolic length of a line segment included in the unit circle cannot be increased.

Transformations of the domains in four-dimensional space by a pair of analytic functions are called pseudo-conformal transformations. As was first pointed out by Poincaré, two arbitrary simply-connected four-dimensional domains cannot be transformed one into the other by a pseudo-conformal transformation. Also Hartogs has shown that every analytic function regular in a simply connected four-dimensional region assumes on the boundary of this region all the values it assumes in the interior, that is, a function cannot be constructed that vanishes at some point in the interior of the domain B and has an absolute value of one on the boundary. An immediate generalization of the Schwarz-Pick lemma therefore cannot be obtained for functions of two complex variables. Therefore, analogous methods to those described above cannot be applied in the case of functions of two complex variables.

To overcome these difficulties the author employs the kernel function $K_B(t, \bar{t})$. In the case of conformal transformations $ds^2 = K_B(z, \bar{z})|dz|^2$ defines a metric which is invariant with respect to conformal transformations. If the domain is simply connected, the metric becomes a hyperbolic metric. Analogously, using the kernel function for functions of two complex variables, the author defines a metric which is invariant with respect to pseudo-conformal transformations. It is clear that if a domain $G \subset B$ then $\lambda_G(t) \leqq \lambda_B(t)$, and since λ is the inverse of the kernel K, $K_G(z, \bar{z}) \geqq K_B(z, \bar{z})$ and hence the metric $K_G(z, \bar{z})|dz|^2 \geqq K_B(z, \bar{z})|dz|^2$. This is a form of the Schwarz-Pick lemma.

The book consists of an introduction and four chapters. In the introduction the author describes his approach to the problem of conformal mapping using orthogonal functions.

Chapter I deals with the characteristics of the space of two complex variables. The author describes domains in the z_1, z_2-space which are of particular importance in the study of analytic functions. In particular, he deals with analytic surfaces, hypersurfaces, and domains with distinguished boundary surfaces. These geometrical studies are not employed in the book, but without an understanding of them one might not fully appreciate the value of the results obtained. These geometrical properties, however, are of fundamental importance in quite another aspect of the theory of functions of two complex variables that the author has developed in other papers. There are several points in Chapter I that would be of particular interest to differential geometers.

Chapter II, which forms the heart of the book, is devoted to the properties of orthogonal functions. The differences between real and analytic orthogonal functions are stressed. The essential and interesting theorem is that the kernel function of complex analytic functions converges at every interior point of the domain. The connection between the theory of orthogonal functions and conformal mapping is described. Special systems of orthogonal functions are also studied.

In Chapter III the author concerns himself with minima problems of the type described above. The problem of interpolation and the application of orthogonal functions to the theory of entire and meromorphic functions are treated. But here the work is sketchy, and in some places only references to papers are made. Furthermore, the author does not indicate very clearly what role the theory of orthogonal functions plays in the study of entire and meromorphic functions. This, of course, could not very well have been accomplished in the space devoted to this topic.

In Chapter IV some further properties of the invariant metric are given, and the differential geometric properties of the Hermitian metric are described.

A companion book to this one was to be written dealing with the invariant metric. We hope that this interesting book will soon appear in this country.

This book is essentially a collection of results by the author and other people working in this field. It is to be regretted that some of the value of this book, as a book, is diminished by the lack of elaboration. In many places results are merely stated without proof. However, a fairly complete bibliography is inserted at the end, and in all instances of the text, references are made to original papers. The material of this book is new and interesting, and it appears that this field is by no means exhausted. Anyone interested in this field would find it very stimulating to read this book.

ABE GELBART

An Introduction to Differential Geometry with Use of the Tensor Calculus. By Luther Pfahler Eisenhart. Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press. 1940. 10+304 pages. $3.50.

The author's *Differential Geometry of Curves and Surfaces*, which was published in 1909, has seen extensive use. Since that time, the tensor calculus has come to play an important role in Riemannian Geometry and in the Theory of Relativity and the author has considered it desirable to rewrite the differential geometry of curves and surfaces in terms of the tensor calculus. The differential geometry treated in the present book is about equivalent to that in the first half of the 1909 book, with the addition of the concept of parallelism in the sense of Levi-Civita.

There are four chapters. The first is concerned principally with the properties of curves in euclidean 3-space, but includes a definition of the parametric equations of a surface and the envelope of a one-parameter family of surfaces, in order to include consideration of the developables associated with a curve. The material of the chapter is for the most part included in the first two chapters of the 1909 book and except for changes in notation due to the use of the summation convention, the order and treatment have not been greatly changed. The principal addition is consideration of the theorem that the envelope of a one-parameter family of non-parallel planes is either a cone, a cylinder, or the tangent surface of a curve.

The second chapter is largely devoted to tensor algebra and calculus. Preparatory to the definition of a tensor, transformations of coordinates in euclidean 3-space and the effects on the fundamental quadratic form of such transformations are considered. Then contravariant and covariant vectors, and scalars, are defined and their geometrical significance is discussed. The definition of a tensor and the development of the algebra of tensors follow. The Christoffel symbols and Riemann tensor are introduced; covariant differentiation is defined; and the chapter closes with a classic existence theorem for a system of partial differential equations.

Chapter III deals with the intrinsic properties of a surface, that is, those properties of a surface which depend only on the coefficients of the first fundamental form. In addition to the fundamental concepts of arc length, angle and area, the concepts of Gaussian curvature, geodesic, geodesic curvature and parallelism in the sense of Levi-Civita are defined by intrinsic methods and studied in detail. Additional material includes the notions of isometric nets, isometric surfaces, differential parameters, conformal correspondence and geodesic correspondence.

Chapter IV deals with properties which depend on the imbedding of the surface in space, or upon the coefficients of the second fundamental quadratic form. The principal topics are: the equations of Gauss and Codazzi; principal radii of curvature and lines of curvature; conjugate nets and asymptotic lines; the spherical representation of a surface; tangential coordinates and lines of curvature. The chapter closes with a brief discussion of spherical and pseudo-spherical surfaces, and minimal surfaces.

The book is well organized, well written, and has excellent illustrations. There is an abundance of exercises of all varieties, some of which illustrate the material treated, others of which extend the theory. A student who has absorbed the contents of the book will know the fundamentals of differential geometry and tensor calculus and should find the passage to the study of n-dimensional Riemannian geometry easy.

The reviewer is not certain that the present text offers the most desirable introduction to the differential geometry of surfaces in three-dimensional euclidean space. The extensive use of tensors brings with it a needed emphasis on the notion of invariance. In the present case it also brings with it an order and an emphasis on symbolic manipulation which tend to obscure the geometry of the situation. For example, the Gaussian curvature is first met as a function depending on the Riemann tensor, which has been previously defined for no very obvious reason. The geometrical significance of the Gaussian curvature becomes clear only much later. As another example, principal directions for a tensor not the fundamental tensor are introduced abruptly and we must wait many pages before it becomes clear as to why anybody thought of them.

The author of a book on differential geometry faces vexing questions as regards analytical rigor. Since functions enter continually in the subject matter, a first question concerns the class (differentiability) of the functions involved. Many authors restrict their considerations to analytic functions, a method which assumes too much but which avoids pitfalls. In the present book we find the following statement on page 3: "Whenever throughout this book we consider any function, it is considered in a domain within which it is continuous in all its variables, together with such of its derivatives as are involved in the discussion." This process of asking the reader to determine the hypotheses of a theorem from the proof can be somewhat misleading. Since the proofs of theorems usually involve other theorems and since these in turn depend on still other theorems, the reader may find it difficult to determine just what his hypotheses are. Since functions are frequently expanded in series, there would appear to be the possibility of confusion between Class C^∞ and analytic. Since results obtained under the assumption of analyticity are used in the derivation of various theorems, these theorems are valid only under the assumption of analyticity, and yet no statement concerning analyticity appears in the proofs of these theorems. It seems to the reviewer that it would have been a happier situation if in each theorem the necessary class of the functions involved had been stated.

There are a number of other analytical details which appear objectionable to the reviewer, but in view of the fact that Fine's *Calculus* and the author's *Coordinate Geometry* are extensively used as references, it would appear that the attainments of the reader are not assumed to be extensive, and consequently the point of rigor should perhaps not be stressed.

Professor Eisenhart's new book is an interesting addition to the literature on differential geometry. It should prove to be highly useful.

GUSTAV A. HEDLUND

Mathematical Logic. By W. V. Quine. New York, Norton, 1940. 8 +348 pp. $4.00.

Unfortunately, the system of logic presented in this book admits the Burali-Forti paradox. This admission renders the system inconsistent. As a result the book fails in its primary purpose and will need serious revision.

Not all portions of the book are affected by the paradox. In particular, the first three and last of the book's seven chapters can survive unchanged (except for very minor details in Chapter 7). Also, much valuable material can be salvaged from the three affected chapters, notably major portions of §§38–40, 43, 44, 47–52.

Quine's first chapter deals with the propositional calculus, the second with the theory of quantification, and the third with certain aspects of the theory of classes, including the theories of identity and description. The treatment of these subjects is very thorough, with good explanations, so that for these subjects the book can be recommended either as a text or as a reference book.

Quine's seventh chapter contains a new proof of Gödel's theorem on the existence of undecidable propositions. The novel feature of Quine's proof is the following. Let L be a logic with a denumerable number of symbols. We can think of L as a system involving only two symbols, namely "x" and an accent, by replacing the denumerable symbols of L by x, x', x'', x''', $\cdots$. So there is no loss of generality in assuming that L has a finite number of symbols, namely $S_1, S_2, \cdots, S_n$. Let expressions of L be finite sequences of the S's (allowing multiple uses). Let us endow $Mxyz$ with the meaning: if x is a single symbol, then x is the next symbol after y in the list $S_1, S_2, \cdots, S_n$; if x is a complex expression, then x is the result of writing y followed by z; if x is not an expression then $x=y$. Then, as Quine shows in detail, a symbolism based on M, four variables, an accent (for producing more variables), the stroke function, and parentheses (for universal quantification) suffices for the usual syntactical discussions about L. In particular, it suffices to define "provable formula" for L. If we take the S's to be the nine symbols of the M-system, then the M-system suffices for its own syntax, and the Gödel theorem follows readily.

BARKLEY ROSSER

p17561

NOTES

The editors of the Bulletin wish to make grateful public acknowledgment of the services rendered by the following persons who have refereed papers: C. R. Adams, R. P. Agnew, Emil Artin, Reinhold Baer, Harry Bateman, E. T. Bell, P. O. Bell, Garrett Birkhoff, L. M. Blumenthal, R. P. Boas, H. F. Bohnenblust, D. G. Bourgin, A. T. Brauer, Richard Brauer, A. B. Brown, Herbert Busemann, E. W. Chittenden, R. V. Churchill, J. A. Clarkson, George Comenetz, A. H. Copeland, Richard Courant, H. S. M. Coxeter, D. R. Curtiss, M. M. Day, J. J. DeCicco, D. M. Dribin, Samuel Eilenberg, Arnold Emch, Aaron Fialkow, W. W. Flexner, Orrin Frink, H. L. Garabedian, J. W. Givens, Saul Gorn, D. W. Hall, Marshall Hall, O. G. Harrold, Einar Hille, Ralph Hull, Dunham Jackson, Nathan Jacobson, R. D. James, B. W. Jones, Mark Kac, Irving Kaplansky, M. S. Knebelman, R. E. Langer, C. G. Latimer, D. H. Lehmer, Walter Leighton, Norman Levinson, E. R. Lorch, E. J. McShane, Saunders MacLane, W. G. Madow, Szolem Mandelbrojt, Morris Marden, G. M. Merriman, E. W. Miller, W. E. Milne, Deane Montgomery, C. B. Morrey, F. J. Murray, John von Neumann, Ivan Niven, Oystein Ore, Gordon Pall, G. H. Peebles, Sam Perlis, R. S. Phillips, George Pólya, Hillel Poritsky, Anatol Rappoport, W. T. Reid, Arthur Rosenthal, A. E. Ross, A. C. Schaeffer, O. F. G. Schilling, I. J. Schoenberg, I. M. Sheffer, N. E. Steenrod, Otto Szász, Gabor Szegö, J. W. Tukey, J. V. Uspensky, R. J. Walker, H. S. Wall, J. L. Walsh, L. E. Ward, L. R. Wilcox, Aurel Wintner, Leo Zippin, Max Zorn, and Antoni Zygmund.

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Board, who have been consulted regarding papers offered for publication in volumes 49 and 50: Garrett Birkhoff, H. R. Brahana, Richard Brauer, H. E. Buchanan, H. B. Curry, Samuel Eilenberg, C. J. Everett, W. K. Feller, L. M. Graves, M. H. Ingraham, Shizuo Kakutani, M. S. Knebelman, B. O. Koopman, D. H. Lehmer, Hans Lewy, N. H. McCoy, E. J. McShane, H. M. MacNeille, F. J. Murray, B. J. Pettis, George Pólya, G. B. Price, H. A. Rademacher, W. C. Randels, P. V. Reichelderfer, A. C. Schaeffer, Peter Scherk, O. F. G. Schilling, Wladimir Seidel, J. J. Stoker, M. H. Stone, D. J. Struik, J. D. Tamarkin, J. M. Thomas, and Louis Weisner.

The summer session arranged at Brown University for advanced instruction and research in mechanics, and for which sixty students were accepted, met with such general approval that the program is being continued during the academic year 1941–1942. Courses are being given in (1) Advanced Dynamics, (2) Elasticity, (3) Fluid Dynamics, (4) Theory of Airflight, (5) Graphical and Numerical Methods in Applied Mathematics, (6) Partial Differential Equations; in addition there are two seminars. More than thirty persons are enrolled. On the staff are Professors J. L. Synge, Richard von Mises, J. D. Tamarkin, Willy Feller, Stefan Bergman, and Willy Prager. The instruction is supported by the Engineering, Sciences, and Management Defense Training Program of the United States Office of Education and by the Carnegie Corporation of New York. A few small fellowships are still available for the second semester. Those desiring information should communicate with the Dean of the Graduate School, Brown University, Providence, Rhode Island.

Yale University offers each year a number of Sterling research fellowships, some of them open to mathematicians. For the year 1942–1943 the normal stipend will be $1800. Applicants must have received the degree of Doctor of Philosophy or its equivalent from approved universities in the United States or foreign countries. In general, preference is given to applicants of not more than thirty-five years of age, but older persons are eligible. Applications should be received before March 1; blanks may be obtained from the Graduate School office, Yale University, New Haven, Connecticut.

Professor T. C. Esty of Amherst College has been given the title professor emeritus.

Mr. L. M. Klauber, vice president and general manager of the San Diego Gas and Electric Company, received an honorary doctorate of laws from the University of California at Los Angeles.

Mr. W. G. Banks of Centenary College has been promoted to an assistant professorship.

Associate Professor J. L. Barnes of Tufts College has been promoted to a professorship.

Dr. P. G. Bergmann has been appointed to an assistant professorship at Black Mountain College, Black Mountain, North Carolina.

Assistant Professor J. G. Bowker of Middlebury College has been promoted to an associate professorship.

Assistant Professor Fannie W. Boyce of Wheaton College, Wheaton, Illinois, has been promoted to an associate professorship.

Dr. J. R. Britton of the University of Colorado has been promoted to an assistant professorship.

Assistant Professor E. A. Cameron of the University of North Carolina has been promoted to an associate professorship.

Dr. Harold Chatland of Montana State University has been promoted to an assistant professorship.

Dr. Max Coral of Wayne University has been promoted to an assistant professorship.

Associate Professor H. B. Curry of Pennsylvania State College has been promoted to a professorship.

Miss Rachel Davison of Houghton College, Houghton, New York, has been promoted to an associate professorship.

Dr. Douglas Derry of the University of Saskatchewan has been promoted to an assistant professorship.

Assistant Professor J. L. Doob of the University of Illinois has been promoted to an associate professorship.

Assistant Professor E. D. Eaves of the University of Tennessee has been promoted to an associate professorship.

Mr. W. F. Eberlein is working as propellor research analyst in the Navy Department.

Associate Professor Nat Edmonson of the Agricultural and Mechanical College of Texas has been promoted to a professorship.

Associate Professor P. D. Edwards of Ball State Teachers College has been promoted to a professorship.

Dr. C. H. Frick of Montana State College has been appointed professor of mathematics at Mary Washington College, Fredericksburg, Virginia.

Assistant Professor D. F. Gunder of the Colorado State College of Agriculture and Mechanic Arts has been promoted to an associate professorship.

Dr. Mary E. Haller of the University of Washington has been promoted to an assistant professorship.

Mr. George Hartnell of the Cheltenham Magnetic Observatory has retired.

Assistant Professor M. A. Heaslet of San José State College has been promoted to an associate professorship.

Miss Gertrude Hendrix of Eastern Illinois State Teachers College has been promoted to an assistant professorship.

Dr. P. G. Hoel of the University of California at Los Angeles has been promoted to an assistant professorship.

Mr. C. W. Hook of Georgia School of Technology has been promoted to an assistant professorship.

Dr. S. B. Jackson of the University of Wisconsin has been appointed to an assistant professorship at the University of Maryland.

Associate Professor G. R. Livingston of San Diego State College has been promoted to a professorship.

Dr. E. R. Lorch of Barnard College, Columbia University, has been promoted to an assistant professorship.

Associate Professor W. G. McGavock of Davidson College has been promoted to a professorship.

Associate Professor H. M. MacNeille of Kenyon College has been promoted to a professorship.

Dr. G. T. Miller of Purdue University has been promoted to an assistant professorship.

Mr. G. R. Mirick has been appointed principal of the high school at Lincoln School of Teachers College, Columbia University.

Dr. A. P. Morse of the University of California has been promoted to an assistant professorship.

Professor L. R. Perkins of Middlebury College has resigned because of ill health.

Dr. George Pólya who has been at Brown University as visiting professor has been appointed to a visiting professorship at Smith College.

Dr. Willy Prager who recently held a professorship of aeronautical theory at the University of Istanbul and who was formerly director of the Aerodynamical Institute at Göttingen has been appointed to a professorship of mechanical engineering at Brown University.

Assistant Professor A. W. Recht of the University of Denver has been promoted to an associate professorship.

Professor H. W. Reddick of Cooper Union is on leave of absence to direct the Defense Training Institute, Brooklyn, New York. Assistant Professor F. H. Miller has been named acting head of the department of mathematics at Cooper Union.

Dr. H. J. Riblet of Adelphi College, Garden City, New York, has been appointed to an assistant professorship at Hofstra College, Hempstead, New York.

Dr. L. D. Rodabaugh has been appointed to an assistant professorship at Butler University.

Dr. M. A. Sadowsky of Illinois Institute of Technology has been promoted to an assistant professorship.

Dr. Raphaël Salem has been appointed to a lectureship at Massachusetts Institute of Technology.

Assistant Professor Nathan Schwid of the Texas College of Mines has been promoted to an associate professorship.

Dr. M. F. Smiley of Lehigh University has been promoted to an assistant professorship.

Assistant Professor C. E. Smith of San Diego State College has been promoted to an associate professorship.

Associate Professor C. E. Springer of the University of Oklahoma has been promoted to a professorship.

Mr. H. S. Stanley has been appointed to an associate professorship at the University of South Carolina.

Assistant Professor W. R. Talbot of Lincoln University, Jefferson City, Missouri, has been promoted to an associate professorship.

Assistant Professor E. W. Titt of the University of Maryland has been appointed to an associate professorship at the University of Texas.

Dr. C. B. Tompkins of Princeton University has been appointed to a visiting lectureship at the University of Wisconsin.

Dr. E. P. Vance of the University of Nevada has been promoted to an assistant professorship.

Dr. C. E. Van Orstrand of the U. S. Geological Survey Department has retired.

Assistant Professor R. W. Veatch of the University of Tulsa has been promoted to an associate professorship.

Associate Professor J. A. Ward of Delta State Teachers College has been promoted to a professorship.

Dr. Alexander Weinstein has been appointed lecturer in the department of applied mathematics at the University of Toronto for the present academic year.

Brother Bernard Alfred Welch of Manhattan College has been promoted to an assistant professorship.

Assistant Professor E. D. Wells of the University of Pittsburgh, Erie Center, has been promoted to an associate professorship.

Dr. John Williamson of Johns Hopkins University has been appointed to an associate professorship at Queens College.

Professor R. S. Burington of Case School of Applied Science has been given leave of absence to continue work in the Bureau of Ordnance of the Navy Department.

Dr. F. W. Dresch of the University of California has been granted leave of absence for 1941–1942 for military service and is with the Bureau of Ordnance, Navy Department.

Associate Professor H. T. Engstrom of Yale University is on leave this year to serve in the Navy Department.

Professor H. H. Ferns of the University of Saskatchewan has been granted leave of absence to serve in the Canadian Officers Training Corps.

Assistant Professor A. L. Foster of the University of California has been granted leave of absence for the spring of 1942.

Dr. Marian W. Thornton is on leave of absence from the Minnesota State Department of Health.

Dr. I. E. Segal has been appointed research assistant at Princeton University.

The following appointments to instructorships are announced: Acadia University: Dr. D. S. Miller; Ground School, Air Corps Advanced Flying School, Kelly Field, Texas: Dr. E. G. H. Comfort; University of Alabama: Dr. Albert Neuhaus; Brown University: Dr. K. L. Nielsen; University of California: Dr. R. W. Shephard; Case

School of Applied Science: Dr. R. H. Sorgenfrey; College of the City of New York: Dr. Olaf Helmer; University of Connecticut: Dr. C. G. A. Nordling; De Paul University: Mr. A. R. Craw; Frostburg State Teachers College, Frostburg, Maryland: Dr. W. N. Hallett; Harvard University: Dr. R. S. Phillips; Hunter College: Dr. W. W. Gutzman; University of Illinois: Dr. H. M. Schwartz; Indiana University Extension, East Chicago, Indiana: Mr. H. E. Burns; James Millikin University: Dr. W. W. Denton; Johns Hopkins University: Dr. J. D. Bankier, Dr. L. I. Wade; University of Kansas: Dr. R. S. Pate; Louisiana State University: Dr. P. H. Anderson; Macalester College, St. Paul, Minnesota: Dr. František Wolf; Michigan State Normal College, Ypsilanti, Michigan: Dr. Edith R. Schneckenburger; North Carolina State College: Dr. C. F. Strobel; Princeton University; Mr. T. W. Anderson, Dr. Ernst Snapper; Rensselaer Polytechnic Institute: Mr. H. K. Brown; University of Saskatchewan; Dr. Max Wyman; United States Naval Academy: Mr. R. C. Morrow, Mr. W. H. Sears; University of Vermont: Dr. D. T. McClay.; Wellesley College: Miss Alberta Schuettler; University of Wisconsin: Dr. Dorothy L. Bernstein.

The death of Dr. B. G. A. Ackermann-Teubner on February 18, 1941, has been reported. He had been a member of the Society since 1907.

Dr. K. D. Swartzel, who was head of the department of mathematics in the University of Pittsburgh from 1922 until his retirement in 1939, died October 30, 1941. For twenty-eight years prior to 1922 he was a member of the department of mathematics in the Ohio State University.

Mr. H. L. Sweet of Phillips Exeter Academy died March 27, 1941.

Professor T. H. Taliaferro of the University of Maryland died September 25, 1941, at the age of seventy years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

Algebra and Theory of Numbers

1. A. A. Albert: *Quadratic forms permitting composition.*

It is proved that a quadratic form over a field F permits composition if and only if it is equivalent in F to the norm form xx^J of an alternative algebra over F such that $x+x^J$ and xx^J are in F. These norm forms are shown to be quadratic forms in 1, 2, 4 or 8 indeterminates, except for the diagonal norm forms (in 2^t indeterminates) of the purely inseparable fields of degree 2^t and exponent 2 over F of characteristic 2. The result has never before been obtained for fields of characteristic 2. Indeed it seems only to have been given completely for algebraically closed fields of characteristic not two. (Received October 20, 1941.)

2. S. P. Avann: *Lattices with arbitrary automorphism groups.*

There exists a partially ordered set having an arbitrary group as its group of automorphisms. Hence there exists a distributive lattice with a given automorphism group. A partially ordered set with cyclic group C_n has at least 3 conjugate sets of n elements for $n \leqq 7$ and at least 2 for $n > 7$. From these facts minimal partially ordered sets, lattices, and distributive lattices with cyclic automorphism groups can easily be obtained. (Received October 25, 1941.)

3. S. P. Avann: *The τ-function and its application to semi-modular lattices.*

Let $\tau[L]$, $\sigma[L]$ be the number of join irreducibles and the number of meet irreducibles, respectively, in a finite lattice L. In a non-distributive Birkhoff lower semi-modular (ξ'') lattice $\sigma > \tau \geqq \rho$, where ρ is rank of lattice. In a finite lattice L the following conditions are equivalent: (1) every element has unique irredundant join irreducible representation; (2) the sublattice generated by the elements covered by an element is a Boolean algebra, for each element of the lattice; (3) L is a finite Jordan-Dedekind chain lattice with $\tau = \rho$; (4) L is a ξ'' lattice with every modular sublattice distributive; (5) L is a ξ'' lattice with no modular non-distributive sublattice with coverings; (6) L is a ξ'' lattice with no modular non-distributive sublattice; (7) L is a ξ'' lattice with $\tau = \rho$. Although some of the equivalences are known, a new method of proof in contrast to combinatorial methods is emphasized. Note (6)→(5) and (3)→(7). (Received October 25, 1941.)

4. Reinhold Baer: *Inverses and zero-divisors.*

It may happen that an element in a ring is both a zero-divisor and an inverse, that

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it possesses a right-inverse but no left-inverse, and that it is neither a zero-divisor nor an inverse. Thus there arises the problem of finding conditions which assure the absence of these paradoxical phenomena; and it is the object of the present note to show that chain conditions on the ideals serve this purpose. The methods employed in this investigation allow the existence of universal units to be proved at the same time. (Received November 25, 1941.)

5. I. A. Barnett and C. W. Mendel: *A property of the traces of a square matrix.*

Let A be a square matrix having as elements n^2 independent variables and let $t_1, t_2, \cdots, t_n$ be the traces of $A, A^2, \cdots, A^n$, respectively. Then the jacobian of the t_i with respect to each set of n variables chosen from the original n^2 variables is never identically zero provided only that at least one of the variables is a diagonal element of A. (Received November 18, 1941.)

6. Salomon Bochner and R. S. Phillips: *Absolutely convergent Fourier series in noncommutative normed rings.*

This is a first attempt to extend Gelfand's analysis of normed rings, with $|x\cdot y| \leq |x|\cdot|y|$, from the commutative to the noncommutative case. If R_1 is any (noncommutative) normed ring, if $G=\{m\}$ is any commutative index group, and if R is the normed ring of sequences $A=\{a_m\}$, $a_m\in R_1$, $m\in G$, with the norm $|A|=\sum_m|a_m|$ and the product $A\cdot B=\sum_p a_{m-p}b_p$, then an element A of R has a left-inverse (two-sided inverse) if and only if for each character $\chi(m)$ on G the element $\sum_m a_m\chi(m)$ has a left-inverse (two-sided inverse) in R_1. The familiar case of $\{m\}$ being the additive group of all integers can also be treated very easily by Wiener's original method. (Received November 22, 1941.)

7. R. H. Bruck: *Certain numerical invariants of polyadics.*

Let $x(i_1, i_2, \cdots, i_p)$, where each index i_α ranges over the natural numbers $1, 2, \cdots$, be a set X of indeterminates over a field K, and denote by X_r the subset of X obtained by restricting the indices to the range $1, 2, \cdots, r$. For each r let there be given a form $F_r(X)$ (with assigned coefficients in K) multilinear in each of the p indices $i_1, i_2, \cdots, i_p$ and containing only the indeterminates of X_r; for example, if $p=2$, $F_r(X)$ could be taken to be the determinant $|x_{ij}|$, $i, j=1, 2, \cdots, r$. The sequence of forms $F_1(X)$, $F_2(X), \cdots$ may be called a rank sequence. Let $A(S_1, S_2, \cdots, S_p)$ be a "polyadic" with components in K, where each label S_j ranges over a finite set of fixed labels and has associated with it a prescribed type of linear transformation. If the identification $x(i_1, i_2, \cdots, i_p)=A(S_{i_1}, S_{i_2}, \cdots, S_{i_p})$ is made, then to each rank sequence corresponds a numerical invariant (or rank) of the polyadic, defined as the first r, possibly infinite, for which $F_r(X)=0$. The procedure is used to derive most of the known ranks of tensors, (ordinary) matrices and n-way matrices. (Received November 24, 1941.)

8. R. H. Bruck: *Isotopy of algebras with a principal unit.*

In the notation of A. A. Albert (abstract 47-9-331; also *Structure of Algebras*, pp. 9–13) the following, where U, V, W are nonsingular linear transformations, are three equivalent definitions of the isotopy of $\mathfrak{A}_0$ and $\mathfrak{A}$: (1) $R^0_x=UR_yW$, $y=x^V$; (2) $T^0_x=VT_zW$, $z=x^U$; (3) $x\circ y=(x^U\cdot y^V)^W$. The present study concerns regular algebras; for example, division algebras and linear closures of finite quasi-groups. $\mathfrak{A}$ is defined to be regular if it contains a left-regular element a and a right-regular element b; here for example, a is left-regular if the equation $ax=0$ implies $x=0$. $\mathfrak{A}$ has an isotope with principal unit if and only if $\mathfrak{A}$ is regular; the transformations which yield such isotopes are of form $U=PR_b^{-1}$, $V=PT_a^{-1}$, $W=P^{-1}$. If $\mathfrak{A}$, $\mathfrak{B}$ are isotopes with principal units and $\mathfrak{A}_1$ is an invariant subalgebra of $\mathfrak{A}$, then $\mathfrak{B}$ contains an invariant subalgebra isotopic to $\mathfrak{A}_1$. Every isotope of a simple algebra with principal unit is a regular simple algebra, but not conversely. In the present paper various applications are considered. It is noted that the isotopes of a division algebra are division algebras, and that a division algebra of order two is isotopic to a field. (Received November 24, 1941.)

9. I. S. Cohen: *A generalisation of Macauley's theorem on unmixed ideals.*

Let $\mathfrak{o}$ be the ring of nonhomogeneous coordinates of the general point of an r-dimensional algebraic variety V over a ground field of characteristic zero. Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$, let $\mathfrak{J}=\mathfrak{o}_{\mathfrak{p}}$ be the quotient ring of $\mathfrak{p}$, and let $\eta_1, \cdots, \eta_s$ be s elements in $\mathfrak{J}$ such that the ideal $\mathfrak{A}=\mathfrak{J}\cdot(\eta_1, \cdots, \eta_s)$ is of dimension less than or equal to $r-s$. It is proved that if $\mathfrak{p}$ represents a simple subvariety of V, then $\mathfrak{A}$ is unmixed, of dimension $r-s$. This result, which is of a local character, implies the following theorem in the large: If an ideal $\mathfrak{B}=\mathfrak{o}\cdot(\zeta_1, \cdots, \zeta_s)$ in $\mathfrak{o}$ is of dimension less than or equal to $r-s$, then any imbedded prime ideal of $\mathfrak{B}$ represents a singular subvariety of V. In the case where V is a linear space (that is, $\mathfrak{o}$ a polynomial ring) these theorems reduce to theorems which have been proved by van der Waerden and Macauley, respectively. (Received November 21, 1941.)

10. R. P. Dilworth: *A theorem on lattice homomorphisms.*

The following theorem is proved: *Let S be a partially ordered set containing a complete, atomic, Boolean algebra B. Then there is an order homomorphism of S onto B leaving B invariant, preserving cross-cuts if they exist, and preserving distributive unions.* For the case in which B contains only two elements, this theorem gives most of the known lattice imbedding theorems. For B finite, this theorem is basic in the study of lattice homomorphisms. (Received October 24, 1941.)

11. R. P. Dilworth: *Dependence relations in a Birkhoff lattice.*

Let P be a set of elements. A relation $\perp$ between elements and subsets of P is a *dependence* relation if (1) $p\perp S+p$, (2) $p\perp S$ and $S\perp T$ implies $p\perp T$, (3) $p\perp S+p'$ implies either $p\perp S$ or $p'\perp S+p$. If P is the set of points of a Birkhoff lattice L it is well known that $p\perp S$ if and only if $p\subseteq \Sigma S$ defines a dependence relation with the above properties. Now let P_k be the elements in L of rank k. Define $r(x)=\rho(x)-k+1$. A subset S of P_k is *independent* if $r(\Sigma T)\geqq n(T)$ for every subset T of S. $n(T)$ denotes the number of elements in T. Define $p\perp S$ if and only if an independent subset T of S exists such that $p+T$ is not independent. It is shown that $\perp$ so defined satisfies (1), (2), and (3). Furthermore the independent sets of P_k are characterized in terms of the lattice structure. Also, a number of applications to imbedding problems are given. (Received October 24, 1941.)

12. Marshall Hall and R. P. Dilworth: *The imbedding problem for modular lattices.*

We prove the following: *There are modular lattices of every dimension greater than* *three which cannot be imbedded in a complemented modular lattice. Every modular lattice of dimension three or less is imbeddable.* The proofs rest on the fact that a modular lattice having but one independent modular functional can be imbedded in a complemented modular lattice only if it can be imbedded in a projective geometry. (Received November 24, 1941.)

13. B. W. Jones: *The number of classes in related genera of quadratic forms.*

For every prime p, there is associated with any genus g_1, of quadratic forms, another genus g_2. An upper limit for the number of classes in the genus of g_2 is given which depends on the number of classes in g, and the automorphs of the forms in the genus g_1. Under certain conditions the upper limit is reached. (Received October 6, 1941.)

14. Irving Kaplansky and O. F. G. Schilling: *Some remarks on relatively complete fields.*

In a previous paper of one of the authors, the following question arose: Under what circumstances can a field K be relatively complete in two inequivalent valuations? In this note it is shown that a necessary and sufficient condition is that K be separably algebraically closed, a result analogous to F. K. Schmidt's theorem on multiply complete fields. It is also proved that if a field K admits only cyclic extensions and is not relatively complete in any valuation, then no finite extension of K can be relatively complete. (Received November 10, 1941.)

15. C. C. MacDuffee: *On the composition of algebraic forms.*

If F is an algebraic field of degree n and class number h, every integral number of F corresponds to a matrix of the form $x_1S_1+x_2S_2+\cdots+x_nS_n$ under the regular representation, where S_i is an $n\times n$ matrix with rational integral elements. Furthermore, every ideal of the ith class corresponds under the Poincaré correspondence to a matrix of the form $x_1A_1^{(i)}+x_2A_2^{(i)}+\cdots+x_nA_n^{(i)}$, and conversely, every matrix of this form corresponds to an ideal of the ith class. The polynomials $f_1=|x_1S_1+\cdots+x_nS_n|$, $f_i=|x_1A_1^{(i)}+\cdots+x_nA_n^{(i)}|$ are n-ary n-ic forms which admit composition in the sense of Gauss, and under this operation form a group isomorphic with the class group, the norm form f_1 being the identity. (Received November 25, 1941.)

16. Saunders MacLane and O. F. G. Schilling: *Groups of algebras over an algebraic number field.*

Let K/F be the join $K'\cup K''$ of two normal subfields K' and K'' over an algebraic number field F. Suppose that S, S', S'' are the groups of algebras, prime to the discriminant of K/F, which are split by the fields K, K' and K'', respectively. This paper treats the question: When does the hypothesis $K=K'\cup K''$ imply the corresponding relation $S=S'\cup S''$ for the groups of algebras? It turns out that this relation will hold if and only if the Galois groups of K, K', K'' satisfy a certain condition on the distribution of the elements of maximal order. In particular, this condition always holds if K is abelian and in a number of other cases. (Received October 2, 1941.)

17. R. S. Pate: *Functional homomorphisms.* I. Preliminary report.

Two groups G and G' are considered for which a functional homomorphism $f(g)$

of G to G' may be defined such that (1) $f(g)$ is a subset of elements of G', (2) every element of G' occurs in some $f(g)$ and (3) $f(g_1)f(g_2) = f(g_1g_2)$. The set f^{-1} consists of all elements x of G such that $f(x) = f(g)$. The set $f^{-1}(1)$ is an invariant subgroup of G and $f^{-1}(g)$ a coset of $f^{-1}(1)$. The set of all distinct $f(g)$ is a group G'/f which is simply isomorphic to $G/f^{-1}(1)$. If $f(1)$ is an invariant subgroup of G' and $f(g)$ a coset of $f(1)$, $f(g)$ is the customary true homomorphism between G and G'. Certain sets of conditions on $f(g)$ which reduce a functional homomorphism to a true one are considered. A rigid group is one such that every functional homomorphism of any group to it is a true homomorphism. A group A is the product of two groups B and C if A can be constructed from B and C by the well known multiple homomorphism method. Certain theorems concerning functional homomorphisms, rigid groups and products are proved. (Received October 23, 1941.)

18. Samuel Perlis: *Normal bases of cyclic fields of prime-power degree.*

Let Z be a separable, normal extension of finite degree over a field F. If a quantity u and its conjugates form a basis of Z/F, u is said to generate a normal basis of Z/F, and it is known that such a basis always exists. The present paper considers the case in which Z/F is cyclic of prime-power degree $n = p^e$. It is shown that if p is the characteristic of F, u generates a normal basis of Z/F if and only if the trace of u in Z/F is not zero, and all such quantities are easily found in terms of the structure theory of these fields as developed by Albert. This theorem is false if the characteristic of F is not p. Parallel to Albert's theory, the paper next assumes that F contains p distinct roots of unity and again determines necessary and sufficient conditions. The methods used throughout are elementary and make no appeal to the theories of algebras or representations. (Received November 24, 1941.)

19. O. F. G. Schilling: *Normal extensions of relatively complete fields.*

Let F be a relatively complete field with perfect residue class field. The author discusses the Hilbert theory of infinite normal extensions K/F. It is shown in special cases that the algebraic structure of K/F depends essentially on the residue class field and the value group of K. The results of the Hilbert theory are then employed for the solution of existence problems of normal fields K/F with assigned algebraic and arithmetic properties. (Received October 8, 1941.)

20. M. F. Smiley and W. R. Transue: *Metric lattices as singular metric spaces.*

The fact that each pair of elements of a metric lattice (G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, p. 41) which are not comparable together with their sum and product form a *pseudo-linear quadruple* (L. M. Blumenthal, *Distance Geometries*, University of Missouri Studies, vol. 13 (1938), p. 48) suggests that the presence of "sufficiently many" pseudo-linear quadruples in a metric space (M, δ) might ensure that lattice operations could be defined in M so that (M, δ) becomes a metric lattice $(M, \delta, <)$. In this note it is shown that, while this statement is not true, the additional assumption of a weak form of either of two five point transitivities of metric betweenness (Everett Pitcher and M. F. Smiley, *Transitivities of betweenness*, this Bulletin, abstract 47-5-196) suffices. (Received November 20, 1941.)

21. R. M. Thrall and C. J. Nesbitt: *On the modular representations of the symmetric group.*

Let S_m denote the symmetric group of degree m. If $m<2p$ the order $m!$ of S_m is divisible by only the first power of the prime number p. The general theory of modular representations is fairly well developed in such cases. Nakayama (*On some modular properties of irreducible representations of symmetric groups* I, Japanese Journal of Mathematics, vol. 17 (1940), pp. 89–108; II, ibid., pp. 411–423) has recently determined the irreducible modular characters for $m<2p$. In the present paper the requirement of irreducibility is relaxed. The structure of the regular representation is given in full. It is shown that there are only a finite number of inequivalent indecomposable modular representations and these are all determined. In the final sections specific matrix forms for these representations are computed for the case $m=p$. (Received November 21, 1941.)

22. T. L. Wade and R. H. Bruck: *The characteristic function of a 2p-tensor.*

Let $A^{(i)}_{(j)}=A^{i_1 \cdots i_p}_{j_1 \cdots j_p}$, $p>1$, be an arbitrary tensor of the type indicated by its indices, and $\delta^{(i)}_{(j)}$ be the numerical tensor defined in abstract 47-5-184. The determinant of $\lambda\delta^{(i)}_{(j)}-A^{(i)}_{(j)}$, a monic polynomial $f(\lambda)$ of degree $N=n^p$ (n the dimension of the coordinate system), is an invariant, which may be termed the characteristic function of $A^{(i)}_{(j)}$. The set, C, consisting of the coefficients of $f(\lambda)$, yields not all, as in the case $p=1$, but only N out of a total of N^2-n^2+1 functionally independent absolute invariants of $A^{(i)}_{(j)}$ (abstract 47-9-335). The members of C are invariants of the connex $A^{(i)}_{(j)}X^{(j)}Y_{(i)}$, in which $X^{(j)}$ undergoes an arbitrary nonsingular linear transformation of N-dimensional space, and $Y_{(i)}$ the contragredient transformation. This property is in fact characteristic of those invariants of $A^{(i)}_{(j)}$ dependent upon the set C. In particular, $X^{(j)}$ may be of the form $x^{j_1}_{1|} \cdots x^{j_p}_{p|}$, where the p vectors $x^{i}_{\alpha|}$ undergo different transformations; similarly for $Y_{(i)}$, where $y^{\alpha|}_{i}$ is contragedient to $x^{i}_{\alpha|}$. For $A^{(i)}_{(j)}$ bisymmetric $f(\lambda)$ factors, the factors being the analogously defined characteristic functions of the simple algebras of which the algebra of bisymmetric tensors is the direct sum. (Received November 24, 1941.)

23. Morgan Ward: *Conditions for a lattice to be a Boolean algebra.*

Let S be a lattice in which if $a\supset c\supset b$, elements w and r exist such that $a\supset w\supset b$, $a\supset r\supset b$, $a=w\cup c$, $b=r\cap c$. Then if every element of S has a unique complement, S is a Boolean algebra. (Received October 22, 1941.)

24. P. M. Whitman: *Note on a certain representation of lattice elements.* Preliminary report.

In a finite lattice L, if a given element A covers exactly r elements, then A can be represented as the join of r or fewer join-irreducible elements. If this L is the lattice of all splittings of some lattice (abstract 46-9-442), then the number cannot be less than r. Under substantial restrictions, the same conclusions can be stated about the lattice of all colonies of cells (of algae, and so on), descended from a certain cell, which precede a chosen colony, where the cells reproduce by subdivision of one cell into two cells. (Received November 24, 1941.)

25. L. R. Wilcox: *Extensions of semi-modular lattices.* III.

The author's result (abstract 47-5-208) is extended to all complemented semimodular lattices of dimension equal to or greater than 4. The following theorem is proved. Let L be left complemented (abstract 47-9-356) with the further property that $b, c \in L$, $bc \neq 0$ implies $(a+b)c = a+bc$ for $a \leq c$; suppose also that there exists in L a chain of length 6. Then there exists a complemented modular lattice Λ containing L order-isomorphically and having the properties (a) $a \in L$, $a \neq 0$, $b \geqq a$ implies $b \in L$, and (b) for $a \in \Lambda$, $b \in L$, $a \leqq b$ there exists $c \in L$ such that c is a complement (in Λ) of a in b. Properties (a), (b) characterize Λ uniquely up to isomorphisms. This theorem is a lattice-theoretic generalization of well known imbeddings of affine and hyperbolic spaces into projective spaces. (Received November 24, 1941.)

26. Leonard Carlitz: *q-Bernoulli numbers and polynomials.*

Rational functions of q are defined by means of $q(qb+1)^m = b^m$ $(m>1)$, where after expansion b^m is replaced by b_m; "polynomials" are defined by $b_m(x) = \sum_{\alpha=0}^{m} C_{m,\alpha} q^{\alpha x}[x]^{m-\alpha} b_\alpha$, where $[x] = (q^x-1)/(q-1)$. Many of the properties of the ordinary Bernoulli numbers and polynomials are readily extended to these quantities; in addition there are certain formulas in the generalized case that are not easily specialized to the case $q=1$. Among possible explicit formulas for b_m may be mentioned $b_m = \sum_{s=0}^{m} 1/[s+1] \sum_{\alpha=0}^{s} (-1)^\alpha \left[{s \atop \alpha}\right] q^{\alpha(\alpha+1-2s)/2} \cdot [\alpha]^m$, which leads at once to a generalized Staudt-Clausen theorem: $b_m = \sum_{s=1}^{m+1} N_s(q)/F_s(q)$ $(m>0)$, where $F_s(q)$ is the cyclotomic polynomial and $\deg N_s < \deg F_s$. (Received November 24, 1941.)

27. Joseph Lehner: *The Ramanujan identities and congruences for powers of eleven.* Preliminary report.

The author proves the existence of a "Ramanujan identity" for the modulus 11^α $(\alpha \geqq 1)$. For $\alpha = 1, 2$, this identity implies the Ramanujan conjecture: $p(n) \equiv 0 \pmod{11^\alpha}$ if $24n \equiv 1 \pmod{11^\alpha}$. The methods used are those of Rademacher's paper *The Ramanujan identities under modular substitution* (to be published in the American Journal of Mathematics). A modification of Hecke's T-operator is used. This operator is defined as follows: $U_{11}F(\tau) = \Sigma_\lambda F(\tau+\lambda/11)$, $\lambda \bmod 11$. If $F(\tau)$ is a modular function belonging to $\Gamma_0(121)$, that subgroup of the modular group defined by $c \equiv 0 \pmod{121}$, then $U_{11}F$ belongs to $\Gamma_0(11)$. Then it can be expressed as a polynomial in $A(\tau)$, $B(\tau)$, certain well known functions which constitute a basis for $\Gamma_0(11)$. By taking F to be $\eta(121\tau)/\eta(\tau)$, where $\eta(\tau)$ is the well known elliptic modular function of Dedekind, we obtain the desired Ramanujan identity for the modulus 11. Identities for higher powers of 11 are then obtained by a two-fold induction, one for even α, the other for odd α. The possibility of proving Ramanujan's conjecture for higher values of α $(\alpha>2)$ is being investigated. (Received November 19, 1941.)

ANALYSIS

28. C. B. Barker: *The Lagrange multiplier rule for two dependent and two independent variables.*

Let $\bar{z}_1(x, y)$ and $\bar{z}_2(x, y)$ be of class C'''' on a closed simply connected region $\bar{G}$ of class C_a''' and minimize (1) $\iint_G f(x, y, z_1, z_2, p_1, p_2, q_1, q_2)dxdy$ among all pairs of functions $z_1(x, y)$ and $z_2(x, y)$ which coincide on the boundary G^* with $\bar{z}_1(x, y)$ and $\bar{z}_2(x, y)$, respectively, and which satisfy (2) $\phi(x, y, z_1, z_2, p_1, p_2, q_1, q_2) = 0$ on G; assume that f

and ϕ are of class C'''' in their arguments everywhere. It is proved in this paper that there exists a unique multiplier $\lambda(x, y)$ of class C' on $\overline{G}$ such that z_1 and z_2 satisfy the equations $(\partial/\partial x)(f_{p_i}-\lambda\phi_{p_i})+(\partial/\partial y)(f_{q_i}-\lambda\phi_{q_i})=f_{z_i}-\lambda\phi_{z_i}$, $i=1, 2$, provided that the pair (z_1, z_2) is "quasi-normal" with respect to the equation (2). The quasi-normality requires (i) that $\phi_{p_1}\phi_{q_2}-\phi_{p_2}\phi_{q_1}\neq 0$ on $\overline{G}$ for the pair (z_1, z_2), (ii) that to any pair of functions ζ_1, ζ_2 which vanish on G^* and satisfy the equation of variation on G shall correspond a 1-parameter family of solutions $z_i(x, y; \mu)$ of (2) which coincide with $z_i(x, y)$ on G^* for each μ near zero, which reduce to z_i for $\mu=0$, and which have the ζ_i as their variations for $\mu=0$, and (iii) that a certain other rather complicated differential expression not be an exact differential on any rectangle on G. (Received November 3, 1941.)

29. E. F. Beckenbach: *Painlevé's theorem and the analytic prolongation of a minimal surface.*

The following generalization of Painlevé's theorem is obtained. If $x_j(u, v)$, $x_j'(u, v)$, $j=1, 2, 3$, are defined, respectively, in contiguous domains D, D' having a rectifiable arc C of boundary in common; if $x_j(u, v)$, $x_j'(u, v)$ are harmonic and satisfy $E=G$, $F=0$ in their respective domains of definition; if $X_j(u, v)$, $X_j'(u, v)$ denote the direction-cosines of the normals to the minimal surfaces S, S' thus determined; and if the above functions remain continuous on C and satisfy the relations $x_j(u, v)=x_j'(u, v)$, $X_j(u, v)=X_j'(u, v)$, $j=1, 2, 3$, for all points on C; then S, S' are analytic prolongations of each other across C. (Received November 22, 1941.)

30. E. F. Beckenbach and Maxwell Reade: *Mean-values and harmonic polynomials.*

It is shown that a necessary and sufficient condition that, for a fixed $n\geqq 3$, a function $f(x, y)$, superficially summable in a finite domain D, assume at each point P of D a value equal to its areal (or peripheral) mean on every regular n-gon whose center is at P and whose circumscribed circle lies together with its interior in D, is that $f(x, y)$ be a harmonic polynomial of degree at most $n-1$. The familiar characteristic property of harmonic functions in terms of circular mean-values appears as a limiting case. (Received November 22, 1941.)

31. Stefan Bergman: *On operators in the theory of partial differential equations and their application.*

There exists for every equation $L(U)\equiv\Delta U+C(z, \bar{z})U=0$, $z=x+iy$, $\bar{z}=x-iy$, a function $E(z, \bar{z}, t)$ such that every solution U, of $L(U)=0$ can be represented in the form $U=P(f)=\int_{-1}^{+1}Ef(\zeta)d\tau$, $\zeta=z(1-t^2)/2$, $d\tau=(1-t^2)^{-1/2}dt$; f being a suitable analytic function of one complex variable ζ. This representation is applied to the solution of boundary value problems. Let $\phi(z, \bar{z})=c$ be a curve k in the xy-plane. If $E(z, \bar{z}, t)=E^*(z, z+\phi(z, \bar{z}), t)$ then $P(f)$ and $H(f)=\int_{-1}^{+1}E^*f(\zeta)d\tau$ assume on k the same values. Thus: if F is the analytic function which assumes on k the given values, V, and $f=h$, is the solution of $H(f)=F$, $U=P(h)$ will be the solution of $L(U)=0$ assuming on k the values V. Using the representation $U=\int_0^{\pi}\int_{-1}^{+1}E(r, t, v)f(\zeta)d\tau dv$ valid for the solutions of $T(U)\equiv U(x, y, z)+C(r^2)U=0$, $r^2=x^2+y^2+z^2$, an analogous method for the solution of boundary value problems of $T(U)=0$ is obtained. Using the last representation the following residue theorem is proved: there exist to every function U_1 two functions U_2 and U_3, $T(U_k)=0$, such that $\int_C(U_1dx+U_2dy+U_3dz)=0$ if C lies on a sphere with the center at origin and can be reduced to a point in the regularity domain of U_k. (Received November 24, 1941.)

32. Salomon Bochner and W. T. Martin: *A class of removable singularities in several complex variables.*

Given a function $f(z_1, \cdots, z_n)$ continuous in a region R and analytic in R except possibly on some exceptional set E in R. Under what conditions on the set E does this imply that f is analytic throughout R? This question is investigated. One sufficient condition is found to be that (measure $E_\epsilon)/\epsilon \to 0$ as $\epsilon \to 0$ where E_ϵ is the set of points in R whose distance from E is less than ϵ. (Received November 24, 1941.)

33. J. W. Bradshaw: *On a certain class of continued fractions.*

Continued fractions which are equivalent to the series $S_k = \sum n^{-k}$ for several values of k are generalized by the addition of a variable term to each of the partial denominators. Series equivalent to the resulting continued fractions are set up by means of two systems of difference equations. Application is made to the calculation of the remainder of S_k after summing a few terms. Some of the continued fractions considered are very rapidly convergent. For example, it has been possible, by means of one of them, to recompute Stieltjes' thirty-place value of S_3. (Received November 21, 1941.)

34. J. L. Doob: *Topics in the theory of Markoff chains.*

Let the matrix function $p_{ij}(t)$ $(t>0)$ $i, j = 1, 2, \cdots$ satisfy the following conditions: $p_{ij}(t) \geqq 0$, $\sum_j p_{ij}(t) = 1$, $p_{ik}(s+t) = \sum_j p_{ij}(s) p_{jk}(t)$. Then the $p_{ij}(t)$ can be interpreted as the transition probabilities of a stochastic (Markoff) process. The properties of the $p_{ij}(t)$ (especially as $t \to \infty$ and $t \to 0$) and the relations between the Markoff process and the matrix function are examined in detail. For example, necessary and sufficient conditions that the Fokker-Planck differential equations in the $p_{ij}(t)$ hold are formulated in terms of the properties of the state function x_t (taking on the value j if the jth state is assumed at time t). (Received October 30, 1941.)

35. H. J. Ettlinger: *The theory of the Riesz integral.*

In this paper the author defines the definite integral following Riesz by means of step functions and, making use of a simple set of axioms, obtains a characterization of the Riemann and the Lebesgue integrals. Other more general integrals are discussed. (Received November 24, 1941.)

36. Abe Gelbart: *On functions of two complex variables with bounded real parts.*

Let $f(z_1, z_2)$ be regular in the interior of a finite four-dimensional domain D^4, belonging to the class $\mathfrak{D}^4$ of domains equivalent to the bicylinder, and having a bounded real part in D. The author obtains an upper bound for $f(z_1, z_2)$ in terms of $A = \max_{(z_1, z_2) \in D} \operatorname{Re} f(z_1, z_2)$, $f(0, 0)$ and the domain only. The upper bound becomes unbounded only when the point $\{z_1, z_2\}$ approaches a two-dimensional surface on the three-dimensional boundary, so that a consequence of the inequality is that a function regular in the interior of the domain D^4 with a bounded real part is regular almost everywhere on the three-dimensional boundary. From this result, an upper bound for $\left|\partial^{m+n} f(z_1, z_2)/\partial z_1^m \partial z_2^n\right|$ is obtained again in terms of A, $f(0, 0)$, and the domain only, for the particular domain bounded by the analytic hypersurfaces $S_1^3 = E[z_1 = re^{i\lambda_1}, 0 \leqq \lambda_1 \leqq 2\pi]$ and $S_2^3 = E[z_1 = re^{i\lambda_2} + p(re^{i\lambda_2}) z_2]$. (Received November 24, 1941).

37. Einar Hille: *On the oscillation of differential transforms.* II.

G. Pólya and N. Wiener have shown that $V[(D^2-\lambda)f(x)] \geq V[f(x)]$ where $f(x+2\pi)=f(x)$, $V[g(x)]$ is the number of sign changes of $g(x)$ in $(0, 2\pi)$, $D=d/dx$, and $\lambda>0$. From this point of departure they proved that if $V[D^{2n}f(x)] \leq N$ for all n, then $f(x)$ is a trigonometric polynomial of degree not greater than $N/2$, as well as other results relating $V[D^{2n}f(x)]$ with the analytical properties of $f(x)$. In the present paper this theory is extended to a large class of linear second order differential operators ϑ. For such operators $V[(\vartheta-\lambda)f(x)] \geq V[f(x)]$, $\lambda>0$, if $f(x)$ satisfies the appropriate boundary conditions. Further, $\liminf_{n\to\infty} V[\vartheta^n f(x)] < \infty$ implies that $f(x)$ is a finite sum of characteristic functions of a corresponding boundary problem. Included in the discussion are the differential operators associated with the names of Bessel, Hermite, Jacobi, Laguerre, Legendre, and Mathieu as well as those of the classical boundary value problems of the Sturm-Liouville type with analytical coefficients. For the Jacobi and Legendre operators partly sharper results have been proved by other methods by G. Szegö in the first paper in this series. (Received November 22, 1941.)

38. Mark Kac: *On convergence of certain series of functions.*

Let $\phi(x)$ be such that (1) $\phi(x+2\pi)=\phi(x)$, (2) $|\phi(x_1)-\phi(x_2)| \leq M|x_1-x_2|^\alpha$ $(0<\alpha\leq 1)$, (3) $\int_0^{2\pi}\phi(x)dx=0$, and let $\{n_k\}$ be a sequence of integers satisfying the gap condition $n_{k+1}/n_k>q>1$ $(k=1, 2, \cdots)$. The following theorem is then proved: If $\sum_{k=1}^{\infty}a_k^2<\infty$, the series $\sum_{k=1}^{\infty}a_k\phi(n_kx)$ converges almost everywhere. (Received November 21, 1941.)

39. R. B. Kershner: *The continuity of functions of many variables.*

It is known that a function of n real variables may be continuous with respect to each of the variables separately in a given region and still have discontinuities. In fact the set of these discontinuities may have full measure and may include all points of an $(n-2)$-dimensional region. In this paper it is shown that the set of discontinuities cannot have Menger dimension greater than $(n-2)$. The principal tool is Baire's theorem and correspondingly a stronger but more involved statement is proved involving the category of the discontinuities. With the assumption of a very little extra smoothness, much less than a Lipschitz condition with respect to the separate variables, it is shown that the discontinuities are nowhere dense. Examples are given incating to what extent the results given are complete. (Received November 25, 1941.)

40. H. N. Laden: *An interpolation polynomial involving derivatives of a prescribed function.*

In an earlier paper (Duke Mathematical Journal, vol. 8 (1941)), the author considered an interpolation polynomial $F_n(x)$ of degree less than or equal to $4n-1$ which takes preassigned values $y_1, y_2, \cdots, y_n$ at prescribed abscissas $x_1, x_2, \cdots, x_n$ and such that $F_n^{(\nu)}(x_i)=0$ $(i=1, 2, \cdots, n;\ \nu=1, 2, 3)$, especially where the abscissas are zeros of classical orthogonal polynomials and $y_i=f(x_i)$ $(i=1, 2, \cdots, n)$, $f(x)$ being an arbitrary continuous function. Earlier conjectures, that some convergence theorems for Laguerre abscissas are "best possible" are shown to be only partially correct. Also, $F_n(0)$ is investigated for Laguerre abscissas, $F_n(\pm 1)$ for Jacobi abscissas. It is shown, further, that for Newton (equidistant) abscissas on $(-1, 1)$, there exist continuous functions on $[-1, 1]$ such that the corresponding $F_n(x)$ does not converge uniformly to $f(x)$ on $[-1, 1]$ as $n\to\infty$. In addition, the case where $F_n^{(\nu)}(x_i)=y_i$ $(\nu=1, 2, 3;$ $i=1, 2, \cdots, n)$ are preassigned not necessarily zero is considered for Jacobi abscissas and for functions with prescribed moduli of continuity. The methods used are due to Féjer, Shohat and Szegö. (Received November 24, 1941.)

41. E. J. McShane: *On Perron integration.*

Perron's definition of integral has the advantage of elegance, and yields very simple proofs of many theorems; but the most general theorem on integration by parts has not been proved for the Perron integral except by the detour of proving the equivalence of the Perron integral and the special Denjoy integral. Here a definition of integral is presented which is shown to be equivalent to that of Perron, but has the advantage of permitting a direct proof of the general theorem on integration by parts. The new definition is closely related in form to that of Perron; the distinction is that Perron's major functions are replaced by pairs (right majors and left majors), and analogously for minor functions. (Received November 24, 1941.)

42. E. J. McShane: *The derivative of the indefinite Lebesgue integral.*

This note contains a proof of the fundamental theorem on the derivative of the indefinite Lebesgue integral which (it is hoped) has the advantages of simplicity and of requiring little preparatory material. (Received November 24, 1941.)

43. P. T. Maker: *The Cauchy theorem for functions on closed sets.*

Let $f(z)$ be defined on E, a bounded, closed set in the complex plane and such that every point z is the limit of two sequences of points lying on curves having noncollinear tangents at z. It is shown that if $f(z)$ has a derivative at each point of E, there is a sequence of coverings $\{R_n\}$ of E, and a continuation, $f^*(z)$, of $f(z)$ to the rest of the plane, for which $\lim_{n\to\infty}\int_{R_n} f^*(z)dz = 0$ and $\lim_{n\to\infty} mR_n = mE$. Conditions weaker than the existence of the derivative are found which give the same result. (Received November 25, 1941.)

44. E. J. Mickle: *Associated double integral variation problems.*

By generalizing the concept of adjoint minimal surfaces, Haar (Mathematische Annalen, vol. 100 (1928), pp. 481–502) has shown that with every extremal surface of a double integral variation problem $J[z]$ in which the integrand function is of the form $F(p, q)$, there can be associated an adjoint extremal surface which is itself an extremal surface of an associated adjoint variation problem. The results of Haar can be extended so that with every extremal surface of the problem $J[z]$ there can be associated twenty-four surfaces such that these surfaces are themselves extremal surfaces, respectively, of twenty-four associated double integral variation problems. The transformations defining the associated problems form a group of order twenty-four. Analogous results can be obtained for parametric double integral variation problems. (Received November 18, 1941.)

45. G. C. Munro: *Systems of linear differential equations with constant coefficients.*

This paper presents a new, simple and direct treatment of systems of linear homogeneous differential equations with constant coefficients. (Received October 9, 1941.)

46. S. B. Myers: *An existence theorem for a self-adjoint system of second-order, linear, homogeneous differential equations.*

In 1929 Morse (Mathematische Annalen, vol. 103 (1930), p. 66) generalized the classical Sturm comparison theorem to n dimensions. One consequence of his results is that if $p_{ij}(x)\eta_i\eta_j \geq A^2\sum \eta_i^2$ and $r_{ij}(x)\eta_i\eta_j \leq B^2\sum \eta_i^2$ for all $(\eta)=(\eta_1, \cdots, \eta_n)$, then there exists a solution $(y)\neq(0)$ of the system of differential equations $d/dx(r_{ij}y_j')+p_{ij}y_j=0$ vanishing at $x=a$ and vanishing later before or at $x=a+\pi B/A$. It is assumed that r_{ij} and p_{ij} are of class C' and $r_{ij}\eta_i\eta_j$ is positive definite. In the present paper the same result is proved under weaker hypotheses: namely, with the above inequalities replaced by $\sum p_{ii} \geq nA^2$, $\sum r_{ii} \leq nB^2$. This result has applications to a problem in differential geometry in the large (Myers, Duke Mathematical Journal, vol. 8 (1941), pp. 401–404). (Received November 21, 1941.)

47. N. M. Oboukhoff: *The historical development of total differential as the principal part of the increment of a function of several variables.*

There is more than formalism in the differential calculus of Leibniz, that has irrevocably been assimilated by mathematics. This is his basic concept of the differential: what Leibniz calls "difference" is actually "differential" in the sense of the principal part of the increment of a function; likewise in Leibniz' hands analysis became a particular system of calculus; Leibniz as logician and mathematician promoted the same form of structure, that of calculus. Also, he used pragmatico-operational approach where logical foundations were not strong enough. We can discern the general outlines of neoclassical analysis in the characteristics of Leibniz' doctrine. The first half of the eighteenth century showed deterioration of the foundations of calculus in contrast to an irresistible impetus of its growth and development of applications. Later, a balance was restored by Lagrange; the foundations of calculus were formulated by him in a kind of synthesis of the two major doctrines, those of Leibniz and of Newton. In the nineteenth century Cauchy, following in the footsteps of Leibniz and Lagrange proved that the total differential could be determined independently of derivatives as the principal part of the increment of a function, although he did not use this term; Weierstrass introduced it. (Received November 24, 1941.)

48. O. G. Owens: *An explicit formula for the solution of the ultrahyperbolic equation in four independent variables.*

By adapting H. Lewy's generalization of the Riemann integration method for a linear hyperbolic equation in two independent variables, an explicit formula is derived which gives the value of the solution of the ultrahyperbolic equation in four independent variables. The formula requires initial values which may be sufficiently differentiated, that is, regular initial values, and also certain Riemann functions. It is shown that there must necessarily be two integro-differential equations which hold for the solution and its normal derivative on the initial surface. These results are carried further in the case of a hyper-plane, where, by means of an example, the equalities are shown to be not identically satisfied. (Received October 22, 1941.)

49. J. F. Paydon and H. S. Wall: *An extension of the Stieltjes continued fraction theory.*

The authors show that if a_n is in the parabola $|z|-R(z)=k/2$, $0<k\leq 1$, and $\sum|b_n|$ diverges, $b_1=1$, $a_{n+1}=1/b_nb_{n+1}$, $(n=1, 2, 3, \cdots)$, then the continued fraction $1/1+a_2t/1+a_3t/1+\cdots$ converges uniformly on the interior of the cardiod $|t|^2=[|t|+R(t)]/2k$. If $\sum|b_n|$ converges the sequences of even and odd approximants converge uniformly to separate limits. The convergence theorem of Stieltjes (a_n real and positive) appears as the limiting case $k\to 0$. The theorem solves the prob-

lem of "limitar-periodic" continued fractions in important cases, for example, if c is not real and less than or equal to $-1/4$ it is easy to determine a circle with center c which bounds a convergence region for $1/1+a_1/1+a_2/1+\cdots$. Also included is a theorem, recently announced by Leighton and Thron (abstract 47-7-308). (Received November 5, 1941.)

50. Edmund Pinney: *Theory of functions on linear topological spaces to Banach spaces.*

This paper deals for the most part with the theory of Banach valued functions defined on a linear topological space. The Riemann integral and the linear topological M-differential are defined, and various of their properties are investigated. Finally, these results are applied in the proof of existence theorems for the abstract differential equation system $df(x, v, u)=F(x, f, dx)$, (where differentiation is with respect to x only), $f(u, v, u)=v$; and for the partial differential equation $g_1(u, v; h)+g_2(u, v; F(u, v, h))=0$. When $F(x, y, z)$ is linear in y, it is shown that under the conditions given, $f(x, v, u)$ is linear in v, and differentiable with respect to all three places. (Received October 24, 1941.)

51. G. Y. Rainich: *Factorisation of polynomials, in a ring, with application to partial differential equations.* Preliminary report.

Matrices have been used for the purpose of obtaining a system of first order equations equivalent to a partial differential equation of the second order (in the same sense that the Cauchy-Riemann equations are equivalent to the Laplace equation) by Brill, Dirac and others. The procedure consists in expressing a second order linear differential operator as a product of two first order operators with matrix coefficients. With a view of extending these results the present paper expresses a quadratic polynomial with indeterminates as variables as the product of a linear and *another quadratic* polynomial; the coefficient field must for this purpose be extended into a noncommutative ring. This ring is studied geometrically by considering its elements as operators on a vector space; upon introduction of coordinates these operators are expressed as matrices. As special cases we obtain the above mentioned solution of Dirac and another solution which leads to relations previously obtained by Duffin. The structure of the general solution is studied in terms of these two special solutions. A generalization in which the second factor is a polynomial of degree higher than the second is indicated. (Received November 24, 1941.)

52. Maxwell Reade: *Some remarks on subharmonic functions.* Preliminary report.

Continuous subharmonic functions are characterized by the property that they are dominated by their circular averages, or mean-values (T. Radó, *Subharmonic Functions*, Berlin, 1937, pp. 7–8). The author investigates the essentiality of the use of circular averages; he uses general regular n-gonal averages instead, for $n\geqq 3$. A typical result is the following. *If* $f(x, y)$ *is continuous in a bounded simply connected domain* D, *and if* $f(x, y)\leqq 1/4h^2\int_{-h}^{h}\int_{-h}^{h}f(x+\xi, y+\eta)d\xi d\eta$, *holds for all* h *sufficiently small, for each point* (x, y) *of* D, *then* $f(x, y)$ *is subharmonic in* D. The converse does not hold, in general, without further hypotheses placed upon $f(x, y)$; a simple counterexample is the (sub)harmonic polynomial $x^4-6x^2y^2+y^4$. However, *if* $\Delta f(x, y)$ *throughout* D, where Δ is the Laplacian operator, *then* $f(x, y)$ *is dominated by its square averages, for* *sufficiently small squares*. The squares used here are for illustrative purposes only; general n-gons are used in the paper. (Received November 17, 1941.)

53. G. E. Reves and Otto Szász: *Some theorems on double trigonometric series.*

The paper generalizes to two variables the theorems of Cantor-Lebesgue and of Fatou-Denjoy-Lusin on trigonometric series, and two theorems of Szász on absolute convergence of Fourier series. The mode of proof is an adaptation of the methods used in the one variable case. (Received November 10, 1941.)

54. R. M. Robinson: *Bounded univalent functions.*

Let $f(z)$ be regular and univalent for $|z|<1$, $|f(z)|<1$ there, and $f(0)=0$. Using the method of Lowner, a detailed study is made of the inequalities involving $|f'(0)|$, $|z_0|$, $|f(z_0)|$, and $|f'(z_0)|$. A typical result is that if $|f'(0)|$ and $|z_0|$ are given, the largest possible value of $|f'(z_0)|$ is attained for a mapping of $|z|<1$ on the unit circle with a radial slit provided $|z_0| \leq 1/2$, but not in all cases. Unbounded univalent functions are considered as limits of bounded functions. Let $F(z)$ be regular and univalent for $|z|<1$, with $F(0)=0$ and $F'(0)=1$. Relations between $|z_0|$, $|F(z_0)|$, and $|F'(z_0)|$ are studied. In particular, sharp bounds for $|F'(z_0)|$ in terms of $|F(z_0)|$ are given. A striking result is that if $|F(z_0)| \leq 1/4$ then $|F'(z_0)| < 2.07$, but that if $|F(z_0)|$ has a given value greater than 1/4, no upper bound for $|F'(z_0)|$ exists. (Received October 21, 1941.)

55. H. M. Schwartz: *On sequences of Stieltjes integrals.*

This paper continues the discussion of convergence criteria for sequences of Riemann-Stieltjes integrals $J_n = \int_a^b f dg_n$ of a former paper (*Sequences of Stieltjes integrals*, this Bulletin, vol. 47 (1941), pp. 947–955); it contains a development of sets NS of necessary and sufficient conditions for the convergence of $\{J_n\}$ for different subclasses of the class F of functions f for which J_n exist (g_n being assumed to be of bounded variation), and it concludes with an indication of the possibility of applying the results to certain linear operations. From results obtained in the aforesaid paper one can conclude that for the class D of functions of F which have no singularities of the second kind, a NS set is given by the requirement that the sequence $\{g_n\}$ be uniformly of bounded variation and converge on all the common continuity points of g_n and on a and b, but it was left an open question whether this set of conditions would suffice for subclasses of F wider than D. It is now shown that the answer to this question is in the negative; thus, for example, for the set of the functions of F which have at most one singularity of the second kind at some fixed point u of (a, b), it is necessary to add to the above conditions the requirement of equicontinuity at u of the set $\{g_n\}$. (Received November 24, 1941.)

56. Otto Szász: *On a theorem of Hardy and Littlewood.*

The theorem under consideration asserts that if an integrable function $f(\theta) = o(\log 1/\theta)^{-1}$, then $\sum_0^n (\nu+1)^{-1} s_\nu^* = o(\log n)$. Here $s_0^*, \cdots, s_n^*$ are the values $|s_0|, |s_1|, \cdots, |s_n|$ rearranged in decreasing order, and s_n is the nth partial sum of the Fourier series for $\theta = 0$. The author proves the same result under a more general assumption; applications are also given. (Received November 10, 1941.)

57. G. Szegö: *On the oscillation of differential transforms.* I.

Let $f(x)$ be a real periodic function with period 2π for which all derivatives $f^{(k)}(x)$ exist, and denote by $2N_k$ the number of the mod 2π distinct sign variations of $f^{(k)}(x)$. The following theorems hold: (1) If $N_k=O(1)$, $f(x)$ is a trigonometric polynomial. (2) If $N_k<k/\log k$, $f(x)$ is an integral function. (3) If $\rho>1$ and $2N_k<(k/\rho)^{1/\rho}$, $f(x)$ is an integral function of order $\rho/(\rho-1)$. (1) is due to G. Pólya and N. Wiener (cf. a forthcoming paper in the Transactions), (2) and (3) are refinements of certain results of these authors. (2) and (3) are best possible results since it can be shown that the conclusions of (2) and (3) are not necessarily valid if $N_k=O(1)$ or $N_k=O(k^{\alpha})$, $\alpha>1/\rho$, respectively. The method used in the present paper is different from that of G. Pólya-N. Wiener. It leads also to a refinement of (2) in which from a certain limitation of N_k the analytic character of $f(x)$ in a certain strip is concluded. Finally, analogous results are obtained by replacing $f^{(k)}(x)$ by $\vartheta^k f(x)$ where $\vartheta=(1-x^2)D^2-2xD$ is Legendre's operator (cf. II of this series by E. Hille). (Received November 22, 1941.)

58. E. W. Titt: *A method for integrating the linear hyperbolic equation in three independent variables.*

This paper is concerned with Cauchy's problem with data given over a surface duly inclined to the characteristic cone. The same choice of coordinate system relative to the characteristic cone is made that the writer has used previously (Annals of Mathematics, (2), vol. 40 (1939), pp. 862–891). A solution of the homogeneous adjoint equation is obtained by integrating the elementary solution twice in a direction exterior to the cone. This quantity has a finite discontinuity in one partial derivative as in the case of Green's function for an ordinary differential equation. Integrating over a region bounded by one nap of a cone and the initial surface a formula is obtained for the unknown integrated over a two-dimensional region. Varying the vertex of the cone in order to obtain a formula for u itself seems simpler than in the case of Volterra's method. (Received November 24, 1941.)

59. W. R. Transue: *Contributions to the theory of subharmonic functions.*

First, by use of the two-constant theorem, it is shown that the maximum of a subharmonic function on a level line, $H=\lambda$, of an harmonic function, H, is a convex function of λ. A variety of facts concerning subharmonic functions are shown to be obtainable from this result. Second, a limitation on the mean of u, logarithmically subharmonic, on the circumference of circles within the unit circle is shown to induce a limitation on the mean of u^2. Finally, it is shown that, by replacing Green's function in the representation method of F. Riesz by $E_p(P, Q)=\log PQ-\log OQ+R[(P/Q)+(1/2)(P/Q)^2+\cdots+(1/p)(P/Q)^p]$, subharmonic functions can be represented in the neighborhood of a point in cases where they do not possess an harmonic majorant in that neighborhood and hence where representation using Green's function is not possible. (Received October 24, 1941.)

60. W. J. Trjitzinsky: *Analytic theory of parametric linear partial differential equations.*

In two recent addresses G. D. Birkhoff brings out the significance of asymptotic developments, in the field of linear partial differential equations, for the domain of quantum mechanical ideas. He introduces formal series and makes a conjecture that 'actual' solutions, of an appropriate type, exist asymptotic to these series. In view of

Birkhoff's earlier work in the asymptotic theory of ordinary differential equations, as well as in consequence of certain considerations of mathematical physics, the truth of this conjecture would offhand appear as rather likely. This fact explains the purpose of the present work—taking a purely mathematical point of view, the author considers linear partial differential equations containing a parameter λ and first establishes existence of formal solutions containing those of Birkhoff as a special case. He then establishes (for second order equations) some general existence theorems, asserting existence of 'actual' solutions, which are functions asymptotic to the formal series. This theory is naturally divided into two parts—one relating to equations of elliptic type, the other referring to those of hyperbolic type. Equations of parabolic type have not been considered in the present work. (Received November 24, 1941.)

61. S. M. Ulam: *A geometrical approach to the theory of representations of topological groups.* Preliminary report.

The theorem of von Neumann on the representations of compact groups by sequences of finite matrices is proved by consideration of geometrical properties of compact convex bodies in the space of continuous (real-valued) functions on a compact space. (The use of Haar measure is avoided.) A number of related results are derived. (Received November 25, 1941.)

62. František Wolf: *On majorants of analytic functions.*

If $f(z)$ is analytic in $|x| < a, |y| < b$ and $|f(z)| \leq M(x)$ where $\int_{-a}^{a} \log^+ \log^+ M(x)dx < \infty$, then to an arbitrary $\delta > 0$ corresponds a ϕ, dependent only on $M(x)$ and δ, but independent on the particular $f(z)$, such that $|f(z)| \leq \phi$ for $|x| < a-\delta, |y| < b-\delta$. This is a generalization of a result of N. Levinson (*Gap and Density Theorems*, p. 127, Theorem XLIII). The theorem is proved by a method used by the author to prove a generalization of Phragmén-Lindelof theorem (Journal of the London Mathematical Society, vol. 14 (1939), p. 208) which becomes a corollary of the above theorem. Another corollary is Nils Sjögren's result (Congrès des Mathématiques Scandinaves à Helsinfors, 1938): If $f(z)$ is analytic in the unit circle and such that $|f(z)| \leq M(\arg z)$ for $1-\epsilon < r < 1$ and $\int_0^{2\pi} \log^+ \log^+ M(\theta) \cdot d\theta < \infty$, then $|f(z)| \leq \phi$ in $|z| \leq 1-\delta < 1$. ϕ does not depend on the particular $f(z)$, but only on $M(\theta)$ and δ. (Received October 25, 1941.)

Applied Mathematics

63. A. E. Engelbrecht: *Circular plates with large deflections.*

The nonlinear system of equations derived by von Kármán is used to obtain a solution for a family of thin circular plates involving radial symmetry and having a uniform moment applied at the periphery. The edge of the plate to which the external moment is applied suffers no displacement normal to the plane of the plate, but is free to move laterally. The solution is effected by expanding the deflection w and the stress function ϕ in terms of a small parameter $\epsilon = h/a$ where h is the plate thickness and a the radius of the plate. By this expansion the nonlinear system reduces to an iterative process for determining the successive terms. Satisfactory numerical results are obtained for the deflection, bending moments and direct planar stresses for plates whose maximum deflection is twice the order of the plate thickness. (Received October 21, 1941.)

64. R. E. Gaskell: *On longitudinal vibrations of a bar.*

A uniform cylindrical bar, built-in at the end $x=0$, has attached to the free end, $x=1$, a weight, W. Each section of the bar has an initial displacement, $f(x)$, and the system is released from rest at $t=0$. The longitudinal vibrations of the bar are found by use of the Laplace transformation in both integral and series form. The solution obtained can be verified, except on characteristic lines, when $f(x)$ is a continuous function with its first three derivatives piecewise continuous and $f^{iv}(x)$ bounded and integrable. (Received November 21, 1941.)

65. A. E. Heins: *On the transformation theory of the solution of partial differential equations.* II. Preliminary report.

A complete transformation theory of the solution of the partial differential equations of mathematical physics in two dimensions was recently considered by the author (abstract 47-5-254). With the aid of the finite Fourier transform, solutions were obtained to the two-dimensional equation of heat conduction under prescribed initial and boundary conditions, when the boundaries were either rectangles or circles. This present paper treats the solution of the same equations when the boundaries are conveniently represented in polar coordinates. Instead of a finite Fourier transform, a finite Hankel transform is employed. The regularity of the finite Hankel transform serves as a condition to eliminate any superfluous boundary elements which do not enter directly into the problem. Some special examples are considered. (Received November 21, 1941.)

66. F. B. Hildebrand: *Note on the integro-differential equation of a problem in the theory of plane stress.*

It has been shown by E. Reissner (Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 300–305) that the stress σ in a stiffener of length L and cross-sectional area A, attached to a semi-infinite elastic sheet in a direction normal to the edge of the sheet, and subjected to a concentrated axial force F, satisfies the Cauchy integro-differential equation $\sigma(x)+A(L)\sigma(L)\{(L-x)^{-1}+K(L,x)\}=\lambda\int_0^L[A(\xi)\sigma(\xi)]'\{(\xi-x)^{-1}+K(\xi,x)\}d\xi$, where $K(\xi,x)=B_1x\xi/(\xi+x)^3+(B_2x+B_3\xi)/(\xi+x)^2+B_4/(\xi+x)$ and λ and the B_i are certain constants depending upon the dimensions and properties of the stiffener and sheet. In the present note, use is made of a classical theorem of Plessner to prove, under certain physically reasonable restrictions on the function $A(x)\sigma(x)$, that a solution of this equation must satisfy the condition $A(L)\sigma(L)=0$, so that the stress is determined by the abbreviated equation $\sigma(x)=\lambda\int_0^L[A(\xi)\sigma(\xi)]'\{(\xi-x)^{-1}+K(\xi,x)\}d\xi$ and the boundary conditions $A(0)\sigma(0)=F$, $A(L)\sigma(L)=0$. (Received November 21, 1941.)

67. J. J. L. Hinrichsen: *Bounds for the libration points in a restricted problem of n bodies.*

Let $n-1$ bodies of equal mass be fixed in the vertices of a regular $(n-1)$-sided polygon and rotate with uniform velocity about their common center of gravity. Consider the motion of an infinitesimal mass, moving in the same plane, attracted by the $n-1$ bodies according to the Newtonian law of attraction. If $\Omega(x, y)$ represents the potential function of the system, bounds are found for the points whose coordinates (x, y) are solutions of $\partial\Omega/\partial x=\partial\Omega/\partial y=0$. For $n>3$, there will exist $3n-2$ such points, one at the center and $n-1$ equally distributed along each of three concentric circles.

They lie on axes of symmetry of the polygon and cannot lie further from the center than $(2n-1)/(n-1)$ of the distance from one of the $n-1$ bodies of equal mass to the center of mass of the other $n-2$ bodies. Approximate positions of the points are obtained for $n=3, 4, 5$. (Received October 25, 1941.)

68. D. L. Holl: *Consolidation of elastic earth layers.*

Biot (Journal of Applied Physics, vol. 12 (1941), p. 155) has established the equations (1) $\nabla^2\overline{U}+k_1$ grad div $\overline{U}-k_2$ grad $\sigma=0$ and (2) $\nabla^2\epsilon=k_3\,\partial\epsilon/\partial t$. For the consolidation of an elastic earth containing pores filled with moisture. The vector $\overline{U}$ is the displacement vector whose div $U=\epsilon$, and σ is the excess hydrostatic water pressure, and k_1, k_2, and k_3 are physical constants. Solutions of (1) and (2) are obtained for a layer infinite in horizontal extent but of finite depth and supported on a rigid surface which is either porous or is impervious to the passage of the imprisoned fluid. Specific solutions are given when the surface loads which cover a rectangular loading area and for axially symmetrical loads are known. (Received October 21, 1941.)

69. Isaac Opatowski: *On the theory of lethal irradiation of microörganisms.* II.

By means of the Laplace transformation three sets of formulas are obtained which give, each one independently of the other one, the vulnerability constants k_i (abstract 47-9-414) and the number of quanta $n+1$ necessary to kill the organism, when the number of killed organisms N_{n+1} is known as a function of the time t. However, the k_i's are uniquely determined only if their relative order of magnitude is known a priori. Two sets of these formulas give the homogeneous product sums $h_r(k_1, \cdots, k_{n+1})$ (D. E. Littlewood, *Theory of Group Characters*, p. 82) whereas a third set gives $h_r(k_1^{-1}, \cdots, k_{n+1}^{-1})$ in terms of the moments of dN_{n+1}/dt. Therefore, the evaluation of k_i's is reduced to the solution of an algebraic equation of degree $n+1$. If $\{k_i\}$ is an arithmetic progression, $N_{n+1}(t)$ reduces to the incomplete beta-function. Biophysically this case corresponds to a progressive destruction of the "sensitive volume" of the microörganism (B. M. Duggar, *Biological Effects of Radiation*, vol. II, p. 1321) by the quanta. The problem here treated is equivalent mathematically to that of finding a differential system of the type stated in the abstract referred to above, which is satisfied by a known function $N_{n+1}(t)$. (Received October 6, 1941.)

70. Willy Prager: *Fundamental theorems of a new mathematical theory of plasticity.*

The main difficulty all attempts at a mathematical theory of plasticity have, so far, had to cope with arises from the assumption that the material will not behave in a plastic manner unless a certain invariant of the stress tensor has reached a given critical value. Two different sets of equations will, therefore, be valid in the plastified and the not yet plastified regions. The problem becomes all the more involved by the fact that the boundary between these regions is not known beforehand but has to be determined so as to secure continuity of stresses. In order to avoid this difficulty, the author has proposed stress-strain relations giving a gradual transition from the elastic to the plastic state (Proceedings of the Fifth International Congress of Applied Mechanics, Cambridge, Massachusetts, 1938, p. 234). These relations have been applied to various problems of plane strain (Revue de la Faculté des Sciences de l'Université d'Istanbul, (A), vol. 5 (1941), p. 215). The present paper contains two variational principles which, in this new mathematical theory of plasticity, play the same role as the principle of least work and Castigliano's principle do in elasticity. (Received November 24, 1941.)

71. J. L. Synge: *On the theory of the airfoil of finite span.*

The problem of the potential flow past an airfoil of finite span is formulated, and then linearized by the usual approximation based on the assumptions that the thickness and curvature of the airfoil are small. The problem is reduced to the solution of an integro-differential equation, which agrees with that obtained by Mattioli in 1939 for a flat plate of no thickness. (Received November 21, 1941.)

72. Feodor Theilheimer: *The potential of curvilinear distributions.*

This paper deals with potentials which are induced by charges distributed along curve segments. It is assumed that the curves and the charge distributions are rational functions of a suitable parameter. Then it can be shown that there exists for every curve a finite number of transcendental functions $p_k(x, y, z)$ such that the potential $h(x, y, z)$ can be represented in closed form by means of these functions $p_k(x, y, z)$ together with algebraic and theta functions. (Cf. Bergmann, Mathematische Annalen, vol. 101 (1929), pp. 534–558.) If the curves can be represented as rational functions of second degree the functions $p_k(x, y, z)$ are connected in a simple way with the elliptic functions, and tables can be used for actual computation. These results supply a method for approximate solution of certain boundary value problems. This method is here applied to an aerodynamic problem taken from the theory of the finite air wing. (Received November 22, 1941.)

73. R. H. Tripp: *Bending of a thin plate having the form of a parallelogram.*

The Lagrange plate equation $N\nabla^4 w = p(x, y)$ is solved for a thin plate whose boundary is a parallelogram with opposite angles $\pi/4$ and $3\pi/4$. Two types of loading are considered: (1) a concentrated load at the center of symmetry and (2) a uniformly distributed load. When all the edges are simply supported the problem becomes that of the solution of two linear infinite systems of equations in infinitely many unknowns of the type $x_i + \sum_j a_{ij} x_j = b_i$. The properties of the determinant $|\delta_{ij} + a_{ij}|$ and b_i are such as to insure the existence and convergence of the solution. Derived results for the moments are discussed. (Received October 21, 1941.)

74. Alexander Weinstein: *On the flexural center and the center of twist.*

These concepts have been treated simultaneously by several authors by Saint-Venant's theory of beams. These authors introduced two essentially different definitions of the flexural center and three essentially different definitions of the center of twist. On the other hand, Sothwell has given a descriptive definition of both centers and has outlined a proof of their coincidence provided that one end of the beam is held by rigid constraints, a situation which cannot be met by Saint-Venant's theory. It is shown in the present paper that the coincidence of both centers can be established by using definitions previously employed by the authors who worked with Saint-Venant's methods. (Received November 21, 1941.)

GEOMETRY

75. L. M. Blumenthal: *Metric characterisation of n-dimensional elliptic space $\mathcal{E}_{nr}$*. Preliminary report.

The objective is the characterization of the elliptic metric by means of relations between mutual distances of points in certain finite subsets of the space. If δ-supplementation denotes the process by which a semi-metric Σ^* arises from a semi-metric Σ of diameter d upon replacing arbitrary distances pq in Σ by $\delta - pq$, $\delta \geqq d$, and identifying points with zero distance in Σ^*, then $\mathcal{E}_{nr} = \sup_{\pi r} S_{nr}$, where S_{nr} is the metrically convex spherical surface of radius r and dimension n. It is proved that semi-metric Σ is congruently contained in $\mathcal{E}_{nr}$ if and only if $p, q \in \Sigma$ implies $pq \leqq \pi r/2$ and there exists a $\sup_{\pi r}\Sigma$ congruently contained in S_{nr}. A semi-metric m-tuple $p_1, p_2, \cdots, p_m$ with $p_i p_j \leqq \pi r/2$ is congruently contained in $\mathcal{E}_{nr}$ if and only if a symmetric square matrix (ϵ_{ij}), $\epsilon_{ii} = 1$, $\epsilon_{ij} = 1$ $(i, j = 1, 2, \cdots, m)$, exists such that the determinant $|\epsilon_{ij} \cos (p_i p_j/r)|$ has rank not exceeding $n+1$ with all nonvanishing principal minors positive. This puts in algebraic form the determination of congruence indices for the $\mathcal{E}_{nr}$, at least with respect to finite semi-metric sets. (Received October 24, 1941.)

76. Nathaniel Coburn: *Unitary curves in unitary space.*

The question of the existence of an arc length parameter for a unitary curve K_1 imbedded in an n-dimensional unitary space K_n is discussed. First, the familiar formula for arc length element (ds^2) is generalized. Then, it is shown that if and only if K_1 possesses a natural parameter, does an arc length parameter which is an analytic function of the curve parameter exist. In fact, ∞^1 such parameters exist; all have the same absolute value but different moduli. If the curve parameter is real (K_1 reduces to X_1), then again ∞^1 such arc length parameters exist. One and only one of these parameters is real and positive; this parameter is the one commonly associated with X_1 in K_n. The remainder of the paper is concerned with determining those K_n into which can be imbedded various classes of K_1 which possess an arc length parameter (such K_1 are denoted by U_1). The principal result is: If the metric tensor of K_n is not of rank one, then those unitary K_1 which satisfy a system of differential equations of the third or higher order in the parameter are not U_1. (Received October 24, 1941.)

77. N. A. Court: *On the theory of the tetrahedron.*

A "quasi-polar" sphere (Q) may be associated with the general tetrahedron (T) having for center the Monge point M of (T) and for the square of its radius one third of the power of M for the circumsphere (O) of (T). The following two propositions may serve as samples of the many properties of (Q): The "quasi-polar" sphere is coaxial with the circumsphere (O) and the twelve point sphere (L) of (T); The polar reciprocal tetrahedron of (T) with respect to the sphere (Q) is circumscribed about the medial tetrahedron of (T). A second sphere (G) may be related to (T) having for center the centroid G of (T) and for the square of its radius one forty-eighth of the sum of the squares of (T). The sphere (G) is orthogonal to (Q) and is coaxial with (Q), (O), and (L). The four spheres having for centers the vertices of (T) and orthogonal to (Q) cut the spheres having for diameters the respective medians of (T) along four circles lying on the same sphere, namely the sphere (G) of (T). In the special case when the tetrahedron becomes orthocentric, the spheres (Q) and (G) become, respectively, the polar sphere and the first twelve point sphere of the orthocentric tetrahedron. (Received November 21, 1941.)

78. J. W. Peters: *The euclidean geometry of the n-dimensional simplex.*

In this paper theorems associated with the triangle and the tetrahedron are extended to the n-dimensional simplex formed by $n+1$ points in a euclidean space of n dimensions. The centroid and Monge point of the simplex as well as the centroids and Monge points of the faces are defined. The following extension of Mannheim's theorem for a tetrahedron is proved. The $n+1$ planes determined by the $n+1$ altitudes of the simplex and the Monge points of the corresponding faces meet in the Monge point of the simplex. A hypersphere on the centroids of the faces is discussed. It is shown that this hypersphere passes through $3(n+1)$ points associated with the simplex and has a number of properties similar to the nine point circle associated with a triangle and with the twelve point sphere associated with a tetrahedron. (Received November 8, 1941.)

79. J. L. Vanderslice: *Invariant theory of vector pencil fields.*

At each point $(x^1, \cdots, x^n)$ of a space subject to general coordinate transformations is associated a pencil of contravariant vectors, $\xi^i(x^1, \cdots, x^n, u)$. The parameter u of the pencil is normalized to give an invariant metric parameter x^0 associated with each coordinate system x^i and transforming like the gauge variable of generalized projective geometry. The principal result is the discovery of an affine connection with components which are functions of x^α $(\alpha=0, 1, \cdots, n)$ and a method of covariant differentiation of tensor functions of x^α. A study of the equivalence problem then leads to a complete set of tensor invariants for the vector pencil field. (Received November 17, 1941.)

80. André Weil and C. B. Allendoerfer: *A general proof of the Gauss-Bonnet theorem.*

The following generalized Gauss-Bonnet theorem was recently proved independently by Allendoerfer and Fenchel: If a closed Riemann manifold R_n of even dimension can be made a subspace of an euclidean space then $\int KdO = 1/2\omega_n\chi$ where K is the total curvature of the manifold, ω_n is the area of an n-dimensional sphere, and the integration is over the manifold. The present paper removes the restriction that R_n be a subspace of an euclidean space. To do this R_n is subdivided into simplices each of which is small enough to have an isometric euclidean imbedding. The method of tubes is applied to these subdivisions separately, special attention being paid to their boundaries. The terms resulting from the boundaries are found to be intrinsic and drop out when the simplices are reassembled to form R_n. (Received November 26, 1941.)

Logic and Foundations

81. E. C. Berkeley: *Application of symbolic logic to punch card operations.*

This paper discusses the analysis of operations with punched tabulating cards, and also to some extent operations with handwritten cards, as taking place in a large life insurance company, for purposes of valuing policies, computing, recording, and summarizing payments, and so on. The chief instrument of analysis is a system of coding, constructed using symbolic logic and other techniques. The system of coding is exhibited in part, and examples of the coding are given. Some related problems of practical and theoretical nature are described and discussed. (Received November 21, 1941.)

82. G. D. W. Berry: *On formalising semantics.*

Two alternative formalizations of semantics may be distinguished. Quine's ***Mathematical Logic*** (New York, 1940) with modifications, provides the logical and syntactical components of each. In each, designation is primitive, denotation and truth defined. Besides including statements specifying that all designated expressions are elements, that only abstracts designate, and that nothing designates more than one thing, the axioms of both systems include all statements formed from 'If — is designated, then '—' designates it' by replacing the blanks with any constant term. The paradoxes of Grelling, Richard, Konig, and Epimenides each requires an hypothesis of the form, '— is designated.' In the first system, which has only one designation-relation, each paradox becomes a ***reductio ad absurdum*** argument for the existence of an undesignated entity. The argument of the Epimenides, in particular, establishes the existence of non-Tarskian statements, or statements equivalent to the denial of their truth. Some axioms of the form '— is designated' are adopted. Others may be added piecemeal, when needed, at the investigator's risk. In the second system, which has infinitely many designation-relations, each statement of this form is an axiom for some appropriate designation-relation. Classes undesignated and statements non-Tarskian relative to one such relation are respectively designated and Tarskian relative to another. (Received November 26, 1941.)

83. Alonzo Church: *On sense and denotation.*

The title is intended to translate Frege's ***Über Sinn und Bedeutung***. The ***denotation*** of a proper name (including descriptions, class abstracts, also sentences as proper names of truth-values) is that of which it is a name. One may ***understand*** a proper name in the sense of knowing its meaning linguistically, yet not know its denotation. This linguistic meaning is the *sense*. In particular, the sense of a sentence is the proposition. Two proper names coinciding in sense must have the same denotation—although to determine the denotation, given the sense, may require settling a question of extra-linguistic fact. If a constituent part of a proper name is replaced by another having the same sense, the sense of the whole is not altered; if it is replaced by another of the same denotation, the denotation of the whole is not altered, but the sense may be. Thus is solved, in particular, Russell's puzzle about "the author of Waverley." Assuming tacitly that names have only one kind of meaning, the denotation, Russell concluded that a logically sound language cannot employ descriptions: sentences containing descriptions must be reconstrued as mere convenient abbreviations of sentences of a different form. This seems to be tenable but is less elegant that Frege's solution of the same problem. (Received November 21, 1941.)

84. Nelson Goodman: *Sequences.*

The only method hitherto available for defining sequences on the basis of class theory results in identifying a sequence of n components with a certain class n types higher than those components. By this method, the sequence $x_1 \cdots x_n$ becomes: $\hat{s}\{(s\in 1: s\subset x_1 . \vee . x_1 \subset s . x_1 \neq \Lambda) \vee (s\in 2: s \subset x_2 . \vee . x_2 \subset s . x_2 \neq \Lambda) \vee \cdots (s \in n: s \subset x_n . \vee . x_n \subset s . x_n \neq \Lambda)\}$. The method works for a sequence of finite length n in a universe containing at least $n+1$ elements of the type of members of components of the sequence. That the components must be classed can be shown to be no actual restriction. Any component x_k of a sequence Q is defined as: $\hat{s}\{s \in p'(K \cap Q) . \vee . p'(K \cap Q)$

$=\Lambda\,.\,s\in s'(K\cap Q)\}$. The above schemata may be supplanted by formal definitions of the class of sequences and of the kth component of any sequence. Every class of classes is correlated to some one sequence by a definable relation we call the "establishment" of the sequence by the class. A class that establishes a sequence having the members of the class, and the null class, as its components is called "self-ordered." (Received November 21, 1941.)

85. S. C. Kleene: *On the interpretation of intuitionistic number theory.*

A closed existential statement $(Ex)A(x)$ can be interpreted from the constructivist standpoint as a partial communication of a more explicit statement which gives a number x such that $A(x)$ holds, together with such further information as is required to complete the meaning of $A(x)$ for that x. Likewise, a closed generality statement $(x)A(x)$ can be interpreted as the assertion of the possibility of describing an effective general method for obtaining, to any given x, such information as will complete the meaning of $A(x)$ for that x. These ideas are made precise in a truth-definition for the class of statements formed from given general recursive predicates of natural numbers by the operations of the predicate calculus. The definition will be used to obtain some metamathematical results for the intuitionistic predicate calculus and number theory. Typical clauses, for closed statements: If $A(n)$ is realized by a, $(Ex)A(x)$ is *realised* by $2^{n}\cdot 3^{a}$. $(x)A(x)$ is *realised* by the Gödel number e of a general recursive function $\phi(x)$ such that, for every n, $\phi(n)$ realizes $A(n)$. $A\rightarrow B$ is *realised* by the Gödel number e of a partial recursive function $\phi(x)$ such that, whenever a realizes A, $\phi(a)$ realizes B. Then A is *realisable*, if there is a number a which realizes it. (Received November 21, 1941.)

86. Barkley Rosser: *The Burali-Forti paradox.*

Four basic principles of the theory of ordinal numbers and well-ordered series are: (1) To every well-ordered series there corresponds a unique ordinal number. (2) The series of ordinal numbers is well-ordered. (3) If x is a term of a well-ordered series S, then the series consisting of all terms of S which precede x is also well-ordered, and has a smaller ordinal number than S. (4) Any ordinal number, α, is the ordinal number of the series of all ordinals which precede α. The Burali-Forti paradox is the statement that these four principles are incompatible. To avoid the paradox, two devices have been tried. One is to adopt some form of a theory of types which will invalidate (4). The other device has the effect of invalidating (1) in certain critical cases. In his book, *Mathematical Logic*, Quine proposed a device which is essentially a combination of the two aforementioned devices. Unfortunately Quine's device does not invalidate (4) and fails to invalidate (1) at certain critical points, and so the Burali-Forti paradox appears in Quine's book. This is shown in detail in the present paper. (Received November 24, 1941.)

87. A. R. Schweitzer: *On the genesis of the algebra of logic in the foundations of geometry.*

Following his theory of geometrical relations (American Journal of Mathematics, vol. 31 (1909), p. 400), the author separates descriptive n-space ($n=1, 2, 3, \cdots$) into 2^{n+1} compartments $C(\ )$, $C(\alpha_i)$, $C(\alpha_i\alpha_j)$, $\cdots$, $C(\alpha_1\alpha_2\cdots\alpha_{n+1})$ by means of two n-simplexes T_1 and T_2 such that $T_1=\alpha_1\alpha_2\cdots\alpha_{n+1}$ with sides prolonged is in the interior of, and cut off by, T_2. Then $\sum(I)$ denotes the reflexive formal sum of all C's; $\sum(\alpha_1)$ is the sum of all C's containing α_1 as argument, and so on; $\sum(\lambda_1)$ is the sum of all C's not containing α_1, and so on. Then $C(\ ) = \sum(\lambda_1\lambda_2 \cdots \lambda_{n+1})$, $C(\alpha_1) = \sum(\alpha_1\lambda_2 \cdots \lambda_{n+1})$ and so on. The C's are replaced by their corresponding expressions in terms of $\sum$ and the subsequent development follows that of an earlier paper reported in this Bulletin (abstract 47-9-430). The preceding provides a representation R_n for the latter exposition; R_2 is equivalent to the so-called Euler diagram which in turn is topologically equivalent to the figure of a triangle in a descriptive plane extended in accordance with the author's system $^2K_2$ (ibid., p. 395, Axiom 12). R_3 is topologically equivalent to the figure of a trifoliate curve inscribed in a circle. Another basis for the algebra of logic in the author's theory is found in a simple modification of his axioms for a linear "permutation" (ibid., p. 370). (Received November 24, 1941.)

STATISTICS AND PROBABILITY

88. R. D. Gordon: *An application of the Cauchy integral to a problem in probability theory.*

The author was consulted about interpretations of certain data on biological populations. The problem aims at obtaining error corrections on the data. The classical basis of the Bayes-Laplace formula is chosen: $(x)(n/x) = (n)(x/n)$ where the notation has the usual meaning in probability theory. The probability (n/x) is known through its generating function. The distribution (x) of the parameter x, an integer, is taken on reasonable grounds to be $(x) = 1/b+1$, for $0 \leq x \leq b$; $(x) = 0$ otherwise. Then $(n) = \sum(x)(n/x)$ is determined. An "observation" results in a value n_0 of the stochastic variable n. The problem: to determine the distribution $(n/n_0) = \sum(n/x)(x/n_0)$. We actually obtain the generating function for the *moments* of (n/n_0). Simple manipulations result in the Cauchy-integral equation $(x)G_{(n/n_0)}(t;\ x) = 1/2\pi i\int_{(\Gamma)} G_{(n)}(t/s) \cdot E(s; x)ds/s$ where the G's represent generating functions for the probability distributions indicated in their subscripts, and $E(s;\ x)$ represents an *unknown* generating function for probabilities (x/n) *with respect to* n. The problem is solved by applying standard generating-function operators to this equation and integrating over the circle $|s| = t$. (Received November 14, 1941.)

89. G. F. McEwen: *On the probability that a ratio of random numbers will depart from a harmonic ratio by less than a given amount.*

The ratio of two numbers is called harmonic if it equals the ratio of two small integers, one figure numbers, for example. In practical applications both numerator and denominator are subject to error and their ratio accordingly departs from perfect harmony. Accordingly, it is necessary to estimate the probability P of getting by chance a ratio departing less than a given amount from perfect harmony. This problem of determining the proportion, among all possible combinations of numbers, that have a ratio departing from perfect harmony less than this amount, is equivalent to that of finding the probability P that the ratio of any two numbers drawn at random will depart by an amount X or less from the harmonic ratio N/M. The solution is $P = 2(M^2/N)X/[1-(M/N)^2X^2]$, where $N > M$. (Received October 27, 1941.)

TOPOLOGY

90. W. W. Flexner: *Noncommutative chains.* II. Preliminary report.

Extending the author's recent work (Duke Mathematical Journal, vol. 8 (1941), pp. 497–505) to a finite convex 3-complex, 2-dimensional noncommutative chains and a "homology group" Π_2 are defined. The 2-chains C are products of basic 2-chains $\Omega = aEa^{-1}$ where E is a 2-cell or its inverse and a is a noncommutative one-chain such that $F\Omega = a(FE)a^{-1}$, where FE is the boundary of E, is a continuous closed "path" beginning and ending at a fixed point ω. The group of 2-chains is subject to the relations $CC'C^{-1} = (FC)C'(FC)^{-1}$. The subgroups $\mathfrak{Z}_2$, of cycles, and $\mathfrak{J}_2$, of bounding cycles, are then defined much as in the commutative case. If $\Pi_2 = \mathfrak{Z}_2/\mathfrak{J}_2$, it is commutative and subdivision invariant. (Received November 24, 1941.)

91. D. W. Hall: *On a theorem of E. E. Betz.*

In a recent paper (American Journal of Mathematics, vol. 63 (1941), pp. 127–135) E. E. Betz has studied compact locally connected continua M which satisfy the following conditions: (a) no two points of M separate M, (b) every simple closed curve J of M separates M into at least two and at most a finite number of complementary domains. He has obtained a rather general accessibility theorem for these continua. Using the result of Betz the present paper shows that if M be any compact locally connected continuum satisfying (a) and (b) and if G is any open subset of M, then there is an open subset R of M contained in G which is homeomorphic with an open 2-cell. (Received November 25, 1941.)

92. R. C. James: *Linearly arc-wise connected topological groups.*

The definitions of arc-wise connected, locally arc-wise connected, and simply connected topological spaces are given in terms of a continuous function with real numbers as arguments and values in the topological space. For topological groups, one can add the restriction that the function be linear (additive and continuous) and obtain definitions of linearly arc-wise connected and locally linearly arc-wise connected topological groups. In this paper certain of these properties are used as conditions for a topological abelian group to be a linear topological space. It is shown that a topological abelian group which is simply connected and locally linearly arc-wise connected either has no nonzero elements of finite order or a dense set of elements of the nth order (for each $n > 1$). Several necessary and sufficient conditions for a topological abelian group T to be a linear topological space are found, two of which are: (1) T be connected, locally linearly arc-wise connected, and have no nonzero points of finite order; (2) T be simply connected, locally linearly arc-wise connected, and have a neighborhood of the identity which contains no elements of the second order. (Received October 24, 1941.)

93. R. C. James: *Normable topological abelian groups.*

Several equivalent definitions have been given of bounded sets and convex neighborhoods of a linear topological space. In this paper these concepts are generalized to topological groups, and are used with linear arc-wise connectedness (connectedness by paths defined by a linear function of a real variable) to find conditions that a topological abelian group be normable and that it be a linear topological space. It is shown that a convex, linearly arc-wise connected topological abelian group which has no nonzero elements of finite order is a convex linear topological space, and that a normable, linearly arc-wise connected topological abelian group is a normable linear topological space. A theorem of Komogoroff's on the normability of linear topological spaces is generalized to linearly arc-wise connected topological abelian groups. (Received October 24, 1941.)

94. Saunders MacLane and Samuel Eilenberg: *A theorem on group extensions.*

An abelian group extension of a group G by an abelian group H is customarily represented either by means of a factor set of H in G, or by means of a multiplication table for the generators of H. The latter depends essentially on the representation of H as a factor group $H = F - B$ of a free group F. The exact equivlaence of these two descriptions of a group extension turns out to be important in the study of homology groups (abstract 47-11-487). It may be stated as follows: If F is a free group, and $H = F - B$, then the group Fact $\{G, H\}$ of (abelian) factor sets of H in G modulo the transformation quantities is isomorphic to the group Hom $\{B, G\}$ of all homomorphisms of B into G, modulo the subgroup of the homomorphisms of B into G which can be extended to F. If G is topological, so is this isomorphism. (Received November 24, 1941.)

95. Harlan C. Miller: *On irreducible continua.*

This paper is concerned with properties of compact continua which are irreducible between some two points, and especially of those which are unicoherent. Some of the results are: If M is a compact atriodic unicoherent continuum, and H and K are proper subcontinua such that $M = H + K$, then H and K are unicoherent. If M is a compact hereditarily decomposable continuum irreducible between some two points, then M contains two mutually exclusive continua, H and K, such that (1) in order that M be irreducible between two points it is necessary and sufficient that one of the points belong to H and the other to K, and (2) $M-(H+K)$ is strongly connected. If M is a compact hereditarily decomposable continuum irreducible from point A to point B, there exists an upper semi-continuous collection, G, of mutually exclusive continua filling up M such that if H and K are the elements of G containing A and B, respectively, then with respect to its elements, G is an arc from H to K; in order that M be unicoherent it is necessary and sufficient that G be a continuous collection. (Received November 25, 1941.)

96. Knox Millsaps: *First order differentials of functions with arguments in topological groups and values in topological abelian groups.*

A first order differential is defined for a function $F(x)$ with arguments in a topological group G and values in a topological abelian group A and increments in the central subgroup of G with a relativized topology and a generation postulate. The fundamental theorems on uniqueness for increments in the central subgroup, continuity, linear combinations, and the iterative function $K(F(x))$, where $K(y)$, has arguments and values in t.a.g.'s and is M_1-differentiable (see Michal, Revista de Ciencias), are then proved. (Received October 24, 1941.)

97. M. E. Shanks: *Irreducible transformations of the discontinuum into continua.* Preliminary report.

A continuous transformation T of the Cantor discontinuum C into a continuum is strongly irreducible on a closed subset homeomorphic to C. Irreducible transformations of C into locally connected continua are studied in particular with special reference to the one-dimensional case. Certain results are obtained which indicate a simple structure for T in special one-dimensional cases. (Received October 24, 1941.)

98. R. H. Sorgenfrey: *Some theorems on co-terminal arcs.*

The following results have been established: If H denotes the sum of two arcs with common end points and K denotes their common part, then every component of $H-K$ is a subset of some simple closed curve lying in $\overline{H-K}$. If three arcs have a common end point and no one of them is a subset of any other, then one of them fails to be a subset of the sum of the other two. If three arcs have two end points in common then there are two of them neither of which is a subset of the sum of the remaining two. (Received November 25, 1941.)

99. P. M. Swingle: *Topological foundations of an uncertain mathematics.*

In this paper a probable origin is given to point sets, as in quantum physics, in a universe of incessantly moving electrons. Philosophical consideration is given to the origin of ideas in such a universe and the conclusion reached that there is little reason for the certainty that exists in traditional mathematics. The nature of a probability topology is considered, a degree of certainty being reached in an abstract topology having an interpretation in what is called a diophantine space, whose points are "cube"-units. (Received October 27, 1941.)

100. P. A. White: *R-regular convergence spaces.*

In this paper a hyperspace K_r of a compact metric space M is considered, which consists of all closed locally γ^s-connected $(s\leq r)$ subsets of M. An infinite sequence $\{p_i\}\subset K$ will be said to converge to $p\subset K$ provided the corresponding subsets $\{P_i\}$ of M converge s-regularly to P, $(s\leq r)$. This definition of convergence is used to define limit point, closed set, and open set in the obvious manner. It is shown that the original definition of limit point is equivalent to an open set definition, and that K will be a regular Hausdorff space satisfying the first axiom of countability with these open sets as neighborhoods. It is also shown that unless M is finite, K can be written as the sum of a countable number of sets which are both open and closed, and that not more than one of these sets can be compact. (Received November 24, 1941.)

101. G. W. Whitehead: *Homotopy groups of spheres and their rotation groups.*

In this paper relations between the homotopy groups of the n-sphere S^n and those of its rotation group R_n are studied. The relationship between these groups is established as follows: to each map $f(S^m\times S^n)\subset S^n$ there is associated a mapping $f^*(S^{m+n+1})\subset S^{n+1}$. This operation is a generalization of one introduced by H. Hopf (Fundamenta Mathematicae, vol. 25 (1935), pp. 427–440), and induces a homomorphism of $\pi_m(R_n)$ into $\pi_{m+n+1}(S^{n+1})$. For $m=1, 2$ this homomorphism is an isomorphism. This isomorphism is used to construct a counterexample to a theorem stated without proof by H. Freudenthal (Proceedings, Akademie van Wetenschappen te Amsterdam, vol. 42 (1939), pp. 139–140). In particular, Freudenthal's construction of maps of S^{2n-1} on S^n with Hopf invariant 1 fails if $n\equiv 2$ (mod 4). (Received November 24, 1941.)

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THE PROBLEM OF BOLZA IN THE CALCULUS OF VARIATIONS

M. R. HESTENES

1. **Introduction.** The problem of Bolza can be described briefly as the most general problem in the calculus of variations for which there exists at the present time a theory of relative maxima and minima that is comparable in completeness to those of the simpler problems in the calculus of variations. This completeness has been brought about in the last few years and it is the purpose of the present paper to discuss in some detail the results that have been achieved and the methods that have been used in obtaining them. It is impossible to give here an adequate discussion of the various aspects of the problem of Bolza. Nor is such a discussion necessary, inasmuch as three excellent reports have been given already which were concerned in whole or in part with the problem of Bolza. The first of these reports, given in 1936 by Professor Bliss [14],[1] is devoted to the study of the evolution of problems in the calculus of variations and, in particular, the evolution of the problem of Bolza. Professor Bliss pointed out that even Euler and Lagrange formulated problems that are of essentially the same generality as the problem of Bolza. Moreover, they derived in a formal way the Euler-Lagrange equations that the solutions of the problems must satisfy. The second report was given in 1937 by Professor Reid [17] and was concerned with boundary value problems and their relations to problems in the calculus of variations. In particular, he pointed out the various relationships between the problem of Bolza and boundary value problems. The third and final report was given in 1938 by Professor McShane [22] who outlined the progress that had been made in the calculus of variations during the preceding twenty-five years. Here the essential achievements in the theory of the problem of Bolza were described but nothing was said as to how these results were obtained. It is the purpose of the present paper to discuss certain interesting and important aspects of the theory of the problem of Bolza that Professors Bliss, Reid and McShane of necessity had to omit or describe inadequately in their reports. In particular I shall describe to you the basic ideas involved in the sufficiency proofs for relative minima. The progress made in

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An address delivered before the Chicago meeting of the Society on April 11, 1941, by invitation of the Program Committee; received by the editors June 2, 1941.

[1] Numbers in brackets refer to the list of papers at the end of this paper.

the technique of sufficiency proofs not only has enabled us to obtain better sufficiency theorems but also has in a sense unified the sufficiency proofs of the fixed end point problems and the variable end point problems.

There are certain phases of the problem of Bolza that I too must omit in the present report. I shall not discuss the parametric case; nor shall I have the occasion to describe the Hamilton Jacobi theory and the interesting approach to the theory of the problem of Lagrange made by Carathéodory. I shall not describe how the theory can be modified so as to take care of discontinuous solutions and many problems that are essentially of Bolza type. Finally I shall omit the important topic of existence theorems for the problems of Bolza and Mayer developed by Graves, McShane and others. References to these and other topics can be found in the papers listed at the end of the present paper.

2. **Formulation of the problem.** The problem of Bolza can be formulated in a number of ways, each of which has its peculiar advantages and disadvantages. One of the most useful formulations is the one given by Professor Bliss [4] in 1932. This formulation closely resembles the original one given by Bolza [1] in 1913 and can be described briefly as follows: Consider a class of arcs

$$(1) \qquad y_i(x), \qquad x_1 \leqq x \leqq x_2;\ i = 1, \cdots, n,$$

in $(xy_1 \cdots y_n)$-space satisfying a prescribed set of differential equations and end conditions

$$(2) \qquad \phi_\gamma(x, y, y') = 0, \qquad \gamma = 1, \cdots, m < n,$$

$$(3) \qquad \psi_\mu[x_1, y(x_1), x_2, y(x_2)] = 0, \qquad \mu = 1, \cdots, p \leqq 2n + 2.$$

We seek to minimize in this class a function of the form

$$(4) \qquad I = g[x_1, y(x_1), x_2, y(x_2)] + \int_{x_1}^{x_2} f(x, y, y')dx.$$

The case when $g=0$ is the so-called problem of Lagrange and when $f=0$ we have the general problem of Mayer as formulated by Bliss. These three problems are equivalent in the sense that each can be transformed into either one of the other two types. Of the three problems, the problem of Bolza appears to be the most convenient.

The formulation of the problem of Bolza just described has been one of the most popular ones in recent years and for many purposes it is the most convenient one. However, there is a second formulation that appears to be more useful for the purposes of the present report.

It is a modification of one given by Professor Morse [3] and can be described as follows: Consider a class of arcs

$$(5)\qquad a_k,\ y_i(x), \qquad x_1 \leqq x \leqq x_2;\ k = 1, \cdots, r;\ i = 1, \cdots, n,$$

in axy-space whose components a_k are independent of x and which satisfy conditions of the form

$$(6)\qquad \phi_\gamma(a, x, y, y') = 0, \qquad \gamma = 1, \cdots, m < n,$$

$$(7)\qquad x_s = x_s(a),\quad y_i(x_s) = y_{is}(a), \qquad s = 1, 2;\ i = 1, \cdots, n.$$

We seek to minimize in this class a function of the form

$$(8)\qquad I = g(a) + \int_{x_1}^{x_2} f(a, x, y, y')dx.$$

This formulation differs from the one first described in that certain components of our arcs, denoted by a_k, are known to be constants. If we consider these constants as functions $a_k(x)$ satisfying the differential equations $a_k'(x) = 0$ and if the a's appearing in the functions $x_s(a)$, $y_{is}(a)$, $g(a)$ are replaced by $a_k(x_1)$, the problem becomes one of the type formulated by Bliss. On the other hand, the first formulation can be transformed into one of the second type by a simple device.

I remarked a moment ago that the formulation of the problem of Bolza just described is a modification of one given by Morse. It differs from the one given by Professor Morse in that we have introduced the parameters a_k in the functions f and ϕ_γ. Although no generality is gained by this change, it has been made for the following two reasons: In the first place in the study of Mayer fields one is led in a natural way to the study of an auxiliary problem in which the parameters a_k appear in the integrand. Thus by introducing these parameters in the original integrand one is able to apply at once to the auxiliary problem the results previously obtained. This procedure is analogous to the one introduced by Bliss in the study of the second variation and simplifies the arguments that have to be made. A second trivial advantage is that the class of problems that are immediate special cases is enlarged, although the class of problems that can be transformed into one of Bolza type is unaltered.

No discussion of the problem of Bolza would be complete if it did not include a consideration of the isoperimetric problem of Bolza [24, 25]. This problem differs from the one just described in that the arcs (5) are required to satisfy, besides equations (6) and (7), a set of isoperimetric conditions

$$I_\rho = g_\rho(a) + \int_{x_1}^{x_2} f_\rho(a, x, y, y')dx = 0, \qquad \rho = 1, \cdots, p. \tag{9}$$

Although this problem can be reduced to one of Bolza type, the present theory for the problem of Bolza does not give adequate sufficiency theorems for this problem. An additional argument is needed. I shall discuss this point later. I would like to point out, however, that once one has obtained a satisfactory theory for the isoperimetric problem of Bolza, these results can be applied at once to a large variety of problems. In particular, they can be applied to the usual isoperimetric problems and to the more complicated problem of minimizing functions of integrals. The latter problem was proposed by Euler and has been studied recently by Brady [23].

3. **Normality.** One of the most important advances in the theory of the problem of Bolza has been in the elimination of undesirable assumptions of normality. The concept of normality arises when one considers the problem of imbedding a particular arc E satisfying equations (6) and (7) in a one-parameter family of arcs

$$a_k(t), \qquad y_i(x, t), \qquad x_1(t) \leqq x \leqq x_2(t), \tag{10}$$

satisfying these equations. If the family (10) contains E for $t=0$, its variations

$$\alpha_k = a_k'(0), \qquad \eta_i(x) = y_{it}(x, 0), \qquad x_1 \leqq x \leqq x_2, \tag{11}$$

along E are solutions of equations of the form

$$\Phi_\gamma(\alpha, x, \eta, \eta') = \phi_{\gamma a_k}\alpha_k + \phi_{\gamma y_i}\eta_i + \phi_{\gamma y_i'}\eta_i' = 0, \quad \gamma = 1, \cdots, m, \tag{12}$$

$$\eta_i(x_s) = c_{isk}\alpha_k, \qquad s = 1, 2;\ i = 1, \cdots, n. \tag{13}$$

Unfortunately, not all solutions α_k, η_i of these equations are necessarily variations (11) of a family (10) that satisfies equations (6) and (7) and contains E for $t=0$. Perhaps the simplest criterion that will insure this property is the existence of a set of $2n$ solutions α_{kj}, $\eta_{ij}(x)$ $(j=1, \cdots, 2n)$ of equations (12) whose determinant

$$\begin{vmatrix} \eta_{ij}(x_1) - c_{i1k}\alpha_{kj} \\ \eta_{ij}(x_2) - c_{i2k}\alpha_{kj} \end{vmatrix}, \qquad j = 1, \cdots, 2n, \tag{14}$$

is different from zero. If this criterion is satisfied, the arc E is said to be *normal* relative to the end conditions (7). It is clear that E will be normal if the determinant (14) can be made different from zero by a set of $2n$ solutions α_{kj}, $\eta_{ij}(x)$ of equations (12) having $\alpha_{kj}=0$. If this stronger condition is satisfied, the arc E is said to be *normal on* *the interval* x_1x_2, inasmuch as this criterion is a condition of normality for the fixed end point problem obtained by holding the a's fast. An excellent discussion of the concept of normality and its consequences has been given by Bliss [21].

Until recently the concept of normality was either ignored or else strong normality assumptions were imposed. In fact, it was customary to suppose that the arc under consideration was normal on every subinterval and in the sufficiency proofs it was further assumed that an extension of the arc had this property. The effect of these normality assumptions was that the arguments used for simpler problems could be carried over to the problem of Bolza with the obvious modifications. Unfortunately the minimizing arcs of many important problems that are of the Bolza type are necessarily abnormal on every subinterval. For example, this is true for the problem of Mayer. Thus the earlier theories for the problem of Bolza were not applicable to these special cases. A further objection to normality assumptions is that it is difficult in a particular case to verify these assumptions. It would be desirable to proceed without the use of hypotheses of normality and much has been done in this direction. The sufficiency theorems that are now known involve no assumptions of normality. Moreover McShane [26] has shown that a minimizing arc necessarily satisfies (with multipliers $l_0 \geqq 0$, $l_\gamma(x)$): (I) the Euler-Lagrange equations and the transversality condition; (II) the condition of Weierstrass; (III) the condition of Clebsch, even if the arc is abnormal. Graves [5] had shown earlier that these results were true for arcs that are normal relative to the end conditions (7) but are not normal on every subinterval as was previously supposed. He also gave rather complicated analogues of the Weierstrass condition for abnormal arcs. As yet it appears to be necessary to assume that the arc under consideration is normal relative to the end conditions in order to obtain an adequate discussion of the fourth necessary condition involving the non-negativeness of the second variation. However, McShane [31] has announced recently that the fourth necessary condition holds when the maximum rank attainable for the matrix (14) is $2n-1$. Moreover he states that by means of an example it can be shown that the fourth necessary condition in its present form does not hold if this determinant is always of rank at most $2n-2$.

4. **Conditions for a minimum.** Consider now a particular arc E without corners that minimizes the function I subject to the conditions (6) and (7). If E is normal, it has associated [24] with it a unique set of multipliers $l_\gamma(x)$ with which it satisfies the *Euler-Lagrange equations*

$$(d/dx)F_{y_i'} = F_{y_i}, \qquad \phi_\gamma = 0, \tag{15}$$

and the *transversality condition*

$$dg + \left[(F - y_i'F_{y_i'})dx + F_{y_i'}\,dy_i\right]_1^2 + \int_{x_1}^{x_2} F_{a_h}da_h dx = 0. \tag{16}$$

Here $F=f+l_\gamma(x)\phi_\gamma$ and equation (16) is an identity in da_h when the *differentials* $dx_1, dy_{i1}, dx_2, dy_{i2}$ are expressed in terms of da_h by means of equations (7). This is known as the *first necessary condition* (I) for E. Moreover E must satisfy the *condition* (II) *of Weierstrass*, that is, at each element (a, x, y, y', l) belonging to E the inequality

$$E(a, x, y, y', l, Y') \geqq 0 \tag{17}$$

must hold for every admissible solution (Y') of the equations $\phi_\gamma(a, x, y, Y')=0$, where

$$\begin{aligned} E = {} & F(a, x, y, Y', l) - F(a, x, y, y', l) \\ & - (Y_i' - y_i')F_{y_i'}(a, x, y, y', l). \end{aligned} \tag{18}$$

From this result one concludes that the further inequality

$$F_{y_i'y_j'}\pi_i\pi_j \geqq 0 \tag{19}$$

must hold on E for every solution (π) of equations $\phi_{\gamma y_i'}\pi_i=0$. This condition is known as the *condition* (III) *of Clebsch*. As a *fourth necessary condition* (IV) the second variation

$$I_2(\alpha, \eta) = b_{hk}\alpha_h\alpha_k + \int_{x_1}^{x_2} 2\omega(\alpha, x, \eta, \eta')dx \tag{20}$$

of I along E must be non-negative for every solution α_h, η_i of equations (12) and (13). The last condition can be replaced by conditions involving broken extremals for the second variation.

A solution

$$a_h, \quad y_i(x), \quad l_\gamma(x), \qquad x_1 \leqq x \leqq x_2, \tag{21}$$

of equations (15) is called an *extremal*. It is said to be *nonsingular* if the determinant

$$\begin{vmatrix} F_{y_i'y_j'} & \phi_{\gamma y_i'} \\ \phi_{\beta y_j'} & 0 \end{vmatrix} \tag{22}$$

is different from zero at each point on it. Every normal minimizing arc E determines a unique extremal, also designated by E, satisfying the necessary conditions described above. When these conditions are suitably strengthened they will insure a minimum. For a proper strong relative minimum it is sufficient that E belongs to a nonsingular extremal satisfying the transversality condition (16); the strengthened conditions: (II_N) of Weierstrass which requires the inequality (17) to hold for all solutions (a, x, y, y', l) of $\phi_\gamma = 0$ in a neighborhood N of those on E, and (IV′) which requires the second variation (20) to be positive for all non-null solutions α_h, $\eta_i(x)$ of equations (12) and (13). If the Weierstrass condition (II_N) is replaced by the Clebsch condition (III) described above, a set of sufficient conditions for a weak relative minimum is obtained. In either case the condition (IV′) can be replaced by conditions involving broken extremals for $I_2(\alpha, \eta)$. A detailed account of these and other conditions can be found in the papers of Bliss [13], Hestenes [10, 16], Morse [11] and Reid [12].

5. **A consequence of the sufficiency conditions.** The sufficiency conditions just described hold even if the arc E is not normal. However, it was pointed out by Professor Bliss that these conditions are essentially a set of sufficiency conditions for a normal problem obtained by enlarging the class of comparison arcs. Thus as far as sufficiency conditions are concerned there is no loss of generality in assuming that the problem is normal. The sufficiency conditions described above also have a further startling consequence. It is clear that our problem will be unaltered if the integral I to be minimized is replaced by an integral of the form

$$J = g(t) + \int_{x_1}^{x_2} F(a, x, y, y')dx,$$

where F is of the form

$$F = f + m_\gamma(a, x, y, y')\phi_\gamma.$$

If an arc E satisfies the sufficiency conditions described above, the multipliers m_γ can be chosen so that E affords at least a proper weak relative minimum to J in the class of all neighboring arcs satisfying the end conditions (7) but not necessarily the differential equations (6). Thus in a sense the differential equations $\phi_\gamma = 0$ can be discarded. The essential step in the proof of this result was supplied by Professor Reid [25], who showed that by taking m_γ of the form $c\phi_\gamma$, where c is a large positive constant, the quadratic form $F_{y_i'y_j'}\pi_i\pi_j$ can be made positive definite along E. It remains to show that these multipliers m_γ can be further modified so that the second variation of J is positive definite for all non-null admissible variations α_h, η_i satisfying the end conditions (13) irrespective as to whether or not they satisfy the differential equations (12). That this can be done is an immediate consequence of the sufficiency proofs for the problem of Bolza given by Reid [25] and Hestenes [24] as we shall see in a moment.

As one would expect, a similar result holds in the finite case. Consider the problem of minimizing a function $f(x) = f(x_1, \cdots, x_n)$ on a surface $\phi_\gamma(x) = 0$ $(\gamma = 1, \cdots, m < n)$. For a proper relative minimum at $(x) = (a)$ it is sufficient that there exist a function $F = f + l_\gamma \phi_\gamma$ such that at $(x) = (a)$ one has $F_{x_i} = 0$, $\phi_\gamma = 0$ and $F_{x_i x_j} h_i h_j > 0$ for every solution $(h) \neq (0)$ of the equations $\phi_{\gamma x_i} h_i = 0$. For a sufficiently large constant c the function $G = F + c\phi_\gamma \phi_\gamma$ will have $G_{x_i} = 0$ and $G_{x_i x_j} h_i h_j$ positive definite at $(x) = (a)$. The point $(x) = (a)$ will accordingly afford a proper minimum to $G(x)$ relative to all points in a neighborhood of $(x) = (a)$. Clearly $G(x) = f(x)$ when $\phi_\gamma = 0$.

6. **Mayer fields.** We come now to the consideration as to how the sufficiency theorems can be established. Consider first the case when $g = 0$ and none of the parameters a_k is present. Then our problem becomes the problem of Lagrange with fixed end points. The method for this case is classical. It consists of constructing a Mayer field. By a *Mayer field* $\mathcal{F}$ is meant a region $\mathcal{F}$ in xy-space together with a set of slope functions and multipliers $p_i(x, y)$, $l_\gamma(x, y)$ on $\mathcal{F}$ such that $\phi_\gamma(x, y, p) = 0$ and the Hilbert integral

$$I^*(C) = \int_C F(x, y, p, l)dx + (dy_i - p_i dx)F_{y_i'}(x, y, p, l)$$

is independent of the path in $\mathcal{F}$, where $F = f + l_\gamma \phi_\gamma$. For any arc C in $\mathcal{F}$ on which $\phi_\gamma = 0$ one has the relation

$$I(C) = I^*(C) + \int_C E(x, y, p, l, y')dx, \tag{23}$$

the last integrand being the Weierstrass E-function (18). For an arc E in $\mathcal{F}$ on which $y_i' = p_i$, the relation $I^*(E) = I(E)$ holds. Thus if an arc C (satisfying $\phi_\gamma = 0$) in $\mathcal{F}$ joins the end points of E, one has $I(C) \geqq I^*(C) = I^*(E) = I(E)$, provided the E-function is non-negative along C. Consequently E is a minimizing arc when the Weierstrass condition (II_N) holds.

Returning now to the problem at hand let us recall the sufficiency proof given by Bliss [4] in 1932. This proof combines results that had been established previously by Mayer and Hahn. It consists of three steps: Given an arc E_0 satisfying the criteria for a minimum it is shown first that there is a neighborhood $\mathcal{F}_1$ of E_0 such that every extremal E in $\mathcal{J}_1$ affords a minimum to $I(C)$ in the class of arcs C in $\mathcal{J}_1$ joining its end points and having $\phi_\gamma = 0$. Next it is proved that $I(E_0)$ is a minimum on the class of extremals in $\mathcal{J}_1$ satisfying the end conditions (7). Finally it is shown that the end points of an arc C in a smaller neighborhood $\mathcal{J}$ of E_0 can be joined by an extremal E in $\mathcal{J}_1$. Hence if C satisfies equations (6) and (7) one has $I(C) \geqq I(E) \geqq I(E_0)$, as desired. A similar method applied to broken extremals was used by Morse [3] in 1931.

The method just described had to be abandoned because the end points of an arc C cannot in general be joined by an extremal E unless restrictive normality conditions are satisfied. For example, the method is not applicable to the problem of Mayer. In order to remedy this situation Bliss and Hestenes [7, 8] devised a method that is applicable to the problem of Mayer, but it too was based on undesirable normality assumptions. A new method accordingly had to be devised. The new method consists essentially of enlarging the concept of Mayer fields. Curiously enough it can be considered to be a generalization of the method used by Bliss, although to the casual observer there may appear to be no connection between the two methods.

Consider now a region $\mathcal{J}$ in axy-space and a set of slope functions and multipliers $p_i(a, x, y)$, $l_\gamma(a, x, y)$ on $\mathcal{J}$. Set

$$J(C) = g(a) + \int_{x_1}^{x_2} F(a, x, y, y')dx, \tag{24}$$

$$J^*(C) = g(a) + \int_{x_1}^{x_2} F^*(a, x, y, y')dx, \tag{25}$$

where $F = f + l_\gamma \phi_\gamma$ and

$$F^* = F(a, x, y, p) + (y_i' - p_i)F_{y_i'}(a, x, y, p).$$

One then has the formula

$$J(C) = J^*(C) + \int_C E(a, x, y, y')dx \tag{26}$$

analogous to (23), in which the last integrand is the Weierstrass E-function $E = F - F^*$. For a solution C of the equations $y_i' = p_i$ the relation $J(C) = J^*(C)$ holds. If $\phi_\gamma = 0$ on C then $J(C) = I(C)$, where $I(C)$ is the function to be minimized. Clearly we can suppose that $I(C) \equiv J(C)$ as far as the problem of Bolza is concerned.

The region $\mathcal{J}$ and the slope functions and multipliers $p_i(a, x, y)$, $l_\gamma(a, x, y)$ will be said [24] to define a *Mayer field* $\mathcal{J}$ if the Hilbert integral $J^*(C)$ is independent of the path in $\mathcal{J}$ in the sense that $J^*(C) = J^*(C')$ for any two arcs C, C' in $\mathcal{J}$ having the same end values $[a, x_1, y(x_1), x_2, y(x_2)]$. A solution a_k, $y_i(x)$ of the equations $y_i' = p_i$ defines an extremal E of $J(C)$ and is called an *extremal of the field* $\mathcal{J}$. The sufficiency proof consists of showing that if an admissible arc E_0 satisfies the criteria for a minimum, it is an extremal of a Mayer field having the following two properties: The Weierstrass E-function $F - F^*$ is non-negative for every solution (a, x, y, y') of $\phi_\gamma = 0$ having (a, x, y) in $\mathcal{J}$. The arc E_0 minimizes $J^*(C)$ in the class of arcs C satisfying the end conditions (7). Thus if an arc C in $\mathcal{J}$ satisfies equations (6) and (7) one has, by virtue of formula (24), the desired relations $J(C) \geqq J^*(C) \geqq J^*(E_0) = J(E_0)$. In fact if our problem has been modified so that the quadratic form $F_{y_i' y_j'} \pi_i \pi_j$ is positive definite along E_0, as Professor Reid has done, then the arc E_0 not only minimizes $J(C)$ in the class of arcs C in $\mathcal{J}$ satisfying conditions (6) and (7), but affords at least a weak relative minimum in the class of all arcs in $\mathcal{J}$ satisfying the end conditions (7). This establishes the remarks made in §5.

I remarked a moment ago that the method just described can be considered to be a generalization of the method used earlier by Bliss [4]. Bliss imposed sufficient normality conditions so that the end points of every arc C in $\mathcal{J}$ can be joined by an extremal E. Under these assumptions the field can be chosen [19] so that the relation $J^*(C) = J(E)$ holds, provided C satisfies the end conditions (7). Thus the inequality $J^*(C) \geqq J^*(E_0)$ becomes the criterion that E_0 minimize $J(E)$ in the class of extremal arcs satisfying the end conditions of our problem.

In order to carry out the method just described it must be known under what conditions the arc E_0 affords a minimum to $J^*(C)$ subject to the end conditions (7). In the theory of Mayer fields we are led therefore to the consideration of an auxiliary minimum problem. In this new problem the parameters a_h will in general appear in the integrand of $J^*(C)$ even if the original function $I(C)$ does not have this property. Thus by introducing the a's in the original integral I one can apply at once the theory of the first and second variation to our new problem. Nothing new is obtained from the first variation of $J^*(C)$. The second variation $J_2^*(\alpha, \eta)$ of $J^*(C)$ along E_0 is the Hilbert integral for a Mayer field for the second variation $I_2(\alpha, \eta)$ along E_0. The slope functions and multipliers $\pi_i(\alpha, x, \eta)$, $\lambda_\gamma(\alpha, x, \eta)$ of this field are the variations along E_0 of the slope functions and multipliers p_i, l_γ of the original field. In this accessory field the second variation $J_2(\alpha, \eta)$ of $J(C)$ along E_0 plays the role of $J(C)$ and is connected to $J_2^*(\alpha, \eta)$ by the formula

$$(27) \qquad J_2(\alpha, \eta) = J_2^*(\alpha, \eta) + \int_{x_1}^{x_2} F_{y_i'y_j'}(\eta_i' - \pi_i)(\eta_j' - \pi_j)dx$$

corresponding to formula (26). This formula in a somewhat modified but equivalent form is known as the Clebsch transformation of the second variation. Thus the theory of Mayer fields has a counterpart in the theory of the second variation. If a Mayer field exists for the second variation, then E_0 can always be imbedded in a Mayer field whose slope functions and multipliers have as their variations along E_0 the slope functions and multipliers of the given Mayer field for the second variation. The existence of Mayer fields therefore has been reduced to the question of existence of Mayer fields for the second variation. A nonsingular extremal satisfying the transversality conditions and on which the second variation is positive is always an extremal of a Mayer field $\mathcal{F}$ having the properties described above.

7. **Expansion proofs.** The question naturally arises as to whether or not Mayer fields are essential in the sufficiency proofs for the problem of Bolza. At present I am familiar with only one sufficiency proof for a strong relative minimum that is not based directly on the theory of Mayer fields. This proof is an expansion proof given by Reid [**18**, **20**, **25**] and is a modification and extension of one given by Levi for free problems. Although the expansion proof does not make direct use of the theory of Mayer fields, there is a close relationship between the two methods. In order to emphasize this relationship I shall interpret for you the methods used in the expansion proofs in such a way that one can see more clearly wherein it differs from the more classical method of constructing Mayer fields.

Consider now an extremal E_0,

$$(28) \qquad a_{h0}, \quad y_{i0}(x), \quad l_{\gamma 0}(x), \qquad x_1 \leqq x \leqq x_2,$$

satisfying the criteria for a strong relative minimum. We can suppose, with Reid, that the integrand f has been modified so that the quadratic form $F_{y_i'y_k'}\pi_i\pi_k$ is positive definite along E_0. Then there is a constant $b > 0$ such that at each element (a, x, y, y', l) in a neighborhood N of those on E_0 the inequality

$$(29) \qquad E(a, x, y, y', l, Y') \geqq b\|Y' - y'\|^2/(1 + \|Y' - y'\|)$$

holds whenever $\phi_\gamma(a, x, y, Y') = 0$, where $\|y'\| = (y_i'y_i')^{1/2}$. This formula also holds whenever (a, x, y, Y', l) is in N, if N is sufficiently small. Let $\pi_i(\alpha, x, \eta)$, $\lambda_\gamma(\alpha, x, \eta)$ be the slope functions and multipliers for a Mayer field for the second variation of I whose Hilbert integral $J_2^*(\alpha, \eta)$ is positive for every admissible variation $(\alpha, \eta) \neq (0, \eta)$ satisfying the end conditions (13). Let $\mathcal{J}$ be a neighborhood of E_0 in *axy*-space and select slope functions and multipliers $p_i(a, x, y)$, $l_\gamma(a, x, y)$ on $\mathcal{J}$ such that $p_i = y_{i0}'$, $l_\gamma = l_{\gamma 0}$ along E_0 and π_i, λ_γ are the variations of p_i, l_γ along E_0. As in the theory of Mayer fields construct the functions $J(C)$, $J^*(C)$ by means of formulas (24), (25). The integral J^* is now no longer independent of the path but its second variation along E_0 is the Hilbert integral $J_2^*(\alpha, \eta)$ of the Mayer field for the second variation. The formulas (26) and (27) hold as before. Consider an admissible arc C in $\mathcal{J}$ satisfying the conditions (6) and (7). By the use of Taylor's formula for $J^*(C)$, the properties of $J_2^*(\alpha, \eta)$ and the inequality (29) one finds after considerable manipulation that the relation

$$J^*(C) - J^*(E_0) + \int_C E dx \geqq 0 \tag{30}$$

holds, provided $\mathcal{J}$ is taken sufficiently small. By the use of equations (26) and $J^*(E_0) = J(E_0)$ this inequality becomes $J(C) - J(E_0) \geqq 0$, as desired.

The expansion proof given by Reid was presented in quite a different manner. However, from the above interpretation it is seen that it has much in common with the theory of Mayer fields. In particular it is seen that the basic property for $J^*(C)$ is that its second variation along E_0 is the Hilbert integral for the second variation of $I(C)$ and is positive for all variations $(\alpha, \eta) \neq (0, \eta)$ satisfying the variational end conditions (13). As far as the problem of Bolza is concerned it appears that the theory of Mayer fields is the simpler. However the expansion method has the advantage that it appears to be applicable to problems for which Mayer fields fail to exist.

8. **Further results.** The inequality (29) has many applications. For example, it can be used to prove the following analogue of Osgood's theorem: Let E_0 be a nonsingular extremal satisfying the criteria for a strong relative minimum described above. There exists a neighborhood $\mathcal{J}$ of E_0 in *axy*-space such that corresponding to every neighborhood $\mathcal{J}$ of E_0 interior to $\mathcal{J}$, there is a constant $\epsilon > 0$ such that the inequality $I(C) \geqq I(E_0) + \epsilon$ holds for every arc C in $\mathcal{J}$, not in $\mathcal{J}'$, which satisfies equations (6) and (7). This result was established by Reid [25] by expansion methods. It can also be proved by the use of Mayer fields with the help of the relations (26) and (29).

There is a second important result that is a consequence of the inequality (29). It is a generalization by Reid [25] of a theorem of Lindeberg which states that if E_0 is a nonsingular extremal (28) satisfying the condition (II_N) of Weierstrass, and ρ is a positive constant, there is a neighborhood $\mathcal{F}$ of E_0 such that $J(C)-J(E_0)>\rho/2$ holds for every admissible arc C in $\mathcal{F}$ for which one has

$$\int_C E(a,\, x,\, y,\, y_0',\, y')dx > \rho,$$

where $y_0'(x)$ are the slope functions of E_0. By the use of this result and one other, that I shall not describe here, Reid was able to obtain an elegant proof of the sufficiency theorems for the isoperimetric problem of Bolza. The sufficiency conditions for this problem are analogous to those for the problem of Bolza. Only two changes need to be made. First, the new function F appearing in the various conditions is now of the form $F=f+\lambda_\gamma(x)\phi_\gamma+\lambda_{m+\rho}f_\rho$, where $\lambda_{m+\rho}$ $(\rho=1, \cdots, p)$ are constants. Similarly g is to be replaced by $g+\lambda_{m+\rho}g_\rho$. The second difference is that the variations α_h, η_i must satisfy besides the conditions (12) and (13), the additional conditions

$$(31)\qquad \begin{aligned} J_{\rho 1}(\alpha,\, \eta) = g_{\rho a_h}\alpha_h &+ \left[f_\rho x_{s a_h}\alpha_h\right]_{s=1}^{s=2} \\ &+ \int_{x_1}^{x_2}\{f_{\rho a_h}\alpha_h + f_{\rho y_i}\eta_i + f_{\rho y_i'}\eta_i'\}dx = 0 \end{aligned}$$

where the functions $J_{\rho 1}(\alpha, \eta)$ are the variations along E_0 of the functions J_ρ appearing in the side conditions (9). A second sufficiency proof for the isoperimetric problem of Bolza has been given by Hestenes [24].

9. **Indices of extremals.** Returning now to the problem of Bolza let us consider a normal nonsingular extremal E_0 satisfying the end conditions (7), the transversality condition (16) and the condition (III) of Clebsch. If the second variation of I is positive along E_0, then E_0 affords at least a proper weak relative minimum to $I(C)$. The question naturally arises as to whether or not E_0 has certain minimizing properties even if the second variation is not positive along E_0. The answer is in the affirmative. The extremal E_0 is a minimizing arc for a normal isoperimetric problem of Bolza obtained by adjoining conditions of the form (9). The least number k of side conditions of this type which will determine an isoperimetric problem of Bolza for which $I(E_0)$ is a minimum will be called the *index* of the extremal E_0 (cf. [15]). The condition that the problem remain normal is an essential one since otherwise one could adjoin the trivial condition $I(C)=I(E_0)$ and the concept of index would not be a useful one. Clearly the index of E_0 is zero if and only if E_0 is a minimizing arc for the original problem of

Bolza. Except in certain degenerate cases the adjoined side conditions can be taken to be of the special form

$$
(32)\qquad
\begin{aligned}
I_1(\alpha_\rho, \eta_\rho; C) = g_{a_h}\alpha_{h\rho} &+ \left[F x_{s a_h}\alpha_{h\rho}\right]_1^2 \\
&+ \int_{x_1}^{x_2} \{F_{a_h}\alpha_h + F_{y_i}\eta_i + F_{y_i'}\eta_i'\}\,dx = 0
\end{aligned}
$$

where $\alpha_{h\rho}$, $\eta_{i\rho}$ $(\rho = 1, \cdots, k)$ form a set of k variations satisfying the equations (12) and (13) and the elements (a, x, y, y') are those belonging to C. The expression $I_1(\alpha, \eta; E_0)$ is the first variation of I along E_0 and is zero for all variations satisfying the conditions (12), (13).

In order to determine the index k of E_0 it is sufficient to select the fewest number of conditions (9) or (32) preserving normality which are such that the second variation $I_2(\alpha, \eta)$ is positive along E_0 for all non-null variations α_h, η_i satisfying the conditions (12), (13) and (31). For then the criteria for a weak relative minimum will be satisfied. It will be seen in the following sections that the second variation in general can be made positive if the variations α_h, η_i are required to satisfy, besides the conditions (12) and (13), a set of k additional conditions of the form

$$
(33)\quad b_{h\gamma}\alpha_{h\rho}\alpha_\gamma + \int_{x_1}^{x_2} \{\omega_{\alpha_h}\alpha_{h\rho} + \omega_{\eta_i}\eta_{i\rho} + \omega_{\eta_i'}\eta_{i\rho}'\}\,dx = 0, \quad \rho = 1, \cdots, k,
$$

where $\alpha_{h\rho}$, $\eta_{i\rho}$ are k solutions of equations (12) and (13). The fewest number of conditions (33) that are effective in this manner is called the ***index*** of $I_2(\alpha, \eta)$ and in general is equal to the index of E_0. This follows because the equations (33) are the variations along E_0 of the equations (32).

10. **Indices of the second variation.** In order to discuss the index of the second variation it is convenient to change our notation somewhat and return to the formulation used by Bliss. We consider the class $\mathfrak{D}$ of all continuous arcs

$$
\eta: \qquad \eta_i(x), \qquad x_1 \leqq x \leqq x_2;\ i = 1, \cdots, n,
$$

having piecewise continuous derivatives that satisfy differential equations of the form

$$
(34)\qquad \Phi_\gamma(\eta, \eta') = M_{\gamma i}\eta_i + N_{\gamma i}\eta_i' = 0, \qquad \gamma = 1, \cdots, m < n,
$$

whose coefficients are continuous functions of x on x_1x_2. On this class there is defined a quadratic function

$$J(\eta) = 2q(\eta) + \int_{x_1}^{x_2} 2\omega(\eta, \eta')dx \tag{35}$$

where $2q$ is a quadratic form in the end values $\eta_i(x_1)$, $\eta_i(x_2)$ of η and

$$2\omega = P_{ij}(x)\eta_i\eta_j + 2Q_{ij}(x)\eta_i\eta'_j + R_{ij}(x)\eta'_i\eta'_j.$$

It is assumed that the functions $P_{ij}=P_{ji}$, Q_{ij}, $R_{ij}=R_{ji}$ are continuous on x_1x_2 and that the determinant

$$\begin{vmatrix} R_{ij} & N_{\beta i} \\ N_{\gamma j} & 0 \end{vmatrix} \tag{36}$$

is different from zero. The bilinear form $J(\xi, \eta)$ associated with $J(\eta)$ is given by the formula

$$J(\xi, \eta) = q_{\eta_{i1}}\xi_i(x_1) + q_{\eta_{i2}}\xi_i(x_2) + \int_{x_1}^{x_2} \{\omega_{\eta_i}\xi_i + \omega_{\eta'_i}\xi'_i\}dx \tag{37}$$

and is identical with the first variation of $J(\eta)$. We have $J(\xi, \eta) = J(\eta, \xi)$ and $J(\eta, \eta) = J(\eta)$. Finally let $\mathcal{E}$ be the class of all variations η in $\mathcal{D}$ satisfying end conditions of the form

$$\Psi_\mu(\eta) = a_{\mu i}\eta_i(x_1) + b_{\mu i}\eta_i(x_2) = 0, \qquad \mu = 1, \cdots, p \leq 2n. \tag{38}$$

We are interested in the properties of $J(\eta)$ on $\mathcal{E}$. Other classes $\mathcal{E}$ are also of interest. The problem of the second variation $I_2(\alpha, \eta)$ of I described in the preceding sections is readily reduced to one of this type.

Many of the results that follow are equally applicable to a quadratic function on an arbitrary linear space. However as we shall be interested in the interpretations of these results in the problem of Bolza, I shall restrict myself to the problem here described. The results here given are a combination of the results obtained by Birkhoff and Hestenes [15] and Hazard [29].

Two arcs ξ and η will be said to be *J-orthogonal* if the relation $J(\xi, \eta) = 0$ holds. An arc ξ will be said to be *J-orthogonal to a subclass* $\mathcal{E}^*$ of $\mathcal{D}$ if it is J-orthogonal to every arc in $\mathcal{E}^*$. J-orthogonal classes are defined similarly. By the *J-complement* of a subclass $\mathcal{E}^*$ of $\mathcal{E}$ in $\mathcal{E}$ will be meant the totality of arcs in η in $\mathcal{E}$ that are J-orthogonal to $\mathcal{E}^*$. Similarly by the J-complement of an arc ξ in $\mathcal{E}$ will be meant the set of all arcs in $\mathcal{E}$ that are J-orthogonal to ξ. The condition that ξ be an extremal is that it be J-orthogonal to the class of all arcs η in $\mathcal{D}$ vanishing at x_1 and x_2. This is equivalent to the definition of extremals given previously if one identifies extremals which differ only in their multipliers. An extremal is seen to satisfy the transversality condition used in the previous sections if and only if it is J-orthogonal to $\mathcal{E}$.

In the last section the index of the second variation was defined in terms of an isoperimetric problem of Bolza. There is a second definition of index that is a more convenient one and which is more easily understood. The index of $J(\eta)$ on $\mathcal{E}$ will be called the J-index of $\mathcal{E}$. By the J-index of a linear subclass $\mathcal{E}^*$ of $\mathcal{D}$ will be meant the dimension of a maximal linear subclass of $\mathcal{E}^*$ on which $J(\eta)$ is negative definite.

This number is easily seen to be well defined. In view of the nonvanishing of the determinant (36) the J-index of $\mathcal{E}$ will be finite if and only if the Clebsch condition (III) holds. We shall be interested only in the case in which J-indices are finite. We shall accordingly assume that the Clebsch condition holds.

In order to see that the index here defined is identical with the one given earlier, let $\xi_1, \cdots, \xi_k$ be a minimal basis for a maximal linear manifold $\mathcal{E}^-$ in $\mathcal{E}$ on which $J(\eta)$ is negative definite. Then an arc η in $\mathcal{E}$ is J-orthogonal to $\mathcal{E}^-$ if and only if the conditions $J(\xi_\alpha, \eta)=0$ $(\alpha=1, \cdots, k)$ hold. It follows from our choice of $\mathcal{E}^-$ that $J(\eta)\geqq 0$ for every arc in $\mathcal{E}$ satisfying these isoperimetric conditions. Moreover no fewer conditions have this property. The two concepts of index are therefore identical. There is a further defining property of the J-index of a class $\mathcal{E}^*$. The J-index of $\mathcal{E}^*$ is equal to the dimension of a maximal linear subclass of $\mathcal{E}^*$ on which $J(\eta)\leqq 0$ and which contains no arc $\eta\neq 0$ that is J-orthogonal to $\mathcal{E}^*$. This is a very useful criterion for the determination of indices.

One of the basic problems is the determination of the relations between the J-index of $\mathcal{E}$ and that of a linear subclass $\mathcal{E}^*$ of $\mathcal{E}$. This problem can be completely solved if $\mathcal{E}^*$ is related to $\mathcal{E}$ in a special way. For want of better terminology, these subclasses will be called *special* subclasses of $\mathcal{E}$. Let $\mathcal{E}^*$ be a linear subclass of $\mathcal{E}$ and denote by E_0^* the set of all arcs in $\mathcal{E}^*$ that are J-orthogonal to $\mathcal{E}^*$. The class $\mathcal{E}^*$ will be called a *special* subclass of $\mathcal{E}$ if every arc η in $\mathcal{E}$ that is J-orthogonal to $\mathcal{E}_0^*$ is a sum $\eta_1+\eta_2$ of an arc η_1 in $\mathcal{E}^*$ and an arc η_2 that is J-orthogonal to $\mathcal{E}^*$. For example the subclass $\mathcal{E}^0$ of all arcs in $\mathcal{E}$ vanishing at x_1 and x_2 is a *special* subclass of $\mathcal{E}$. Here the condition that $\mathcal{E}^0$ be a *special* subclass of $\mathcal{E}$ is equivalent to the condition that the end points of every arc η in $\mathcal{E}$ that is J-orthogonal to the extremals that vanish at x_1 and x_2 can be joined by an extremal. Every linear subclass of $\mathcal{E}$ containing $\mathcal{E}^0$ is a special subclass of $\mathcal{E}$. In fact any linear subclass of $\mathcal{E}$ of finite dimension or which is the J-complement of such a class is a special linear subclass of $\mathcal{E}$.

The principal result upon which the determination of indices and upon which many comparison theorems are based is the following one: Let $\mathcal{E}^*$ be a special linear subclass of $\mathcal{E}$ and as before denote by $\mathcal{E}_0^*$ the class of all arcs in $\mathcal{E}^*$ that are J-orthogonal to $\mathcal{E}^*$. Let m be the dimension of a maximal linear subclass of $\mathcal{E}_0^*$ whose non-null arcs are not J-orthogonal to $\mathcal{E}$. If the J-indices of $\mathcal{E}^*$ and its J-complement in $\mathcal{E}$ be denoted by the symbols k^* and k', respectively, then the J-index k of $\mathcal{E}$ is given by the formula $k=k^*+k'+m$.

As a first example of the application of this result let us consider the problem of determining the J-index of the class $\mathcal{E}^0$ of all arcs η in $\mathfrak{D}$ vanishing at x_1 and x_2. For convenience of description it will be assumed that $\eta\equiv 0$ is the only extremal that vanishes identically on any subinterval of x_1x_2. This is a normality condition and need not be assumed if one modifies the definition of conjugate points given below. A point x_3 on $x_1<x<x_2$ will be said to be conjugate to x_1 if there is an extremal $\eta\neq 0$ that vanishes at x_1 and x_3. The number k of linearly independent extremals in a maximal set that vanish at x_1 and x_3 will be called the order of x_3 as a conjugate point of x_1. Let $\xi_1, \cdots, \xi_m$ be a maximal set of linearly independent arcs in $\mathcal{E}^0$ that are identically zero on a subinterval x_3x_2 of x_1x_2 and are identical with an extremal on x_1x_3. The number of these arcs is equal to the sum of the orders of the conjugate points of x_1 between x_1 and x_2. These arcs form the basis of a *special* subclass $\mathcal{E}^*$ of $\mathcal{E}^0$ whose index is zero. No arc $\xi\neq 0$ in $\mathcal{E}^*$ is J-orthogonal to $\mathcal{E}^0$. Moreover the index of its J-complement in $\mathcal{E}^0$ is also zero. The J-index of $\mathcal{E}^0$ is accordingly equal to the dimension m of $\mathcal{E}^*$ and hence is equal to the sum of the orders of the conjugate points of x_1 between x_1 and x_2.

Returning now to the original class $\mathcal{E}$ we observe that its J-index is the sum of three quantities. The first of these quantities is the index of the class $\mathcal{E}^0$ just described, that is, the sum of the orders of the conjugate points of x_1 between x_1 and x_2. The second quantity is the J-index of the class of extremals in $\mathcal{E}$, that is, the J-index of the J-complement of $\mathcal{E}^0$ in $\mathcal{E}$. The third quantity is the dimension of a maximal linear class of extremals vanishing at x_1 and x_2 which contains no extremal $\xi\neq 0$ that satisfies the transversality conditions for $\mathcal{E}$.

As a third example consider the class $\mathcal{E}^*$ of all arcs in $\mathcal{E}$ forming the J-complement of the class of all arcs vanishing at a set of points $t_0=x_1<t_1< \cdots <t_r<t_{r+1}=x_2$. These points can be chosen so close together that the index of the J-complement of $\mathcal{E}^*$ is zero and such that every arc in $\mathcal{E}^*$ that is J-orthogonal to $\mathcal{E}^*$ is also J-orthogonal to $\mathcal{E}$. The J-index of $\mathcal{E}$ is therefore identical with that of $\mathcal{E}^*$.

Moreover the class $\mathcal{E}^*$ is composed of the totality of broken extremals in $\mathcal{E}$ with corners at the points $t_1, \cdots, t_r$. This result identifies our index with the one introduced by Morse.

Examples of this type can be multiplied in many ways so as to obtain comparison and oscillation theorems which relate the J-indices of various subclasses of $\mathcal{E}$. Other oscillation and comparison theorems are also easily obtained. For example, if a second quadratic function $J^*(\eta)$ satisfies the condition $J^*(\eta) \geqq J(\eta)$ on $\mathcal{E}$, the J^*-index of $\mathcal{E}$ cannot exceed its J-index.

Boundary value problems arise in the calculus of variations when one wishes to obtain relationships between $J(\eta)$ and a second quadratic functional $J^*(\eta)$. The function $J^*(\eta)$ is usually taken to be of the form

$$J^*(\eta) = 2q^*(\eta) + \int_{x_1}^{x_2} K_{ij}(x)\eta_i\eta_j dx.$$

We shall assume that $J^*(\eta)$ is positive on $\mathcal{D}$ although this assumption is not essential. An arc $\xi \neq 0$ in $\mathcal{E}$ will be called a *characteristic arc* if its J^*-complement in $\mathcal{E}$ is identical with its J-complement in $\mathcal{E}$. If ξ is a characteristic arc, there is a value σ such that $J(\xi, \eta) = \sigma J^*(\xi, \eta)$ on $\mathcal{E}$. In fact $\sigma = J(\xi)/J^*(\xi)$, and ξ is an extremal for $J - \sigma J^*$. The number of linearly independent characteristic arcs in a maximal set determining the same characteristic value σ will be called the order of σ. It can be shown that the J-index of $\mathcal{E}$ is equal to the sum of the orders of the negative characteristic values. Thus it is seen that the theory of indices here presented is closely related to boundary value problems.

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University of Chicago

THE ANALYSIS OF LINEAR TRANSFORMATIONS

F. J. MURRAY

1. **Introduction.** If T is a closed additive transformation with domain dense (abbreviated c.a.d.d.) between Hilbert spaces, there exist two resolutions of the identity, $E_1(\lambda)$ and $E_2(\lambda)$ such that $E_2(\lambda)T \subset TE_1(\lambda)$. These resolutions are defined for $0 \leqq \lambda < \infty$, and if $0 < \alpha < \beta$ T is an isomorphism, when contracted to the range[1] of $E_1(\beta) - E_1(\alpha)$. This is important in applications, since only under an isomorphism is the convergence of a sequence of elements equivalent to the convergence of the images. This property compensates for the lack of compactness of Hilbert space.

These resolutions also permit us to express T as a denumerable sum of such isomorphisms.[2] Each isomorphism in turn can be expressed in terms of the values of T on an orthonormal set complete in a certain subspace. Thus T is analysed into components which determine it by addition and closure. Another interesting property of these resolutions associated with T is the fact that if f is such that for every $g \in \mathfrak{M}_1(\lambda)$ (the range of $E_1(\lambda)$), $|f+g| \geqq |f|$ and if Tf exists then $|Tf+Tg| \geqq |Tf|$.

When one considers a c.a.d.d. T in a general reflexive Banach space,[3] many properties lose their significance, but those mentioned above do not. Since they indicate a complete analysis of such transformations one is led to consider the possibility of generalizing the notion of a resolution of the identity and the association of two of these with a c.a.d.d. T.

At least five such generalizations are possible. However, the complete analysis given above cannot be carried through in general linear spaces at present because of various unsolved problems of these spaces. In the present talk, we show the dependence of this analysis upon these problems and classify the problems from this point of view. It is hoped that this will result in a more systematic development of the theory of linear spaces.

2. **Projections.** The difficulties appear in attempting to generalize the notion of a projection. Let us consider the usual notion of a pro-

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An address delivered before the Washington meeting of the Society on May 2, 1941, by invitation of the Program Committee; received by the editors July 18, 1941.

[1] Cf. [11, chap. 9], or [8, §5, pp. 312–318, in particular Theorem VI, p. 315].

[2] Cf. [8, loc. cit.].

[3] We shall follow the notation of [1]. A reflexive space is one such that $\overline{\mathfrak{B}} = \mathfrak{B}$.

jection in Hilbert space and endeavor to express its properties in terms which generalize to Banach spaces. This can be most readily done in terms of the range $\mathfrak{M}$ of E and the set $\mathfrak{N}$, of f's such that $Ef=0$. The following properties are present: 1. For every f in $\mathfrak{H}$, there is a g in $\mathfrak{M}$, ($g=Ef$) and an h in $\mathfrak{N}$, such that $f=g+h$. 2. $\mathfrak{M}\cdot\mathfrak{N}=\{\theta\}$. 3. $\mathfrak{M}$ and $\mathfrak{N}$ are closed additive sets.[4] 4. h in $\mathfrak{N}$ and g in $\mathfrak{M}$, imply $|h+g|\geqq|h|$.

In Hilbert space, given $\mathfrak{M}$, these properties[5] determine a unique self-adjoint E, with domain $\mathfrak{H}$ and bound 1. In suitably restricted spaces given a closed additive set, $\mathfrak{M}$, 1 and 4 will determine $\mathfrak{N}$.[6] Property 2 will hold but not 3 since $\mathfrak{N}$ in general will not be additive.[7] One may then abandon 4. If we are given $\mathfrak{M}$, let us call an $\mathfrak{N}$ satisfying 1, 2, 3, a complement to $\mathfrak{M}$. If $\mathfrak{M}$ has a complement, $\mathfrak{N}$, the corresponding E is a bounded linear transformation[8] with $E^2=E$. Such a transformation we will call a bounded projection on $\mathfrak{M}$. Since 4 is lost, E is not unique for $\mathfrak{M}$, but this is not the worst difficulty. It can be shown that there exist manifolds $\mathfrak{M}$, which do not possess bounded projections.[9]

One may, however, be willing to substitute for the boundedness of E just closure. If we substitute for 1, the Property 1′: for a dense set of f's, $f=g+h$, $g\in\mathfrak{M}$, $h\in\mathfrak{N}$ ($Ef=g$), we will call an $\mathfrak{N}$ satisfying 1′, 2, 3, a quasi-complement to $\mathfrak{M}$. The corresponding E is closed and if we have a c.a.d.d. E with $E^2=E$, and range $\mathfrak{M}$, then $\mathfrak{M}$ has a quasi-complement.[10] The following problem arises:

Problem I. *In a linear space, does every $\mathfrak{M}$ possess a quasi-complement?*[11]

If $\mathfrak{M}$ does not have a complement it must be "infinite," that is, infinite-dimensional and having no finite complement. We also ask:

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[4] Properties 2 and 3 insure the uniqueness and additivity of the resolution given in 1.

[5] The Condition 4 implies that $\mathfrak{M}$ and $\mathfrak{N}$ are orthogonal, since $|h+tg|^2\geqq|h|^2$ for any complex t implies $(h, g)=0$. Thus E is the orthogonal projection on $\mathfrak{M}$. Cf. [15, Theorem 1.23, p. 22 and Definition 2.16, p. 70] or [11, chap. 2, §5, Theorem VI and chap. 6, §1, Definition 1].

[6] Cf. Appendix I, remark preceding Lemma 4.

[7] In Appendix I, $\mathfrak{N}$ is the set of h_0's such that $F(h_0)=|F|\cdot|h_0|$ for an $F\in\mathfrak{M}^\perp$. We may make this correspondence more precise by letting $|F|=|h_0|$. Even when this correspondence is one-to-one as say in L_p or l_p ($p\neq 2$) it is not linear and $\mathfrak{N}$ the image of a linear set will not be linear in general.

[8] Cf. [10, proof of Lemma 1.11].

[9] Cf. [10].

[10] The proof of these statements is similar to that of [10, Lemma 1.11].

[11] These problems are of greatest interest when $\mathfrak{B}$ is reflexive.

PROBLEM II. *Do there exist Banach spaces such that every infinite manifold has no complement (quasi-complement)?*

Since the Banach space $\mathfrak{B}$ and any closed additive subset $\mathfrak{M}$ in it are abelian groups, we may consider $\mathfrak{B}/\mathfrak{M}$ the quotient group of $\mathfrak{B}$ by $\mathfrak{M}$. This is the set of classes of elements of $\mathfrak{B}$, mod $\mathfrak{M}$, that is, two elements f_1 and f_2 in $\mathfrak{B}$ belong to the same class $\mathfrak{c}$, if $f_1-f_2\in\mathfrak{M}$. $\mathfrak{B}/\mathfrak{M}$ is a normed linear space if addition and scalar multiplication of classes result from the corresponding operations[12] of $\mathfrak{B}$ mod $\mathfrak{M}$ and $|\mathfrak{c}| = \text{g.l.b.}(|f|, f\in\mathfrak{c})$. If $\mathfrak{B}$ is reflexive, $\mathfrak{B}/\mathfrak{M}$ is complete.[13] We say that $\mathfrak{B}$ has been resolved abstractedly into $\mathfrak{M}$ and $\mathfrak{B}/\mathfrak{M}$.

In Hilbert space, each class $\mathfrak{c}$ of $\mathfrak{H}/\mathfrak{M}$ contains a unique h of $\mathfrak{M}^{\perp}$, with $|h| = |\mathfrak{c}|$ and this correspondence $h\sim\mathfrak{c}$ is linear. In a reflexive $\mathfrak{B}$, the existence for every $\mathfrak{M}$ of a complement with these properties would imply that $\mathfrak{B}$ is a Hilbert space.[14] However, if additivity is dropped, we have for reflexive convex spaces an orthogonal $\mathfrak{N}$ as we have mentioned above.

3. **Bounded transformations.** Let us now consider the association of projections with bounded transformations. If T is bounded and $\mathfrak{M}$ and $\mathfrak{N}$ are complements then T takes $\mathfrak{M}$ into an additive set $T\mathfrak{M}$ and $\mathfrak{N}$ into $T\mathfrak{N}$. If $f=g+h$, $g\in\mathfrak{M}$, $h\in\mathfrak{N}$, $Tf=Tg+Th$. Thus every element Tf of $\mathfrak{R}_T$, the range of T, can be expressed as a sum of elements of $T\mathfrak{M}$ and $T\mathfrak{N}$, $f'=g'+h'$. If T^{-1} exists, we see that this resolution is unique, for $g'=TET^{-1}f'$.

Thus E has the corresponding operation TET^{-1} in the range space. But even in Hilbert space, this transformation may be unbounded and may even have no closed extension.[15] Thus a bounded T may not preserve the relation of complementation or quasi-complementation, even when T^{-1} exists.

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[12] The statement about the norm is the most interesting but is not particularly difficult. The closure of $\mathfrak{M}$ insures that if $|\mathfrak{c}| = 0$, $\mathfrak{c}$ contains θ and is precisely $\mathfrak{M}$, the 0-class for addition, modulo $\mathfrak{M}$. Obviously $|a\mathfrak{c}| = |a|\cdot|\mathfrak{c}|$ and the triangle inequality is obtained by a simple ϵ-proof.

[13] Cf. Appendix I, Lemma 6.

[14] Since $h=(1-E)f$ and $|h|\leq|f|$, we see that $1-E$ has bound 1. But $(1-E)^* = 1-E^*$, which would also have bound 1. We have supposed that $\mathfrak{B}$ is reflexive and thus to every $\mathfrak{M}'\subset\overline{\mathfrak{B}}$, we can find a $\mathfrak{M}\subset\mathfrak{B}$, such that $\mathfrak{M}^{\perp}=\mathfrak{M}'$. Hence every $\mathfrak{M}'\subset\overline{\mathfrak{B}}$ has a projection of bound 1 on it. This implies that $\overline{\mathfrak{B}}$ is a Hilbert space and that $\overline{\mathfrak{B}}=\mathfrak{B}$ (cf. [4] or [13]).

[15] Cf. Appendix II.

However, for the special resolutions $E_1(\lambda)$ and $E_2(\lambda)$ associated with T as in §1, in Hilbert space, we do have that complementation is preserved.[16] Also in Hilbert space, if $\mathfrak{M}$ is such that T is an isomorphism on $\mathfrak{M}$, then a bounded T preserves complementation for at least one complement.[17] Thus we are interested in the $\mathfrak{M}$, for which T preserves complementation.

To obtain a precise statement of our problem; let us find the exact conditions under which the equation $TE_1=FT$ determines a transformation F. Now let g be given such that there is an f for which $Tf=g$. Then $TE_1f=Fg$ defines a transformation F, if $Tf_1=Tf_2$ implies $TE_1f=TE_2f$ or if $T(f_1-f_2)=\theta$ implies $TE_1(f_1-f_2)=\theta$. Thus we will have an F determined by T and E if E projects the set of zeros of T, $\mathfrak{N}_T$, onto part of itself. If $\mathfrak{R}_{E_1}\supset\mathfrak{N}_T$ or $\mathfrak{N}_{E_1}\supset\mathfrak{N}_T$ then this condition is certainly fulfilled.

Now a bounded projection, E_1, will be said to be associated with a bounded T (abbreviated E_1 b.A.b. T) if $\mathfrak{R}_{E_1}\supset\mathfrak{N}_T$ or $\mathfrak{N}_E\supset\mathfrak{N}_T$ and if the corresponding F has a bounded extension (which will be a bounded projection E_2). We suppose that $\mathfrak{R}_T$ is dense in $\mathfrak{B}_2$.

PROBLEM III_1. *Given a bounded T, find the set of bounded projections associated with it.*

One can associate a closed projection with a bounded T in an analogous manner (E_1 c.A.b. T) and one has a corresponding Problem III_2.

4. **Closed transformations.** Let us consider the similar problems for c.a.d.d. transformations. A closed transformation is characterized by the fact that the "graph," the set of pairs $\{f, Tf\}$ in $\mathfrak{B}_1\oplus\mathfrak{B}_2$, f in $\mathfrak{D}_T$, is closed.[18]

Given a projection E_1 the resolution in $\mathfrak{B}_1$, $f=g+h$, yields a resolution of the graph of a bounded T, $\{f, Tf\}=\{g, Tg\}+\{h, Th\}$. The graph of a bounded transformation with domain $\mathfrak{B}_1$ is isomorphic to $\mathfrak{B}_1$, and thus this latter resolution will be determined by a projection $E_{1,2}$ which is defined by the equation $E_{1,2}\{f, Tf\}=\{E_1f, TE_1f\}$.

The situation is different in the case of a c.a.d.d. T. Since $\mathfrak{D}_T$ is no longer $\mathfrak{B}_1$, it may not include $\mathfrak{R}_{E_1}$ whether E_1 is bounded or closed. Thus if E is a one-dimensional projection on an element not in $\mathfrak{D}_T$, the resolution cannot be imported into the graph.

This difficulty appears even in Hilbert space. When one is forced to deal with it in defining reducibility, one appends the condition:

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[16] Cf. [8, §5, pp. 312–318].

[17] Cf. Appendix III.

[18] The graph is discussed in [16, pp. 299–301], [8, pp. 302–307], [9, pp. 88–91], and [11, chap. 4].

"$f \in \mathfrak{D}_T$ implies $E_1 f \in \mathfrak{D}_T$." When this condition is applied to either a bounded or closed[19] E_1 in $\mathfrak{B}_1$, one sees that $E_{1,2}$ must be defined throughout the graph and hence is bounded.[20]

On the other hand, if one starts with a bounded projection $E_{1,2}$ in the graph, $E_{1,2}\{f, Tf\} = \{g, Tg\}$, then the equation $E_1 f = g$ may not determine a transformation with a closed extension.[21] Even if both E_1 and $E_{1,2}$ are bounded, however, the F defined by the equation $TE = FT$ may not have a closed extension.[18]

DEFINITION. *A bounded projection, E_1, will be said to be associated with a c.a.d.d. T, (E_1 b.A.c. T) if $\mathfrak{R}_E \supset \mathfrak{R}_T$ or $\mathfrak{N}_E \supset \mathfrak{N}_T$ and E_2 is bounded. A closed projection will be said to be associated with a c.a.d.d. T (E_1 c.A.c. T) if $\mathfrak{R}_E \supset \mathfrak{R}_T$ or $\mathfrak{N}_E \supset \mathfrak{N}_T$, $E_{1,2}$ is bounded and E_2 has a closed extension.*

We then have Problems III_3 and III_4 analogous to III_1 and III_2, with T closed rather than bounded.

5. **Abstract resolutions.** We have discussed the preservation by T of complementation and quasi-complementation. Let us now consider abstract resolutions.

If T is c.a.d.d. and $\mathfrak{M}$ is a manifold such that $\mathfrak{D}_T$ is dense in $\mathfrak{M}$, and $\mathfrak{M}'$ is the closure of the set of Tf's with f in $\mathfrak{M}$, then we can consider a transformation T from $\mathfrak{B}_1/\mathfrak{M}$ to $\mathfrak{B}_2/\mathfrak{M}'$, defined as follows. Let $\mathfrak{c} \in \mathfrak{B}_1/\mathfrak{M}$ and $f \in \mathfrak{c}$. Then $Tf \in \mathfrak{c}' \in \mathfrak{B}_2/\mathfrak{M}'$. Now if f_1 is also in $\mathfrak{c}$, $Tf_1 \in \mathfrak{c}'$, and thus $\mathfrak{c}'$ depends only on $\mathfrak{c}$ and we define $T_{\mathfrak{M}}\mathfrak{c} = \mathfrak{c}'$.

Now $T_{\mathfrak{M}}$ is readily seen to be additive and its domain is dense in $\mathfrak{B}_1/\mathfrak{M}$. In general, however, it is not closed,[22] although for T bounded,

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[19] For E_1 closed the existence of $E_1 f$ for f in $\mathfrak{D}_T$ is implied.

[20] When $E_{1,2}$ is defined by the equation $E_{1,2}\{f, Tf\} = \{E_1 f, TE_1 f\}$ and E_1 satisfies the given condition, then $E_{1,2}$ is bounded. For let $\mathfrak{M}_{1,2}$ be the set of pairs $\{f, Tf\}$ for which f is in $\mathfrak{M}_1$, the range of E_1, and let $\mathfrak{N}_{1,2}$ be the set of $\{g, Tg\}$ with $g \in \mathfrak{N}_1$. We prove that $E_{1,2}$ is bounded by showing that $\mathfrak{M}_{1,2}$ and $\mathfrak{N}_{1,2}$ are complementary (cf. Footnote 8). The Conditions 2 and 3 on $\mathfrak{M}_{1,2}$, $\mathfrak{N}_{1,2}$ follow from the corresponding conditions on $\mathfrak{M}_1$, $\mathfrak{N}_1$ and 1 follows from the given condition on E_1.

[21] This is true even in Hilbert space. Let $\mathfrak{H}$ be realized as the space $\mathfrak{L}_2$ on the interval $(0, 2\pi)$. The set of pairs, $\{\exp(inx), -n\exp(inx)\}$, $n = 0, \pm 1, \pm 2, \cdots$ and $\{\exp(x) - \exp(2\pi - x), i(\exp(x) + \exp(2\pi - x))\}$, are orthogonal. Let $\mathfrak{T}$ denote the manifold determined by these pairs in $\mathfrak{H} \oplus \mathfrak{H}$. It is readily seen that $\mathfrak{T}$ is the graph of a transformation T, that is, if $\{\theta, g\}$ is in $\mathfrak{T}$, then $g = \theta$. Now if $E_{1,2}$ is the projection of $\mathfrak{T}$ onto the manifold determined by the pair $\{\exp(x) - \exp(2\pi - x), i(\exp(x) + \exp(2\pi - x))\}$ then the corresponding E_1 has no closed extension. This is an immediate consequence of the fact that E_1 is zero on the dense set determined by the $\exp(inx)$ for $n = 0, \pm 1, \pm 2, \cdots$.

[22] Cf. Appendix IV.

$T_{\mathfrak{M}}$ is bounded. On the other hand,[13] $T_{\mathfrak{M}}$ and T on $\mathfrak{M}$ do not determine T.

But suppose T and $\mathfrak{M}$ are such that if f is in $\mathfrak{D}_T$ and in $\mathfrak{c} \in \mathfrak{B}_1/\mathfrak{M}$, then $\mathfrak{c}$ contains a minimal element h, which is also in $\mathfrak{D}_T$ and Th is minimal in $T_{\mathfrak{M}}\mathfrak{c}$. We say then that T preserves minimality relative to $\mathfrak{M}$. For reflexive convex spaces, $T_{\mathfrak{M}}$ itself is then closed.[14]

We will say that the abstract resolution of a manifold $\mathfrak{M}$ is associated with a closed T if T preserves minimality relative to $\mathfrak{M}$.

In Hilbert space, the resolutions associated with T are precisely those which break T, that is, those which reduce[15] $(T^*T)^{1/2}$. Thus this criterion determines the manifolds more precisely than the previous ones.

PROBLEM IV$_1$ (IV$_2$). *Given a bounded T (c.a.d.d. T), find the abstract resolutions associated with it.*

In Hilbert space the answer to these problems is known. In general linear spaces the answer is not so interesting but we will consider other generalizations of the manifolds which reduce $(T^*T)^{1/2}$.

Notice that if the abstract resolution with respect to $\mathfrak{M}$ is associated with T, the latter can be regarded as resolved into linear components of T on $\mathfrak{M}$ and $T_{\mathfrak{M}}$. This notion is particularly interesting for $\mathfrak{M} = \mathfrak{N}_T$, whose abstract resolution is associated with T.

6. **Bases.** We now consider the general problem of reducing T to linear components, rather than to the special case in which there are only two components. One such resolution is given by any basis for the graph and thus we may consider the basis problem.

Banach defines a basis [1, p. 110] as a sequence of elements $\{f_i\}$ such that to each $f \in \mathfrak{B}$, there exists a unique sequence of numbers η_i such that $f = \sum_{i=1}^{\infty} \eta_i f_i$. This definition implies that $\eta_i = F_i(f)$ is an additive linear functional of f [1, p. 111].

PROBLEM V. *Does every separable Banach space have a basis?*

The graph of a bounded transformation is isomorphic to the domain space and thus a basis in $\mathfrak{B}_1$ determines a basis in $\mathfrak{T}$, and hence a resolution into irreducible linear components. For a closed transformation one must consider the graph directly.

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[13] A two-dimensional example will make this clear. Let T_α be defined by the equation $T_\alpha\{x, y\} = \{x+\alpha y, y\}$. If $\mathfrak{M}$ is the set of vectors $\{x, 0\}$, one sees that T_α on $\mathfrak{M}$ and $T_{\mathfrak{M}}$ do not determine T_α.

[14] Cf. Appendix V.

[15] Cf. [8, §5, pp. 312–318].

For a bounded T, given a subsequence of a basis $\{f_{i_n}\}$ such that $\sum \eta_{i_n} f_{i_n}$ converges for every f one would consider the corresponding projection $Ef = \sum \eta_{i_n} f_{i_n}$. However, it is not known whether every infinite subsequence of a basis series is convergent. When every such subsequence is convergent, we will call the basis an absolute basis. The existence question for these is also important and we shall call it V_2. We also have this problem:

PROBLEM VI_1. *Is every basis an absolute basis?*

7. **Resolutions of the identity.** A basis can be regarded as a special case of a resolution of the identity. One way of defining the latter notion is given by Lorch [5, Definition 4, p. 226]. This is the most restrictive way. A family of projections $E(\lambda)$ is said to constitute a resolution of the identity: 1. if 0 and 1 are in the family; 2. if $\lambda \geqq \mu$, $E(\lambda)E(\mu) = E(\mu) = E(\mu)E(\lambda)$; 3. if there exists a constant K, such that for any real numbers a_i, b_i, $(i = 1, \cdots, n)$ with $a_1 \leqq b_1 \leqq \cdots \leqq a_i \leqq b_i \leqq \cdots \leqq a_n$, b_n and for any complex numbers μ_i with $|\mu_i| \leqq 1$, the bound of the operator $\sum_{\alpha=1}^{n} \mu_\alpha (E(b_\alpha) - E(a_\alpha))$ does not exceed K.

One may point out that if one replaces the condition $|\mu_i| \leqq 1$ by $|\mu_i| = 1$, the new Condition 3′ is equivalent[24] to 3. This type of resolution will be referred to as an absolutely bounded resolution of the identity. Other definitions of a resolution of the identity may be obtained by replacing the Condition 3 or 3′ by any one of the following: 3″. There exists a K such that for every λ, the bound of $E(\lambda)$ is less than or equal to K. 3‴. The $E(\lambda)$ exist and are bounded except possibly on a denumerable, nowhere dense set of points. 3^{IV}. Each $E(\lambda)$ is closed. The corresponding resolutions will be termed, respectively, bounded, essentially bounded, and closed. To make our existence problem definite we also introduce the definition.

If $f \in \mathfrak{B}$, and $a_1, b_1, \cdots, a_n, b_n$ are as in Condition 3 above, let $\mathfrak{A}_f$ denote the set of g's for which there are λ_α's such that $g = \sum_{\alpha=1}^{n} \lambda_\alpha (E(b_\alpha) - E(a_\alpha))f$. If there is an f, such that $\mathfrak{A}_f$ is dense, then we say that the resolution of the identity has a simple spectrum.

PROBLEM VII_1. *Does every separable Banach space contain an absolutely bounded resolution of the identity with a simple spectrum?*

For bounded, essentially bounded or closed resolutions, we have the corresponding Problems VII_2, VII_3, VII_4.

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[24] A proof of the equivalence of 3 and 3′ can be outlined as follows: If $|\mu| \leqq 1$ (μ real), then $|f + \mu g| \leqq \max(|f+g|, |f-g|)$. This can be used to show that for a given set of μ_i's and an f, we can find a set of ϵ_i's with $|\epsilon_i| = 1$, such that $|\sum \mu_i (E(b_i) - E(a_i))f| \leqq |\sum \epsilon_i (E(b_i) - E(a_i))f|$. This can be used to show that the least K of 3′ is greater than or equal to the least K of 3.

Now corresponding to a basis, we can find a bounded resolution of the identity.[27] A basis is absolute if and only if the corresponding resolution is absolutely bounded.[28] Thus we may generalize Problem VI_1 to the following problem:

PROBLEM VI_2. *Is every bounded resolution of the identity absolutely bounded?*

8. **Resolutions and transformations.** We need to consider only resolutions such that $E_1(\lambda)=0$, if $\lambda<0$. A resolution $E(\lambda)$ will be said to be associated with a transformation T: 1. if $E(0)$ is the manifold of zeros of T. 2. If the resolution $E(\lambda)$ is absolutely bounded, bounded or essentially bounded and $E(\lambda)$ is bounded, then T and $E(\lambda)$ are associated as in Problems III_1 or III_2. If the resolution is considered simply as closed, then T and $E(\lambda)$ are associated as in Problems III_3 or III_4. 3. If the resolution $E(\lambda)$ is absolutely bounded, bounded or closed and $0<\alpha<\beta$ then T is an isomorphism on the range of $E(\beta)-E(\alpha)$. If $E(\lambda)$ is essentially bounded and $0<\alpha<\beta$ and there is no singular point between α and β, then T is an isomorphism on the range of $E(\beta)-E(\alpha)$.

A special case is that in which T takes the range of $E(\beta)-E(\alpha)$ into itself. (Here certain of the above restrictions may be ignored.) Lorch has shown that if T is associated in this way with an absolutely bounded resolution, then one has an operational calculus for T [5] and the converse of this is quite simple.

On the other hand one can show that there exist bounded transformations which are not associated with any absolutely bounded or even bounded resolution of the identity.[29] However we may still ask this question:

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[27] Given a basis, $x_1, x_2, \cdots$, with the corresponding functionals, $f_1, f_2 \cdots$, a resolution of the identity can be obtained as follows: Let $\{\lambda_i\}$ be a sequence of positive numbers such that $0<\lambda_1<\lambda_2<\cdots<1$, and $\lim_{i\to\infty}\lambda_i=1$. Let $n(\lambda)=0$, for $\lambda<\lambda_1$, $n(\lambda)=n$ if $\lambda_n\leqq\lambda<\lambda_{n+1}$ and for $\lambda\geqq 1$, $n(\lambda)=\infty$. Then for $n(\lambda)=n$ let $E(\lambda)y=\sum_{i=1}^{n}f_i(y)x_i$ Since the partial sums of a convergent biorthogonal series are uniformly bounded, the $E(\lambda)$'s are readily proved to constitute a resolution of the identity.

[28] In the construction of the preceding footnote given any finite subset $x_1', \cdots, x_m'$ of the $x_1, x_2, \cdots$ and corresponding functionals, $f_1', \cdots, f_m'$, we can find a set a_i, b_i such that $\sum(E(b_i)-E(a_i))y=\sum_{i=1}^{m}f_i'(y)x_i'$. Thus if the corresponding resolution is absolutely bounded, the partial sums of the subsequences are also uniformly bounded and the basis is absolute. The proof of the converse is longer but not essentially more difficult. In $\sum\mu_i(E(b_i)-E(a_i))$ we consider the real and imaginary parts of the sum. In either part, we can replace the corresponding part of μ_i by either $+1$ or -1 and at most increase the bound. Thus it can be shown that the bound of the given sum does not exceed the sum of the bounds of four sums in the form $\sum E(b_i)-E(a_i)$. The uniform bound of the latter is of course determined by the uniform bounds of the arbitrary partial sums if the given basis is absolute (cf. [6, pp. 564–566]).

Problem VIII$_1$. *Does every bounded transformation possess an essentially bounded resolution?*[30]

A similar Problem VIII$_2$ arises when we simply ask for a closed resolution. A closed transformation determines a bounded transformation from the graph to the domain space $A\ \{f, Tf\} = f$. It is readily seen that if A possesses a resolution, T will also. Thus we need not consider closed transformations.

9. **Abstract resolutions and transformations.** One may also attempt to define the notion of association, relative to the abstract resolutions on a family $\mathfrak{M}(\lambda)$ of manifolds. Here again we need only concern ourselves with bounded T's and a tentative definition is: 1. $\mathfrak{M}(0)$ is $\mathfrak{N}_T$. 2. $\mathfrak{M}(\lambda) \subset \mathfrak{M}(\mu)$ for $\lambda < \mu$. 3'. $\mathfrak{M}(\lambda)$ is associated with T as in Problem IV.

What is desired here is a generalization of the notion of the resolution of the absolute value of the transformation and a consideration of examples with finite-dimensional domain spaces shows that here this definition does not yield the desired result. Nevertheless in these cases the usual "choosing the maximum" construction does yield a family $\mathfrak{M}(\lambda)$ for $0 < \lambda < C$, C being the bound of T with the Properties 1, 2 and 3. T has bound λ on $\mathfrak{M}(\lambda)$, and 4. The family $\mathfrak{M}(\lambda)$ is maximal, that is, if $f \notin \mathfrak{M}(\lambda)$, there exists a $\lambda' \geqq \lambda$, for which $f \in \mathfrak{M}(\lambda')$ implies that 2 or 3 is false.

For our general discussion, we introduce the following definitions: $\mathfrak{M}$ will be said to be an over-λ-manifold if $f \in \mathfrak{M}$ implies $|Tf| \geqq \lambda |f|$. $\mathfrak{N} \subset \overline{\mathfrak{B}}$ will be said to be a coverover-λ-manifold if $|Tf| \geqq \lambda |f|$ implies that there is an F in $\mathfrak{N}$ with $F(f) \neq 0$. Similarly we define an under-λ-manifold and coverunder-λ-manifold.

In Hilbert space there are manifolds which are both over-λ-manifolds and coverover-λ-manifolds. Since in more general spaces, these are in different spaces, one can't expect this result to generalize readily. However it may happen that $\mathfrak{M}(\lambda)$, for $\lambda < 0$, may have a complement or quasi-complement which is an over-λ'-manifold for $\lambda' > 0$. We say then that the resolution $\mathfrak{M}(\lambda)$ is conjugated and thus we have this problem:

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[29] Cf. Appendix VI.

[30] If an e.b. resolution is associated with T, it is not in general unique. However if the answer to Problem VII$_1$ is yes and one e.b. resolution exists for T, then there is also one with simple spectrum. In general, each Problem VIII can be referred to the projection situation for $\mathfrak{N}_T$ and the answers to Problems V or VII.

PROBLEM IX. *Do there exist transformations T which do not possess a conjugated manifold family?*

One can establish without great difficulty, the existence of a family of manifolds $\mathfrak{M}(\lambda)$ with the Properties[31] 1, 2, 3, 4. The definitions yield that if $\mathfrak{M}$ is a maximal under-λ-manifold, $\mathfrak{M}^{\perp}$ is a minimal cover-over-λ-manifold.

10. **Comment.** The above discussion shows clearly the approach to the analysis of a specific transformation T. For instance, let us consider the problem of determining the projections E, which are associated with a given T as in Problem III_1. Suppose T^{-1} exists. This association is equivalent to the statement "TET^{-1} is bounded." Suppose the bound is k. Let $\mathfrak{M}$ be an under-λ-manifold for T and an over-σ-manifold for E. Then $E\mathfrak{M}$ must be an under-$k\lambda/\sigma$-manifold for T. For a bounded E a similar condition is sufficient. Thus the problem can be referred to the λ-manifold.

PROBLEM X. *Given T, find the various λ-manifolds.*

While this discussion does not present an exhaustive list of the problems one meets in this field, the others are apparent from the given considerations.

APPENDIX I

We establish certain results which are essential in the theory of abstract resolutions. We consider a vector space $\mathfrak{B}$ and $\mathfrak{M}$ a closed linear set in $\mathfrak{B}$. Let $\mathfrak{B}/\mathfrak{M}$ be as in §2 and $\mathfrak{B}_{\mathfrak{M}}$ denote the conjugate space to $\mathfrak{B}/\mathfrak{M}$.

LEMMA 1. *$\mathfrak{M}^{\perp}$ and $\mathfrak{B}_{\mathfrak{M}}$ are isomorphic.*

Let us correspond to a functional G in $\mathfrak{B}_{\mathfrak{M}}$, a functional F on $\mathfrak{B}$ defined by means of the equation $F(f)=G(\mathfrak{c})$ if $f\in\mathfrak{c}$. One can readily show that F is linear with $|F|=|G|$ and $F\in\mathfrak{M}^{\perp}$. On the other hand, given an $F\in\mathfrak{M}^{\perp}$, we see that $F(f)$ is a function of the class, modulo $\mathfrak{M}$, of f, that is, $F(f)=G(\mathfrak{c}), f\in\mathfrak{c}$, for some $G\in\mathfrak{B}_{\mathfrak{M}}$. Thus the correspondence, $F\sim G$ is one-to-one between $\mathfrak{M}^{\perp}$ and $\mathfrak{B}_{\mathfrak{M}}$ and metric preserving. Finally one observes that it is additive and hence $\mathfrak{B}_{\mathfrak{M}}$ and $\mathfrak{M}^{\perp}$ are isomorphic.

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[31] Suppose that the bound of T is 1. To obtain the manifolds $\mathfrak{M}(\lambda)$, we proceed as follows: It is easy to prove that if $\mathfrak{M}$ is under-λ, there is a maximal under-λ-manifold which contains it. Thus we may take $\mathfrak{M}(1/2)$ as a maximal under-1/2-manifold which contains $\mathfrak{R}_T$, $\mathfrak{M}(3/4)$ as a maximal under-3/4-manifold which contains $\mathfrak{M}(1/2)$, $\mathfrak{M}(1/4)$ as a maximal under-1/4-manifold contained in $\mathfrak{M}(1/2)$ and containing $\mathfrak{R}_T$, and so on. For other values of λ between 0 and 1, $\mathfrak{M}(\lambda)$ is the intersection of the manifolds defined for larger values of λ. The resulting family can be shown to have the Properties 1, 2, 3, 4.

LEMMA 2. *Suppose $\mathfrak{B}$ is reflexive. Given an element $c_0\in\mathfrak{B}/\mathfrak{M}$, there exists an $h_0\in\mathfrak{B}$ such that if $F\in\mathfrak{M}^{\perp}$ corresponds to $G\in\mathfrak{B}_{\mathfrak{M}}$ as in Lemma* 1, *then $F(h_0)=G(c_0)$ and $|h_0|=|c_0|$.*

Each c_0 in $\mathfrak{B}/\mathfrak{M}$ determines a linear functional $K_0(G)$ of the $G\in\mathfrak{B}_{\mathfrak{M}}$, with $K_0(G)=G(c_0)$. Correspondingly, we have an H_0 defined on $\mathfrak{M}^{\perp}$ such that $H_0(F)=K_0(G)$. Now H_0 has an extension H defined on $\overline{\mathfrak{B}}$ with $|H_0|=|H|$ and such that for $F\in\mathfrak{M}^{\perp}$, $H_0(F)=H(F)$ [1, corollary p. 29]. (For the extension giving complex homogeneity refer to [2]. A similar proof is given in [9] for $\mathfrak{B}=L_p$ which can be readily generalized.) Since $\mathfrak{B}$ is reflexive, there exists an $h_0\in\mathfrak{B}$ such that $F(h_0)=H(F)$ for all $F\in\overline{\mathfrak{B}}$. Thus if $F\sim G$ as in Lemma 1, $G(c_0)=K_0(G)$ $=H_0(F)=H(F)=F(h_0)$ and $|c_0|=|K_0|=|H_0|=|H|=|h_0|$.

LEMMA 3. *The h_0 of Lemma 2 is in c_0 and minimal in c_0, that is, h_0 is such that $|h_0+g|\geqq|h_0|$ for every g in $\mathfrak{M}$.*

If $f\in c_0$ and $F\in\mathfrak{M}^{\perp}$, we have a $G\in\mathfrak{B}_{\mathfrak{M}}$, such that $F(f)=G(c_0)$ $=F(h_0)$. But $F(f)=F(h_0)$ or $F(f-h_0)=0$ for every $F\in\mathfrak{M}^{\perp}$ implies $f-h_0\in(\mathfrak{M}^{\perp})^{\perp}$. Now $(\mathfrak{M}^{\perp})^{\perp}=\mathfrak{M}$ since $\mathfrak{B}$ is reflexive. (Cf. [9, proof of Theorem 1.3, p. 85].) Thus $f-h_0\in\mathfrak{M}$, which implies, since $f\in c_0$, that $h_0\in c_0$. Since $|h_0|=|c_0|$, $|h_0|\leqq|f|$ for every $f\in c_0$. Furthermore h_0+g is in c_0 for $g\in\mathfrak{M}$ and thus we have shown the second statement.

A space is said to be strictly convex, if $|f+g|<|f|+|g|$ when f and g are not linearly dependent.

LEMMA 4. *The minimal h_0 is unique, if $\mathfrak{B}$ is strictly convex.*

If $|c_0|=0$, the minimal element is precisely θ. Suppose $|c_0|\neq 0$ and that h_0 and h are distinct minimal elements. It is readily seen that h_0 and h are linearly independent and that also $\frac{1}{2}(h_0+h)$ is in c_0. Since strict convexity implies that $|\frac{1}{2}(h_0+h)|<\frac{1}{2}|h_0|+\frac{1}{2}|h|=|c_0|$, we have a contradiction and only one minimal element is possible.

J. Clarkson has shown that if $\mathfrak{B}$ is uniformly convex (cf. [3, Definition 1, p. 396]) then each c_0 contains a minimal element h_0. This result is unpublished and is less general than that given above (cf. [8] or [12]). In this connection one should also mention the result of Clarkson [3, p. 413], that any separable Banach space can be assigned an equivalent norm, which is strictly convex.

In the cases in which the minimal element is unique, conditions 1 and 4 of §2, determine precisely the set $\mathfrak{N}$ of minimal elements.

LEMMA 5. *The set of elements h_0, minimal relative to $\mathfrak{M}$, is the set of elements h for which there is an $F\neq 0$ and $\in\mathfrak{M}^{\perp}$, such that $|F|\cdot|h| = F(h)$.*

Suppose $F\neq 0$ and $\in\mathfrak{M}^{\perp}$ and h are such that $|F|\cdot|h| = F(h)$. Thus if $g\in\mathfrak{M}$, $F(g)=0$ and $|F|\cdot|h+g| \geqq F(h+g)=F(h)=|F|\cdot|h|$. Hence $|h+g|\geqq|h|$ and h is minimal relative to $\mathfrak{M}$. On the other hand suppose h_0 is minimal relative to $\mathfrak{M}$ and $h_0\in c_0$. There is a $G_0\in\mathfrak{B}_{\mathfrak{M}}$, such that $G_0\neq 0$ and $|G_0|\cdot|c_0| = G_0(c_0)$. (Cf. [1, chap. 4, §2, Theorem 3, p. 55].) By Lemma 1, we have an $F\in\mathfrak{M}^{\perp}$, with $|F|=|G_0|$ and $F(h_0)=G_0(c_0)=|G_0|\cdot|c_0|=|F|\cdot|h|$. Thus h_0 is an h for which there is an $F\in\mathfrak{M}^{\perp}$ such that $F(h_0)=|F|\cdot|h_0|$. This completes the proof of the lemma.

Another result is this lemma:

LEMMA 6. *If $\mathfrak{B}$ is reflexive, then $\mathfrak{B}/\mathfrak{M}$ has the same property and in particular is complete.*

PROOF. Let $t(G)$ be a linear functional on the set of linear functionals on $\mathfrak{B}/\mathfrak{M}$. Correspondingly there is a functional $t'(F)$ on $\mathfrak{M}^{\perp}$. This can be extended to $\bar{\mathfrak{B}}$ and hence $t'(F)=F(k)$ for some k of $\mathfrak{B}$. If c is the class of k, we have $t(G)=t'(F)=F(k)=G(c)$. Thus $\mathfrak{B}/\mathfrak{M}$ is the set of linear functionals on a complete space and is readily proved to be complete.

APPENDIX II

Let $\mathfrak{M}$ and $\mathfrak{N}$ be orthogonal complements in $\mathfrak{H}$. Let T be bounded, $\mathfrak{R}_T$ dense and such that T^{-1} exists. We shall show that there exist T's, $\mathfrak{M}$'s and $\mathfrak{N}$'s such that the closure $\mathfrak{M}'$ and $\mathfrak{N}'$ of $T\mathfrak{M}$ and $T\mathfrak{N}$ are neither complementary nor quasi-complementary and if E is the projection on $\mathfrak{M}$, TET^{-1} does not have a closed extension.

Suppose we have a complete orthonormal set $\phi_0, \phi_1, \cdots$ and a sequence of numbers $\lambda_i>0$, with $\sum_{i=0}^{\infty}\lambda_i<\infty$. Let $\beta_i=(1-\lambda_i^2)^{1/2}$. Define a transformation T'' by means of the equation $T''\phi_i=\lambda_i(\lambda_i\phi_{i+1}+\beta_i\phi_0)$ and let T' be defined as the least additive extension of T''. (Cf. [**15**, p. 45, Theorem 2.10] or [**11**, Definition 4, p. 32].)

T' is bounded. For the domain of T' consists of those f's in the form $\sum_{i=0}^{n}a_i\phi_i$ and each a_i is such that $|a_i|\leqq|f|$. Then $|T'f| = |\sum_{i=0}^{n}a_iT\phi_i| \leqq \sum_{i=0}^{n}|a_i|\,|T\phi_i| \leqq \sum_{i=0}^{n}|f|\cdot\lambda_i \leqq |f|\sum|\lambda_i| \leqq C|f|$. Since T' is bounded, it possesses a bounded extension, T with domain the full space. (Cf. [**15**, p. 57, Theorem 2.23] or [**11**, p. 10, Theorem II].)

For T, we have that $Tf=\theta$ implies $f=\theta$. For let $f=\sum_{i=0}^{\infty}a_i\phi_i$. Then $Tf=\sum_{i=0}^{\infty}a_i\lambda_i(\lambda_i\phi_{i+1}+\beta_i\phi_0)=(\sum_{i=0}^{\infty}\lambda_i\beta_ia_i)\phi_0+\sum_{i=1}^{\infty}\lambda_i^2a_{i-1}\phi_i$. Thus Tf

$=\theta$ implies $\lambda_i^2 a_{i-1}=0$ for $i=1, 2, \cdots$ or $a_i=0$ for $i=0, 1, \cdots$. Hence $f=\theta$ and T has an inverse.

$\Re_T$ is dense, for $\Re_T$ contains $\lambda_i\phi_{i+1}+\beta_i\phi_0$ and since $\lambda_i\rightarrow 0$, $\beta_i\rightarrow 1$, the closure $[\Re_T]$ must contain ϕ_0. Subtracting $\beta_i\phi_0$, we can show that $[\Re_T]$ also contains ϕ_{i+1} for $i=0, 1, \cdots$. Hence $[\Re_T]$ contains a complete orthonormal set and is the full space. (Cf. [11, p. 21].)

Let $\mathfrak{M}$ be the closed linear manifold determined by the ϕ_i's with even subscripts, $\mathfrak{N}$ by those with odd subscripts. $\mathfrak{M}$ and $\mathfrak{N}$ are orthogonal complements. It is readily seen that $[T\mathfrak{M}]$ and $[T\mathfrak{N}]$ both contain ϕ_0 and thus are not even quasi-complementary.

If E is the orthogonal projection on $\mathfrak{M}$, $TET^{-1}=F$ does not have a closed extension. For let $f_k=\lambda_{2k}\phi_{2k+1}+\beta_{2k}\phi_0$, then $Ff=f$ and $f_k\rightarrow\phi_0$. Hence if F has a closed extension $[F]$ then $[F]\phi_0$ would exist and equal ϕ_0. On the other hand, if $g_k=\lambda_{2k+1}\phi_{2k+2}+\beta_{2k}\phi_0$, we have $Fg_k=\theta$ and $g_k\rightarrow\phi_0$. Thus if $[F]$ exists, we would have $[F]\phi_0=\theta$. This contradiction shows that $[F]$ does not exist.

While the above argument is given only for Hilbert space, it can be readily extended to any space having a basis, $\phi_0, \phi_1, \cdots$ for which there is a k such that $f=\sum_{i=0}^{\infty}a_i\phi_i$ implies $|a_i|\leqq k|f|$.

APPENDIX III

Let T be a bounded transformation from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ or in $\mathfrak{H}$ and let $\mathfrak{M}$ be a closed linear manifold such that T is an isomorphism on $\mathfrak{M}$. We shall show that T preserves complementation for at least one complement to $\mathfrak{M}$.

Let $WH=T$ be the canonical resolution of T where H is self-adjoint definite, W partially isometric. (Cf. [16, Theorem 7, p. 307] or [8, Theorem 1.24, p. 312] or [11, chap. 9, p. 95].) Let H have the form $\int_0^{\infty}\lambda dE_1(\lambda)$. We suppose, as we may, that $\mathfrak{H}_2$ is precisely the closure of the range of T.

LEMMA 1. *There exists a c.a.d.d. transformation* T'' *from* $\mathfrak{H}_2$ *to* $\mathfrak{H}_1$, *whose domain is* $\Re_T$ *and whose range is the range of* $1-E_1(0)$ *and which is such that* $TT''\subset 1$, $T''T=1-E_1(0)$.

If $h\in\Re_T$, there is a unique g in $\Re_{1-E_1(0)}$ such that $Tg=h$. $h\in\Re_T$ implies that there is an f such that $Tf=h$. Let $g=(1-E_1(0))f$. $TE_1(0)=0$ implies $Tf=Tg=h$. This g is uniquely determined by h. for if $Tg_1=h=Tg_2$, we have that $T(g_1-g_2)=\theta$. Thus g_1-g_2 is in the range of $E_1(0)$ and $1-E_1(0)$ and hence is θ. Let $T''h=g$,

From the above, we see that if h is in $\Re_T$, then $TT''h=Tg=h$ and thus $TT''\subset 1$. For g in $\Re_{1-E_1(0)}$, $T''Tg=Th=g$ and if f is in

$\mathfrak{N}_T=\mathfrak{N}_{1-E_1(0)}$, $T''Tf=T''\theta=\theta$. The linearity of $T''T$ yields that $T''T=1-E_1(0)$.

LEMMA 2. *Let F be a projection $\mathfrak{M}=\mathfrak{R}_F$, $\mathfrak{N}=\mathfrak{N}_F$, $\mathfrak{N}\supset\mathfrak{N}_T$. If TFT'' is bounded, its closure is a projection and $\mathfrak{M}'$ and $\mathfrak{N}'$ are complementary.*

$F_2=TFT''$ has domain $\mathfrak{R}_T$ which is dense. Thus $[F_2]$ has domain $\mathfrak{H}_2$. Since $\mathfrak{N}_F\supset\mathfrak{N}_1$, $FE_1(0)=0$ and $F(1-E_1(0))=F$. Thus $TFT''TFT''=TF(1-E_1(0))FT=TF^2T''=TFT''$. Thus $F_2F_2=F_2$. This yields $[F_2]^2=[F_2]$, since F is bounded. Thus $[F_2]$ is a projection.

$\mathfrak{R}_{F_2}=T\mathfrak{M}$ for $F_2=TFT''$ implies $\mathfrak{R}_{F_2}\subset T\mathfrak{M}$ while if $g\in T\mathfrak{M}$, $g=Tf$, $f\in\mathfrak{M}$ and

$$F_2g=TFT''Tf=TF(1-E(0))f=TFf=Tf=g.$$

Thus for $g\in T\mathfrak{M}$, $F_2g=g$. Thus $\mathfrak{R}_{F_2}=T\mathfrak{M}$ and $\mathfrak{R}_{[F_2]}=\mathfrak{M}'$.

A similar discussion will show $\mathfrak{N}_{F_2}=T\mathfrak{N}$ since $\mathfrak{N}\supset\mathfrak{N}_T$. Now if $g\in\mathfrak{R}_T=\mathfrak{D}_{F_2}$, then $g=h_1+h_2$, $h_1\in T\mathfrak{M}$, $h_2\in T\mathfrak{N}$ and $F_2g=h_1$. Now if $[F_2]f=\theta$, there must be a sequence of such g's approaching f with the corresponding h_1's approaching θ. Thus f is also the limit of the h_2's and hence is in $\mathfrak{N}'$. Thus $\mathfrak{N}'=\mathfrak{N}_{[F_2]}$.

Since F is bounded these facts show that $\mathfrak{M}'$ and $\mathfrak{N}'$ are complementary.

LEMMA 3. *Let T be an isomorphism on $\mathfrak{M}$. Then there exists a $\lambda_0>0$, such that $1-E_1(\lambda_0)$ when contracted to $\mathfrak{M}$ is also an isomorphism.*

Since T is an isomorphism on $\mathfrak{M}$, there exists a constant C such that for $f\in\mathfrak{M}$, $|Tf|\geqq C|f|$. (Cf. [1, chap. 3, Theorem 5, p. 41] or [11, p. 50, Theorem IV].) This may be written

$$|TE_1(C/2)f+T(1-E_1(C/2))f|^2\geqq C_1^2|f|^2.$$

Since T preserves orthogonality for manifolds which reduce H [8, §5, Definition 1, p. 312 and Theorem IV, p. 313], we get

$$|TE_1(C/2)f|^2+|T(1-E_1(C/2))f|^2\geqq C^2|f|^2.$$

This implies, if C_1 is the bound of T,

$$(C/2)^2|E_1(C/2)f|^2+C_1^2|(1-E_1(C/2))f|^2\geqq C^2|f|^2.$$

Using $|E_1(C/2)f|^2=|f|^2-|(1-E_1(C/2))f|^2$, we get

$$(C_1^2-C^2/4)|(1-E_1(C/2))f|^2\geqq(3/4)C^2|f|^2.$$

Thus there exists a constant K such that $|(1-E_1(C/2))f|^2\geqq K^2|f|^2$ for all f in $\mathfrak{M}$. Since $1-E(C/2)$ is bounded this establishes the result.

Let B be the contraction of $1-E_1(C/2)$ with domain $\mathfrak{M}$. From the above we see that B^{-1} exists and has bound K^{-1}. B is closed since it is defined on a closed linear manifold. Since B^{-1} exists and is bounded $\overline{\mathfrak{M}}=\mathfrak{R}_B$ is also closed. $\overline{\mathfrak{M}}\subset\mathfrak{R}_{1-E(C/2)}$, $\subset\mathfrak{R}_{1-E_1(0)}$, and $\overline{\mathfrak{M}}^{\perp}\supset\mathfrak{N}_T$.

Lemma 4. *$TE_1(\lambda_0)T''$ and $T(1-E_1(\lambda_0))T''$ have bound* 1.

We have $TE_1(\lambda_0)=E_2(\lambda_0)T$ and $T(1-E_1(\lambda_0))=(1-E_2(\lambda_0))T$ (cf. [**8**, §5, Definition 1, p. 312 and Theorem IV, p. 313]). Applying T'' first we get $TE_1(\lambda_0)T''\subset E_2(\lambda_0)$ and $T(1-E_1(\lambda_0))T''\subset 1-E_2(\lambda_0)$ and hence $TE_1(\lambda_0)T''$ and $T(1-E_1(\lambda_0))T''$ have bound 1.

Lemma 5. *If $\overline{\mathfrak{M}}=\mathfrak{R}_{\bar{E}}$ is included in the range of $1-E_1(\lambda_0)$, $\lambda_0>0$, then $T\bar{E}T''$ is bounded.*

If $\bar{E}$ is the orthogonal projection on $\overline{\mathfrak{M}}$, $\bar{E}=\bar{E}(1-E_1(\lambda_0))$. The transform is $T\bar{E}T''=T\bar{E}(1-E_1(\lambda_0))T''=T\bar{E}(1-E_1(0))(1-E_1(\lambda_0))T''$ $=T\bar{E}T''T(1-E_1(\lambda_0))T''$. T is an isomorphism on the range of $1-E_1(\lambda_0)$ and hence the complementation of $\overline{\mathfrak{M}}$ and $\mathfrak{H}\ominus\mathfrak{M}_1(\lambda_0)\ominus\overline{\mathfrak{M}}$ relative to $\mathfrak{H}\ominus\mathfrak{M}_1(\lambda_0)$ must be preserved as a complementation relation relative to the range of $1-E_2(\lambda_0)$. Thus $T\bar{E}T''$ is bounded relative to the range of $1-E_2(\lambda_0)$ which includes that of $T(1-E_1(\lambda_0))T''$. Since $T(1-E_1(\lambda_0))T''$ is also bounded, $T\bar{E}T''T(1-E_1(\lambda_0))T''=T\bar{E}T''$ is bounded.

Lemma 6. *$TB^{-1}T''$ exists and is bounded.*

First we establish that the domain of $TB^{-1}T''$ is $T\overline{\mathfrak{M}}$. For $\mathfrak{D}_{TB^{-1}T''}$ $=\mathfrak{D}_{B^{-1}T}$. Since T and T'' are in an inverse relationship relative to $\mathfrak{R}_{1-\bar{E}_1(0)}$, which includes $\overline{\mathfrak{M}}$, $T\overline{\mathfrak{M}}=T\mathfrak{R}_B=T\mathfrak{D}_{B^{-1}}=\mathfrak{D}_{B^{-1}T''}$.

For $f\in\overline{\mathfrak{M}}\subset\mathfrak{R}_{1-E_1(\lambda_0)}$, we have $|Tf|\geqq\lambda_0|f|$. Since T'' and T are inversial for $g\in T\overline{\mathfrak{M}}$, $|T''g|\leqq(1/\lambda_0)|g|$. Since B^{-1} and T are bounded this yields the desired result.

Lemma 7. *$B^{-1}\bar{E}$ is a projection with range $\mathfrak{M}$ and manifold of zeros $\overline{\mathfrak{M}}^{\perp}$.*

Since $\bar{E}\leqq 1-E_1(\lambda_0)$ and $B\subset 1-E_1(\lambda_0)$, we have $\bar{E}B^{-1}g=g$ for g in $\mathfrak{R}_{\bar{E}}=\overline{\mathfrak{M}}=\mathfrak{R}_B=\mathfrak{D}_{B^{-1}}$. Thus $\bar{E}B^{-1}\subset 1$ and $(B^{-1}\bar{E})(B^{-1}\bar{E})=B^{-1}\bar{E}$. $B^{-1}\bar{E}$ is bounded and has domain the full space since $\mathfrak{R}_{\bar{E}}=\mathfrak{D}_{B^{-1}}$. Hence it is a projection. Since $\mathfrak{R}_{\bar{E}}=\mathfrak{D}_{B^{-1}}$, $\mathfrak{R}_{B^{-1}\bar{E}}=\mathfrak{R}_{B^{-1}}=\mathfrak{D}_B=\mathfrak{M}$. $\mathfrak{N}_{B^{-1}\bar{E}}=\mathfrak{N}_{\bar{E}}$ $=\overline{\mathfrak{M}}^{\perp}$.

We are now in a position to obtain our result. For $TB^{-1}\bar{E}T''$ $=TB^{-1}(1-E_1(0))\bar{E}T''=TB^{-1}T''T\bar{E}T''=(TB^{-1}T'')(T\bar{E}T'')$ which is bounded by Lemmas 5 and 6. Since $\overline{\mathfrak{M}}^{\perp}\supset\mathfrak{N}_T$, Lemma 2 shows that T preserves the complementation of $\mathfrak{M}$ and $\overline{\mathfrak{M}}^{\perp}$.

APPENDIX IV

We give an example in Hilbert space of a closed T, whose domain includes a cl. 1 $\mathfrak{M}$, for which $T_{\mathfrak{M}}$ is not closed nor has a closed extension. We first prove the following lemma.

LEMMA. *Let* $\{\{a_1^{(k)}, a_2^{(k)}, \cdots\}\}$ *denote a sequence of sequences with the property that* $\sum_{i=1}^{\infty}|a_1^{(k)}|^2 \to 0$, *as* $k \to \infty$, *and for a* K *independent of* k, $\sum_{i=1}^{\infty}|a_i^{(k)} i 2^i|^2 < K$. Then $\sum_{i=1}^{\infty}|a_i^{(k)}| i \to 0$, *as* $k \to \infty$.

PROOF. No clarity is lost if we suppress k. Let $c_i = a_i i 2^i$. By hypothesis, $\sum_{i=1}^{\infty}|c_i|^2 < K$. Now $\sum_{i=N+1}^{\infty} i |a_i| = \sum_{i=N+1}^{\infty} 2^{-i}|c_i| < K2^{-N}3^{1/2}$, by Schwarz's inequality. Our hypotheses also imply that for every N, $\sum_{i=1}^{N}|a_i| i \to 0$. These results imply that lim sup $\sum_{i=1}^{\infty}|a_i| \cdot i < K2^{-N}3^{1/2}$ for every N. Hence $\sum_{i=1}^{\infty}|a_i| i \to 0$.

Let $\phi_0, \phi_1, \cdots$ be a complete orthonormal set in $\mathfrak{H}$, let $\psi_k = \phi_{2k-1}$, for $k = 1, 2, \cdots$ and $\omega_k = \phi_{2k-2}$ for $k = 1, 2, \cdots$. We define a transformation T'' by the equations $T'' \psi_k = \phi_k$, $T''\omega_k = -k2^k\phi_k + k\phi_0$, $k = 1, 2, \cdots$. T'' has a linear extension T'. f is in the domain of T', if $f = \sum_{i=1}^{\infty} a_i \omega_i + \sum_{i=1}^{\infty} b_i \psi_i$ and $T'f = (\sum_{i=1}^{\infty} i a_i)\phi_0 + \sum_{i=1}^{\infty}(b_i - i a_i 2^i)\phi_i$.

The condition that T' have a closed extension T is that whenever $\{f^{(k)}\}$ is a sequence in the domain of T' such that $|f^{(k)}|^2 \to 0$ and $T'f^{(k)} \to g$, then $g = \theta$. Suppose we have such a sequence of f's and let $g = \sum_{i=0}^{\infty} c_i \phi_i$. We shall show that $g = \theta$.

To do this we first show that $c_0 = 0$. Suppressing the superscript we let f determine a_i and b_i as above. Then $\sum_{i=1}^{\infty} i a_i \to c_0$. Now we have since $f \to \theta$ and $Tf \to g$, that $\sum_{i=1}^{\infty}|a_i|^2 + \sum_{i=1}^{\infty}|b_i|^2 \to 0$ and that there is a K such that $\sum_{i=1}^{\infty}|b_i - i a_i 2^i|^2 < K$. Since the $\sum_{i=1}^{\infty}|b_i|^2$ are bounded we may even infer $\sum_{i=1}^{\infty}|i a_i 2^i|^2 < K_1$ for a constant K_1. Our lemma now tells us that $\sum_{i=1}^{\infty} i a_i \to 0$ and thus $c_0 = 0$.

Since for $i \geqq 1$, we have $(b_i - i2^i a_i) \to c_i$, we must also have $c_i = 0$ for $i \geqq 1$. Hence $g = \theta$. Thus T' has a closed extension T.

If $\mathfrak{M}$ is the closed linear manifold determined by the ψ_k, $T\mathfrak{M}$ is the set determined by $\phi_1, \phi_2, \cdots$. Let $\mathbf{c}_k$ denote the class of $(1/k)\omega_k$ relative to $\mathfrak{M}$, which is also the class of $h_k = 2^k\psi_k + (1/k)\omega_k$. Now $Th_k = T'h_k = 2^k\phi_k - 2^k\phi_k + \phi_0 = \phi_0$. Hence $Th_k = \phi_0$. Let $\mathbf{c}_0$ be the class of ϕ_0 relative to $T\mathfrak{M}$. Then $|\mathbf{c}_k| = 1/k$ and hence $\mathbf{c}_k \to \theta$. Also $T_{\mathfrak{M}}\mathbf{c}_k = \mathbf{c}_0 \to \mathbf{c}_0$, with $|\mathbf{c}_0| = 1$ since ϕ_0 is in $(T\mathfrak{M})^{\perp}$. Hence $T_{\mathfrak{M}}$ does not have a closed extension.

APPENDIX V

We suppose that each $\mathfrak{B}_i$ is separable, regular ($\mathfrak{B}_i$ reflexive), and strictly convex. If T is closed and preserves minimality relative to $\mathfrak{M}$, then $T_{\mathfrak{M}}$ is closed.

For suppose $\{c_\iota\}$ is a sequence of $\mathfrak{B}_1/\mathfrak{M}$ such that c_ι and $T_{\mathfrak{M}}c_\iota$ are convergent to c and c', respectively. Let h_ι be the minimal element of c_i, h that of c, g that of c'.

There is a linear functional G defined on $\mathfrak{B}_1/\mathfrak{M}$ such that $G(c) = |G|\cdot|c|$. As in Appendix I, there corresponds an $F\in\mathfrak{M}^\perp$ with $|F|=|G|$, $F(h)=G(c)=|G|\cdot|c|=|F|\ |h|$. Also $F(h_\iota)=G(c_\iota) \to G(c)=F(h)$.

The $|h_\iota|=|c_\iota|$ are bounded. Thus there is a subsequence of the h_ι which is weakly convergent to an h' with $|h'|\leqq \lim |h_\iota| = \lim |c_\iota| = |c| = |h|$ or $|h'|\leqq|h|$. Also $|F|\cdot|h'|\geqq F(h')=F(h)=|F|\cdot|h|$. It follows that $|h|=|h'|$ and $F(h')=F(h)$. It can be shown that this implies that $h=h'$, if the space is strictly convex.[22]

Thus if $\{c_\iota\}$ is convergent to c, then a subsequence of their minimal elements is weakly convergent to the minimal element of c. Thus we can choose a subsequence of the h_ι's which converges weakly to h and of these a further subsequence can be chosen so that the Th_ι approach g. Thus $\{h, g\}$ is the weak limit of a subsequence of the $\{h_\iota, Th_\iota\}$. Since the graph is a closed linear set, it is weakly closed. Thus $\{h, g\}$ is in the graph of T and $Th=g$. Hence $T_{\mathfrak{M}}c$ exists and equals c' and $T_{\mathfrak{M}}$ is closed.

APPENDIX VI

We give here an example of a transformation, which is not associated with any absolutely bounded resolution in the sense of Lorch or with any bounded resolution in the sense of §8.

For if $T=\int_{-C}^{C}\phi(\lambda)dE(\lambda)$, in Lorch's notation for an absolutely bounded resolution, $E(\lambda)$, then the set $\mathfrak{N}_T$ of zeros of T has a projection of bound K on it. This is also true if, in the sense of §8, T is associated with a bounded resolution. Thus it will be sufficient for our purposes if we construct a T such that $\mathfrak{N}_T$ does not possess a bounded projection.

Let us consider the construction given in [10, p. 152] of a manifold without a projection and in particular the terminology used there.

We regard l_p as a sum $\sum\oplus l_{p,2^n}$. Let F_n be the projection of l_p on the nth term in this sum, that is, on the $l_{p,2^n}$. Let E_n be the projection of $l_{p,2^n}$ on $\mathfrak{M}_n$. Let λ_n be such that $\lambda_n(F_n-E_nF_n)$ has bound 1. It can readily be shown that $T=\sum_{n=0}^{\infty}\lambda_n(F_n-E_nF_n)$ has bound 1 and that the zeros of T form the set $\mathfrak{P}$, which has no bounded projection on it.

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[22] One shows that if h is such that $F(h)=|F|\cdot|h|$ and g is such that $F(g)=0$, then $|h+g|\geqq|g|$. For $|F|\cdot|h+g|\geqq|F(h+g)|=F(h)=|F|\cdot|h|$. Thus h and h' are minimal relative to the set of g's for which $F(g)=0$. Since $F(h'-h)=F(h')-F(h)=0$, they are in the same class relative to this set. For a strictly convex space this minimum in a class is unique.

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COLUMBIA UNIVERSITY

CONFORMAL GEOMETRY OF ONE-PARAMETER FAMILIES OF CURVES

ROBERT COLEMAN, JR.

A single regular analytic arc in the plane has no conformal differential invariants. The conformal theory of curvilinear angles was initiated by Kasner,[1] and has been elaborated by him and others. The present paper is concerned with conformal differential invariants of a real one-parameter family of regular analytic arcs in the plane. We assume that the family is defined in some region R of the (x, y)-plane by an equation of the form: $u(x, y)=$constant, where u is a single-valued function which satisfies the conditions: (1) u is analytic in the region R, (2) u assumes real values for real values of x and y, (3) $u_x^2+u_y^2$ does not vanish in R. By a conformal transformation we shall mean a real conformal transformation, nonsingular in R. Our principal results are: When a family $u=c$ is transformed conformally into a family $U=c$, the parameters of the two families being the same, the quantity $\Delta\equiv(u_{xx}+u_{yy})/(u_x^2+u_y^2)$, and certain conformally invariant derivatives of Δ are unaltered. There exist rational functions of Δ and these derivatives which are independent of the parameter in terms of which the family $u=$constant is expressed. We obtain a geometric interpretation of the invariants and apply the results to a generalization of isothermal families.

1. **The invariants.** Let $U(X, Y)=c$ be a one-parameter family in the (X, Y)-plane. Let this plane be mapped conformally on the (x, y)-plane by the transformation

$$(1.1)\qquad X = X(x, y), \qquad Y = Y(x, y),$$

where

$$(1.2)\qquad X_x = Y_y, \quad X_y = -Y_x, \quad X_{xx} + X_{yy} = Y_{xx} + Y_{yy} = 0.$$

The family $U(X, Y)=c$ is transformed into $u(x, y)=c$ where $u(x, y) \equiv U[X(x, y), Y(x, y)]$. By differentiating this last identity we obtain:

$$(1.3)\qquad U_X X_x + U_Y Y_x = u_x, \qquad U_X X_y + U_Y Y_y = u_y.$$

These equations together with (1.2) give

$$(1.4)\qquad u_x^2 + u_y^2 = J(U_X^2 + U_Y^2), \quad \text{where } J \equiv X_x^2 + X_y^2.$$

Received by the editors April 6, 1941.

[1] Proceedings of the International Congress at Cambridge, 1912.

Differentiating the first of (1.3) with respect to x and the second with respect to y, and making use of (1.2) we obtain

$$u_{xx} + u_{yy} = J(U_{XX} + U_{YY}). \tag{1.5}$$

This equation with (1.4) gives

$$\frac{u_{xx} + u_{yy}}{u_x^2 + u_y^2} = \frac{U_{XX} + U_{YY}}{U_X^2 + U_Y^2}. \tag{1.6}$$

Hence the quantity $\Delta \equiv (U_{XX} + U_{YY})/(U_X^2 + U_Y^2)$ is unaltered.

Let dS denote the linear element of the (X, Y)-plane, and ds that of the (x, y)-plane. Under the mapping (1.1) we have

$$dS^2 = J ds^2. \tag{1.7}$$

Let $Q(X, Y)$ be a differentiable function defined in a region R' of the (X, Y)-plane, C' any arc with continuous tangent in R'. Let dQ/dS denote the directional derivative of Q along C' at some point P'. Let C and P be the images of C' and P' under the mapping (1.1), and suppose that under the mapping we have $Q[X(x, y), Y(x, y)] \equiv q(x, y)$, then

$$\left(\frac{dQ}{dS}\right)_{P'} = J^{-1/2}\left(\frac{dq}{ds}\right)_P.$$

This equation together with (1.4) gives

$$\frac{1}{(U_X^2 + U_Y^2)^{1/2}} \frac{dQ}{dS} = \frac{1}{(u_x^2 + u_y^2)^{1/2}} \frac{dq}{ds}. \tag{1.8}$$

Equation (1.8) holds in particular when d/dS denotes differentiation along a curve of the family $U = c$. In what follows we give d/dS this meaning. The curve $U = c_0$ is so oriented that if θ denotes the angle from the positive direction of the X-axis to the positive direction of $U = c_0$, then

$$\cos\theta = U_Y[U_X^2 + U_Y^2]^{-1/2}, \qquad \sin\theta = -U_X[U_X^2 + U_Y^2]^{-1/2}.$$

Since angles are preserved by the mapping (1.1), equation (1.8) holds if the derivatives are taken along orthogonal trajectories of $U = c$ and $u = c$. The trajectories are so oriented that the angle from the positive direction of $U = c_0$ to the positive direction of its orthogonal trajectory is $\pi/2$. We shall use the symbol d/dN to denote differentiation along an orthogonal trajectory of $U = c$.

Suppose that $I(U_X, U_Y, U_{XX}, \cdots)$ is any differentiable function of the derivatives of the first n orders of $U(X, Y)$ which is unaltered by the mapping (1.1), that is,

$$I(U_X, U_Y, U_{XX}, \cdots) \equiv I(u_x, u_y, u_{xx}, \cdots).$$

Then from (1.8) we see that the quantities

$$\frac{1}{[U_X^2 + U_Y^2]^{1/2}} \frac{dI}{dS}, \qquad \frac{1}{[U_X^2 + U_Y^2]^{1/2}} \frac{dI}{dN}$$

will also be unaltered; furthermore, they depend only on the derivatives of U of the first $n+1$ orders. We have seen that Δ is unaltered. Hence from Δ we may obtain an infinite number of other invariant functions of the derivatives of U by repeated application of the operators

$$\frac{d}{d\lambda} \equiv \frac{1}{[U_X^2 + U_Y^2]^{1/2}} \frac{d}{dS}, \qquad \frac{d}{d\lambda_N} \equiv \frac{1}{[U_X^2 + U_Y^2]^{1/2}} \frac{d}{dN}.$$

If $Q(X, Y)$ is any p times differentiable function we will write

$$\frac{d^pQ}{d\lambda^p} \equiv Q_{(p\lambda)}, \qquad \frac{d^pQ}{d\lambda_N^p} \equiv Q_{(p\lambda_N)}.$$

From (1.4) and (1.7) we obtain $[U_X^2 + U_Y^2]dS^2 = (u_x^2+u_y^2)ds^2$. Suppose that P_1', P_2' are two points on the same curve $U = c_0$ and P_1, P_2 are the corresponding points on the curve $u = c_0$. Since the transformation is nonsingular, by hypothesis, we have

$$\int_{P_1'}^{P_2'} [U_X^2 + U_Y^2]^{1/2} dS = \int_{P_1}^{P_2} (u_x^2 + u_y^2)^{1/2} ds$$

where the integrals are taken along $U = c_0$ and $u = c_0$, respectively. We have this theorem:

THEOREM 1. *When a family $U = c$ is transformed by a conformal transformation into a family $u = c$, the quantities*

$$\int_{P_1'}^{P_2'} (U_X^2 + U_Y^2)^{1/2} dS; \ \Delta, \ \Delta_{(p_1\lambda)(p_2\lambda_N)(p_3\lambda)} \cdots \tag{1.9}$$

are invariant.

It is to be noted that the quantities (1.9) are expressed in terms of a particular parameter, and are not independent of changes of parameter. For isothermal families, Δ and its derivatives in (1.9) vanish identically when the parameter is suitably chosen. For other families we have this theorem.

THEOREM 2. *For non-isothermal families, there exist rational functions of the quantities* (1.9) *which are independent of the parameter in terms of which the family is expressed.*

To establish this theorem we note first of all that since $U_X^2+U_Y^2\neq 0$ by hypothesis, X, Y may be expressed in terms of U, V where $V=k$ gives the orthogonal trajectories of $U=c$. We have

$$
\begin{aligned}
\frac{\partial}{\partial U} &\equiv \left(X_U\frac{\partial}{\partial X}+Y_U\frac{\partial}{\partial Y}\right)\\
&= \frac{1}{U_X^2+U_Y^2}\left[U_X\frac{\partial}{\partial X}+U_Y\frac{\partial}{\partial Y}\right]\equiv\frac{d}{d\lambda_N},\\
-\mu\frac{\partial}{\partial V} &= -\mu\left(X_V\frac{\partial}{\partial X}+Y_V\frac{\partial}{\partial Y}\right)\\
&= \frac{1}{U_X^2+U_Y^2}\left[U_Y\frac{\partial}{\partial X}-U_X\frac{\partial}{\partial Y}\right]\equiv\frac{d}{d\lambda},
\end{aligned}
\tag{1.10}
$$

where

$$
\mu\equiv\frac{V_Y}{U_X}=-\frac{V_X}{U_Y}.
$$

Suppose now that the parameter of a non-isothermal family is changed by means of the equation $\bar{U}=k(U)$. Then if we denote by $\Delta(P')$ the value of Δ at a point P' we have

$$
\phi\bar{\Delta}(P')=\frac{d}{dU}(\log\phi)+\Delta(P'),
\tag{1.11}
$$

where $\phi=dk/dU$ and the bar refers to the parameter $\bar{U}$. With a point $P'(U, V)$ we may associate a second point P_0' on the same curve $U=c_0$ as follows. Let $V=k_0$ be some *fixed* orthogonal trajectory cutting every curve $U=c$ in the region R'; then P_0' is the point (U, k_0). From (1.1) we have

$$
\phi[\bar{\Delta}(P')-\bar{\Delta}(P_0')]=\Delta(P')-\Delta(P_0').
\tag{1.12}
$$

From (1.10) we obtain

$$
\phi\frac{d}{d\bar{\lambda}_N}\equiv\phi\frac{\partial}{\partial\bar{U}}=\frac{\partial}{\partial U}=\frac{d}{d\lambda_N},
\tag{1.13}
$$

$$
\phi\frac{d}{d\bar{\lambda}}\equiv-\bar{\mu}\phi\frac{\partial}{\partial V}=-\mu\frac{\partial}{\partial V}\equiv\frac{d}{d\lambda}.
\tag{1.14}
$$

Now let $L(P'; P_0')$ be any rational function of Δ and a finite number of the derivatives in (1.9). Let $\bar{L}(P'; P_0')$ denote the same function of the corresponding quantities for the parameter $\bar{U}$, and suppose that L and $\bar{L}$ satisfy

$$\phi^n \bar{L}(P'; P_0') = L(P'; P_0'), \tag{1.15}$$

then we can obtain two other such functions involving the derivatives in (1.9) up to the $(p+1)$th order if L involves them up to the pth order. For (1.14) and (1.15) give $\phi^{n+1}\bar{L}_{\bar{\lambda}} = L_\lambda$. Hence L_λ is one such function. Secondly, (1.13) and (1.15) give

$$\phi^{n+1}\bar{L}_{\bar{U}} + n\phi^n \bar{L} \frac{d}{dU} (\log \phi) = L_U.$$

This, by virtue of (1.11) and (1.15) is equivalent to

$$\phi^{n+1}\bar{M} = M,$$

where

$$M \equiv L_U + nL\Delta(P'), \qquad \bar{M} \equiv \bar{L}_{\bar{U}} + n\bar{L}\bar{\Delta}(P'),$$

so that M is a second such function. Now from (1.12), $\Delta(P') - \Delta(P_0')$ is one function of the type L above. Hence for each of the derivatives in (1.9) we can obtain a function rational in it and the lower derivatives which satisfies a relation of the type (1.15). The function ϕ in (1.15) can be eliminated as follows. We have seen that

$$\lambda(P') \equiv \int_{P_0'}^{P'} \left[U_X^2 + U_Y^2\right]^{1/2} dS,$$

where the integral is taken along a curve $U = c_0$, is invariant. Clearly

$$\bar{\lambda}(P') = \phi\lambda(P')$$

so that

$$\bar{\lambda}^n \bar{L} = \lambda^n L.$$

This completes the proof.

2. Geometric interpretation. We have seen that when a family $u(x, y) = c$ is transformed conformally into a family $U(X, Y) = c$, the quadratic form

$$(u_x^2 + u_y^2)(dx^2 + dy^2) \tag{2.1}$$

is invariant. We may associate with the family $u = c$ a surface Σ, whose linear element is (2.1). Σ undergoes an isometric transformation when the (x, y)-plane undergoes a conformal transformation. The surface Σ is in conformal correspondence with the (x, y)-plane, corresponding points having the same coordinates (x, y), so that the family $u(x, y)=c$ in the plane is the conformal image of a family $u(x, y)=c$ in Σ. Suppose that we have a family $U(X, Y)=c$ in the (X, Y)-plane, with associated surface Σ'. If there is an isometric mapping between Σ and Σ' carrying $u=c$ into $U=c$, then this mapping induces a conformal transformation between the two planes which carries the family $u=c$ into $U=c$. Hence we have the following theorem:

THEOREM 3. *A necessary and sufficient condition that two families be conformally equivalent is that they admit parameters such that their associated surfaces are isometric, with the images of the families in the surfaces corresponding.*

From (1.10) we obtain

$$dx^2 + dy^2 = \frac{1}{u_x^2 + u_y^2}(du^2 + \mu^{-2}dv^2). \tag{2.2}$$

Hence the linear element of Σ is

$$d\sigma^2 = du^2 + \mu^{-2}dv^2. \tag{2.3}$$

Using a standard formula for the geodetic curvature of a curve $u=c$ in Σ we obtain[3]

$$\frac{1}{\rho} = \left(\log\frac{1}{\mu}\right)_u. \tag{2.4}$$

By virtue of (1.10) this is equivalent to

$$\frac{1}{\rho} = \left(\log\frac{1}{\mu}\right)_u = \Delta. \tag{2.5}$$

As a consequence of this equation we have this theorem:

THEOREM 4. *The geodetic curvature of a curve $u=c_0$ through a point P of Σ is measured by the value of Δ at P. The operators $d/d\lambda$, $d/d\lambda_N$ denote differentiation with respect to the arc length of the curves $u=c$ and their orthogonal trajectories respectively.*

If K denotes the Gaussian curvature of Σ we have $K=-\mu[\mu^{-1}]_{uu}$. By virtue of (2.5) this is equivalent to

$$-K = \Delta^2 + \Delta_u. \tag{2.6}$$

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[3] Eisenhart, *Differential Geometry*, 1909, p. 134.

THEOREM 5. *The Gaussian curvature of Σ and its various rates of change along the orthogonal net determined by $u=c$ is measured by rational functions of Δ and its derivatives in* (1.9).

If two families are conformally equivalent, then for a suitable choice of parameters, the equations $U(X, Y)=u(x, y)$, $V(X, Y)=v(x, y)$, where $V=K$, $v=k$ give the orthogonal trajectories, define a conformal correspondence. If the associated surface of $U=c$ has the linear element $dU^2+M^{-2}dV^2$, then by (2.3) we have that $M(U, V)\equiv\mu(U, V)$. Conversely, if $M(U, V)\equiv\mu(U, V)$, the families are equivalent by Theorem 3. The function μ is an integrating factor of $-u_y dx+u_x dy=0$.

THEOREM 6. *Let the families $u=c$, $v=k$ form an orthogonal net. Let $\overline{U}(X, Y)=C$, $\overline{V}(X, Y)=K$ form a second such net. If $\mu(u, v)$ is an integrating factor of $-u_y dx+u_x dy=0$, then a necessary and sufficient condition for the conformal equivalence of the families $u=c$, $\overline{U}=C$ is that there exist parameters $U=F(\overline{U})$, $V=G(\overline{V})$ such that $-U_Y dX+U_X dY=0$ admits an integrating factor of the form $\mu(U, V)$.*

The reciprocal relationship between a family and its associated surface is given by this next theorem:

THEOREM 7. *The orthogonal trajectories of a family $u=c$ are conformal images of a system of geodesics of the surface Σ associated with $u=c$. Conversely, if Σ is a real surface which admits conformal representation on the plane such that a family of its geodesics corresponds to the orthogonal trajectories of a family $u=c$, then the family $u=c$ can be so parameterized that the associated surface is isometric with Σ.*

The first part of the theorem is an immediate consequence of equation (2.3). Now suppose $u(x, y)=c$ is a family in the plane, with orthogonal trajectories $v(x, y)=k$. Let Σ be a real surface which can be mapped conformally onto the plane so that a system of its geodesics $\tilde{v}=b$ go over into the family $v=k$. We may take on Σ a system of geodesic parameters consisting of $\tilde{v}=b$ and their orthogonal trajectories, $\tilde{u}=a$, and the linear element of Σ takes the form $d\tilde{u}^2+\zeta(\tilde{u}, \tilde{v})d\tilde{v}^2$. The conformal correspondence between Σ and the plane may be written $\tilde{v}=f(v)$, $\tilde{u}=g(u)$. But these equations define a change of parameters in the (x, y)-plane, so that if we change the parameters of the families $u=c$ and $v=k$ in accordance with these equations we have $dx^2+dy^2=\xi(\tilde{u}, \tilde{v})[d\tilde{u}^2+\zeta(\tilde{u}, \tilde{v})d\tilde{v}^2]$. Because of (2.2)

$$\xi(\tilde{u}, \tilde{v}) = \frac{1}{\tilde{u}_x^2+\tilde{u}_y^2}, \qquad \zeta(\tilde{u}, \tilde{v}) = [\mu(\tilde{u}, \tilde{v})]^{-2}.$$

Hence if we express the family $u=c$ in the form $\bar{u}=a$ we see from (2.3) that the linear element of the associated surface is given by $d\sigma^2=d\bar{u}^2+\zeta(\bar{u}, \bar{v})d\bar{v}^2$. This completes the proof.

3. Generalization of isothermal families. Isothermal families are characterized by the fact that they admit parameters such that $\Delta\equiv 0$, so that when an isothermal family is referred to such a canonical parameter the associated surface is of constant zero curvature. We shall determine those families which admit parameters such that the associated surface is of constant curvature. Such families are of special character and will be referred to as *families of constant curvature.*

Given a family $u(x, y)=c$. Let the parameter be changed to $\bar{u}=h(u)$. Denote by K and $\bar{K}$ the curvatures of the corresponding associated surfaces. By use of equations (1.11), (1.13), and (2.6) we obtain

$$[\phi(u)]^2\bar{K} = K - \Delta \frac{d}{du}\,[\log \phi(u)] - \frac{d^2}{du^2}\,[\log \phi(u)], \qquad \phi \equiv \frac{dh}{du}. \tag{3.1}$$

It follows from this equation that when a family of constant curvature is referred to an arbitrary parameter, Δ satisfies an equation of the form

$$\Delta_u + \Delta^2 + \Delta \frac{d}{du}(\log \phi) + \frac{d^2}{du^2}(\log \phi) + a\phi^2 = 0$$

where a is a constant. This equation together with the fact that

$$\Delta \equiv \frac{u_{xx}+u_{yy}}{u_x^2+u_y^2}, \qquad \frac{\partial}{\partial u} \equiv \frac{1}{u_x^2+u_y^2}\left[u_x\frac{\partial}{\partial x}+u_y\frac{\partial}{\partial y}\right]$$

shows that the function $u(x, y)$ must satisfy a special differential equation of the third order. The family $u=c$ is consequently of special character. We may distinguish three classes of families of constant curvature: (1) flat families, which admit a parameter such that $K=0$; (2) spherical families, non-isothermal families which admit a *real* parameter such that $K=1$; (3) pseudo-spherical families, non-isothermal families which admit a *real* parameter such that $K=-1$. A family of constant curvature belongs to one and only one of these three classes. This fact is a consequence of (3.1) and the fact that a family $u=c$ for which $\Delta\equiv f(u)$ is isothermal.[3] The determination of all families of constant curvature is given in the following theorems.

THEOREM 8. *The orthogonal trajectories of any one-parameter family*

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[3] Eisenhart, op. cit., p. 96.

of straight lines is a flat family. Conversely, every flat family can be reduced conformally to the orthogonal trajectories of some one-parameter family of straight lines.

THEOREM 9. *The orthogonal trajectories of any non-isothermal one-parameter family of circles of the form* $(x-c_1)^2+(y-c_2)^2=1+c_1^2+c_2^2$ *is a spherical family. Conversely, every spherical family can be reduced conformally to the orthogonal trajectories of some one-parameter family of such circles.*

THEOREM 10. *The orthogonal trajectories of any non-isothermal one-parameter family of circles of the form* $x^2+(y-c_1)^2=c_2^2$ *is pseudo-spherical. Conversely, every pseudo-spherical family can be reduced conformally to the orthogonal trajectories of some one-parameter family of such circles.*

The proofs of these theorems are quite similar. Consider, for example, Theorem 9. Let Σ denote the unit sphere $\xi^2+\eta^2+\zeta^2=1$. If Σ be referred to its minimal[4] lines, $\alpha=c$, $\beta=d$, then any real one-parameter family of its geodesics is given by $C\alpha\beta+(A-iB)\alpha+(A+iB)\beta-C=0$, where A, B, C are real functions of a real variable t. The equations $X+iY=\alpha$, $X-iY=\beta$ define a real conformal correspondence between Σ and the (X, Y)-plane, the family of geodesics of Σ corresponding to $C(X^2+Y^2)+2AX+2BY=C$. In general $C\neq 0$, so that the images of the geodesics of Σ are the circles

$$\left(X+\frac{A}{C}\right)^2+\left(Y+\frac{B}{C}\right)^2=1+\left(\frac{A}{C}\right)^2+\left(\frac{B}{C}\right)^2. \tag{3.2}$$

From Theorem 7 it follows that the parameter of the orthogonal trajectories of the circles (3.2) can be so chosen that the associated surface for these trajectories is isometric with Σ. This proves the first part of Theorem 9. Suppose, conversely, that we are given a spherical family $u=c$, referred to a parameter such that $K=1$, then Σ is the associated surface. By Theorem 7 the orthogonal trajectories of $u=c$ are conformal images of a family of geodesics of Σ. As before we may map Σ conformally on the (X, Y)-plane so that its geodesics go over into the circles (3.2). This mapping clearly induces a conformal correspondence between the (x, y)-plane and the (X, Y)-plane so that the orthogonal trajectories of $u=c$ go over into the circles (3.2). This completes the proof. The proofs of the two remaining theorems are entirely analogous.

COLUMBIA UNIVERSITY

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[4] Eisenhart, op. cit., p. 109.

SOME THEOREMS ON SUBSERIES

J. D. HILL[1]

1. **Absolutely convergent series.** A simple calculation reveals that the arithmetic mean value of all subsums (including the void sum) of a given finite sum $s_n = u_1 + u_2 + \cdots + u_n$ is equal to $s_n/2$. In this section we shall show (see Theorem 1 below) that an integral mean value can be found, consistent with the preceding, for the sums of all infinite subseries of a given absolutely convergent series $\sum u_k = s$. We begin by defining a one-to-one correspondence between the set of all infinite subseries of a given absolutely convergent real series $\sum u_k = s$, and the set of all points on the interval $I \equiv (0 < \xi \leqq 1)$. If ξ is any point of I then ξ admits a unique *nonterminating* binary representation of the form

$$(1.1)\qquad \xi = 0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_k \cdots$$

where

$$(1.2)\qquad \alpha_{k_i} = 1 \quad (1 \leqq k_i < k_{i+1};\ i = 1, 2, 3, \cdots); \quad \alpha_k = 0 \quad \text{otherwise.}$$

To the point ξ shall correspond the infinite subseries $\sum_i u_{k_i}$. Conversely, if $\sum_i u_{k_i}$ $(1 \leqq k_i < k_{i+1})$ is a given infinite subseries of $\sum u_k$, we shall place it in correspondence with the point ξ of I defined by (1.1) and (1.2).

We now define a function $\phi(\xi)$ by setting $\phi(0) \equiv 0$ and

$$(1.3)\qquad \phi(\xi) \equiv \sum_{k=1}^{\infty} \alpha_k u_k, \qquad 0 < \xi \leqq 1,$$

where $0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_k \cdots$ is the nonterminating binary representation of ξ. In view of the above correspondence the set of all functional values $\phi(\xi)$ for ξ on I is evidently identical with the set of the sums of all infinite subseries of $\sum u_k$. This fact leads us to investigate the integrability of the function $\phi(\xi)$ and we find that the following lemma holds.

LEMMA 1. *The integral*

$$(1.4)\qquad \int_0^1 \phi(\xi)\,d\xi$$

exists in the sense of Riemann, and has the value $s/2$.

Received by the editors February 25, 1941.

[1] The author is indebted to the referee for valuable comments.

Before proceeding with the proof we may observe that this lemma furnishes a generalization of the fact mentioned above for finite sums. We may therefore express Lemma 1 in the form of the following theorem.

THEOREM 1. *In the sense of the integral* (1.4), *the mean value of the sums of all infinite subseries of a given absolutely convergent series is equal to one-half the sum of the series.*

PROOF OF LEMMA 1. We introduce the partial sums of the series in (1.3) and thus form a sequence of functions $\{\phi_n(\xi)\}$ for $n=1, 2, 3, \cdots$ where $\phi_n(0) \equiv 0$ and

$$\phi_n(\xi) \equiv \sum_{k=1}^{n} \alpha_k u_k, \qquad 0 < \xi \leq 1. \tag{1.5}$$

For each fixed n it is possible to choose the set of digits $\alpha_1, \alpha_2, \cdots, \alpha_n$ in 2^n distinct ways. We denote these choices by $\alpha_{1i}, \alpha_{2i}, \cdots, \alpha_{ni}$ $(i=1, 2, 3, \cdots, 2^n)$. Then for each fixed i the set of all numbers ξ of the form $0.\alpha_{1i}\alpha_{2i}\cdots\alpha_{ni}\alpha_{n+1}\alpha_{n+2}\cdots\alpha_k\cdots$ (nonterminating) comprises the interval $I_{ni} \equiv (0.\alpha_{1i}\alpha_{2i}\cdots\alpha_{ni} < \xi \leq 0.\alpha_{1i}\alpha_{2i}\cdots\alpha_{ni}111\cdots)$ of length 2^{-n}. The intervals I_{ni} $(i=1, 2, \cdots, 2^n)$ are mutually disjoint and collectively exhaust the interval I. On the interval I_{ni} the function $\phi_n(\xi)$ has the constant value $\sum_{k=1}^{n}\alpha_{ki}u_k$. Therefore $\phi_n(\xi)$ is a step function.

Since $|\phi_n(\xi)| \leq \sum |u_k|$ and $|\phi(\xi)-\phi_n(\xi)| \leq \sum_{k>n}|u_k|$ for all $n=1, 2, 3, \cdots$ and all ξ $(0 \leq \xi \leq 1)$, it follows that $\phi(\xi)$ is the uniform limit of a uniformly bounded sequence of step functions. This implies that (1.4) exists as a Riemann integral. To find its value we notice that $\phi(\xi)+\phi(1-\xi)=\phi(1)=s$ for all values of ξ except those in the denumerable set T composed of all points having the form $k\cdot 2^{-n}$ for $k=0, 1, 2, \cdots, 2^n$; $n=1, 2, 3, \cdots$. If we denote by S the set $I-T$, we have in the sense of Lebesgue,

$$\int_S \phi(\xi)d\xi + \int_S \phi(1-\xi)d\xi = s.$$

Since each integral on the left has the same value as the integral (1.4), the proof is complete.

By considering the series of real and imaginary parts it is easily seen that Lemma 1, and hence Theorem 1, remains valid for absolutely convergent series of complex terms.

In addition to the properties of the function $\phi(\xi)$ already mentioned we may in passing call attention to some further properties that it possesses. In the first place it is apparent that each of the step functions $\phi_n(\xi)$ is continuous everywhere in the interval $0 \leq \xi \leq 1$ except perhaps at points of T, and that each is continuous on the left everywhere in this interval. On account of the uniform convergence of $\{\phi_n(\xi)\}$ to $\phi(\xi)$ it is clear that $\phi(\xi)$ possesses the same properties. Moreover, it is not difficult to see that at each point of T, say $\xi_0 = 0.\alpha_1\alpha_2 \cdots \alpha_n 0111 \cdots$, the saltus $|\phi(\xi_0) - \lim_{\xi \to \xi_0 +}\phi(\xi)|$ is equal to $|u_n - \sum_{k>n} u_k|$. It follows at once that $\phi(\xi)$ is continuous everywhere if and only if u_k is of the form $a \cdot 2^{-k}$ $(k = 1, 2, 3, \cdots)$, in which case $\phi(\xi) = a\xi$. Finally, by means of the property $\phi(\xi) + \phi(1 - \xi) = s$ for ξ in S, we can easily establish the equation

$$\frac{1}{2\delta}\int_{1/2-\delta}^{1/2+\delta} \phi(\xi)d\xi = \frac{s}{2}, \qquad 0 < \delta \leq 1/2.$$

This shows that the mean value of the function $\phi(\xi)$ is $s/2$ in every subinterval of (0, 1) whose midpoint is 1/2.

2. **Conditionally convergent series.** Throughout this section $\sum u_k$ will denote a conditionally convergent real series. For series of this type the corresponding sequence of functions $\{\phi_n(\xi)\}$ defined by (1.5) will again be a sequence of step functions. Moreover, if $\sum_i u_{k_i}$ $(k_i < k_{i+1})$ denotes formally a given infinite subseries of $\sum u_k$ it is clear that the behavior as $n \to \infty$ of the sequence $\{\sum_{i=1}^{n} u_{k_i}\}$ is identical with the behavior of the corresponding sequence $\{\phi_n(\xi)\}$, where ξ is defined by (1.1) and (1.2). In studying the character of subseries we may therefore confine our attention to the sequence $\{\phi_n(\xi)\}$.

An interesting subset of I is the set G of all points ξ which correspond to convergent subseries of $\sum u_k$. We shall prove that G is a set of the first category by establishing the following stronger result.

THEOREM 2. *For all points ξ of I except those in a set H of the first category we have*

$$\liminf_{n} \phi_n(\xi) = -\infty, \qquad \limsup_{n} \phi_n(\xi) = +\infty. \tag{2.1}$$

PROOF. We recall the sets T and S as defined above in the proof of Lemma 1. We may regard S itself as a metric space S^* by conserving the euclidean notion of distance. It is clear that S^* is of the second category on itself. Furthermore, since all points of discontinuity for the step function $\phi_n(\xi)$ are included in the set T it follows that each of these functions is continuous on S^*.

Let A denote the subset of S^* of all points ξ for which $\limsup_n \phi_n(\xi) < \infty$. The points of A correspond to all subseries whose partial sums are bounded from above, except those in correspondence with points of T.

The set A is of the first category on S^*. To establish this fact let A_m $(m=1, 2, 3, \cdots)$ be that subset of A of all points ξ such that $\phi_n(\xi)\leqq m$ for all $n=1, 2, 3, \cdots$. Then $A=\sum_{m=1}^{\infty}A_m$, and each of the sets A_m is closed in S^* since $\phi_n(\xi)$ is continuous in S^*. To obtain a contradiction we assume that A is of the second category. Then at least one of the sets A_m, say A_μ, must be such that its closure, namely A_μ itself, contains all points of S^* which lie in a certain subinterval of I. We may assume this subinterval to be of the form

$$0.\beta_1\beta_2\cdots\beta_p<\xi<0.\beta_1\beta_2\cdots\beta_p+2^{-p}, \tag{2.2}$$

in the binary scale. We now define the point $\xi_1=0.\beta_1\beta_2\cdots\beta_p\gamma_{p+1}\gamma_{p+2}\cdots\gamma_k\cdots$ by setting $\gamma_k=1$ if $u_k\geqq 0$ and $\gamma_k=0$ if $u_k<0$ $(k=p+1, p+2, \cdots)$. By a familiar property of conditionally convergent series, infinitely many of the γ_k are 0 and infinitely many are 1. Since the nonterminating representation of each point in T is ultimately comprised wholly of 1's, it is clear that ξ_1 belongs to S^* and moreover, lies in the interval (2.2). On the other hand we have $\lim_n \phi_n(\xi_1)=+\infty$, since the subseries corresponding to ξ_1 is divergent to $+\infty$. It follows that ξ_1 cannot belong to A_μ, and this contradiction completes the argument.

From the fact just established we conclude that A is likewise of the first category on I.

We now let B denote the subset of S^* of all points ξ for which $\liminf_n \phi_n(\xi)>-\infty$. The set B evidently coincides with the set A defined with respect to the series $\sum(-u_k)$. Thus B is also of the first category on I.

Finally, the set T, being denumerable, is of the first category on I, and we observe that at each point of T the sequence $\{\phi_n(\xi)\}$ is convergent.

It follows that $H=A+B+T$ is a set of the first category, and that all points of I for which at least one of the relations (2.1) fails to hold are contained in the set H. This completes the proof. As a concluding remark we may observe that the set G, defined above, is of the first category since it is a subset of H.

Theorem 2 was suggested by a theorem of the same general type established recently by Agnew[2] in connection with rearrangements of conditionally convergent series. The domain space I of Theorem 2 occupies the role played by Agnew's metric space E in which a point $x=\{k_i\}$ is a rearrangement of the sequence $(1, 2, 3, \cdots)$ of positive integers and the distance (x, y) between the points $x = \{k_i\}$ and $y = \{h_i\}$ of E is given by the formula of Fréchet

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[2] Agnew, *On rearrangements of series*, this Bulletin, vol. 46 (1940), pp. 797–799.

$$(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{| k_i - h_i |}{1 + | k_i - h_i |}. \tag{2.3}$$

The analogous approach for subseries of conditionally convergent series may be employed to yield a further theorem (see Theorem 3 below) of the same nature as Theorem 2. To this end we denote by D the metric space in which a point x is a strictly increasing infinite sequence $\{k_i\}$ of positive integers and the distance (x, y) between the points x and $y = \{h_i\}$ is given by (2.3).

Unlike the space E of Agnew, the space D is complete, and therefore, by the Baire theorem, of the second category. The proof of completeness is entirely straightforward and may be left to the reader. It is likewise a simple matter to verify that the sequence $\{x_n\} \subset D$ converges to $x_0 \in D$, where $x_n = \{k_i^n\}$ for $n = 0, 1, 2, \cdots$, if and only if integers N_i exist such that $k_i^n = k_i^0$ for all $n > N_i$ $(i = 1, 2, 3, \cdots)$. We shall use this fact presently.

To each x in D there corresponds an infinite subseries $\sum_i u_{k_i}$ of $\sum u_k$, and, of course, conversely. If for each x in D we set

$$f_n(x) = \sum_{i=1}^{n} u_{k_i}, \qquad n = 1, 2, 3, \cdots \tag{2.4}$$

we may then state the following analogue of Theorem 2.

THEOREM 3. *For all points x of D except those in a set W of the first category we have*

$$\liminf_n f_n(x) = -\infty, \qquad \limsup_n f_n(x) = +\infty. \tag{2.5}$$

PROOF. Let U denote the set of all x in D for which $\limsup_n f_n(x) < \infty$, and let U_m $(m = 1, 2, 3, \cdots)$ denote the subset of U on which $f_n(x) \leq m$ for all $n = 1, 2, 3, \cdots$. Then $U = \sum_{m=1}^{\infty} U_m$. Moreover, each of the sets U_m is closed. For let $x_0 = \{k_i^0\}$ be any point of the derived set U_m', and let $\{x_p\}$ be an arbitrary sequence in U_m converging to x_0. If $x_p = \{k_i^p\}$ for $p = 1, 2, 3, \cdots$, then, by the remark made above, there exist integers N_i such that $k_i^p = k_i^0$ for all $p > N_i$ $(i = 1, 2, 3, \cdots)$. If we let $P_n = \max (N_1, N_2, \cdots, N_n)$, then $f_n(x_p) = f_n(x_0)$ for all $p > P_n$. Since $f_n(x_p) \leq m$ $(n, p = 1, 2, 3, \cdots)$ it follows that $f_n(x_0) \leq m$ $(n = 1, 2, 3, \cdots)$. Thus x_0 belongs to U_m and U_m is therefore closed.

If we assume that U is of the second category at least one of the closed sets U_m, say U_μ, must contain a sphere $K = [(x, x_0) \leq r]$ of positive radius r. Let the center x_0 be the sequence $\{k_s^0\}$, and let s be chosen so large that $2^{-s-1}+2^{-s-2}+\cdots<r$. Now a point $X=\{j_s\}$ of D exists such that $\lim_n f_n(X)=+\infty$, and such that $j_{s+1}>k_s^0$. If we define x_1 as $(k_1^0, k_2^0, \cdots, k_s^0, j_{s+1}, j_{s+2}, \cdots)$, then x_1 belongs to K and $\lim_n f_n(x_1)=+\infty$. Consequently x_1 cannot be a point of U, and this contradiction establishes U as a set of the first category.

In a similar fashion it may be shown that the set V of all x in D for which $\liminf_n f_n(x) > -\infty$ is likewise a set of the first category. Hence if we set $W=U+V$ the theorem follows.

Finally, let $\sum u_k$ be a convergent series of complex terms for which $\sum|u_k|=+\infty$, and for this series let $\phi_n(\xi)$ $[f_n(x)]$ be defined as in (1.5) [(2.4)]. We may consider the series of real and imaginary parts in the light of Theorem 2 [Theorem 3] and thus show that the set of all ξ on I [x in D] for which we have $\limsup_n |\phi_n(\xi)| < \infty$ [$\limsup_n |f_n(x)| < \infty$] is a set of the first category.

MICHIGAN STATE COLLEGE

A FORMULA FOR THE DIRECT PRODUCT OF CROSSED PRODUCT ALGEBRAS

SAUNDERS MACLANE AND O. F. G. SCHILLING

1. **Introduction.** In this note we wish to present a uniform treatment of certain properties of crossed products. A crossed product over any field F is an algebra determined by a finite, separable, normal extension N of F, with a Galois group Γ, and a certain factor set[1] h of elements $h_{S,T}$ in N, for automorphisms S and T in Γ. The crossed product (N, Γ, h) consists of all sums $\sum u_S z_S$, where the coefficients z_S lie in N, and the fixed elements u_S have the multiplication table

$$(1)\qquad u_S u_T = u_{ST} h_{S,T}, \qquad z u_S = u_S z^S, \qquad z \text{ in } N.$$

Let K be a normal subfield of N, corresponding to the subgroup Δ of the Galois group Γ. A factor set g in N is called *symmetric* in Δ if $g_{S,T}=g_{U,V}$ whenever SU^{-1} and TV^{-1} are in Δ.

Presented to the Society, May 3, 1941; received by the editors March 31, 1941.

[1] Definitions are given in A. A. Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939. Theorems cited below without explicit source all refer to this work.

THEOREM 1. *A crossed product* (N, Γ, h) *is split by a normal subfield* K *of* N *if and only if its factor set* h *is associate to a* Δ*-symmetric factor set* g, *where* Δ *is the subgroup of* Γ *corresponding to* K.

Part of this result may be stated more explicitly, using the factor group Γ/Δ as the Galois group of K over F.

THEOREM 2. *A factor set* g *is* Δ*-symmetric if and only if there is a factor set* G *of elements* $G_{\sigma,\tau}$ *in* K, *where* σ, τ *are in* Γ/Δ, *such that, for* S *in the coset* σ, T *in the coset* τ,

$$g_{S,T} = G_{\sigma,\tau}. \tag{2}$$

Furthermore the corresponding crossed products are similar,

$$(N, \Gamma, g) \sim (K, \Gamma/\Delta, G). \tag{3}$$

An equivalent of these theorems was stated by Deuring.[2] Since they were not used in Albert's Colloquium Lectures, Deuring's somewhat obscure proof is apparently the only one available. We give here a new proof. It is based on the simple observation that the standard proof[3] of the formula

$$(N, \Gamma, g) \times (N, \Gamma, h) \sim (N, \Gamma, gh) \tag{4}$$

can be extended to treat the case $(N, \Gamma, h)\times(K, \Gamma/\Delta, G)$. From this formula we obtain the theorems above, as well as a general formula for the direct product of two crossed products built on *any* two normal fields. In a systematic treatment, this proof has the advantage that it involves practically no more trouble than the proof of the ordinary product formula (4), and includes this as a special case.

2. Idempotents of matric subalgebras.

It seems convenient to use the following restatement of known results about possible total matric subalgebras of a simple algebra.

THEOREM 3. *Let the unity element of a simple algebra* A *be represented as a sum* $1 = e_1 + e_2 + \cdots + e_t$ *of pairwise orthogonal idempotents. Then* A *has a total matric subalgebra* M *with a basis* e_{ij}, $i, j = 1, \cdots, t$, *having the usual multiplication table,* $e_{ij}e_{jk} = e_{ik}$, $e_{ij}e_{mk} = 0$ *for* $j \neq m$, *and so constructed that* $e_i = e_{ii}$, *for* $i = 1, \cdots, t$, *if and only if* A *has for each* i *an automorphism which maps* e_1 *on* e_i.

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[2] M. Deuring, *Algebren*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 4, no. 1 (1935), pp. 62–64.

[3] Given in Albert, partly in Theorem 2.27 and partly in Theorem 5.6, and, originally, in somewhat different form, in H. Hasse, *Theory of cyclic algebras over an algebraic number field*, Transactions of this Society, vol. 34 (1932), pp. 191–194.

PROOF. If A contains M, then $A = M \times C$, where C is the centralizer of M (Theorem 1.17). The algebra M has an inner automorphism mapping e_{11} on e_{ii}, and this can be extended, so as to be the identity on C, to all of A. This proves the necessity of our condition.

Conversely, suppose each idempotent e_i has the form e^σ, where $e = e_1$, σ is a suitable automorphism. Decompose e into primitive pairwise orthogonal idempotents of A (Theorem 2.16), as $e = f_1 + \cdots + f_r$. One then computes that the unity of A can be decomposed into the pairwise orthogonal primitive idempotents $1 = \sum f_j^\sigma$, where the sum is taken over all $j = 1, \cdots, r$ and over all σ needed to give the idempotents e^σ. The structure theorem (Theorem 3.19) then asserts that $A = M \times D$, where M is a total matric algebra with basis e_{ij}, and where the diagonal elements e_{kk} are the given idempotents f_j^σ. Each of the original idempotents e_i is the sum of exactly r idempotents f_j^σ. Hence in M we select a subalgebra consisting of those matrices which are constructed from blocks of $r \times r$ scalar matrices. This subalgebra is itself a total matric algebra, and its idempotents are the given e_i.

3. **The product formula.** We now prove the following theorem:

THEOREM 4. *If K is a normal subfield of a normal field N, belonging to the subgroup Δ, then the direct product of two crossed products to N and K is given by*

$$(N, \Gamma, h) \times (K, \Gamma/\Delta, G) \sim (N, \Gamma, h\tilde{g}), \tag{5}$$

where $\tilde{g}$ is the Δ-symmetric factor set obtained from the given set G for Γ/Δ by the extension (2).

PROOF. Let C denote the direct product on the left of (5); since C has a unity element, we may regard F as a subfield of C. By the definition of a direct product, C is generated by a subalgebra isomorphic to (N, Γ, h), which we can identify with this algebra, and another subalgebra isomorphic to (but not identical with) the second factor $(K, \Gamma/\Delta, G)$; so we may write[4] C as

$$C = (N, \Gamma, h) \times (K', \Gamma/\Delta, G'), \tag{6}$$

where the subfields K' and N have only the elements of F in common, where K is equivalent to K' over F under a correspondence $y \leftrightarrow y'$, and where G' is the map of G under this isomorphism. Each coset of

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[4] In (5), we simply use the ordinary convention, writing the direct product of two algebras not necessarily disjoint; in (6) we have C represented more explicitly as the direct product of two of its subalgebras. The distinction is a familiar one.

Γ/Δ may then be interpreted as an isomorphism of K' under the natural correspondence

$$(7) \qquad (y')^\sigma = (y^S)', \qquad y \text{ in } K, S \text{ in the coset } \sigma.$$

The crossed product (N, Γ, h) is determined by the formulas (1), while $(K', \Gamma/\Delta, G')$ is determined by similar formulas

$$v_\sigma v_\tau = v_{\sigma\tau} G'_{\sigma,\tau}; \qquad y' v_\sigma = v_\sigma y'^\sigma, \qquad y \text{ in } K.$$

The automorphisms S and σ may be extended to the whole algebra C of (6) by the formulas

$$(8) \qquad a^S = u_S^{-1} a u_S, \qquad a^\sigma = v_\sigma^{-1} a v_\sigma,$$

for any a in C. In a direct product, any term in one factor commutes with any term in the other, hence S leaves fixed all elements of K' and σ leaves fixed all elements of N.

The direct product C contains the commutative subalgebra $N \times K'$; since N is separable, this algebra is semi-simple (chap. 3, §7) and as such is the direct sum of fields L_i with idempotents e_i. Since $1 = \sum e_i$, L_i has the form $e_i(N \times K')$. Let L be one of these fields, with idempotent e. Then the mapping $z \to ez$ carries the elements z of N homomorphically into L; since both are fields, this must be an isomorphism of N to part of L. For similar reasons, $y' \to ey'$ maps K' isomorphically on part of L. These two mappings agree on the common subfield F of N and K'. Therefore L contains the two fields eK and eK', which are equivalent over eF because K and K' are equivalent over F. Since K is normal over F, this implies that $eK = eK'$. This identity means that for each element y in K there exists an element y^* in K' such that $ey = ey^*$, and such that the mapping $y \to y^*$ is an equivalence of K to K' over F. Now two mappings $y \to y'$ and $y \to y^*$ of K to K' can differ only by an automorphism σ of K' over F, so that we may write $y' = y^{*\sigma}$. One may then compute that the replacement of e by the idempotent e^σ simply replaces y^* by y' in the equation $ey = ey^*$. Furthermore, e^σ is the unity element of the field L^σ, which is a direct summand of $N \times K'$ because σ, as defined by (8), is an automorphism of this algebra. Now change the notation, writing L for L^σ, e for e^σ; we then have in $N \times K'$ a direct summand L with unity e such that

$$(9) \qquad ey = ey', \qquad y \text{ in } K,$$

where $y \to y'$ is the given equivalence of K to K'. Furthermore, $L = eN$, though we do not need this fact.

The idempotents e^σ, for σ in Γ/Δ, are all distinct. For suppose this were not the case; then $e^\sigma = e$ for some $\sigma \neq 1$, so that $e^\sigma y = ey = ey'$

$=e^\sigma y'$. On the other hand, one computes by (9) that

$$e^\sigma y = v_\sigma^{-1} e v_\sigma y = v_\sigma^{-1}(ey)v_\sigma = v_\sigma^{-1}(ey')v_\sigma = e^\sigma y'^\sigma, \tag{10}$$

so that $e^\sigma y' = e^\sigma y'^\sigma$. Since the correspondence $y' \rightarrow e^\sigma y'$ is one-one, this gives $y' = y'^\sigma$ for all y', which means that σ is the identity, contrary to assumption. The distinct idempotents e^σ belong to k distinct summands L^σ of $N \times K'$, where k is the degree of K over F. Since each summand L^σ has at least the degree of N, these summands include all the direct summands of $N \times K'$. Hence every primitive idempotent in $N \times K'$ is one of the idempotents e^σ.

If S is in the coset σ, then $(e^S)^\sigma = e$; for one may compute the effect of multiplying $(e^S)^\sigma$ by an element x^S of K, getting

$$e^{S\sigma} x^S = e^{S\sigma} x^{S\sigma} = (ex)^{S\sigma} = (ex')^{S\sigma} = e^{S\sigma} x'^{S\sigma} = e^{S\sigma} x'^\sigma = e^{S\sigma}(x^S)',$$

where the last transformation uses the definition (7) of the automorphism σ. Since any element y of K can be written in the form $y = x^S$, this proves that $e^{S\sigma} y = e^{S\sigma} y'$, for every y. On the other hand, $e^{S\sigma}$ is a primitive idempotent, hence is e^τ for some τ in Γ/Δ. As in (10), one then computes that $e^{S\sigma} y = e^\tau y = e^\tau y'^\tau$. Compared with the previous equation, this means that $y'^\tau = y'$, hence that $\tau = 1$, hence that $e^{S\sigma} = e$, as asserted.

The conclusion $e^{S\sigma} = e$ may be reinterpreted in terms of the definitions (8) of the extended automorphisms S and σ. It then becomes the assertion that e commutes with the product $u_S v_\sigma$. If S is in the coset σ, we write w_S for $u_S v_\sigma$, and have

$$e w_S = w_S e, \qquad w_S = u_S v_\sigma. \tag{11}$$

The idempotents $e, e^\sigma, e^\tau, \cdots$ of $N \times K'$ are all conjugate in the given algebra C of (6); hence Theorem 3 provides a total matric subalgebra M of C of degree k and with basis e_{ij}, where $e_{11} = e$, $e_{22} = e^\sigma$, $\cdots$. This algebra is a direct factor of C (Theorem 1.17); so

$$C = (N, \Gamma, h) \times (K', \Gamma/\Delta, G) = M \times B, \tag{12}$$

where B is the C-centralizer of M. By the structure of a total matric algebra eCe will be a subalgebra equivalent to B. This subalgebra contains a subfield $eNe = eN$ isomorphic to N, with automorphisms $ez \leftrightarrow ez^S$, and also contains elements $ew_S e = ew_S$ of (11), one for each automorphism. The multiplication table for these elements may be computed, using the fact that $eG = eG'$; it is

$$(ez)(ew_S) = (ew_S)(ez^S), \qquad (ew_S)(ew_T) = (ew_{ST})(eh_{S,T}G_{\sigma,\tau}),$$

where S and T lie respectively in the cosets σ and τ. Since the whole algebra eCe has the same degree as N, this means simply that eCe is a crossed product to eN and the factor set $eh_{S,T}G_{\sigma,\tau}$ (Albert, p. 67). Therefore (12) proves that C is similar to a crossed product B of the desired form.

4. **Properties of symmetric factor sets.** We now return to the proof of Theorem 2 of the introduction. Given a factor set $\mathfrak{G}$ of elements in K, one proves at once that the definition (2) for $\mathfrak{g}$ does yield a factor set for Γ. Conversely, if a given factor set $\mathfrak{g}$ is Δ-symmetric, the associativity conditions $g_{S,TR}g_{T,R}=g_{ST,R}(g_{S,T})^R$ for R in Δ become

$$g_{S,T}g_{T,1} = g_{ST,1}(g_{S,T})^R.$$

But the associativity conditions with $R=1$ make $g_{T,1}=g_{ST,1}$, so the result above becomes $(g_{S,T})^R=g_{S,T}$; hence each element $g_{S,T}$ of the factor set lies in the subfield K. One may then define $G_{\sigma,\tau}$ by (2), and show that $\mathfrak{G}$ is a factor set for K. For $\mathfrak{G}$ so defined, the formula of Theorem 4 gives

$$(N, \Gamma, \mathfrak{g}^{-1}) \times (K, \Gamma/\Delta, \mathfrak{G}) \sim (N, \Gamma, \mathfrak{g}^{-1}\,\mathfrak{g}) \sim (N, \Gamma, 1) \sim 1.$$

Multiplying by $(N, \Gamma, \mathfrak{g})$, one concludes that $(K, \Gamma/\Delta, \mathfrak{G})\sim(N, \Gamma, \mathfrak{g})$, as in the formula (3) of Theorem 2.

Theorem 1 now follows formally from Theorem 2. For, if an algebra $(N, \Gamma, \mathfrak{h})$ is split by the normal subfield K, it is similar[5] to a crossed product $(K, \Gamma/\Delta, \mathfrak{G})$, and by Theorem 2 the latter algebra is in turn similar to $(N, \Gamma, \mathfrak{g})$, where $\mathfrak{g}$ is the Δ-symmetric extension of $\mathfrak{G}$. But $(N, \Gamma, \mathfrak{h})\sim(N, \Gamma, \mathfrak{g})$ gives $\mathfrak{h}\sim\mathfrak{g}$ (Theorem 5.5), so $\mathfrak{h}$ is the associate of a Δ-symmetric factor set $\mathfrak{g}$, as asserted. Conversely, if $\mathfrak{h}$ is associate to a Δ-symmetric factor set $\mathfrak{g}$, then

$$(N, \Gamma, \mathfrak{h}) \sim (N, \Gamma, \mathfrak{g}) \sim (K, \Gamma/\Delta, \mathfrak{G}),$$

and the latter algebra is indeed split by K. This completes the proof of Theorem 1.

5. **Arbitrary direct products.** Now we consider two crossed products to any two given fields K and K' which are finite, separable, and normal over a common base field F. The composite $N=K\cup K'$ of K and K' is uniquely determined; we may regard K and K' as subfields of N. The Galois group Γ of N/F is determined in terms of the groups Σ and Σ' of K/F and K'/F as follows.[6]

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[5] Because any normal simple algebra A split by a field K normal over F is similar to a crossed product to this field K. This well known fact is contained in the proof of Theorem 5.1.

[6] This result is known, although explicit citations are rare. To prove it, observe that any S induces and is determined by σ and σ', and then count the number of al-

LEMMA. *Let a field N be the join of two subfields K and K', each finite, separable, and normal over a common base field F, so that N is also finite, separable, and normal over F. If σ, σ' are automorphisms of K, K', respectively, which have the same effect on each element of the intersection $K \cap K'$, then there exists one and only one automorphism S of N/F which induces the given automorphisms σ and σ'. Every automorphism S of N/F may be obtained in this way, and the correspondence $S \leftrightarrow (\sigma, \sigma')$ maps the Galois group of N/F isomorphically on a subgroup of the direct product of the Galois groups of K/F and K'/F.*

Our most inclusive result on direct products now is the following theorem:

THEOREM 5. *Let (K, Σ, G) and (K', Σ', G') be any two given crossed products to fields K and K' normal over F. Let $S \leftrightarrow (\sigma, \sigma')$ and $T \leftrightarrow (\tau, \tau')$ be any two automorphisms of the composite field, determined, as in the lemma, in terms of automorphisms of K and K', and extend the given factor sets G and G' to factor sets for $K \cup K'$ by the formulas*

$$g_{S,T} = G_{\sigma,\tau}, \quad g'_{S,T} = G'_{\sigma',\tau'}, \quad S \to (\sigma, \sigma'), \quad T \to (\tau, \tau'). \tag{13}$$

Then the direct product of the two given crossed products is

$$(K, \Sigma, G) \times (K', \Sigma', G') \sim (K \cup K', \Gamma, gg'). \tag{14}$$

In the special case when K and K' are disjoint, the Galois group of $K \cup K'$ is just the direct product of the two groups Σ and Σ' and the formulas (13) mean simply that the matrix of gg' is the Kronecker product of the matrices G and G'. This case has already been considered by one of us.[7] In the case when K is a subfield of K', the formula (14) specializes to the formula derived in Theorem 4. This special case gives a proof of (14) in general, for observe that $(\sigma, \sigma') \to \sigma$ maps Γ homomorphically on Σ, so that the formula (13) really extends G to be a factor set for Γ which is symmetric relative to a suitable subgroup Δ. Therefore $(K, \Sigma, G) \sim (K \cup K', \Gamma, g)$ by (3). The analogous result for K' then gives (14).

HARVARD UNIVERSITY AND
THE UNIVERSITY OF CHICAGO

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lowable pairs (σ, σ'). If K, K' and $K \cap K'$ have over F the respective degrees k, k', and d, then the degree of $K \cup K'$ over F is kk'/d, and the number of pairs (σ, σ') which agree on the intersection $K \cap K'$ is also kk'/d. Hence every (σ, σ') is realized as an automorphism S of $K \cup K'$.

[7] O. F. G. Schilling, *The structure of certain rational infinite algebras*, Duke Mathematical Journal, vol. 3 (1937), p. 305.

CONNECTED AND DISCONNECTED PLANE SETS AND THE FUNCTIONAL EQUATION $f(x)+f(y)=f(x+y)$

F. B. JONES

Cauchy discovered before 1821 that a function satisfying the equation

$$f(x) + f(y) = f(x + y)$$

is either continuous or totally discontinuous.[1] After Hamel showed the existence of a discontinuous function satisfying the equation,[2] many mathematicians have concerned themselves with problems arising from the study of such functions.[3] However the following question seems to have gone unanswered: Since the plane image of such a function (the graph of $y=f(x)$) must either be connected or be totally disconnected, must the function be continuous if its image is connected? The answer is *no*.[4] The utility of this answer is at once apparent. For if $f(x)$ is totally discontinuous, its image obviously contains neither a continuum nor (in view of Darboux's work) a bounded connected subset even if the image itself is connected. As a matter of fact, if $f(x)$ is discontinuous but its image is connected, then the image, its complement, or some simple modification thereof, serves to illustrate rather easily many of the strange and non-intuitive properties of connected sets now illustrated by numerous complicated examples scattered through the literature. Thus this class of sets is a useful tool in studying connectedness and disconnectedness. A few illustrations are given, particularly in connection with

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Presented in part to the Society, November 23, 1940, under the title *Totally discontinuous linear functions whose graphs are connected*; received by the editors April 2, 1941.

[1] *Cours d'Analyse de l'École Royale Polytechnique*, part 1, *Analyse Algébrique*, 1921. This is Volume 3 of the 2d Series of Cauchy's *Complete Works* published by Gauthier-Villars et Fils, Paris, 1897, p. 99. Darboux in his paper, *Sur la composition des forces en statique*, Bulletin des Sciences Mathématiques, vol. 9 (1875), p. 281, showed (using Cauchy's methods) that if $f(x)$ is bounded in some interval, then $f(x)$ is continuous and of the form Ax.

[2] G. Hamel, *Eine Basis aller Zahlen und die unstetigen Losungen der Funktionalgleichung*: $f(x+y)=f(x)+f(y)$, Mathematische Annalen, vol. 60 (1905), pp. 459–462.

[3] See in particular the early volumes of Fundamenta Mathematicae.

[4] It is odd that Sierpinski overlooked this, since about the time he published his papers on this subject he also published in Volume 1 of Fundamenta Mathematicae an example of a connected punctiform subset of the plane. And at this time he raised with Mazurkiewicz the question of the existence in the plane of a connected set containing no bounded connected subset.

linearly ordered metric spaces. However, such sets are of no use in connection with inner limiting sets.[5]

Convention. Throughout this paper f is used to denote a single-valued real function of a variable, whose range is the set of all real numbers, such that if x and y are real numbers (distinct or not) then $f(x)+f(y)=f(x+y)$. The graph of the equation $y=f(x)$ in a cartesian plane E will be denoted by I_f and called the ***image*** of f (in E). A ***vertical line*** in E will be understood to mean the graph of an equation of the form $x=a$, where a is a real constant.

1. Preliminary theorems.

The following two properties are easily established:[6] (1) $f(rx)=rf(x)$ if r is zero or rational (positive or negative) and (2) if three vertices of a parallelogram in E belong to I_f, then the fourth vertex also belongs to I_f.

THEOREM 0. *For each f, I_f is either connected or totally disconnected.*

THEOREM 1. *If f is discontinuous, then I_f is dense in E.*

THEOREM 2. *Suppose that f is discontinuous. In order for I_f to be connected, it is necessary and sufficient that I_f intersect every continuum in E not lying wholly in a vertical line.*

PROOF. Suppose that I_f is connected and that M is a continuum in E not lying wholly in a vertical line. Then M contains a compact subcontinuum M_1 containing two points P_1 and P_2 which lie in distinct vertical lines. Let D denote the connected domain of E lying between these two vertical lines. It follows from Theorem 1 that if M_1 contains a domain, then I_f contains a point of M_1. On the other hand, if M_1 contains no domain, then $D-D\cdot M_1$ has more than one component. Again by Theorem 1, the segment of I_f between (but not including) P_1 and P_2 contains a point of every component of $D-D\cdot M_1$ and hence contains a point of M_1. Therefore the condition is necessary.

To see that the condition is sufficient, suppose that I_f is the sum of two mutually separate sets H and K. Let D denote a component of $E-\overline{K}$, let B denote a point of K, and let ω denote the point at infinity. The outer boundary (in $E+\omega$) of D with respect to B is a compact continuum M lying in $E+\omega$ and containing no point of I_f.[7] Hence $M-\omega$ is a continuum[8] in E which separates D from B in E. Consequently either $M-\omega$ is an entire vertical line or $M-\omega$ is not a subset of a vertical line. In either case $M-\omega$ contains a point of I_f, which is a contradiction.

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[5] For constructing connected inner limiting sets see Theorem 118 on p. 309 of R. L. Moore's *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932, and Theorem 3 of E. W. Miller's *Some theorems on continua*, this Bulletin, vol. 46 (1940), pp. 150–157. Moore has inadvertently omitted the stipulation that the elements of the postulated collection be mutually exclusive.

[6] See the works referred to in Footnote 2

THEOREM 3. *There exists a function f such that f is discontinuous and I_f is not connected.*

PROOF. Let $\alpha, \beta, \gamma, \cdots$ denote a Hamel basis for the real numbers.[9] Hence every real number x can be expressed *uniquely* in the form $x=a\alpha+b\beta+c\gamma+\cdots$ where the numbers $a, b, c, \cdots$ are either zero or rational and at most a finite number of them are different from zero. Hamel has shown that f may be arbitrarily defined for each of the numbers $\alpha, \beta, \gamma, \cdots$ provided that if $x=a\alpha+b\beta+c\gamma+\cdots$ then $f(x)=af(\alpha)+bf(\beta)+cf(\gamma)+\cdots$. So let $f(\alpha)=1$ and $f(\beta)=f(\gamma)=f(\delta)=\cdots=0$. Since $a, b, c, \cdots$ are rational, it follows that for each real number x, $f(x)$ is either rational or zero. Hence I_f is totally disconnected.

THEOREM 4. *There exists a function f such that I_f intersects every perfect set in E not lying in the sum of a countable collection of vertical lines.*

PROOF. Since the collection of all perfect sets in E not lying in the sum of a countable collection of vertical lines is of power c (the power of the continuum),[10] there exists a well ordering Γ of this collection such that the number of elements of Γ preceding an element of Γ is less than c. Let (x_1, y_1) denote a point of the first element of Γ such that $x_1 \neq 0$. Define $f(x_1)$ to be y_1; and if $x=r_1x_1$, where r_1 is zero or rational, define $f(x)$ to be $r_1f(x_1)$. Each element of Γ must contain points of c distinct vertical lines, and $f(x)$ is so far defined for less than c values of x. So let (x_2, y_2) denote a point of the second element of Γ such that $x_2 \neq 0$ and $f(x_2)$ is not defined. Define $f(x_2)$ to be y_2; and if $x=r_1x_1+r_2x_2$, where r_1 and r_2 are zero or rational, define $f(x)$ to be $r_1f(x_1)+r_2f(x_2)$. In general, this process may be continued c times this way: If γ is an element of Γ such that $f(x)$ has been explicitly defined (as already indicated) by some point of each set of Γ preceding γ in Γ, then let (x_γ, y_γ) denote a point of γ such that $x_\gamma \neq 0$ and $f(x_\gamma)$ has not been defined. Define $f(x_\gamma)$ to be y_γ and if $x = r_1x_1 + r_2x_2 + \cdots$, where not more than a finite number of the rational numbers $r_1, r_2, r_3, \cdots$ are different from zero, define $f(x)$ to be $r_1f(x_1) + r_2f(x_2) + \cdots$. By arguments similar to those of Hamel, it may be shown that this process defines a single-valued function f. Evidently I_f intersects every perfect set in E not lying in a countable number of vertical lines.

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[7] See p. 193, Theorem 23, of R. L. Moore's *Foundations of Point Set Theory*, loc. cit.

[8] Ibid., p. 195, Theorem 25.

[9] G. Hamel, loc. cit.

[10] Sierpinski's *Introduction to General Topology*, translated by C. Cecilia Krieger, The University of Toronto Press, Toronto, 1934, p. 63.

THEOREM 5. *There exists a function f such that f is discontinuous but I_f is connected.*[11]

Theorem 5 follows from Theorems 2 and 4.

THEOREM 6. *If f is discontinuous, I_f is punctiform.*[12]

PROOF. If I_f contained a nondegenerate continuum, then I_f would contain a bounded nondegenerate continuum and f would, therefore, be continuous for some value of x. But if f were continuous for some value of x, f would be continuous for *all* values of x, which is contrary to hypothesis.

THEOREM 7. *Suppose that G is a collection of subsets of E such that every vertical translation in E of a set of G produces a set which also belongs to G. If I_f contains a point of every element of G, then I_f does not contain an element of G.*

Theorem 7 may be easily proved by an indirect argument.

THEOREM 8. *If the subset M of E is punctiform, then $E - M$ is connected and locally connected.*[13]

2. Properties of I_f when f is discontinuous but I_f is connected. Let f be discontinuous, let I_f be connected, and for simplicity let I denote I_f. The following properties of I follow almost immediately from the preceding theorems and the elementary properties of f.

Notation. The symbol ω will be used to denote the point at infinity. If M is a point set and b is a real number, $M+b$ denotes the point set obtained by adding b to the ordinate of each point of M, the abscissa remaining unchanged.

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[11] If f is discontinuous and I_f is connected, by Theorem 2, I_f must intersect every *continuum* not lying in the sum of a countable collection of vertical lines But I_f need *not* intersect every *perfect set* not lying in the sum of a countable collection of vertical lines. I hope to include such an example in a paper pertaining more specifically to function theory and the Hamel basis.

[12] A set is said to be *punctiform* if it contains no nondegenerate continuum.

[13] See the argument on pp. 236 and 237 of Knaster and Kuratowski's, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255.

PROPERTY 1. *Both I and $E-I$ are connected and, hence, neither separates the plane.*

PROPERTY 2. (1) *I contains no nondegenerate continuum,*[14] *and* (2) *$E-I$ contains no continuum not lying in a vertical line.*

PROPERTY 3. *The set I contains no bounded* (*nondegenerate*) *connected subset.*[15]

PROPERTY 4. *Let I^+ denote the set of all points of I with positive ordinates. Although I^+ is totally disconnected and every quasi-component of I^+ is degenerate, I^+ is quasi-connected.*[16]

PROPERTY 5. *Let L denote a non-vertical line in E. Then $I-I\cdot L$ is totally disconnected and every quasi-component of $I-I\cdot L$ is degenerate but $(I-I\cdot L)+\omega$ is biconnected.*[17]

PROPERTY 6. *Let H denote an interval of I. Then H is punctiform, connected, and irreducible between its end points.*[18]

Remark. By Theorems 4 and 7, f exists so that I need not contain a perfect set. If this were the case, the sets $(I-I\cdot L)+\omega$ and H in Properties 5 and 6 respectively would contain no perfect subset of E or $E+\omega$.[19]

PROPERTY 7. *Let K denote $\sum(I+r)$, where r ranges* (*vertically*) *over the set of rational numbers. Then both K and $E-K$ are punctiform, connected and locally connected sets.*[20]

Remark. If, as is shown to be possible by Theorems 4 and 7, I contains no perfect set, it follows from Theorem 7 that the set K in

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[14] Cf. Sierpinski, *Sur un ensemble punctiforme connexe*, Fundamenta Mathematicae, vol. 1 (1920), pp. 7–10.

[15] Cf. Mazurkiewicz, *Sur l'existence d'un ensemble plan connexe ne contenant aucun sous-ensemble connexe, borné*, Fundamenta Mathematicae, vol. 2 (1921), pp. 96–103.

[16] Cf. §3 of Mazurkiewicz, *Sur les ensembles quasi-connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 201–205.

[17] Cf. R. L. Wilder, *A point set which has no true quasi-components which becomes connected upon the addition of a single point*, this Bulletin, vol. 33 (1927), pp. 423–427. Cf. Example α of §5 of Knaster and Kuratowski, *Sur les ensembles connexes*, loc. cit.

[18] Cf., ibid., §5, Example β.

[19] Cf., ibid., §5, Examples γ and δ. For a function whose image is a punctiform, connected, inner limiting (G_δ) set, see page 306 of Kuratowski and Sierpinski, *Les fonctions de classe* 1 *et ensembles connexes punctiformes*, Fundamenta Mathematicae, vol. 3 (1922), pp. 303–313.

[20] Cf. R. L. Moore, *A connected and regular point set which contains no arc*, this Bulletin, vol. 32 (1926), pp. 331–332; R. L. Wilder, *A connected and regular point set which has no subcontinuum*, Transactions of this Society, vol. 29 (1927), pp. 332–340.

Property 7 contains no perfect set.[21] Furthermore, since the set of real numbers contains a subset R such that every perfect subset of the real numbers contains a number in R and a number not in R, it is clear that if I contains no perfect subset of E and the range of r is R (instead of the rational numbers), then the sets K and $E-K$ in Property 7 are both connected and locally connected but neither contains a perfect subset of E.[22]

PROPERTY 8. *Let S denote a space whose points are the points of I and in which "limit point" has the same meaning that it does in E. Then* (1) *S is metric, connected, convex, and separable, but contains no compact nondegenerate continuum*; (2) *S is linearly ordered and continuous with respect to this order*; (3) *S is the sum of countably many totally disconnected, arbitrarily small domains*; (4) *if F denotes the points of S belonging to a circle in E and M denotes the points of S which are on or inside this circle in E, then although M contains no nondegenerate quasi-component, M is not the sum of two nonvacuous mutually separate sets one of which contains F*; (5) *S contains a totally disconnected closed set of which not every point is a limit point of its complement.*

Property 8 gives rise to a number of questions. Particularly, is every point of a totally disconnected closed subset of a connected and linearly ordered *complete* metric space a limit point of its complement?

3. **Plane geometry.** If one defines a line (in the cartesian plane) to be the set of all points (x, y) satisfying an equation of either the form $x=a$ (where a is a constant) or the form $y=f(x)+mx+b$ (where m and b are constants which may be different for different lines, but f is the same function for all lines of this type and I_f is connected), then one gets a curious approximation to euclidean plane geometry. In this geometry translation would be a rigid motion but rotation would not. Also in this geometry a triangle would *cut* the plane but would not *separate* the plane.

THE UNIVERSITY OF TEXAS

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[21] Cf. Knaster and Kuratowski, *A connected and connected im kleinen point set which contains no perfect set*, this Bulletin, vol. 33 (1927), pp. 106–109.

[22] Cf. Mazurkiewicz, *Sur la décomposition d'un domaine en deux sous-ensembles punctiformes*, Fundamenta Mathematicae, vol. 3 (1922), pp. 65–75. It follows from Theorem 8 that Bernstein's (Berichte der Saechsischen Akademie der Wissenschaften, Leipzig, vol. 60 (1908), pp. 325–338) decomposition of the plane into two mutually exclusive sets such that every perfect set contains a point of each of them, as random as this process is, always produces two connected and locally connected sets. This fact is certainly known to many mathematicians and has probably been pointed out elsewhere.

A REMARK ON THE CARDINAL OF LIMIT SPACES

WALTER S. SNYDER

In this note we show that any nonvacuous set can be topologized to form a compact L^* space,[1] and thus the cardinal of such spaces is unrestricted.

If S is normally ordered and the order has a last element we may define lim sup $\{a_k\}$ to be the first element of the order which precedes only a finite number of the terms of $\{a_n\}$, and lim inf $\{a_n\}$ to be the first element of the order which is preceded by, or equals, infinitely many terms of $\{a_n\}$. We term a sequence $\{a_n\}$ convergent if and only if lim inf $\{a_n\}=$lim sup $\{a_n\}$, and the common element is defined to be the limit. The reader will verify that this defines a compact L^* space. Our result thus appears as a consequence of the well known Zermelo theorem to the effect that any set can be normally ordered.

Our result can be obtained, however, without the use of normal ordering, though the topologization is perhaps less natural.

Let S be a given set and x_0 a fixed element of S. We define the class of convergent sequences together with their limits to consist of:

(a) All sequences $\{a_n\}$ for which $a_n=a$ for sufficiently large n, and the limit is a.

(b) All sequences $\{a_n\}$ in which x_0 is the only element of S occurring infinitely often (including the case in which no element of S occurs infinitely often), and the limit is defined to be x_0 in either case. It is easily verified that this defines a compact L^* space in S.

OHIO STATE UNIVERSITY

Received by the editors April 11, 1941.

[1] An L^* space is a limit space in which certain sequences are designated as convergent, and a limit assigned to each in such a way that

(1) $a, a, a, \cdots$ converges to a.

(2) "$\{a_n\}$ converges to a" implies "every subsequence of $\{a_n\}$ converges to a."

(3) If $\{a_n\}$ does not converge to a there exists a subsequence of $\{a_n\}$, say $\{b_n\}$, such that no subsequence of $\{b_n\}$ converges to a. The space is compact if every sequence has a convergent subsequence. (See Kuratowski, *Topologie*, vol. 1, p. 76, or Fréchet, *Les Espaces Abstraits*.)

A NOTE ON REPRESENTATION BY POLYGONAL NUMBERS

L. W. GRIFFITHS

1. **Introduction.** The universal functions of polygonal numbers of order $m+2$ were determined in an earlier paper.[1] In each universal function the number n of variables was at most $m+2$. The minimum value N of n, for each integer $m \geqq 3$, will be proved in this paper to be the integer N defined by (3). It will also be proved that there is a unique universal function having $n=N$ if and only if $m=3$, 4, $2^{N-2}-2$, $2^{N-2}-1$. A universal function having $n=N$ is given by the integers (4). At least one universal function different from this function is given by the integers (6) if $m \neq 3$, 4, $2^{N-2}-2$, $2^{N-2}-1$, and if (7) and (8) hold.

2. **Proofs.** In the notations of the paper to which reference has been made m was an arbitrary but fixed integer greater than or equal to 3. The coefficients $a_1, a_2, \cdots, a_n$ in the universal functions were positive integers to be determined, and $1 \leqq a_1 \leqq \cdots \leqq a_n$. Also, by definition, $w_k = a_1 + \cdots + a_k$ $(1 \leqq k \leqq n)$. It was proved that no function is universal if $w_n < m+2$, and that if $w_n = m+2$ then the function f is universal if and only if f is one of the following:

$$(1) \qquad (1, 1, 1, 1, 1) \text{ or } (1, 1, 1, 2), \qquad \text{with } m = 3 \text{ and } w_n = m + 2,$$

$$(2) \; (1, 1, 1, a_4, \cdots, a_n),\ w_n = m + 2 > 5,\ a_k \leqq w_{k-1} - 1 \ (4 \leqq k \leqq n), \\ \text{but } n > 5 \text{ and } a_5 \neq 3 \text{ if } a_4 = 1.$$

Thus $N=4$ if $m=3$. It will be proved that if $m>3$ then the minimum m is the integer N uniquely defined by

$$(3) \qquad 2^{N-3} - 1 < m \leqq 2^{N-2} - 1.$$

Consider the sequence of integers $1, 1, 1, 2, 2^2, \cdots, 2^{i-3}, \cdots$ in which the ith term is 2^{i-3} if $i \geqq 3$. The sum of the first i terms is $2^{i-2}+1$ if $i \geqq 3$. Now let f be a universal function which satisfies (2). Then it is easily proved by induction that $a_k \leqq 2^{k-3}$ and $w_k \leqq 2^{k-2}+1$ $(3 \leqq k \leqq n)$. Hence if $n<N$ this would imply in particular that $m+2 = w_n \leqq 2^{n-2}+1 \leqq 2^{N-3}+1$. This contradiction of (3) when $n<N$ shows that $n \geqq N$. Furthermore $N \geqq 5$ since $m>3$ in (3).

Presented to the Society, April 11, 1941, under the title *The minimum number of variables in universal functions of polygonal numbers*; received by the editors April 11, 1941.

[1] L. W. Griffiths, *A generalization of the Fermat theorem on polygonal numbers*, Annals of Mathematics, (2), vol. 31 (1930), pp. 1–12.

That N is the minimum value of n will be proved now, by exhibiting a function satisfying (2) and having $n=N$. This function has as its coefficients the integers

$$1,\ 1,\ 1,\ 2,\ 2^2,\ \cdots,\ 2^{N-4},\ m+1-2^{N-3} \tag{4}$$

after they have been arranged in order of increasing magnitude. This can be done because by (3) with $N \geqq 5$ there is a unique integer J such that $0 \leqq J \leqq N-3$ and $2^J \leqq m+1-2^{N-3} < 2^{J+1}$. It is easily verified that the rearranged integers (4) satisfy all the conditions in (2).

The function defined by (4) is the only function satisfying (2) and having $n=N$ if $m=4$. This is also true if

$$m = 2^{N-2}-1 \quad \text{or} \quad 2^{N-2}-2. \tag{5}$$

Then $m+1-2^{N-3}$ is 2^{N-3} or $2^{N-3}-1$ respectively, and the integers (4) require no rearrangement. Now if f is a function which satisfies (2) with $n=N$ and is different from this exhibited function, then there is a coefficient $a_j < 2^{j-3}$ with $j \leqq N-1$. Then $w_{N-1} \leqq 2^{N-3}$ and $a_N \leqq 2^{N-3} -1$, and hence $m+2=w_N \leqq 2^{N-2}-1$. This contradicts (5).

The function defined by (4) is not the only function satisfying (2) and having $n=N$ if $m \neq 4,\ 2^{N-2}-2,\ 2^{N-2}-1$. This will be proved by finding conditions on the integers b and c which are necessary and sufficient that the sequence

$$1,\ 1,\ 1,\ 2,\ 2^2,\ \cdots,\ 2^{N-5},\ b,\ c \tag{6}$$

yields, after rearrangement in order of increasing magnitude, the coefficients of a function satisfying (2). Then values of b and c will be exhibited such that these conditions are satisfied and that the set of integers (6) is not the set (4).

Let b be any integer such that

$$(m+1-2^{N-3})/2 \leqq b \leqq (m+1-2^{N-4})/2,\ b \leqq 2^{N-4}, \tag{7}$$

and let c be defined by

$$c = m+1-2^{N-4}-b. \tag{8}$$

It is easily seen that (8) is equivalent to the sum of the integers (6) being $m+2$, and that the second inequality in (7) is equivalent to $b \leqq c$. If in $b \leqq c$ the integer b is replaced by its value from (8), and if (3) is applied, it is seen that $c \geqq 1+2^{N-5}$. Hence c is the last integer after rearrangement of (6). Now the first inequality in (7) is equivalent, by (8), to the fact that $c+1$ is at most the sum of the preceding integers in (6). By the last inequality in (7) it is easily verified that the rearranged integers (6) satisfy the remaining conditions in (2).

Now, if m is odd, then $(m+1-2^{N-3})/2$ is an integer which may be taken as the value of b since it satisfies the conditions (7). It is easily seen that the set (6), in which $b=(m+1-2^{N-3})/2$ and c is determined by (8), is not the set (4). However, if m is even and not equal to 2^{N-3}, then $b=(m+2-2^{N-3})/2$ and c determined by (8) are integers which satisfy (7) and yield a set (6) which is not the set (4). But if $m=2^{N-3}$, then $N\geqq 6$ and $b=2$ satisfies the conditions (7) on b and yields a set (6) which is not the set (4).

An interesting choice of integers b and c is that given by $b=(m+1-2^{N-4})/2$ if m is odd and less than or equal to $(2^{N-4}+2^{N-3}-1)$, but by $b=(m-2^{N-4})/2$ if m is even and less than or equal to $(2^{N-4}+2^{N-3})$. Then (6) require no rearrangement, and c is respectively b or $b+1$. The resulting integers (6) differ from (4) when $m\neq 2^{N-4}+2^{N-3}-1,\ 2^{N-4}+2^{N-3}$.

Northwestern University

HAUSDORFF METHODS OF SUMMATION WHICH INCLUDE ALL OF THE CESÀRO METHODS

H. L. GARABEDIAN

1. **Introduction.** The transformation[1]

$$\sigma_m = \sum_{n=0}^{m} C_{m,n}\Delta^{m-n}c_n\cdot s_n,$$

where $c_n=\int_0^1 u^n d\phi(u)$ and $\{s_n\}$ is a given sequence, defines a regular method of summation of the sequence $\{s_n\}$ provided that $\phi(u)$ is of bounded variation on the interval $0\leqq u\leqq 1$, continuous at $u=0$, and

$$\phi(u) = \begin{cases} 0 & \text{if } u = 0, \\ 1 & \text{if } u = 1, \\ \frac{1}{2}[\phi(u-0)+\phi(u+0)] & \text{if } 0 \leqq u < 1. \end{cases}$$

If these conditions of regularity are fulfilled the sequence $\{c_n\}$ is said to be a ***regular moment sequence*** (briefly a ***regular sequence***), the mass function $\phi(u)$ is said to be a ***regular mass function***, and the method of summation involved is called a *Hausdorff method of summation* ([1] or [2]) and is designated by the symbol $[H, \phi(u)]$.

Presented to the Society, February 22, 1941; received by the editors April 8, 1941.

[1] To define the symbolism used here we write $C_{m,n}=m(m-1)\cdots(m-n+1)/n!$, $C_{m,0}=1$; $\Delta^t x_j = x_j - C_{t,1}x_{j+1}+C_{t,2}x_{j+2}+\cdots$.

By properly specializing the mass function $\phi(u)$ it is possible to obtain various well known methods of summation. For example, the mass function $\phi(u)=1-(1-u)^{\alpha}$, $\alpha>0$, defines Cesàro summability (C, α). The regular sequence associated with this mass function is $c_n=1/C_{\alpha+n,n}$, $(n=0, 1, 2, \cdots)$.

It is the object of this note to exhibit new Hausdorff methods of summation which have the rare property of including Cesàro summability of all real and positive orders.

2. **A mass function $\phi(u)$ independent of σ for which $[H, \phi(u)]\supset(C, \sigma)$, $\sigma>0$.** In order to exhibit a regular mass function $\phi(u)$ independent of σ for which $[H, \phi(u)]\supset(C, \sigma)$, $\sigma>0$, we write a theorem first stated by E. Hille and J. D. Tamarkin [3] which provides necessary and sufficient conditions in order that $[H, \phi(u)]\supset(C, n)$ $(n=1, 2, 3, \cdots)$. This theorem has recently been restated and proved in a paper by E. Hille, H. S. Wall, and this writer [4].

THEOREM 1. *Necessary and sufficient conditions in order that* $[H, \phi(u)]\supset(C, n)$, $n\geqq 1$, *are*

(i) $\phi(u)$ *is absolutely continuous and has absolutely continuous derivatives of order equal to or less than* $n-1$ *for* $0<u\leqq 1$,

(ii) $\phi^{(n-1)}(u)$ *has a finite right-hand derivative* $\phi_r^{(n)}(u)$ *for* $0<u<1$, *and a finite left-hand derivative* $\phi_l^{(n)}(u)$ *for* $0<u\leqq 1$,

(iii) $\phi(u)$, $u\phi'(u), \cdots$, $u^{n-1}\phi^{(n-1)}(u)$, $u^n\phi_r^{(n)}(u)$, $u^n\phi_l^{(n)}(u)$ *are of bounded variation on the interval* $(0, 1)$,

(iv) $1-\phi(u)$, $\phi'(u), \cdots$, $\phi^{(n-1)}(u)$ *tend to zero as* $u\to 1$,

(v) $\phi(u)$, $u\phi'(u), \cdots$, $u^{n-1}\phi^{(n-1)}(u)$, $u^n\phi_r^{(n)}(u)$, $u^n\phi_l^{(n)}(u)$ *tend to zero as* $u\to 0$.

The simplest regular mass function which we have been able to find which fulfills all of the five conditions of Theorem 1 for all values of $n\geqq 1$ is the function

$$\phi_a(u) = 1 - \exp\left\{u/(u-1)\right\}. \tag{2.1}$$

We observe that this function is monotone increasing on the interval (0, 1). If we modify (2.1) by writing

$$\phi(u) = \left[1 - \exp\left\{-\alpha u/(1-u)^{\beta}\right\}\right]^{\gamma}, \qquad \alpha, \beta, \gamma > 0,$$

we obtain a mass function which also fulfills the conditions of Theorem 1 for all $n\geqq 1$.

3. **A theorem due to Hausdorff** [1, p. 90]. If a regular mass function $\phi(u)$ is continuous on the interval (0, 1), and if furthermore we set $u=e^{-s}$, then the associated regular sequence takes the form

$$c_n = \int_0^\infty e^{-nx}\Phi(x)dx, \qquad n = 0, 1, 2, \cdots, \tag{3.1}$$

where $\Phi(x)=e^{-x}\phi'(e^{-x})=u\phi'(u)$. Hausdorff has proved the following theorem, involving the function $\Phi(x)$ instead of the mass function $\phi(u)$, which is closely related to Theorem 1 of the preceding section.

THEOREM 2. *If the functions $\Phi(x)$, $\Phi'(x)$, $\Phi''(x)$, $\cdots$ exist for $x>0$, if they are absolutely integrable on the interval $(0, \infty)$, and if they tend to zero as $x\to 0$, then the sequence (3.1) defines a regular method of summation which includes all of the Cesàro methods.*

The hypotheses of this theorem imply that all of the functions involved tend to zero as $x\to+\infty$. With this observation it is readily verified that the mass function

$$\phi(u) = \int_0^u \frac{1}{t}\,\Phi\left(\log\frac{1}{t}\right)dt,$$

when $\Phi(x)$ satisfies the conditions of Theorem 2, fulfills the conditions of Theorem 1 for all $n\geqq 1$.

Examples due to Hausdorff of functions which fulfill the conditions of Theorem 2 are:

$$\Phi(x) = A\ \exp\left\{-\,ax - b/x\right\}x^{\lambda-1}, \qquad a,\,b > 0;\ \lambda \text{ arbitrary},$$

$$\Phi(x) = Be^{-(b/x)}x^{\lambda-1}, \qquad b > 0;\ \lambda < 0,$$

where A and B are *normalising factors* which make $c_0=1$. If in particular we have $\Phi(x)=e^{-1/4x}x^{-3/2}/(2\pi^{1/2})$, the associated moment sequence is $c_n=e^{-n^{1/2}}$, $(n=0, 1, 2, \cdots)$, and the associated regular mass function is

$$\phi_b(u) = \frac{1}{2\pi^{1/2}}\int_0^u \frac{1}{t}\exp\left\{1/4\log t\right\}\left(\log\frac{1}{t}\right)^{-3/2}dt.$$

Like the function $\phi_a(u)$ of (2.1), the function $\phi_b(u)$ is a monotone increasing function on the interval (0, 1).

In comparing the mass functions $\phi_a(u)$ and $\phi_b(u)$ we notice that $\phi_a(u)$ is much simpler in form than $\phi_b(u)$. On the other hand the regular sequence associated with $\phi_b(u)$ has a simple form, while the regular sequence associated with $\phi_a(u)$ cannot be computed in elementary terms.

4. The difference matrix. In a recent paper [5] H. S. Wall and this writer have discussed properties of the *difference matrix*

$\Delta = (\Delta^m c_n)$, where $\{c_n\}$ is a regular sequence. In particular it was observed that

$$\Delta^m c_n = \int_0^1 u^n (1 - u)^m d\phi(u), \tag{4.1}$$

and hence that the sequence $(\Delta^m c_0, \Delta^m c_1, \Delta^m c_2, \cdots)$ is an *essentially regular sequence*,[2] the associated mass function being $\int_0^u (1-t)^m d\phi(t)$. We observe now for the first time that m need not be a positive integer, the integral in (4.1) affording a definition for the fractional differences involved here.

If, for example, the mass functions $\phi_a(u)$ and $\phi_b(u)$ are inserted in the difference matrix new methods of summation are generated which have the property of including all of the Cesàro methods. The method of summation thus generated by the mass function $\phi_c(u)$ has the mass function

$$\phi_c(u) = A \int_0^u (1 - t)^{c-1} \exp \{t/(t - 1)\} \, dt, \qquad c \geqq 0,$$

where A is a normalizing factor which insures that $\phi_c(1) = 1$. On the basis of known results arising from inclusion problems in the difference matrix [5] we make the conjecture that summability $[H, \phi_c(u)]$ increases in efficiency with increasing c.

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NORTHWESTERN UNIVERSITY

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[2] An essentially regular sequence may be normalized or made regular by dividing every term of the sequence by its first term.

A CHARACTERIZATION OF THE RADICAL OF AN ALGEBRA

SAM PERLIS

1. The first main result. We shall prove the following result.

THEOREM 1. *Let F be any field and A an algebra over F with a unity element. Then the radical of A consists of all elements h such that $g+h$ is regular for every regular g.*

Let H be the set of all elements h defined in the theorem. It is easy to see that H is a linear set over F. We shall prove now that if A is simple, $H=0$.

Let g and g_1 be any regular elements of A and h be in H. Then $g_1^{-1}g+h$ is regular so that $g+g_1h$ is regular. Hence g_1h is in H and similarly hg_1 is in H. An arbitrary element a of A has[1] the form $a=\sum_{i=1}^{n} g_i$ with regular elements g_i so that $ah=\sum g_ih$ is a sum of elements g_ih of H. Thus ah, and similarly ha, is in H so that H is an ideal of A. If $H\neq 0$ then $H=A$ since A is simple. But A contains the regular element -1, and $(-1)+1$ is not regular so that 1 cannot be in H, whence $H\neq A$. Hence $H=0$.

Next we shall prove that $H=0$ whenever A is semi-simple. Now $A=A_1+A_2+\cdots+A_t$ where the A_i are simple, and each x of A has a unique expression $x=a_1+a_2+\cdots+a_t$ with a_i in A_i. Further, x is regular if and only if each a_i is a regular element of A_i. Let $g=g_1+\cdots+g_t$ be regular, $h=h_1+\cdots+h_t$ be in H, so that $g+h=(g_1+h_1)+\cdots+(g_t+h_t)$. Then $g+h$ is regular for every regular g if and only if g_i+h_i is regular in A_i for every regular g_i of A_i. By the proof above for simple algebras every $h_i=0$ so that $h=0$ and $H=0$.

In considering the case of a general algebra A, we show first that the radical R is contained in H. Let g be regular and r lie in R. Then $g+r$ is regular if and only if $1+g^{-1}r$ is regular. Now $g^{-1}r$ is in R, $(g^{-1}r)^t=0$ for some integer t, $(g^{-1}r)^{2t+1}+1=1$. If λ is an indeterminate, $\lambda+1$ is a factor of $\lambda^{2t+1}+1$ so that $g^{-1}r+1$ is a factor of $(g^{-1}r)^{2t+1}+1=1$; hence, $g^{-1}r+1$ is regular, $g+r$ is regular, r is in H, and R is contained in H.

It remains to prove that R contains H. Since $A-R$ is semi-simple, the set H_0 defined for $A-R$, similarly to H for A, is the zero set. If g is regular in A and h is in H, the class $[g+h]$ in $A-R$ is a regular ele-

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Presented to the Society, April 12, 1941; received by the editors April 11, 1941.

[1] K. Shoda, Mathematische Annalen, vol. 107 (1933), pp. 252–258.

ment of $A-R$. But $[g+h]=[g]+[h]$, and $[g]$ varies over all[2] regular elements of $A-R$ so that $[h]$ must be in $H_0=[0]=R$. Hence h is in R, H is contained in R, $H=R$, and the proof of the theorem is complete.

2. **Extension to arbitrary algebras.** The theorem above is applicable to algebras A_0 without a unity element in the sense that by adjoining a unity element to A_0 we do not alter its radical. To see this, let R_0 be the radical of A_0 and R the radical of the corresponding algebra A with a unity element. Every element of A has the unique form $a=\alpha+a_0$ with α in F, a_0 in A_0. If r_0 is in R_0 then $ar_0=\alpha r_0+a_0r_0$ is a sum of elements of R_0, ar_0 is nilpotent, r_0 is properly nilpotent in A, so that $R_0 \leqq R$. Conversely, let r be in R so that $r^t=0$ for some integer t, $r=\sigma+s_0$ with σ in F, s_0 in A_0, $r^t=\sigma^t+s_1=0$ with s_1 in A_0. Then $s_1=0$, $\sigma=0$ so that $r=s_0$ in A_0. But r is properly nilpotent in A, hence in A_0, hence r is in R_0, $R \leqq R_0$, $R=R_0$.

While the fact just proved enables one to apply Theorem 1 to arbitrary algebras, nevertheless it is desirable to obtain a criterion not dependent on the unity element, as Professor Marshall Hall has pointed out to the author. The remainder of this section is devoted to this purpose.

If A is an algebra without a unity element, the symbol A' will be used throughout the paper to denote the algebra obtained from A by adjoining a unity element. If A has a unity element, A' is defined to be A.

DEFINITION. *An element x of an algebra will be called "quasi-regular" in case there is an element y in the algebra such that*

$$x+xy+y=0, \tag{1}$$

and then y will be called the "quasi-inverse" of x.

Since $(1+x)(1+y)=1+x+xy+y$, we see at once that if an element x of A is quasi-regular in A, then $1+x$ is regular in A'; and conversely, if $1+x$ is regular in A' for x in A, then x is quasi-regular in A', and actually in A as the following result shows.

LEMMA 1. *Let A be an algebra over F. If $A=A'$, a quantity of A is regular if and only if it has the form $1+x$ where x is quasi-regular. If $A \neq A'$, a quantity of A' is regular if and only if it is expressible as $\alpha(1+x)$ where α is a nonzero element of F and x is a quasi-regular element of A.*

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[2] For, if $[b]$ is any regular element of $A-R$, then $[b][c]=[1]$, $bc=1+r$ with r in R. Since we have already proved $R \leqq H$ it follows that $1+r$ is regular, bc is regular, so that b is regular.

The case $A=A'$ is completed by the remark above the statement of the lemma. In the case $A\neq A'$ the same remark shows that if x is quasi-regular in A, then $1+x$ is regular and so is $\alpha(1+x)$. Conversely, let $g=\alpha+x_0$ be any regular element of A' with α in F and x_0 in A. We readily find that $\alpha\neq 0$ so that $g=\alpha(1+x)$, $1+x$ is regular, and

$$(1+x)(\beta+y)=\beta+(\beta x+xy+y)=1$$

for some β in F and y in A. Since $\beta x+xy+y$ is in A and β is in F, we must have $\beta=1$, $x+xy+y=0$, so that x is quasi-regular.

Observe that this lemma provides unique expressions for the regular quantities of A'.

If x is quasi-regular in A, its quasi-inverse is the unique element y such that $1+y$ is the inverse of $1+x$. Moreover,

$$(1+x)(1+y)=1=(1+y)(1+x),$$

whence $xy=yx$. Finally, the inverse $1+y$ of the regular element $1+x$ is known to be a polynomial in $1+x$ so that y is a polynomial in x.

LEMMA 2. *If x is a quasi-regular element of an algebra A, its quasi-inverse is unique, is a polynomial in x, and commutes with x.*

We now obtain the following main criterion.

THEOREM 2. *Let A be an algebra over a field F. Then an element r of A is in the radical of A if and only if $x+\alpha r$ is quasi-regular in A for every x of A which is quasi-regular and every α of F.*

If x is quasi-regular and r is in the radical, αr is in the radical of both A and A', $1+x$ is regular in A', and thus $1+x+\alpha r$ is regular by Theorem 1. By Lemma 1 the element $x+\alpha r$ of A must be quasi-regular. Conversely, suppose that r is an element of A with the property stated in the theorem. Any regular element g of A' has the form $g=\alpha+\alpha x$, $\alpha\neq 0$ in F, and x quasi-regular in A. By hypothesis $x+\alpha^{-1}r$ is quasi-regular so that

$$h=1+x+\alpha^{-1}r$$

is regular, and $\alpha h=g+r$ is regular. By Theorem 1 the element r is in the radical of A', hence in the radical of A.

A common characterization of the radical is that it consists of zero and all properly nilpotent elements. It may be noticed that this characterization is strongly in contrast with the present ones which are phrased in terms of addition rather than multiplication and are concerned with preserving regularity rather than nilpotency.

3. **Some applications.** As an application let us consider bound algebras which have been studied by M. Hall.[3] In his Theorem 3.6, Hall found that a bound algebra A contains three pairwise orthogonal idempotents e_1, e_2, e_3 with the following properties. If $A_i=e_iAe_i$ $(i=1, 2, 3)$ and R is the radical of A, then

(1) A is the supplementary sum $A=A_1+A_2+(A_3, R)$;

(2) $Re_1=0$, $e_2R=0$;

(3) A_1 and A_2 are semi-simple.

We shall prove:

I. *If A has a unity element e, then $e=e_1+e_2+e_3$.*

For proof, let $e_0=e-(e_1+e_2+e_3)$ so that e_0 is either zero or an idempotent orthogonal to each e_i. By property (1) we have $e_0=a_1+a_2+(a_3+r)$ with a_i in A_i, r in R, and we may always assume, without loss of generality, that either $a_3=0$ or else a_3 is not in R. By the orthogonality of the e_i and property (2) we have $e_0e_1=0=a_1$, $e_2e_0=0=a_2$. Now $e_0=a_3+r$, $e_0e_3=0=a_3+re_3$, $a_3=-re_3$ in R so that $a_3=0$. Then $e_0=r$ in R so that e_0 cannot be idempotent and we have $e_0=0$.

Next we again assume that A has a unity element and prove:

II. *If $x=a_1=a_2+(a_3+r)$ is any element of A, a_i in A_i, r in R, then x is regular if and only if each a_i is regular in A_i.*

Since r is in R, x is regular if and only if $y=a_1+a_2+a_3$ is regular. If $z=b_1+b_2+(b_3+s)$ is an element of A, b_i in A_i, s in R, then

$$yz = a_1b_1 + a_2b_2 + a_3b_3 + s_1, \qquad s_1 \text{ in } R,$$

and $yz=e=e_1+e_2+e_3$ if and only if $a_1b_1=e_1$, $a_2b_2=e_2$, $a_3b_3=e_3-s_1$. Hence a_1 and a_2 must be regular in their respective algebras. The third equation above shows that s_1 must be in A_3 as well as in R, hence in the radical of A_3. Then e_3-s_1 is a regular element of A_3, a_3b_3 is regular in A_3, so that a_3 likewise must be regular in A_3. Conversely, if each a_i has an inverse b_i in A_i, it is clear that $b_1+b_2+b_3$ is the inverse of $a_1+a_2+a_3$.

An analogue of II not requiring a unity element will now be obtained.

III. *Let $x=a_1+a_2+(a_3+r)$ be an element of a bound algebra A. Then x is quasi-regular if and only if each a_i is quasi-regular in A_i.*

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[3] *The position of the radical in an algebra*, Transactions of this Society, vol. 48 (1940), pp. 391–404. A "bound algebra" A with radical R is an algebra with the property that if $xR=Rx=0$ for x in A, then x is in R. Hall has reduced the structure theory of arbitrary algebras to that of bound algebras and semi-simple algebras.

By Theorem 2 the element x is quasi-regular if and only if $x-r=y=a_1+a_2+a_3$ is quasi-regular. If each a_i has a quasi-inverse b_i in A_i, then $z_0=b_1+b_2+b_3$ has the property that $yz_0=a_1b_1+a_2b_2+a_3b_3$ so that $y+yz_0+z_0=0$, and y is quasi-regular. Conversely, suppose that y has a quasi-inverse $z=b_1+b_2+(b_3+s)$ with s in R and b_i in A_i. Then

$$(2) \qquad y + yz + z = 0 = \sum_{i=1}^{3} (a_i + a_ib_i + b_i) + ys + s,$$

$$(3) \qquad a_3 + a_3b_3 + b_3 + ys + s = 0, \qquad a_i + a_ib_i + b_i = 0$$

for $i=1, 2$. Thus a_1 and a_2 are quasi-regular in A_1 and A_2, respectively, and the quantity

$$(4) \qquad c = a_3 + a_3b_3 + b_3 = -s - ys$$

is in A_3, $e_3c=c=-e_3s-e_3ys=-e_3s-a_3s=-(e_3+a_3)s$. Since y commutes with its quasi-inverse (Lemma 2), we have $y+zy+z=0$ and as in (4) we are led to the equation

$$(5) \qquad c' = a_3 + b_3a_3 + b_3 = -s - sy.$$

Hence $c'=c'e_3=-s(e_3+a_3)$. We have proved

$$(6) \qquad c = -(e_3 + a_3)s, \qquad c' = -s(e_3 + a_3).$$

Now $(e_3+a_3)(e_3+b_3)=e_3+c=e_3-(e_3+a_3)s$ so that

$$(7) \qquad (e_3 + a_3)(e_3 + b_3 + s) = e_3.$$

Likewise, by forming $(e_3+b_3)(e_3+a_3)=e_3+c'$ we find

$$(8) \qquad (e_3 + b_3 + s)(e_3 + a_3) = e_3.$$

Since e_3 is the unity element of A_3, the results (7) and (8) show that the element e_3+a_3 of A_3 is not a divisor of zero in A_3. Thus e_3+a_3 is a regular element of A_3 so that a_3 must be quasi-regular. This completes the proof.

One may observe that property (3) is an immediate consequence of the fact III and Theorem 2. Suppose that r_1 is in the radical of A_1 and $x=a_1+a_2+(a_3+r)$ is any quasi-regular element of A. Then a_1 is quasi-regular in A_1, $a_1+\alpha r_1$ is quasi-regular in A_1 for every α of F, and $x+\alpha r_1=(a_1+\alpha r_1)+a_2+(a_3+r)$ has the requisite form for a quasi-regular element of A. By Theorem 2 the quantity r_1 is in R as well as in A_1. But then $r_1=0$ by property (1). Thus A_1 is semi-simple, and similarly A_2 is semi-simple.

UNIVERSITY OF MICHIGAN

AN EXTENSION OF A THEOREM OF WITT

BURTON W. JONES

1. **Introduction.** If $\mathfrak{u}_1, \cdots, \mathfrak{u}_n$ is a set of vectors such that $\mathfrak{u}_i\mathfrak{u}_j = \mathfrak{u}_j\mathfrak{u}_i$ are numbers of a field K for $i, j = 1, 2, \cdots, n$, all linear combinations of these vectors with coefficients in K constitute a *vector space*

$$\mathfrak{S} = \langle \mathfrak{u}_1, \cdots, \mathfrak{u}_n \rangle$$

over K and the symmetric matrix $\mathfrak{A} = (\mathfrak{u}_i\mathfrak{u}_j) = (a_{ij})$ is the *multiplication table* for the *basis* $\mathfrak{u}_1, \cdots, \mathfrak{u}_n$. The inner product of two vectors $\sum x_i\mathfrak{u}_i$ and $\sum y_i\mathfrak{u}_i$ is the bilinear form

$$\sum (\mathfrak{u}_i\mathfrak{u}_j)x_iy_j = \sum a_{ij}x_iy_j$$

and the *norm* of a vector is the inner product of a vector and itself; it can be expressed as a quadratic form.

If $\mathfrak{T}$ is a nonsingular transformation with coefficients in K and $(\mathfrak{u}_1, \cdots, \mathfrak{u}_n)\mathfrak{T} = (\mathfrak{v}_1, \cdots, \mathfrak{v}_n)$, the $\mathfrak{v}$'s will constitute a new basis of the same space $\mathfrak{S}$ and the multiplication table for the new matrix is $\mathfrak{T}'\mathfrak{A}\mathfrak{T}$. This has the same effect on the matrix of the quadratic form $\sum a_{ij}x_ix_j$ as the transformation $(x_1, \cdots, x_n)' = \mathfrak{T}(y_1, \cdots, y_n)'$. The quadratic forms f_1 and f_2 are *equivalent* (in K) if one may be taken into the other by a nonsingular transformation with coefficients in K. Then the corresponding vector spaces are said to be *equivalent* (in K). We write $f_1 \cong f_2$ and $\mathfrak{S}_1 \cong \mathfrak{S}_2$.

It should be noted, in passing, that two vector spaces may be equivalent without being identical. For example, if $n = 3$ and

$$\mathfrak{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

it is true that $\langle \mathfrak{u}_1, \mathfrak{u}_2 \rangle \cong \langle \mathfrak{u}_2, \mathfrak{u}_3 \rangle$. However, an isomorphism may be established between two sets of vectors having the same multiplication table.

Two vectors $\mathfrak{u}$ and $\mathfrak{v}$ are *orthogonal* if $\mathfrak{u}\mathfrak{v} = 0$. Two vector spaces are orthogonal if every vector of one is orthogonal to every vector of the other. Two subspaces, $\mathfrak{S}_1$ and $\mathfrak{S}_2$, of $\mathfrak{S}$ are *complementary* if every vector of $\mathfrak{S}$ is the sum of a vector of $\mathfrak{S}_1$ and a vector of $\mathfrak{S}_2$. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are complementary orthogonal subspaces of $\mathfrak{S}$ we write

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Presented to the Society, September 5, 1941; received by the editors April 17, 1941.

$\mathfrak{S}=\mathfrak{S}_1+\mathfrak{S}_2$. This is a direct sum if $\mathfrak{S}$ has no *radical*, that is, if its multiplication table is nonsingular.

Ernst Witt[1] proved a theorem which we shall state in two different ways. K is a field of characteristic not equal to 2 and the spaces have no radicals.

THEOREM A. *If* $\mathfrak{S}_1$, $\mathfrak{S}_2$ *and* $\mathfrak{S}_3$ *are vector spaces over* K *and* $\mathfrak{S}_2$ *and* $\mathfrak{S}_3$ *are orthogonal to* $\mathfrak{S}_1$, *then* $\mathfrak{S}_1+\mathfrak{S}_2\cong\mathfrak{S}_1+\mathfrak{S}_3$ *implies* $\mathfrak{S}_2\cong\mathfrak{S}_3$.

THEOREM B. *If* f *is a quadratic form in* $x_1, \cdots, x_r$ *and* g *and* h *are quadratic forms in* $x_{r+1}, \cdots, x_n$ *(with coefficients in* K*) then* $f+g\cong f+h$ *implies* $g\cong h$.

If the field K is replaced by a ring R we may make definitions analogous to those above. The vector space then becomes a vector lattice, $\mathfrak{L}$ (in the old-fashioned sense), and the transformations $\mathfrak{C}$ of the bases must, together with their inverses, have elements in the ring. Witt's restriction of convenience that the space shall have no radical is not necessary here except that *any result stated in terms of quadratic forms assumes that the forms are not equivalent to forms of fewer variables.*

This paper proves that Witt's result also holds for vector lattices over any ring of p-adic integers for which p is odd. We shall call such a ring an *odd p-adic integer ring* and denote it by R_p. The case $p=2$ presents difficulties all its own which we hope to resolve in a later paper. The completion of such a result would establish the theorem that if f is a quadratic form in $x_1, \cdots, x_r$ and g and h quadratic forms in $x_{r+1}, \cdots, x_n$, then $f+g$ and $f+h$ are of the same genus if and only if g and h are.

The machinery which Witt set up for fields breaks down completely in at least two essential points when applied to R_p. Hence our Lemmas 3 and 4 have no analogues in Witt's theory.

It will be recalled that if a, b and c are integers in a p-adic field, $a\equiv b \pmod{c}$ means that $(a-b)/c$ is a p-adic integer; in other words, the highest power of p dividing c is a divisor of the highest power of p dividing $a-b$. Also it is true that if a and b are p-adic integers and if for q an arbitrary power of p there is a p-adic integer x such that $ax\equiv b \pmod{q}$ then there is a p-adic integer x such that $ax=b$. When we say that a set of vectors are *linearly independent* or *dependent* we mean independence or dependence $\pmod{p}$.

It was surmised by a referee and has been established by the author that with only trivial and obvious modifications the lemmas and *final*

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[1] *Theorie der quadratischen Formen in beliebigen Körpern*, Journal fur die reine und angewandte Mathematik, vol. 176 (1937), pp. 31–48.

result of this paper hold equally well for vector lattices over any ring of $\mathfrak{P}$*-adic integers when* $\mathfrak{P}$ *is any ideal prime to 2 in a field of algebraic numbers.* The multiplication tables (that is, the matrices of the quadratic forms) as well as the transformations of bases and their inverses will have, of course, integers of the ring as elements.

2. **Lemmas.** We now prove the following lemmas:

LEMMA 1. *Let an n-dimensional lattice* $\mathfrak{L}$ *with coefficients in a* p*-adic integer ring be defined by the vectors* $\mathfrak{u}_1, \cdots, \mathfrak{u}_n$, *let* $\mathfrak{v}_1, \cdots, \mathfrak{v}_r$ *be* r *linearly independent* (mod p) *vectors of this lattice. Then there exist vectors* $\mathfrak{v}_{r+1}, \cdots, \mathfrak{v}_n$ *defining a complementary orthogonal lattice to* $\langle \mathfrak{v}_1, \cdots, \mathfrak{v}_r\rangle$ *in* $\mathfrak{L}$ *if and only if the highest power of* p *dividing the determinant of the first* r *columns of matrix* $\mathfrak{A}$ (*or* $\mathfrak{B}$) *below is a divisor of the* g.c.d. *of all determinants formed by replacing one of the first* r *columns by one of the last* n.

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{v}_1^2 & \cdots & \mathfrak{v}_1\mathfrak{v}_r & \mathfrak{v}_1\mathfrak{u}_1 & \cdots & \mathfrak{v}_1\mathfrak{u}_n \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \mathfrak{v}_r\mathfrak{v}_1 & \cdots & \mathfrak{v}_r^2 & \mathfrak{v}_r\mathfrak{u}_1 & \cdots & \mathfrak{v}_r\mathfrak{u}_n \end{pmatrix},$$

$$\mathfrak{B} = \begin{pmatrix} \mathfrak{v}_1^2 & \cdots & \mathfrak{v}_1\mathfrak{v}_r & \mathfrak{v}_1\mathfrak{v}_{r+1}^0 & \cdots & \mathfrak{v}_1\mathfrak{v}_n^0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \mathfrak{v}_r\mathfrak{v}_1 & \cdots & \mathfrak{v}_r^2 & \mathfrak{v}_r\mathfrak{v}_{r+1}^0 & \cdots & \mathfrak{v}_r\mathfrak{v}_n^0 \end{pmatrix}$$

where $\mathfrak{v}_{r+1}^0, \cdots, \mathfrak{v}_n^0$ *define a complementary* (*not necessarily orthogonal*) *space to* $\langle \mathfrak{v}_1, \cdots, \mathfrak{v}_r\rangle$ *in* $\mathfrak{L}$. *For this lemma it is not necessary that* p *be odd.*

PROOF. First note that there is indeed a complementary lattice $\langle \mathfrak{v}_{r+1}^0, \cdots, \mathfrak{v}_n^0\rangle$. That one may use $\mathfrak{A}$ or $\mathfrak{B}$ follows from the fact that the last $n-r$ columns of the latter are linear combinations with coefficients in the ring of the last n columns of the former and the last n columns of the former are linear combinations of the n columns of the latter.

Set

$$\mathfrak{v}_k = \sum_{i=1}^{r} b_{ki}\mathfrak{v}_i + \mathfrak{v}_k^0, \qquad k = r+1, \cdots, n.$$

Then

$$\mathfrak{v}_k\mathfrak{v}_j = \sum_{i=1}^{r} b_{ki}\mathfrak{v}_i\mathfrak{v}_j + \mathfrak{v}_j\mathfrak{v}_k^0, \qquad j = 1, \cdots, r.$$

For any k we can choose r integers b_{ki} so that $\mathfrak{v}_k\mathfrak{v}_j=0$ for $j=1, \cdots, r$ if and only if the conditions of the theorem hold, using matrix $\mathfrak{B}$.

LEMMA 2. *Every n-dimensional lattice $\mathfrak{L}$ in a ring R_p of p-adic integers (p odd) has a basis $\mathfrak{u}_1, \cdots, \mathfrak{u}_n$ such that*

$$\mathfrak{L} = \langle\mathfrak{u}_1\rangle + \cdots + \langle\mathfrak{u}_n\rangle.$$

This is a rather well known result.[2]

In the lemmas that follow, $\mathfrak{u}_1, \cdots, \mathfrak{u}_n$ is a canonical basis of a lattice $\mathfrak{L}$ over the p-adic integer ring, p odd, that is, $\mathfrak{L}=\langle\mathfrak{u}_1\rangle+\cdots+\langle\mathfrak{u}_n\rangle$. The c's are integers of the ring.

LEMMA 3. *If $\mathfrak{v}=c_2\mathfrak{u}_2+\cdots+c_n\mathfrak{u}_n$ and $c_{i+1}\mathfrak{u}_{i+1}^2\equiv 0 \pmod{c_i\mathfrak{u}_i^2}$ for $i=2, 3, \cdots, n-1$, there exists a $\mathfrak{v}_0=c_{02}\mathfrak{u}_2+\cdots+c_{0n}\mathfrak{u}_n$ such that $\mathfrak{v}_0\mathfrak{v}=0$ and $\mathfrak{v}_0$ has an orthogonal complementary space in $\langle\mathfrak{u}_2, \cdots, \mathfrak{u}_n\rangle$ unless for each k such that $2\leqq k\leqq n-1$ one of the following holds:*

1. $c_{k+1}\mathfrak{u}_{k+1}^2\equiv 0 \pmod{pc_k\mathfrak{u}_k^2}$ and $c_k\equiv 0 \pmod{pc_{k+1}}$. This implies $\mathfrak{u}_{k+1}^2\equiv 0 \pmod{p^2\mathfrak{u}_k^2}$.

2. $\mathfrak{u}_k^2/\mathfrak{u}_{k+1}^2$ and c_k/c_{k+1} are units and

$$1+\frac{c_k^2\mathfrak{u}_k^2}{c_{k+1}^2\mathfrak{u}_{k+1}^2}\equiv 0 \pmod{p}.$$

Furthermore, such a $\mathfrak{v}_0$ can be found if 2 holds for two successive values of k.

PROOF. Throughout this proof it is understood that $2\leqq k\leqq n-1$. Choose $\mathfrak{v}_0=c\mathfrak{u}_k+\mathfrak{u}_{k+1}$. Using Lemma 1, we seek to determine c so that (1) $\mathfrak{v}_0\mathfrak{v}=cc_k\mathfrak{u}_k^2+c_{k+1}\mathfrak{u}_{k+1}^2=0$ and (2) $\mathfrak{v}_0\mathfrak{u}_{k+1}=\mathfrak{u}_{k+1}^2$ and $\mathfrak{v}_0\mathfrak{u}_k=c\mathfrak{u}_k^2$ are $\equiv 0 \pmod{\mathfrak{v}_0^2}$ where $\mathfrak{v}_0^2=\mathfrak{u}_{k+1}^2+c^2\mathfrak{u}_k^2$.

1. Suppose $c_{k+1}\mathfrak{u}_{k+1}^2\equiv 0 \pmod{pc_k\mathfrak{u}_k^2}$ and $c_{k+1}\equiv 0 \pmod{c_k}$. Choose c so that (1) holds and have

$$c^2\mathfrak{u}_k^2=\frac{c_{k+1}}{c_k}\,\frac{c_{k+1}\mathfrak{u}_{k+1}^2}{c_k\mathfrak{u}_k^2}\,\mathfrak{u}_{k+1}^2\equiv 0 \pmod{p\mathfrak{u}_{k+1}^2}$$

which implies $\mathfrak{v}_0^2\not\equiv 0 \pmod{p\mathfrak{u}_{k+1}^2}$. Also $c\mathfrak{u}_k^2=-c_{k+1}\mathfrak{u}_{k+1}^2/c_k\equiv 0 \pmod{\mathfrak{u}_{k+1}^2}$ and hence condition (2) holds.

2. Suppose $c_{k+1}\mathfrak{u}_{k+1}^2/(c_k\mathfrak{u}_k^2)$ is a unit. Choose c so that (1) holds and

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[2] See, for example, Lemma 8 of C. L. Siegel's *Über die analytische Theorie der quadratischen Formen*, Annals of Mathematics, (2), vol. 36 (1935), pp. 527–606. In the statement of this lemma there is a misprint. R_p should be replaced by G_p. The corresponding theorem for $\mathfrak{P}$-adic integers is in the third paper of the same series, Annals of Mathematics, (2), vol. 38 (1937), p. 240.

see that c is a unit. If $\mathfrak{u}_k^2 \equiv 0 \pmod{p\mathfrak{u}_{k+1}^2}$ or $\mathfrak{u}_{k+1}^2 \equiv 0 \pmod{p\mathfrak{u}_k^2}$, (2) is seen to hold. We then have difficulty only if the first two parts of Condition 2 of the theorem hold and if in addition $\mathfrak{u}_{k+1}^2 + c^2\mathfrak{u}_k^2 \equiv 0$ (mod $p\mathfrak{u}_k^2$). Then (1) implies

$$c^2\mathfrak{u}_k^2 = \frac{c_{k+1}^2\mathfrak{u}_{k+1}^4}{c_k^2\mathfrak{u}_k^2}$$

and thus

$$1 + \frac{c_{k+1}^2\mathfrak{u}_{k+1}^2}{c_k^2\mathfrak{u}_k^2} \equiv 0 \pmod{p}.$$

For the final remark notice that

$$1 + \frac{c_k^2\mathfrak{u}_k^2}{c_{k+1}^2\mathfrak{u}_{k+1}^2} \equiv 1 + \frac{c_{k+1}^2\mathfrak{u}_{k+1}^2}{c_{k+2}^2\mathfrak{u}_{k+2}^2} \equiv 0 \pmod{p}$$

implies

$$1 - \frac{c_k^2\mathfrak{u}_k^2}{c_{k+2}^2\mathfrak{u}_{k+2}^2} \equiv 0 \pmod{p},$$

and replace $\mathfrak{u}_{k+1}$ by $\mathfrak{u}_{k+2}$ to have the existence of $\mathfrak{v}_0$.

Remark. Condition 1 of the above lemma may be weakened but only by making the statement more complex and less manageable. That there is not always a $\mathfrak{v}_0$ orthogonal to $\mathfrak{v}$ and having a complementary orthogonal space is shown by the following example:

Let $\mathfrak{u}_2^2 = 1$, $\mathfrak{u}_3^2 = p^2$ and $\mathfrak{v} = p\mathfrak{u}_2 + \mathfrak{u}_3$. We show that no $\mathfrak{v}_0 = b_2\mathfrak{u}_2 + b_3\mathfrak{u}_3$ exists for which $\mathfrak{v}_0\mathfrak{v} = 0$ and which has an orthogonal complementary lattice in $\langle \mathfrak{u}_2, \mathfrak{u}_3 \rangle$. Now $\mathfrak{v}_0\mathfrak{v} = 0$ implies $b_2 = -pb_3$ and since $\mathfrak{v}_0$ can have a complementary orthogonal lattice only if b_3 is prime to p, we take $b_3 = 1$ and have $\mathfrak{v}_0 = -p\mathfrak{u}_2 + \mathfrak{u}_3$. Now $(g_1\mathfrak{u}_2 + g_2\mathfrak{u}_3)\mathfrak{v}_0 = -g_1p + g_2p^2 = 0$ implies $g_1 \equiv 0 \pmod{p}$. Hence

$$\begin{vmatrix} -p & 1 \\ g_1 & g_2 \end{vmatrix} \equiv 0 \pmod{p}$$

and $\mathfrak{v}_0$ has no complementary orthogonal lattice.

LEMMA 4. *If* $\langle \mathfrak{v}_1, \cdots, \mathfrak{v}_n \rangle \cong \langle \mathfrak{u}_1, \cdots, \mathfrak{u}_n \rangle$ *are two canonical lattices with* $\mathfrak{v}_1^2 = \mathfrak{u}_1^2$ *and*

$$\mathfrak{v}_1 = c_1\mathfrak{u}_1 + c_2\mathfrak{u}_2 + \cdots + c_n\mathfrak{u}_n$$

while $\mathfrak{v}_1-c_1\mathfrak{u}$, *has the property that, for each k for which $2\leqq k\leqq n-1$, either 1 or 2 of Lemma 3 holds and if 2 does not hold for two successive values of k, then*

$$\langle\mathfrak{v}_2,\cdots,\mathfrak{v}_n\rangle\cong\langle\mathfrak{u}_2,\cdots,\mathfrak{u}_n\rangle.$$

PROOF. The lemma is obvious if $n=2$. Henceforth assume $n\geqq 3$. By renumbering $\mathfrak{u}_2,\cdots,\mathfrak{u}_n$, if necessary, we can have $c_{i+1}\mathfrak{u}_{i+1}^2\equiv 0$ (mod $c_i\mathfrak{u}_i^2$) for $i=2,\cdots,n-1$ as in Lemma 3.

We may write $\mathfrak{v}_i=\sum_{j=1}^{n}c_{ij}\mathfrak{u}_j$ where $c_{1j}=c_j$ and the determinant of the coefficients is prime to p. Write

$$A_i=\begin{vmatrix} c_{11} & \cdots & c_{1i}\\ \cdot & \cdots & \cdot\\ c_{i1} & \cdots & c_{ii}\end{vmatrix}.$$

We know $A_n\not\equiv 0$ (mod p). Assume (what we shall soon prove) that $c_1=c_{11}\not\equiv 0$ (mod p). We now show that we may renumber $\mathfrak{v}_2,\cdots,\mathfrak{v}_n$, if necessary, to make $A_i\not\equiv 0$ (mod p) for $2\leqq i\leqq n$. Assume $A_{i-1}\not\equiv 0$ (mod p); the Laplace expansion of A_n shows that the matrix $\mathfrak{M}_i$ composed of the first i columns of A_n is of rank i. The first $i-1$ rows of $\mathfrak{M}_i$ are linearly independent and not all the remaining $n-i+1$ rows of $\mathfrak{M}_i$ are linearly dependent on them.

Notice that $\mathfrak{v}_i\mathfrak{v}_j=c_{i1}c_{j1}\mathfrak{u}_1^2+\cdots+c_{in}c_{jn}\mathfrak{u}_n^2$ and $\mathfrak{v}_i\mathfrak{v}_j=0$ if $i\neq j$.

We next proceed to prove some preliminary results.

I. $c_2\mathfrak{u}_2^2\equiv 0$ (mod $c_1\mathfrak{u}_1^2$). Suppose the contrary were the case, that is, $c_1\mathfrak{u}_1^2\equiv 0$ (mod $pc_2\mathfrak{u}_2^2$). We have, by Lemma 3, two cases to consider.

Case 1. If $c_i\mathfrak{u}_i^2\equiv 0$ (mod $pc_2\mathfrak{u}_2^2$) and $c_2\equiv 0$ (mod pc_i) for all $i>2$, then $\mathfrak{v}_1\mathfrak{v}_i=0$ implies $c_{i2}c_2\mathfrak{u}_2^2\equiv 0$ (mod $pc_2\mathfrak{u}_2^2$) and hence $c_{i2}\equiv 0$ (mod p) for $i>1$. Since if $n\geqq 3$, $c_2\equiv 0$ (mod p); and hence $A_n\equiv 0$ (mod p), which is false.

Case 2. If $c_i\mathfrak{u}_i^2\equiv 0$ (mod $pc_2\mathfrak{u}_2^2$) and $c_2\equiv 0$ (mod pc_i) for $i>3$ while Condition 2 of Lemma 3 holds for $k=2$, then $\mathfrak{v}_1\mathfrak{v}_i=0$ implies $c_{i2}c_2\mathfrak{u}_2^2+c_{i3}c_3\mathfrak{u}_3^2\equiv 0$ (mod $pc_2\mathfrak{u}_2^2$) (for $i=1$, Condition 2 of Lemma 3 implies the congruence). Hence $A_n\equiv 0$ (mod p), contrary to fact.

II. $c_1=c_{11}\not\equiv 0$ (mod p). (This shows $A_i\not\equiv 0$ (mod p), $i=1,\cdots,n$.) We know $\mathfrak{v}_1^2-c_1^2\mathfrak{u}_1^2\equiv 0$ (mod $c_2\mathfrak{u}_2^2$). Hence $\mathfrak{v}_1^2-c_1^2\mathfrak{u}_1^2=\mathfrak{u}_1^2-c_1^2\mathfrak{u}_1^2\equiv 0$ (mod $c_1\mathfrak{u}_1^2$) and $\mathfrak{u}_1^2\equiv 0$ (mod $c_1\mathfrak{u}_1^2$).

III. For $k>r\geqq 1$ it is true that

$$c_{k1}\mathfrak{u}_1^2\equiv\cdots\equiv c_{kr}\mathfrak{u}_r^2\equiv 0\ (\mathrm{mod}\ \mathfrak{u}_{r+1}^2).$$

If $r\geqq 2$ we know that in all cases of the previous lemma $\mathfrak{u}_{r+1}^2\equiv 0$ (mod $\mathfrak{u}_r^2$). Hence for $r\geqq 1$ we have for each $k>r$,

$$\mathfrak{v}_k\mathfrak{v}_j \equiv c_{k1}c_{j1}\mathfrak{u}_1^2 + \cdots + c_{kr}c_{jr}\mathfrak{u}_r^2 \equiv 0 \pmod{\mathfrak{u}_{r+1}^2} \qquad \text{for } j = 1, \cdots, r,$$

and since $A_r \not\equiv 0 \pmod{p}$ we have the desired result.

IV. If $\mathfrak{u}_{r+1}^2 \equiv 0 \pmod{p\mathfrak{u}_r^2}$ then $\mathfrak{u}_{r+1}^2 \equiv 0 \pmod{p\mathfrak{u}_i^2}$ for $2 \leqq i \leqq r$ and III implies

$$c_{k2} \equiv \cdots \equiv c_{kr} \equiv 0 \pmod{p} \qquad \text{for each } k > r.$$

V. $c_{k1} \equiv 0 \pmod{p}$ for all $k > 2$ or > 3 according as $\mathfrak{u}_3^2 \equiv 0 \pmod{p\mathfrak{u}_2^2}$ or not. A glance at Lemma 3 shows

$$\mathfrak{v}_k\mathfrak{v}_1 \equiv c_{k1}c_1\mathfrak{u}_1^2 + c_{k2}c_2\mathfrak{u}_2^2 + c_{k3}c_3\mathfrak{u}_3^2 \equiv 0 \pmod{pc_2\mathfrak{u}_2^2}.$$

If $\mathfrak{u}_3^2 \equiv 0 \pmod{p\mathfrak{u}_2^2}$ then $c_3\mathfrak{u}_3^2 \equiv 0 \pmod{pc_2\mathfrak{u}_2^2}$ and IV shows c_{k2}, and hence by I, c_{k1} are congruent to 0 (mod p) for $k > 2$. Otherwise $\mathfrak{u}_4^2 \equiv 0 \pmod{p\mathfrak{u}_3^2}$ while $\mathfrak{u}_3^2/\mathfrak{u}_2^2$ and c_3/c_2 are units; then IV implies c_{k2} and c_{k3} and hence c_{k1} are congruent to 0 (mod p) for $k > 3$. This also holds for all $k \geqq 2$, if perchance $c_2\mathfrak{u}_2^2 \equiv 0 \pmod{p\mathfrak{u}_1^2}$.

We now have two cases to consider corresponding to the two cases of Lemma 3. In what follows i is some fixed one of $2, 3, \cdots, n$. In the first place we assume that either $i = 2$ and Case 1 holds for $k = 2$, or $i > 2$ and Case 1 holds for $k = i$ and $k = i - 1$; we then show that $\langle \mathfrak{v}_i \rangle \cong \langle \mathfrak{u}_i \rangle$. In the second place we assume that Case 2 holds for $k = i$; then, with one exception which is dealt with separately, we show that $\langle \mathfrak{v}_i, \mathfrak{v}_{i+1} \rangle \cong \langle \mathfrak{u}_i, \mathfrak{u}_{i+1} \rangle$.

Case 1. Suppose $c_{i+j}\mathfrak{u}_{i+j}^2 \equiv 0 \pmod{pc_i\mathfrak{u}_i^2}$, $c_i \equiv 0 \pmod{pc_{i+j}}$ and $\mathfrak{u}_{i+j}^2 \equiv 0 \pmod{p\mathfrak{u}_i^2}$ for all j for which $1 \leqq j \leqq n - i$, while $c_i\mathfrak{u}_i^2 \equiv 0 \pmod{pc_{i-j}\mathfrak{u}_{i-j}^2}$, $c_{i-j} \equiv 0 \pmod{pc_i}$ and $\mathfrak{u}_i^2 \equiv 0 \pmod{p\mathfrak{u}_{i-j}^2}$ for all j for which $i - 2 \geqq j \geqq 1$. If $i > 2$ take $r = i - 1$ and $k = i$, in IV above and have $c_{i2} \equiv \cdots \equiv c_{i,i-1} \equiv 0 \pmod{p}$ and V shows that $c_{i1} \equiv 0 \pmod{p}$. Hence $A_i \not\equiv 0 \pmod{p}$ implies $c_{ii} \not\equiv 0 \pmod{p}$. If $i = 2$, $n \geqq 3$ implies $c_3 \equiv 0 \pmod{p}$ and $A_2 \not\equiv 0 \pmod{p}$ implies $c_{22} \not\equiv 0 \pmod{p}$. Hence in both cases $c_{ii} \not\equiv 0 \pmod{p}$.

In III above put $r = i - 1$, $k = i$ and have

$$c_{i1}\mathfrak{u}_1^2 \equiv \cdots \equiv c_{i,i-1}\mathfrak{u}_{i-1}^2 \equiv 0 \pmod{\mathfrak{u}_i^2}.$$

But $\mathfrak{v}_i^2 \equiv c_{i1}(c_{i1}\mathfrak{u}_1^2) + \cdots + c_{ii}(c_{ii}\mathfrak{u}_i^2) \pmod{p\mathfrak{u}_i^2}$ and hence we have shown $\mathfrak{v}_i^2 \equiv 0 \pmod{\mathfrak{u}_i^2}$. If $i > 2$ we have shown above that $c_{i1} \equiv \cdots \equiv c_{i,i-1} \equiv 0 \pmod{p}$ and hence $\mathfrak{v}_i^2 \equiv c_{ii}^2\mathfrak{u}_i^2 \pmod{p\mathfrak{u}_i^2}$. If $i = 2$, $\mathfrak{v}_1\mathfrak{v}_2 \equiv c_1c_{21}\mathfrak{u}_1^2 + c_2c_{22}\mathfrak{u}_2^2 \equiv 0 \pmod{pc_2\mathfrak{u}_2^2}$ and $c_1 \not\equiv 0 \pmod{p}$ implies $c_{21} \equiv -c_2c_{22}\mathfrak{u}_2^2/(c_1\mathfrak{u}_1^2)$ (mod $pc_2\mathfrak{u}_2^2/\mathfrak{u}_1^2$). Hence

$$c_{21}^2\mathfrak{u}_1^2 \equiv c_2\frac{c_2\mathfrak{u}_2^2}{c_1^2\mathfrak{u}_1^2}c_{22}^2\mathfrak{u}_2^2 \pmod{pc_2\mathfrak{u}_2^2}$$

and $c_2 \equiv 0 \pmod{p}$ shows $c_{21}^2\mathfrak{u}_1^2 \equiv 0 \pmod{p\mathfrak{u}_2^2}$. In both cases, then, we have $\mathfrak{v}_i^2 \equiv c_{ii}^2\mathfrak{u}_i^2 \pmod{p\mathfrak{u}_i^2}$ which implies $\mathfrak{v}_i^2 \equiv c_{ii}^2\mathfrak{u}_i^2 \pmod{q}$ is solvable for q, an arbitrary power of p. Hence there exists a unit b in R_p such that $\mathfrak{v}_i^2 = b^2\mathfrak{u}_i^2$.

Case 2. Suppose $3 \leqq i+1 \leqq n$, $\mathfrak{u}_i^2/\mathfrak{u}_{i+1}^2$ and c_i/c_{i+1} are units, $1 + c_i^2\mathfrak{u}_i^2/(c_{i+1}^2\mathfrak{u}_{i+1}^2) \equiv 0 \pmod{p}$, $c_{i+j}\mathfrak{u}_{i+j}^2 \equiv 0 \pmod{pc_i\mathfrak{u}_i^2}$, $c_i \equiv 0 \pmod{pc_{i+j}}$, $\mathfrak{u}_{i+j}^2 \equiv 0 \pmod{p\mathfrak{u}_i^2}$ for all j for which $n-i \geqq j \geqq 2$. Also $c_i\mathfrak{u}_i^2 \equiv 0 \pmod{pc_{i-j}\mathfrak{u}_{i-j}^2}$; $\mathfrak{u}_i^2 \equiv 0 \pmod{p\mathfrak{u}_{i-j}^2}$ for all j such that $1 \leqq j < i-1$. Recall I and II above.

Taking $r = i-1$ in IV and using V we have

$$c_{k1} \equiv \cdots \equiv c_{k\,i-1} \equiv 0 \pmod{p} \qquad \text{for } k = i,\ i+1 \text{ and } i > 2.$$

This with $A_{i+1} \not\equiv 0 \pmod{p}$ implies

$$B_i = \begin{vmatrix} c_{ii} & c_{i\,i+1} \\ c_{i+1\,i} & c_{i+1\,i+1} \end{vmatrix} \not\equiv 0 \pmod{p}.$$

The last holds even if $i=2$ unless $n=3$ and $c_3 \not\equiv 0 \pmod{p}$ since $n>3$ implies $c_2 \equiv c_3 \equiv 0 \pmod{p}$. We postpone this exceptional case.

Take $r = i-1$ in III above and have

$$c_{k1}\mathfrak{u}_1^2 \equiv \cdots \equiv c_{k\,i-1}\mathfrak{u}_{i-1}^2 \equiv 0 \pmod{\mathfrak{u}_i^2} \qquad \text{for } k = i,\ i+1.$$

Now

$$\mathfrak{v}_k^2 \equiv c_{k1}(c_{k1}\mathfrak{u}_1^2) + \cdots + c_{k\,i+1}(c_{k\,i+1}\mathfrak{u}_{i+1}^2) \pmod{p\mathfrak{u}_i^2},$$

and

$$c_{k1} \equiv \cdots \equiv c_{k\,i-1} \equiv 0 \pmod{p} \qquad \text{for } k = i,\ i+1 \text{ and } i > 2$$

implies

$$\mathfrak{v}_i^2 \equiv c_{ii}(c_{ii}\mathfrak{u}_i^2) + c_{i\,i+1}(c_{i\,i+1}\mathfrak{u}_{i+1}^2) \pmod{p\mathfrak{u}_{i+1}^2},$$

$$\mathfrak{v}_{i+1}^2 \equiv c_{i+1\,i}(c_{i+1\,i}\mathfrak{u}_i^2) + c_{i+1\,i+1}(c_{i+1\,i+1}\mathfrak{u}_{i+1}^2) \pmod{p\mathfrak{u}_{i+1}^2},$$

$$\mathfrak{v}_i\mathfrak{v}_{i+1} \equiv c_{i+1\,i}(c_{ii}\mathfrak{u}_i^2) + c_{i+1\,i+1}(c_{i\,i+1}\mathfrak{u}_{i+1}^2) \equiv 0 \pmod{p\mathfrak{u}_{i+1}^2}.$$

The argument in Case 1 for $i=2$ may be used here to show that the three congruences above hold even when $i=2$ except in the case postponed above. In fact, if $\mathfrak{u}_2^2 \equiv 0 \pmod{p\mathfrak{u}_1^2}$, III implies $c_{21} \equiv c_{31} \equiv 0 \pmod{p}$, $B_2 \not\equiv 0 \pmod{p}$ and the argument used in Case 1 carries through to show that the three congruences hold. We thus postpone the case in which $n=3$, $c_2c_3 \not\equiv 0 \pmod{p}$ and $\mathfrak{u}_2^2/\mathfrak{u}_1^2$ is a unit.

Divide the three congruences by $\mathfrak{u}_i^2$; let $d = \mathfrak{u}_{i+1}^2/\mathfrak{u}_i^2$. The resulting situation is covered by the next lemma which shows that there are p-adic integers a, b, c, e such that $ae-bc$ is prime to p and

$$\mathfrak{v}_i^2 = (a\mathfrak{u}_i + b\mathfrak{u}_{i+1})^2, \qquad \mathfrak{v}_{i+1}^2 = (c\mathfrak{u}_i + e\mathfrak{u}_{i+1})^2,$$
$$(a\mathfrak{u}_i + b\mathfrak{u}_{i+1})(c\mathfrak{u}_i + e\mathfrak{u}_{i+1}) = 0.$$

Hence $\langle \mathfrak{v}_i, \mathfrak{v}_{i+1}\rangle \cong \langle \mathfrak{u}_i, \mathfrak{u}_{i+1}\rangle$.

It remains to consider the postponed case $n=3$, $c_2c_3\not\equiv 0 \pmod{p}$ and $\mathfrak{u}_2^2/\mathfrak{u}_1^2$ is a unit. Then $c_2^2\mathfrak{u}_2^2+c_3^2\mathfrak{u}_3^2\equiv 0 \pmod{p\mathfrak{u}_1^2}$ implies that $\mathfrak{v}_1^2\equiv c_1^2\mathfrak{u}_1^2\equiv 0 \pmod{p\mathfrak{u}_1^2}$ and hence $c_1\equiv 0 \pmod{p}$, contrary to fact.

LEMMA 5. *If the congruences*

$$x^2 + dy^2 \equiv g_1 \pmod{p}, \quad z^2 + dt^2 \equiv g_2 \pmod{p}, \quad xz + dyt \equiv g_3 \pmod{p},$$

with $d\not\equiv 0 \pmod{p}$, *have solutions* x_0, y_0, z_0, t_0 *with* $x_0t_0-z_0y_0\not\equiv 0 \pmod{p}$, *then there are p-adic integers* x, y, z, t *such that* $x^2+dy^2=g_1$, $z^2+dt^2=g_2$, $xz+dyt=g_3$ *and* $xt-zy\not\equiv 0 \pmod{p}$.

PROOF. To prove the lemma, assume that x_0, y_0, z_0, t_0 is a solution of the congruences with p replaced by q. We seek an x, y, z, t so that

$$\begin{aligned}(x_0+qx)^2+d(y_0+qy)^2 &\equiv g_1 \pmod{pq},\\ (z_0+qz)^2+d(t_0+qt)^2 &\equiv g_2 \pmod{pq},\\ (x_0+qx)(z_0+qz)+d(y_0+qy)(t_0+qt) &\equiv g_3 \pmod{pq}.\end{aligned}$$

That is,

$$\begin{aligned}x_0x+dy_0y &\equiv h_1 \pmod{p},\\ z_0z+dt_0t &\equiv h_2 \pmod{p},\\ x_0z+z_0x+d(y_0t+t_0y) &\equiv h_3 \pmod{p},\end{aligned}$$

for some integers h_1, h_2 and h_3. The matrix of the coefficients on the left is

$$\begin{Vmatrix} x_0 & dy_0 & 0 & 0 \\ 0 & 0 & z_0 & dt_0 \\ z_0 & dt_0 & x_0 & dy_0 \end{Vmatrix}$$

which is of rank 3 $\pmod{p}$.

3. Final results.

We have the following theorems.

THEOREM 1. *If* $\langle \mathfrak{v}_1, \cdots, \mathfrak{v}_n\rangle$ *and* $\langle \mathfrak{u}_1, \cdots, \mathfrak{u}_n\rangle$ *are two equivalent lattices in a ring* R_p *of p-adic integers* (p *odd*) *and if* $\mathfrak{v}_1^2=\mathfrak{u}_1^2\neq 0$ *and* $\mathfrak{v}_1\mathfrak{v}_i=\mathfrak{u}_1\mathfrak{u}_i=0$, $i\neq 1$, *then*

$$\langle \mathfrak{v}_2, \cdots, \mathfrak{v}_n\rangle \cong \langle \mathfrak{u}_2, \cdots, \mathfrak{u}_n\rangle.$$

PROOF. Assume the theorem true for all lesser values of n. Also, $\langle \mathfrak{v}_2, \cdots, \mathfrak{v}_n\rangle$ and $\langle \mathfrak{u}_2, \cdots, \mathfrak{u}_n\rangle$ may be considered to be in canonical form. Lemma 4 establishes the theorem unless Lemma 3 applies. We then have the existence of a vector $\mathfrak{v}_0$ such that $\mathfrak{v}_0\mathfrak{v}_1 = \mathfrak{v}_0\mathfrak{u}_1 = 0$ and $\mathfrak{v}_0$ has an orthogonal complementary space in $\langle \mathfrak{u}_2, \cdots, \mathfrak{u}_n\rangle$ and hence such a space, $\langle \mathfrak{v}_0\rangle^*$, in $\mathfrak{L} = \langle \mathfrak{u}_1, \cdots, \mathfrak{u}_n\rangle$, that is, $\mathfrak{L} = \langle \mathfrak{v}_0\rangle + \langle \mathfrak{v}_0\rangle^*$. Then $\langle \mathfrak{v}_0\rangle^* \cong \langle \mathfrak{v}_1\rangle + \mathfrak{V} \cong \langle \mathfrak{u}_1\rangle + \mathfrak{U}$ where[3] $\mathfrak{V}$ and $\mathfrak{U}$ are the complementary orthogonal spaces of $\langle \mathfrak{v}_1\rangle$ and $\langle \mathfrak{u}_1\rangle$, respectively, in $\langle \mathfrak{v}_0\rangle^*$. By the hypothesis of the induction $\mathfrak{V} \cong \mathfrak{U}$ and $\mathfrak{L} \cong \langle \mathfrak{v}_0\rangle + \langle \mathfrak{v}_1\rangle + \mathfrak{V} \cong \langle \mathfrak{v}_0\rangle + \langle \mathfrak{u}_1\rangle + \mathfrak{U}$ implies

$$\langle \mathfrak{v}_2, \cdots, \mathfrak{v}_n\rangle = \langle \mathfrak{v}_0\rangle + \mathfrak{V} \cong \langle \mathfrak{v}_0\rangle + \mathfrak{U} = \langle \mathfrak{u}_2, \cdots, \mathfrak{u}_n\rangle.$$

THEOREM 2. *If lattices $\mathfrak{L}_1$ and $\mathfrak{L}_2$ over a ring R_p of p-adic integers (p odd) have no radical, then*

$$\mathfrak{L}_3 + \mathfrak{L}_4 = \mathfrak{L}_1 \cong \mathfrak{L}_2 = \mathfrak{L}_3 + \mathfrak{L}_5$$

implies

$$\mathfrak{L}_4 \cong \mathfrak{L}_5.$$

This theorem is easily established by induction using Lemma 2 and Theorem 1. It may also be stated in terms of quadratic forms in a manner analogous to Theorem B.

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[3] The existence of $\mathfrak{V}$ (and similarly of $\mathfrak{U}$) follows from Lemma 1 since $\mathfrak{v}_1^2$ is a divisor of the product of $\mathfrak{v}_1$ by any vector of $\mathfrak{L}$ and hence of $\langle \mathfrak{v}_0\rangle^*$.

EXPANSIONS IN SERIES OF NON-ORTHOGONAL FUNCTIONS

R. V. CHURCHILL

1. **Introduction.** Let $\phi_i(x)$ $(i=1, 2, \cdots)$ be the normalized characteristic functions of the Sturm-Liouville problem

$$\frac{d}{dx}\left(R\frac{d\phi}{dx}\right) + (\lambda P + Q)\phi = 0,$$

$$A_0\phi'(0) + B_0\phi(0) = 0, \qquad A_1\phi'(1) + B_1\phi(1) = 0,$$

in which the functions P, Q, and R are continuous, and $R>0$, $P>0$, when $0 \leqq x \leqq 1$. The set of functions $\{\phi_i(x)\}$ is closed with respect to the class $L^2(0, 1)$, in the sense that Parseval's relation,

$$\int_0^1 Pf^2 dx = \sum_1^\infty \left[\int_0^1 Pf\phi_i dx\right]^2,$$

is satisfied by every function f of that class. This fact can be deduced readily from a theorem by Kellogg[1] on the completeness of the set of solutions of the self-adjoint problem of the second order. It can also be obtained from a result found by Dixon.[2]

In terms of two functions f and G of the class $L^2(0, 1)$, Parseval's relation can be written

$$\int_0^1 PfGdx = \sum_1^\infty \int_0^1 Pf\phi_i dx \int_0^1 PG\phi_i dx = \sum_1^\infty c_i \int_0^1 PG\phi_i dx,$$

where c_i are the Fourier constants of f. If $G=g/P$ when $0<x<t$ and $G\equiv 0$ when $t<x<1$, where the function g belongs to $L^2(0, 1)$, it follows that

$$\int_0^t fgdx = \sum_1^\infty c_i \int_0^t g\phi_i dx, \qquad 0 \leqq t \leqq 1. \tag{1}$$

Presented to the Society in two parts, the first under the title *On an extension of the Sturm-Liouville development* on April 14, 1939, and the second under the title *Term-wise integration of Sturm-Liouville expansions* on April 12, 1940; received by the editors April 25, 1941.

[1] O. D. Kellog, *Note on closure of orthogonal sets*, this Bulletin, vol. 27 (1920), pp. 165–169.

[2] A. C. Dixon, *On the series of Sturm-Liouville, as derived from a pair of fundamental integral equations instead of a differential equation*, Philosophical Transactions of the Royal Society of London, (A), vol. 211 (1912), pp. 411–432; pp. 431, 432.

The convergence of this series is uniform with respect to t $(0\leqq t\leqq 1)$. For, according to Cauchy's inequality,

$$\left|\sum_{m}^{n} c_i \int_0^t g\phi_i dx\right| \leqq \left[\sum_{m}^{n} c_i^2 \sum_{m}^{n}\left(\int_0^t g\phi_i dx\right)^2\right]^{1/2}, \qquad n > m.$$

But the Parseval relation is true for the function G, so that

$$\sum_{1}^{\infty}\left(\int_0^t g\phi_i dx\right)^2 = \int_0^t \frac{g^2}{P}dx \leqq \int_0^1 \frac{g^2}{P}dx.$$

Since the series $\sum_1^\infty c_i^2$ converges, the uniform convergence of the series in (1) becomes apparent.

Hence *the generalized Fourier series* $\sum_1^\infty c_i\phi_i(x)$ *corresponding to* f *can be multiplied by* g *and integrated term by term to give the uniformly convergent representation* (1).

It is our purpose to use this integration process to derive an expansion in series of the characteristic functions of the following problem:

$$\begin{aligned} &y'' + (\lambda + q)y = 0,\\ (2)\qquad &a_0y(0) + y'(0) - b_0y''(0) = 0,\\ &a_1y(1) + y'(1) + b_1y''(1) = 0, \end{aligned}$$

in which the function q is continuous in the interval (0, 1); the a's and b's are constants, and $b_0\geqq 0$, $b_1\geqq 0$. Problems of this type arise in a variety of boundary problems in partial differential equations.[3]

In view of the differential equation, the boundary conditions can be written

$$\begin{aligned} (3)\qquad &y'(0) + [a_0 + b_0q(0) + b_0\lambda]y(0) = 0,\\ &y'(1) + [a_1 - b_1q(1) - b_1\lambda]y(1) = 0, \end{aligned}$$

thus showing how the parameter λ is involved in those conditions.[4] If λ_i and λ_j are two distinct characteristic numbers of the system (2), and y_i, y_j the corresponding characteristic functions, it follows by a familiar procedure that

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[3] See, for example, R. E. Langer, *A problem in diffusion or in the flow of heat for a solid in contact with a fluid*, Tôhoku Mathematical Journal, vol. 35 (1932), pp. 260–275; S. Timoshenko, *Vibration Problems in Engineering*, 1928, pp. 209, 217. Expansion theorems for the case $q\equiv 0$ will be found in the first of these references; also see A. Kneser, *Integralgleichungen*, 1922, pp. 117–121.

[4] Theories have been developed, under hypotheses that are quite different from those with which we are concerned here, applying to general boundary value problems which have the form (2) as a special case. See, for instance, W. T. Reid, American Journal of Mathematics, vol. 54 (1932), pp. 769–790, and J. Tamarkin, Mathematische Zeitschrift, vol. 27 (1927), pp. 1–54.

$$\frac{d}{dx}(y_j y_i' - y_i y_j') = (\lambda_j - \lambda_i) y_i y_j. \tag{4}$$

Hence, in view of (3), the characteristic functions of (2) satisfy the modified orthogonality relation

$$B(y_i, y_j) = 0, \qquad i \neq j, \tag{5}$$

where

$$B(y_i, y_j) = \int_0^1 y_i y_j \, dx + b_0 y_i(0) y_j(0) + b_1 y_i(1) y_j(1).$$

2. **An associated Sturm-Liouville problem.** Since $b_0 \geqq 0$ and $b_1 \geqq 0$, problem (2) has an infinite set of real characteristic numbers λ_i ($i = 0, 1, 2, \cdots$; $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$), and the corresponding characteristic function y_i has exactly i zeros in the interval $0 < x < 1$.[5] According to the conditions (3), none of the functions vanishes at either boundary. We assume henceforth that the sign of $y_0(x)$ has been so adjusted that $y_0 > 0$ when $0 \leqq x \leqq 1$.

Now let the functions Y_i be defined as follows:

$$Y_i(x) = \frac{d}{dx}\left[\frac{y_i(x)}{y_0(x)}\right], \qquad i = 1, 2, \cdots.$$

Then, in view of (4),

$$\frac{d}{dx}(y_0^2 Y_i) = (\lambda_0 - \lambda_i) y_0 y_i,$$

and therefore

$$\frac{d}{dx}\left[\frac{1}{y_0^2}\frac{d}{dx}(y_0^2 Y_i)\right] + (\lambda_i - \lambda_0) Y_i = 0. \tag{6}$$

Replacing y_i' by $y_0 Y_i + y_0' y_i / y_0$ and y_i'' by $y_0 Y_i' + 2 y_0' Y_i + y_0'' y_i / y_0$ in the first boundary condition

$$a_0 y_i(0) + y_i'(0) - b_0 y_i''(0) = 0,$$

we find that the resulting coefficient of $y_i(0)$ is $[a_0 y_0(0) + y_0'(0) - b_0 y_0''(0)]/y_0(0)$. This vanishes because y_0 satisfies the boundary condition. Proceeding in the same way to express the second boundary condition of (2) in terms of Y_i, and rewriting the differential equation (6), we find that the functions $Y_i(x)$ ($i = 1, 2, \cdots$) are the characteristic functions of the following Sturm-Liouville problem:

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[5] See, for instance, E. L. Ince, *Ordinary Differential Equations*, 1927, pp. 231–235.

$$
\begin{aligned}
&\frac{d}{dx}\left(y_0^2\frac{dY_i}{dx}\right)+\left[\lambda_i y_0^2-\lambda_0 y_0^2-2(y_0')^2+2y_0y_0''\right]Y_i=0,\\
(7)\qquad &b_0y_0(0)Y_i'(0)-\left[y_0(0)-2b_0y_0'(0)\right]Y_i(0)=0,\\
&b_1y_0(1)Y_i'(1)+\left[y_0(1)+2b_1y_0'(1)\right]Y_i(1)=0.
\end{aligned}
$$

The characteristic numbers $\lambda_1, \lambda_2, \cdots$ of problem (2) are clearly the characteristic numbers of (7) that correspond to the characteristic functions $Y_1, Y_2, \cdots$. Except possibly for λ_0, every characteristic number of (2) must also belong to this new system; but since all those of (7) are known to be real, all characteristic numbers of (2) must be real.

In order to show that $Y_i(x)$ are the only characteristic functions of (7), let $U(x)$ denote a solution corresponding to the value μ of the parameter written as λ_i in (7). Let $u(x)$ be defined as follows:

$$u(x)=y_0(x)\left[\int_0^x U(x)dx+C\right],$$

where C is a constant yet to be determined. Since $d(u/y_0)/dx=U$, it is easily seen that

$$\frac{u''+(\mu+q)u}{y_0}=\frac{1}{y_0^2}\frac{d}{dx}(y_0^2U)+(\mu-\lambda_0)\frac{u}{y_0}\cdot$$

The derivative of the second member of this equation is zero, according to (6); therefore

$$u''+(\mu+q)u=Ky_0,$$

where K is some constant; also,

$$U'+2\frac{y_0'}{y_0}U+(\mu-\lambda_0)\left(\int_0^x Udx+C\right)=K. \tag{8}$$

If $\mu\neq\lambda_0$, there is one value of C for which $K=0$; for if $x=0$ in (8),

$$C=-\frac{1}{\mu-\lambda_0}\left[U'(0)+2\frac{y_0'(0)}{y_0(0)}U(0)\right].$$

By reversing the steps used to obtain the boundary conditions in (7), we find that $u(x)$ satisfies the boundary conditions of the original problem (2). Since $u''+(\mu+q)u=0$ when C has the above value, μ must be one of the characteristic numbers λ_i $(i=1, 2, \cdots)$ of (2).

Now suppose $\mu = \lambda_0$. Let the corresponding function U be denoted by U_0. According to (8),

$$U_0'(x) + 2\frac{y_0'(x)}{y_0(x)}U_0(x) = U_0'(0) + 2\frac{y_0'(0)}{y_0(0)}U_0(0). \tag{9}$$

Since U_0 satisfies the first boundary condition of (7), it follows that

$$\begin{aligned} \frac{1}{y_0^2}\frac{d}{dx}(y_0^2U_0) &= \frac{U_0(0)}{b_0} && \text{if } b_0 \neq 0, \\ &= U_0'(0) && \text{if } b_0 = 0, \end{aligned}$$

and, after multiplying through by y_0^2 and integrating, we obtain

$$\begin{aligned} y_0^2(1)U_0(1) &= U_0(0)\left[y_0^2(0) + \frac{1}{b_0}\int_0^1 y_0^2dx\right] && \text{if } b_0 \neq 0, \\ &= U_0'(0)\int_0^1 y_0^2dx && \text{if } b_0 = 0. \end{aligned} \tag{10}$$

Suppose $b_0 \neq 0$. In view of the boundary conditions of (7), equation (9) with $x=1$ reduces to $U_0(0)/b_0 = -U_0(1)/b_1$, if $b_1 \neq 0$. Therefore $U_0(0)$ and $U_0(1)$ must be of opposite sign, in contradiction to (10). If $b_1 = 0$ then $U_0(1) = 0$; but $U_0(0) \neq 0$ so that (10) is again contradicted.

When $b_0 = 0$ we find in like manner that $U_0'(0) = -U_0(1)/b_1$, if $b_1 \neq 0$, and this is contrary to (10). If $b_1 = 0$ also, then $U_0(1) = 0$, and since $U_0'(0) \neq 0$ the contradiction again appears.

Therefore λ_0 cannot be a characteristic number of the system (7), and the following theorem is established:

THEOREM 1. *Let* $\lambda_0, \lambda_1, \lambda_2, \cdots$ *be the set of characteristic numbers of problem* (2), *and* $y_0, y_1, y_2, \cdots$ *the corresponding characteristic functions* ($y_0 > 0$). *Then for the Sturm-Liouville problem* (7), *the complete set of characteristic numbers consists of* $\lambda_1, \lambda_2, \cdots$, *and the corresponding characteristic functions* $Y_1, Y_2, \cdots$ *are related to those of the first problem in the following way*:

$$y_i(x) = y_0(x)\left(\int_0^x Y_i(x)dx + C_i\right), \qquad i = 1, 2, \cdots,$$

where

$$C_i = -\frac{1}{\lambda_i - \lambda_0}\left[Y_i'(0) + 2\frac{y_0'(0)}{y_0(0)}Y_i(0)\right] = \frac{y_i(0)}{y_0(0)}.$$

3. **The expansion theorem.** Let $f(x)$ be such that the function $d(f/y_0)/dx$ belongs to the class $L^2(0, 1)$. The generalized Fourier series corresponding to the latter function, with respect to the functions Y_i, is

$$\sum_1^\infty A_i Y_i(x),$$

where

$$A_i \int_0^1 y_0^2 Y_i^2 dx = \int_0^1 y_0^2 \frac{d}{dx}\left(\frac{f}{y_0}\right) Y_i dx, \qquad i = 1, 2, \cdots .$$

The formula (1), with $g \equiv 1$, applies to this series. Therefore, in view of the above relation between Y_i and y_i, we can write

$$\int_0^x \frac{d}{dx}\left(\frac{f}{y_0}\right) dx = \sum_1^\infty A_i \left[\frac{y_i(x)}{y_0(x)} - \frac{y_i(0)}{y_0(0)}\right], \qquad 0 \leq x \leq 1. \tag{11}$$

This is an expansion in terms of the non-orthogonal characteristic functions of problem (2). The convergence of the series is uniform.

In order to improve the form of this result, we now assume that the derivative $f'(x)$ exists and is bounded when $0 \leq x \leq 1$, and that it belongs to the class $L(0, 1)$. The derivative of f/y_0 then belongs to $L^2(0, 1)$, and (11) can be written

$$y_0(0) f(x) - f(0) y_0(x) = \sum_1^\infty A_i [y_0(0) y_i(x) - y_i(0) y_0(x)]. \tag{12}$$

We can multiply all terms here by $y_0(x)$ and integrate over the interval $(0, 1)$. The member on the right then becomes, in view of the modified orthogonality property (5),

$$-\sum_1^\infty A_i \{B(y_0, y_0) y_i(0) + b_1 y_0(1) [y_0(0) y_i(1) - y_i(0) y_0(1)]\}.$$

Using equation (12) with $x = 1$ here, we can write this series as

$$-B(y_0, y_0) \sum_1^\infty A_i y_i(0) - b_1 y_0(1) [y_0(0) f(1) - f(0) y_0(1)],$$

so that the result of our operations upon (12) becomes

$$y_0(0) B(f, y_0) - f(0) B(y_0, y_0) = -B(y_0, y_0) \sum_1^\infty A_i y_i(0).$$

Hence the series appearing here converges and can be written

$$\sum_{1}^{\infty} A_i y_i(0) = f(0) - A_0 y_0(0), \tag{13}$$

where

$$A_0 = \frac{B(f, y_0)}{B(y_0, y_0)}.$$

The expansion (12) becomes, in view of (13),

$$f(x) = \sum_{0}^{\infty} A_i y_i(x), \qquad 0 \leqq x \leqq 1.$$

The convergence is uniform, since the series in (13) converges and the series in (12) is uniformly convergent.

To express the formula for A_i $(i=1, 2, \cdots)$ in terms of y_i, we write

$$\int_0^1 y_0^2 \frac{d}{dx}\left(\frac{f}{y_0}\right) Y_i dx = \left[\frac{f}{y_0}(y_0 y_i' - y_i y_0')\right]_0^1 + (\lambda_i - \lambda_0)\int_0^1 f y_i dx.$$

Eliminating the boundary values of the derivatives with the aid of (3), we find that the second member here is equal to

$$(\lambda_i - \lambda_0) B(f, y_i).$$

Finally, by replacing f by y_i, we obtain at once the corresponding expression for $\int_0^1 y_0^2 Y_i^2 dx$. Therefore

$$A_i = \frac{B(f, y_i)}{B(y_i, y_i)}, \qquad i = 0, 1, 2, \cdots. \tag{14}$$

We have included $i=0$ here since this formula agrees with the one above for A_0.

Our result can be stated as follows:

THEOREM 2. *Let $f(x)$ be any function whose derivative $f'(x)$ exists throughout the interval* $(0, 1)$ *and for which $f'(x)$ is bounded and L-integrable in that interval. Then $f(x)$ admits the representation*

$$f(x) = \sum_{0}^{\infty} A_i y_i(x), \qquad 0 \leqq x \leqq 1,$$

where $y_i(x)$ are the characteristic functions of the differential system (2), *and the coefficients A_i are defined by* (14). *Moreover the convergence of this series to $f(x)$ is uniform in the interval* $0 \leqq x \leqq 1$.

THE UNIVERSITY OF MICHIGAN

ON CONVEX SETS IN LINEAR NORMED SPACES

TRUMAN BOTTS

M. Eidelheit has proved[1] this theorem.

THEOREM. *In a linear normed space two convex bodies (that is, convex sets with inner points) having no common inner points are separated*[2] *by a plane.*

The purpose of this note is to present a quite different and somewhat simpler proof of this result.[3]

It is known[4] for linear normed spaces that

(1) *Through every boundary point of a convex body there passes a plane supporting the body.*

A convex cone with the point x_0 as vertex is defined as a convex body C containing at least one point $x \neq x_0$ and such that for each such point x in C,

$$ax + (1 - a)x_0 \in C, \qquad a \geqq 0.$$

It is easily seen that

(2) *Every supporting plane of a convex cone C passes through the vertex x_0 of the cone.*

For, let $L(x) - b = 0$, where $L(x)$ is a linear functional and b is a constant, define a plane of support of C passing through a boundary point y of C. Suppose for definiteness that

$$L(x) - b \leqq 0$$

holds for all points x in C. Then since every point of the form $ay + (1 - a)x_0$ $(a \geqq 0)$ is a boundary point of C,

$$L(ay + (1 - a)x_0) - b \leqq 0, \qquad a \geqq 0,$$

Presented to the Society, September 5, 1941; received by the editors May 6, 1941.

[1] M. Eidelheit, *Zur Theorie der konvexen Mengen in linearen normierten Räumen*, Studia Mathematica, vol. 6 (1936), pp. 104–111.

[2] Two sets are separated by a plane provided they lie in opposite closed half-spaces of the plane.

[3] Added in proof: There has recently been brought to my attention another proof of Eidelheit's theorem by S. Kakutani, Proceedings of the Imperial Academy of Japan, vol. 13 (1937), pp. 93–94. The first part of the present proof is closely related to the first part of Kakutani's proof.

[4] See S. Mazur, *Über konvexen Mengen in linearen normierten Räumen*, Studia Mathematica, vol. 4 (1933), p. 74.

whence by the linearity of L

$$a(L(y) - b) + (1 - a)(L(x_0) - b) \leqq 0, \qquad a \geqq 0.$$

But $L(y) - b = 0$. Hence

$$(1 - a)(L(x_0) - b) \leqq 0, \qquad a \geqq 0,$$

and since the factor $(1-a)$ can change sign, we must have

$$L(x_0) - b = 0.$$

Let K_1 and K_2 be convex bodies with no common inner points. Let x_1 and x_2 be inner points of K_1 and K_2, respectively. There are unique boundary points x_1' and x_2' of K_1 and K_2 on the segment $\overline{x_1x_2}$. Consider the (perhaps degenerate) segment

$$L = \overline{x_1' x_2'}.$$

Let L_1 be the set of all points $x \in L$ such that no segment joining x to an inner point of K_1 contains an inner point of K_2. The set L_1 is non-empty, since $x_1' \in L_1$. Furthermore, since the complement of L_1 in L is clearly open in L, L_1 is closed in L. Likewise, if L_2 is analogously defined, L_2 is a non-empty set closed in L.

Suppose there exists a point $x \in L - (L_1 + L_2)$. Then it is possible to join x to inner points y_1 and y_2 of K_1 and K_2 such that there exist inner points z_1 and z_2 of K_1 and K_2 lying on the (open) segments

$$\overline{xy_2}, \qquad \overline{xy_1},$$

respectively. But then the segments

$$\overline{y_1z_1}, \qquad \overline{y_2z_2}$$

intersect in a point z which is interior to both K_1 and K_2, contradicting the hypothesis. Hence the supposition that $L - (L_1 + L_2)$ is non-empty is false, and $L = L_1 + L_2$.

It now follows from the connectedness of L that there exists a point $x_0 \in L_1 \cdot L_2$. Let C_1 and C_2 be the sets consisting of all points on rays from x_0 through the inner points of K_1 and K_2, respectively.

(3) *The sets C_1 and C_2 are convex cones having no common points except x_0.*

The fact that C_1 and C_2 are convex cones is evident. If C_1 and C_2 had a common point $y \neq x_0$, then on the ray from x_0 through y there would have to be a point interior to K_1 and a point interior to K_2, contradicting $x_0 \in L_1 \cdot L_2$.

Take the point x_0 to be the origin θ. Let C_1^- denote the reflection in the point θ of the cone C_1: that is, C_1^- is the set of all points of the form $-x$, $x \in C_1$. The set C_1^- is a convex cone with θ as vertex. Let C denote the set of all points of the form

$$ax + (1 - a)y,$$

where $x \in C_2$, $y \in C_1^-$, and $0 \leqq a \leqq 1$.

(4) *The set C is a convex cone with θ as vertex. Furthermore, C contains no point interior to C_1.*

The first statement in (4) is easily verified. That the second holds is seen as follows. Any point of C is of the form

$$z = ax + (1 - a)y, \qquad x \in C_2,\ y \in C_1^-,\ a \in [0, 1].$$

Let x_1 be an inner point of C_1. From (3) it follows that on the segment $\overline{xx_1}$ there is a boundary point x_1' of C_1. Now by (1) C_1 has a supporting plane H_1 passing through x_1'. Since the point x_1' of H_1 is on the segment $\overline{x_1x}$, the points x_1 and x lie in opposite closed half-spaces of H_1. Since x_1 is interior to C_1, x_1 lies in the half-space of H_1 containing C_1. Hence the set C_1 and the point x lie in opposite closed half-spaces of H_1; that is, C_1 and x are separated by H_1. By (2) $\theta \in H_1$, so that C_1 and y are separated by H_1. Hence H_1 separates the sets C_1 and $\overline{xy}$, and the point z, which is contained in $\overline{xy}$, cannot be interior to C_1.

From (4) and (1) it now follows that C has a plane of support H passing through the point θ. But H is then a plane of support of C_2. Likewise it is a plane of support of C_1^- and hence of C_1. Since C_1^- and C_2 are on the same side of H, C_1 and C_2 are separated by H. Therefore K_1 and K_2 are separated by H.

UNIVERSITY OF VIRGINIA

FORCED OSCILLATIONS OF CONTINUOUS DYNAMICAL SYSTEMS

W. H. INGRAM

Introduction. The system of differential equations and boundary conditions

$$\text{(a)} \qquad \frac{\partial u_i}{\partial x} = \Sigma a_{ij}(x)\frac{\partial u_j}{\partial t} + \Sigma b_{ij}(x)u_j, \quad \Sigma\{\alpha_{ij}u_j(a) + \beta_{ij}u_j(b)\} = 0$$

($i, j = 1, 2, \cdots, n$, $a \leqq x \leqq b$) has application in the theory of the electrical transmission line, the diffusion of heat along thin rods and around thin rings and, when some of the u's are employed to designate rates of change of other u's, to vibrating strings, bars, air columns and other dynamical systems. The system of total differential equations

$$\text{(b)} \qquad Y'(x) = (\mu\mathcal{A} + \mathcal{B})Y, \qquad \mathcal{W}_a Y(a) + \mathcal{W}_b Y(b) = 0,$$

where $\mathcal{A} = (a_{ij})$, $\mathcal{B} = (b_{ij})$, $\mathcal{W}_a = (\alpha_{ij})$, $\mathcal{W}_b = (\beta_{ij})$, and where Y is a columnar matrix of n elements each a function of x, may be obtained as the result of the Bernoulli-Taylor substitution $u_i(x, t) = e^{\mu t}y_i(x)$ into (a).

The system (b) has been the starting point for many researches centered around the problem of expressing an arbitrary function f or, more generally, a set of functions $\{f_i\}$, in terms of its characteristic solutions. A solution of this problem in the simple case, having application to the uniform dissipationless vibrating string, was first obtained by Daniel Bernoulli about the year 1732 and a solution having application to the nonuniform string was first obtained by Liouville[1] one hundred years later. A purportedly more rigorous treatment of Liouville's problem was given by Kneser[2] in 1904. Since that date a great many papers have appeared, having to do with the system (b) under one restriction or another, the most comprehensive of which are the papers by Bliss,[3] who obtained uniform convergence in his expansion theorem by requiring (b) to be "definitely" self-adjoint and by imposing a restriction on the functions f_i, and Birkhoff and Langer[4] who considered the general case.

Received by the editors May 7, 1941.

[1] Liouville, Journal de Mathématiques Pures et Appliquées, vol. 1 (1836), pp. 253, 269.

[2] Kneser, Mathematische Annalen, vol. 58 (1904), p. 108.

[3] Bliss, Transactions of this Society, vol. 28 (1926), p. 576.

[4] Birkhoff and Langer, Proceedings of the American Academy of Arts and Sciences, vol. 58 (1923), p. 100.

In their paper, Birkhoff and Langer show how the biorthogonal properties of the characteristic solutions of the system (b) and of its adjoint can be used to obtain the coefficients c_k in the representation $F(x) = \Sigma c_k Y_k(x)$, where the columnar matrices Y_k are the characteristic solutions. This representation, with the scalar c_k's computed as stated, is one which well might be expected to be as satisfactory from the viewpoint of convergence as it appears to be from the viewpoint of computation, but it is admittedly formal in general. Whether there are any problems of practical importance that can be treated successfully by Birkhoff and Langer's theory and that, at the same time, do not come within the scope of the paper by Bliss is not known. It does not seem possible to find a proof that the coefficients computed with the use of the adjoint characteristic functions give an expansion that approximates the function in any satisfactory sense in the general case. Use of the characteristic solutions themselves, or of their real parts, gives an expansion which represents the function in the sense of least squares in certain cases but, since the characteristic solutions of the system (b) do not in general form an orthogonal set, the computation of the coefficients requires here the solution of a presumably infinite system of linear equations in the coefficients. The *Bernoulli-Fourier problem*, that is, the problem of system response to an arbitrary external stress impressed upon a continuous dissipative dynamical system, essentially an expansion problem, thus has had no solution in the general case up to the present. For a *nearly* arbitrary stress, that is, for a stress for which the Burkhardt transform can be represented by a linear combination of characteristic vector solutions of (b) assumed to have a Green's matrix, a solution is given in the following pages, not only to the Bernoulli-Fourier problem for a system (a) having a Green's matrix but also to the *Heaviside problem*, that is, the problem of the behavior of a dynamical system subsequent to the sudden imposition of the external forces. The two problems are treated in §§1 and 2, respectively.

Notation. A column matrix, or vector, of elements $u_{1\cdot}, u_{2\cdot}, \cdots, u_{n\cdot}$ is designated $u_{\cdot}:$. A line matrix of elements $v_{\cdot 1}, v_{\cdot 2}, \cdots, v_{\cdot n}$ is designated $\cdot v_{\cdot}$. By $\mathcal{W}$, $[w]$, or $(w_{\cdot\cdot})$ will be understood a square matrix with element w_{ij} in the ith row and jth column. In vector equations, a column of zeros forming the columnar matrix $0:$ is written simply 0.

The complex number $\mu = \alpha + \iota\omega$ is called a *modal number*. The μ's usually are called characteristic numbers but Hilbert prefers this name for their reciprocals. Since the μ's enumerate the modes of motion of a dynamical system, the name modal number seems preferable to "proper number," "Eigenwert" or "characteristic number."

1. Sustained external forces. A matrix of elements $G_{ij}(x, \xi)$ which are continuous in x except possibly at $x=\xi$ and which have continuous first derivatives in x except at a finite number of points in the square $a \leqq x \leqq b$, $a \leqq \xi \leqq b$ is called a Green's matrix if it satisfies the steady-state equations for the system (a) and if the elements have the property: $G_{jk}(\xi+0, \xi)-G_{jk}(\xi-0, \xi)=\delta_{jk}$. The existence and uniqueness of the Green's matrix for the system (b) in which $\mathcal{A}(x)$ and $\mathcal{B}(x)$ are matrices of piecewise continuous elements can be established by a slight and obvious generalization of the proof given by Birkhoff and Langer[8] where the elements of $\mathcal{A}$ and $\mathcal{B}$ are assumed to be continuous. A necessary and sufficient condition for the existence of a Green's matrix is that the system (b) with $\mu=0$ have no solution other than $Y \equiv 0$.

The derivative of the Burkhardt transform of an integrable vector requires a more detailed discussion. After both sides of the equation

$$\int_a^b \mathcal{G}(x, \xi) U(\xi){:}\, d\xi = \int_a^{x-0} \mathcal{G}(x, \xi) U(\xi){:}\, d\xi - \int^{x+0} \mathcal{G}(x, \xi) U(\xi){:}\, d\xi$$

in which $U{:}$ is any columnar matrix of integrable elements continuous at x have been differentiated, there results

$$\frac{d}{dx}\int_a^b \mathcal{G}(x, \xi) U(\xi){:}\, d\xi = \int_a^b \frac{d\mathcal{G}(x, \xi)}{dx} U(\xi){:}\, d\xi + \mathcal{G}(x, x-0) U(x-0){:}$$
$$- \mathcal{G}(x, x+0) U(x+0){:},$$

an equation in which the last two terms, as a consequence of the continuity of the G's, may be replaced by

$$\mathcal{G}(x+0, x) U(x){:} - \mathcal{G}(x-0, x) U(x){:} = \mathfrak{I} U(x){:}$$

to give the identity

$$\frac{d}{dx}\int_a^b \mathcal{G}(x, \xi) U(\xi){:}\, d\xi = \int_a^b \frac{d}{dx}\mathcal{G}(x, \xi) U(\xi){:}\, d\xi + U(x){:} \tag{1.1}$$

which will be referred to as Lemma I. It may be remarked parenthetically that the more general result

$$\frac{d^{p+1}}{dx^{p+1}}\int_a^b \mathcal{G}(x, \xi) U(\xi){:}\, d\xi = \int_a^b \frac{d^{p+1}}{dx^{p+1}}\mathcal{G}(x, \xi) U(\xi){:}\, d\xi + U(x){:}$$

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[8] Ibid., pp. 66–70. Every system (b) has an adjoint in the sense of Bliss but not in the sense of Birkhoff and Langer. In particular, the electrical transmission line grounded at both ends has no adjoint in the latter sense.

holds when

$$G_{ij}(x, \xi),\ dG_{ij}(x, \xi)/dx,\ \cdots,\ d^{\nu-1}G_{ij}(x, \xi)/dx^{\nu-1}$$

$(i, j=1, 2, \cdots, n)$ all are continuous in x and when

$$\left.\frac{d^{\nu}G_{ij}(x, \xi)}{dx^{\nu}}\right|_{\xi-0}^{\xi+0} = \delta_{ij}$$

and this generalization has application in those cases (for example, the vibrating string) where fewer equations of individually higher order are taken instead of (a) and for which U: satisfies the equation

$$U(x, t): = \int_a^b \mathcal{G}(x, \xi)\{\mathcal{A}_1(\xi)\dot{U}(\xi, t): + \mathcal{A}_2(\xi)\ddot{U}(\xi, t): + \cdots\}d\xi.$$

For the existence of modal functions in such cases it is necessary, as can be shown, to assume certain relationships between the $\mathcal{A}$'s with the result that the analysis is not more but less general than that of the equation (1.2) below. In any case, one arrives at the equation (1.8) after the usual Bernoulli-Taylor substitution.

When (a) has a Green's matrix, any solution of the equation

$$(1.2)\qquad U(x, t): = \int_a^b \mathcal{G}(x, \xi)\left\{\mathcal{A}(\xi)\,\frac{\partial U(\xi, t):}{\partial t} + E(x, t):\right\}d\xi$$

is a solution of the nonhomogeneous extension of (a)

$$(\mathrm{c})\qquad \begin{gathered} \frac{\partial u_i}{\partial x} = \Sigma a_{ij}(x)\,\frac{\partial u_j}{\partial t} + \Sigma b_{ij}(x)u_j + e_i(x, t), \\ \Sigma\{\alpha_{ij}u_j(a) + \beta_{ij}u_j(b)\} = 0, \end{gathered}$$

as may be verified from the properties[4] of the Green's functions and by virtue of Lemma I. Hence when the arbitrary external force and corresponding response are assumed to have the forms

$$(1.3)\quad E(x, t): = \begin{Bmatrix} e_1(x)\cos\omega_0 t \\ e_2(x)\cos\omega_0 t \\ \vdots \\ e_n(x)\cos\omega_0 t \end{Bmatrix},\quad U(x, t): = \begin{Bmatrix} u_1(x)\cos(\omega_0 t - \beta_1) \\ u_2(x)\cos(\omega_0 t - \beta_2) \\ \vdots \\ u_n(x)\cos(\omega_0 t - \beta_n) \end{Bmatrix},$$

appropriate to a sustained periodic disturbance in a dissipative system governed by (c), substitution into the equivalent equation (1.2) yields the system of integral equations

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[4] Bliss, loc. cit., p. 571.

$$\left.\begin{aligned} W(x):\, &= \omega_0 \int_a^b \mathcal{G}(x,\xi)\mathcal{A}(\xi)X(\xi):d\xi + F(x):, \\ X(x):\, &= -\omega_0 \int_a^b \mathcal{G}(x,\xi)\mathcal{A}(\xi)W(\xi):d\xi, \end{aligned}\right\} \tag{1.4}$$

obtained on letting t take on the values 0 and $\pi/2\omega_0$, respectively, and where, as matters of notation

$$W(x):\, = \left\{\begin{matrix} u_1(x)\cos\beta_1 \\ u_2(x)\cos\beta_2 \\ \cdot \\ \cdot \\ u_n(x)\cos\beta_n \end{matrix}\right\}, \qquad X(x):\, = \left\{\begin{matrix} u_1(x)\sin\beta_1 \\ u_2(x)\sin\beta_2 \\ \cdot \\ \cdot \\ u_n(x)\sin\beta_n \end{matrix}\right\}, \tag{1.5}$$

$$F(x):\, = \int_a^b \mathcal{G}(x,\xi)E(\xi,0):d\xi, \tag{1.6}$$

that is, $F(x):$ is the Burkhardt transform of $E(x, 0):$.

The substitution $u_i(x, t) = \epsilon^{\alpha t}\cos(\omega t - \beta_i)y_i(x)$, appropriate to a free oscillation, into (1.2) with $E:\equiv 0$ gives the homogeneous system

$$\left.\begin{aligned} W(x):\, &= \alpha \int_a^b \mathcal{K}(x,\xi)W(\xi):d\xi + \omega \int_a^b \mathcal{K}(x,\xi)X(\xi):d\xi, \\ X(x):\, &= -\omega \int_a^b \mathcal{K}(x,\xi)W(\xi):d\xi + \alpha \int_a^b \mathcal{K}(x,\xi)X(\xi):d\xi, \end{aligned}\right\} \tag{1.7}$$

where $\mathcal{K}(x, \xi) = \mathcal{G}(x, \xi)\mathcal{A}(\xi)$.

The system (1.4) may be written as a single vector equation

$$U(x):\, = \mu \int_a^b \mathcal{K}(x,\xi)U(\xi):d\xi + F(x):, \tag{1.8}$$

where $U(x): = W(x): - \iota X(x):$ and $\mu = \iota\omega_0$ and the system (1.7) may be written in the same way if $\mu = \alpha + \iota\omega$, $F(x):\equiv 0$.

If $\phi_r(x):$ is a solution of the homogeneous specialization of (1.8), that is, if $\phi_r(x):$ satisfies the equation

$$\phi_r(x):\, = \mu_r \int_a^b \mathcal{K}(x,\xi)\phi_r(\xi):d\xi \tag{1.9}$$

with $\mu_r = \alpha_r + \iota\omega_r$ for one or more distinct modal numbers μ_r $(r = 1, 2, \cdots)$ and if $F(x):$ can be expressed in the form

$$F(x):\, = \sum_{r=1}^{l} c_r\phi_r(x):, \tag{1.10}$$

where c_r is a scalar multiplier, then a particular solution of equation (1.8) is given by the expression

$$U(x){:} = \sum_{r=1}^{l} \frac{c_r\mu_r}{\mu_r - \mu}\,\phi_r(x){:}, \tag{1.11}$$

where l may be any natural number no matter how great.

When $n=1$ and the kernel is symmetric, the infinite sums corresponding to (1.10) and (1.11) are known to be convergent in view of results obtained by Schmidt,[7] and equation (1.11) becomes equivalent to the particular solution of the integral equation of the second kind with symmetric kernel given by him. When the kernel is not symmetric the convergence of the infinite sums in question to the function $F(x){:}$ has not been established in any example and, as has been shown by Kowalewski,[8] the equation (1.9) may have no nonzero solution at all. It can be shown that when the solution $\cdot\cdot\psi_s$ of the equation following has the property expressed by the equation $\psi_s{:} = \bar{\phi}_s{:}$, the infinite series $\Sigma c_r\phi_r(x){:}$ converges uniformly and absolutely and represents the Burkhardt[9] transform $F(x){:}$ in the sense of least squares, but this relationship between the ψ's and ϕ's, although holding in trivial cases, does not hold in general.

If solutions $\cdot\cdot\psi_1, \cdot\cdot\psi_2, \cdots, \cdot\cdot\psi_l$ of the equation

$$\cdot\cdot\psi_s(x) = \mu_s \int_a^b \cdot\cdot\psi_s(\xi)K(\xi, x)\,d\xi \tag{1.12}$$

exist corresponding to the modal numbers $\mu_1, \mu_2, \cdots, \mu_l$ of equation (1.9), then we have the following theorem:

THEOREM I. *If the series* $\sum_{r=1}^{l} c_r\phi_r(x){:}$ *is designated by* $F(x){:}$, *then*

$$c_r = \int_a^b \cdot\cdot\psi_r(x)F(x){:}\,dx \Big/ \int_a^b \cdot\cdot\psi_r(x)\phi_r(x){:}\,dx.$$

This follows immediately from the easily proved biorthogonality of the sets $\{\cdot\cdot\psi_r\}$, $\{\phi_r{:}\}$. Thus if equation (1.9) with subscript r is premultiplied on both sides by $\cdot\cdot\psi_s(x)$ and equation (1.12) with subscript s is postmultiplied on both sides by $\phi_r(x){:}$ there result, after integration,

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[7] Erhard Schmidt, Mathematische Annalen, vol. 63 (1907), p. 454.

[8] Cf. Love, *Integral Equations*, p. 117.

[9] H. Burkhardt, *Sur les fonctions de Green relatives à un domaine d'une dimension*, Bulletin de la Société Mathématique de France, vol. 22 (1894), p. 71. The name "Burkhardt transform" for an expression of the form $\int_a^b G(x, \xi)e(\xi)d\xi$ is due to H. Bateman.

$$\begin{aligned}\int_a^b \cdot\cdot\psi_s(x)\phi_r(x):dx &= \mu_r \int_a^b \int_a^b \cdot\cdot\psi_s(x)\mathcal{K}(x, \xi)\phi_r(\xi):d\xi dx,\\ \int_a^b \cdot\cdot\psi_s(x)\phi_r(x):dx &= \mu_s \int_a^b \int_a^b \cdot\cdot\psi_s(\xi)\mathcal{K}(\xi, x)\phi_r(x):d\xi dx,\end{aligned} \tag{1.13}$$

two equations in which the double integrals obviously must be zero when $\mu_s \neq \mu_r$ since they are of equal value.

As a corollary to Theorem I we have this statement: *If* $F(x)$: *is an arbitrary vector and a set of constants* $c_1, c_2, \cdots, c_l$ *are computed with the formula of Theorem* I, *then the series sums to the function or else the function is equal to no linear combination whatever of the* $l\phi$'s.

This corollary says nothing about the possibility of expressing an arbitrary function, or even the Burkhardt transform of an arbitrary function, as a uniformly convergent series in terms of the modal functions, but it is nevertheless of practical importance and is quite adequate to our theoretical needs in those more usual cases where only a limited number of modal functions are known. Suppose $F(x)$: has been computed from (1.6) for a given force function $E(x, 0)$: and that a set of constants $c_1, c_2, \cdots, c_l$ has been computed by the formula of Theorem I. Then the function $D(x)$: given by the equation

$$D(x): = \Sigma c_r \mu_r \mathcal{A}(x)\phi_r(x): \tag{1.14}$$

is to be compared with $E(x, 0)$: and if sufficiently close to $E(x, 0)$: from a physical standpoint to permit such a restatement of the problem then the function $D(x)$: is taken as the initially given force function and a particular solution of the equations of motion is given by (1.11) with the c's as in (1.14). If $D(x)$: is deemed an inadequate representation of $E(x, 0)$: and the situation is not improved by enlargement of the sets $\phi_1, \phi_2, \cdots, \phi_l, \psi_1, \psi_2, \cdots, \psi_l$ then there is no solution possible of the kind considered.

2. The Heaviside problem. It was pointed out by Bateman[10] in 1910 that the Laplace transform of the function u satisfying a partial differential equation of the form

$$\frac{\partial u}{\partial t} = f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)u$$

and linear boundary conditions independent of the time will satisfy the same boundary conditions and in general a simpler differential equation provided the value of u at time $t=0$ be known, and application of the theory was made in a heat conduction problem and in a physical problem involving a system of linear differential equations. Since then the Laplace transformation has been resorted to in treatments by many authors[11] of similar differential equations governing vibration and diffusion problems in which external forces are suddenly imposed at time $t=0$ at one or more regions thereof. In the present section a solution of the problem of the behavior of a dynamical system having a Green's matrix following the sudden application of an external force is obtained as an inversion of a series solution of an integral equation satisfied by the Laplace transform.

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[10] H. Bateman, Proceedings of the Cambridge Philosophical Society, vol. 15 (1910), p. 423.

When the substitution $u_i(x, t)=\epsilon^{at} \cos (\omega t-\beta_i)y_i(x)$ is appropriate to the system (a), $u_i(x, t)$ has the Laplace transform $u_i^*(x, \mu)$ and (c) may be written

$$
\begin{aligned}
\frac{du_i^*(x, \mu)}{dx} &= \Sigma\{\mu a_{ij}(x) + b_{ij}(x)\}u_j^*(x, \mu) \\
&\qquad + \Sigma a_{ij}(x)u_j(x, 0) + e_i^*(x, \mu),
\end{aligned} \tag{2.1}
$$

$$
\Sigma\{\alpha_{ij}u_j^*(a, \mu) + \beta_{ij}u_j^*(b, \mu)\} = 0,
$$

where, as matters of notation,

$$
u_i^*(x, \mu) = \int_0^\infty \epsilon^{-\mu t}u_i(x, t)dt, \quad e_i^*(x, \mu) = \int_0^\infty \epsilon^{-\mu t}e_i(x, t)dt. \tag{2.2}
$$

Assuming, as in Heaviside,

$$
u_i(x, t) = 0, \qquad -\infty < t < 0, \tag{2.3}
$$

$$
\frac{de_i(x, t)}{dt} = 0, \qquad t > 0, \tag{2.4}
$$

equations (2.1) may be written as the vector differential system

$$
\frac{dU_0^*:}{dx} = \{\mu\mathcal{A}(x) + \mathcal{B}(x)\}U_0^* + \mu^{-1}E_0(x):, \tag{2.5}
$$

$$
[\alpha]U_0^*(a, \mu): + [\beta]U_0^*(b, \mu): = 0,
$$

where $E_0(x)$: stands for $E(x, 0+)$: with matric elements $e_i(x, 0+)$ $(i=1, 2, \cdots, n)$.

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[11] The bibliography in Doetsch's book, *Theorie und Anwendung der Laplace-Transformation*, omits reference to papers and books by van der Pol, Philosophical Magazine, vol. 8 (1929), p. 861; Dalzell, Proceedings of the Physical Society, vol. 42 (1930); Humbert, *Le Calcul Symbolique*, Paris, 1929; Dahr, *Operational Calculus*, Stockholm, 1935, and the paper by Bateman.

Now $\mathcal{G}(x, \xi)$ satisfies, identically in ξ, the system

$$(2.6)\qquad \frac{d\mathcal{G}(x, \xi)}{dx} = \mathcal{B}(x)\,\mathcal{G}(x, \xi), \quad [\alpha]\,\mathcal{G}(a, \xi) + [\beta]\,\mathcal{G}(b, \xi) = 0$$

as before and $\int_a^b \mathcal{G}(x, \xi)\{\mu\mathcal{A}(\xi)U^*(\xi): + \mu^{-1}E_0(\xi):\}d\xi$ satisfies (2.5) because of the property (1.1); hence equivalent to (2.5) there is the equation

$$(2.7)\qquad U_0^*(x): = \mu^{-1}F_0(x): + \mu\int_a^b \mathcal{G}(x, \xi)\mathcal{A}(\xi)U_0^*(\xi):d\xi,$$

where

$$(2.8)\qquad F_0(x): = \int_a^b \mathcal{G}(x, \xi)E(\xi, 0+):d\xi.$$

Any solution of (2.7) satisfies (2.5) as may be directly verified.

When $F_\tau(x)$: can be represented by the sum $\sum_{r=1}^{l} c_r(\tau)\phi_r(x):$, we have

$$(2.9)\qquad U_0^*(x, \mu): = \sum \frac{\mu_r c_r(0)}{(\mu_r - \mu)\mu}\,\phi_r(x):$$

a particular solution of equation (2.7), by the result (1.11), and

$$(2.10)\qquad U_0(x, t): = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\mu t}U_0^*(x, \mu):d\mu$$

by the Riemann-Mellin inversion formula, it being assumed that the μ's have negative real part. Dependence of the initial state of the system on the zero of time is indicated by the subscript τ.

When the constant vector E: initially applied at time $t=0$ is abruptly increased at time $t=\tau$ by the increment $\Delta E_\tau(x)$:, the total response at any time $t>\tau$ is given, in accordance with (2.10), by the equation

$$(2.11)\qquad U(x, t): = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \{e^{\mu t}U_0^*(x, \mu): + e^{\mu(t-\tau)}\Delta U_\tau^*(x, \mu):\}d\mu$$

in which

$$(2.12)\qquad \Delta U_\tau^*(x, \mu) = \sum \frac{\mu_r \Delta c_r(\tau)}{(\mu_r - \mu)\mu}\,\phi_r(x):.$$

It follows, as in the usual applications of Duhamel's theorem, that the response to a variable vector force $E(x, t)$:, applied abruptly at time $t=0$ but varying in a continuous manner thereafter, is given by the equation

$$
(2.13)\qquad \begin{aligned} U(x,\, t):\, = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\Big\{ & e^{\mu t}U_0^*(x,\, \mu): \\ & + \int_0^t e^{\mu(t-\tau)}\,\frac{\partial U_\tau^*(x,\, \mu):}{\partial\tau}\,d\tau\Big\}\,d\mu \end{aligned}
$$

in which

$$
(2.14)\qquad U_\tau^*:\, = \sum\frac{\mu_r c_r(\tau)}{(\mu_r-\mu)\mu}\,\phi_r(x):
$$

as before; τ is the time of application of the external force. This result is rigorous when $E(x,\, t):$ can be expressed in the form $\Sigma c_r(t)\mu_r\mathcal{A}(x)\phi_r(x):$ and when $l<\infty$. From physical considerations it seems probable that it is more generally correct.

Of particular interest is the case where $E(x,\, t):$ is given by

$$
(2.15)\qquad E(x,\, t):\, = e_0\sin\,(\Omega t-\beta)\eta(x):,\ \int_a^b\mathcal{G}(x,\, \xi)\eta(\xi):\,d\xi=\sum c_r\phi_r(x):.
$$

Equation (2.13) here takes the form

$$
(2.16)\qquad U_\tau^*(x,\, \mu):\, = \sum e_0\sin\,(\Omega t-\beta)\frac{\mu_r c_r}{(\mu_r-\mu)\mu}\,\phi_r(x):
$$

so that we have

$$
(2.17)\qquad \begin{aligned} U(x,\, t):\, = & \sum\mu_r c_r\phi_r(x):\frac{e_0\sin\beta}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\frac{e^{\mu t}}{(\mu-\mu_r)\mu}\,d\mu \\ & + \sum\mu_r c_r\phi_r(x):\frac{e_0\Omega}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\frac{e^{\mu t}}{(\mu_r-\mu)\mu}\int_0^t e^{-\mu\tau}\cos\,(\Omega\tau-\beta)\,d\tau\,d\mu \end{aligned}
$$

a solution which reduces, after computation of the integrals, to

$$
(2.18)\qquad \begin{aligned} U(x,\, t):\, = e_0\sum\mu_r c_r\phi_r(x):\, \Big\{ & \frac{\mu_r\sin\beta-\Omega\cos\beta}{\mu_r^2+\Omega^2}\,e^{\mu_r t} \\ & + \frac{\mu_r}{\mu_r^2+\Omega^2}\sin\,(\Omega t-\beta)+\frac{\Omega}{\mu_r^2+\Omega^2}\cos\,(\Omega t-\beta)\Big\} \end{aligned}
$$

a result in which the steady-state and damped oscillations are easily identifiable and in which the characteristic phenomena of resonance are exhibited.

NEW YORK, N. Y.

NOTE ON THE COEFFICIENTS OF OVERCONVERGENT POWER SERIES

J. L. WALSH

M. B. Porter gave the first known example of an overconvergent power series, that is to say, of a power series in the complex variable with finite radius of convergence such that a suitable sequence of partial sums converges uniformly in a region containing in its interior both points inside and points outside the circle of convergence. Bourion has recently published[1] a general exposition of the theory of overconvergence to which the reader is referred for further historical and technical details.

Ostrowski established the surprising result that a power series $\sum_{n=0}^{\infty} a_n z^n$ of which the partial sums $s_{m_k} = \sum_{n=0}^{m_k} a_n z^n$ exhibit overconvergence, can be expressed as the sum of a power series $\sum_0^{\infty} a_n' z^n$ with a larger radius of convergence and a power series of the form

$$(1) \qquad \sum_0^{\infty} a_n'' z^n, \qquad a_n'' = 0, \text{ whenever } m_k < n \leqq n_k$$

where n_k and λ are suitably chosen, with $m_k < \lambda n_k$, $0 < \lambda < 1$. Here we have $a_n = a_n' + a_n''$, $a_n' \cdot a_n'' = 0$; the partial sums $s_{m_k}''(z) = \sum_{n=0}^{m_k} a_n'' z^n$ of (1) also exhibit overconvergence.

It is the object of the present note to employ methods already known in the literature to make Ostrowski's result slightly more precise, especially to indicate that in series (1) the gaps cannot be uniquely defined with abrupt initial and terminal elements impossible of alteration by Ostrowski's process of writing the series as the sum of a series with a larger radius of convergence and a series with larger gaps which exhibits overconvergence. The moduli of the coefficients a_n'' must taper off gradually before the gap (m_k, n_k), and must increase gradually after the end of the gap; this remark is to be understood first in the sense that there is an upper limit to the moduli of the coefficients near the ends of a gap, a limit which increases as one moves away from the gap.

Presented to the Society April 27, 1940, under the title *Note on overconvergent power series*; received by the editors January 30, 1941, and, in revised form July 26, 1941.

[1] *L'Ultraconvergence dans les Séries de Taylor*, Actualités Scientifiques et Industrielles, no. 472, Paris, 1937.

THEOREM 1. *Let the series*

$$\sum_{n=0}^{\infty} c_n z^n \tag{2}$$

whose radius of convergence is unity:

$$\limsup_{n\to\infty} |c_n|^{1/n} = 1, \tag{3}$$

have the gaps $(m_1, n_1), (m_2, n_2), \cdots$ *in the sense that* $c_n = 0$ *whenever* $m_k < n \leqq n_k$, *and let the sequence of partial sums* $s_{m_k}(z) \equiv \sum_{n=0}^{m_k} c_n z^n$ *exhibit overconvergence. If* $R_0 > 1$ *is arbitrary, there exists* σ *depending on* R_0 *with* $0 < \sigma < 1$ *such that*

$$\limsup_{\mu_k\to\infty} \left[|c_{\mu_k}|, \ \mu_k \leqq m_k \right]^{1/\mu_k} \leqq R_0^{\sigma(\limsup m_k/\mu_k)-1}; \tag{4}$$

if $r_0 < 1$ *is arbitrary, there exists* τ *depending on* r_0 *with* $\tau > 1$ *such that*

$$\limsup_{\nu_k\to\infty} \left[|c_{\nu_k}|, \ \nu_k > n_k \right]^{1/\nu_k} \leqq r_0^{\tau(\liminf n_k/\nu_k)-1}. \tag{5}$$

The only novelty in Theorem 1 is its emphasis on (4) and (5) for the series (2) which overconverges and which possesses gaps, rather than for a series which overconverges and into which gaps may be introduced by Ostrowski's process; compare Bourion loc. cit., chap. 1, §2.

With the general notation

$$f(z) = \sum_0^{\infty} c_n z^n, \qquad s_m(z) = \sum_0^{m} c_n z^n, \qquad r_m(z) = f(z) - s_m(z),$$

Cauchy's inequality yields

$$[\max |s_n(z)|, \text{ for } |z| = R_0 > 1] \geqq |c_0|, \ |c_1| R_0, \cdots, \ |c_n| R_0^n; \tag{6}$$

$$[\max |r_n(z)|, \text{ for } |z| = r_0 < 1] \geqq |c_{n+1}| r_0^{n+1}, \ |c_{n+2}| r_0^{n+2}, \cdots. \tag{7}$$

Under the hypothesis of Theorem 1 we have for suitable σ and τ (these inequalities follow from the fact of overconvergence by the use of a suitable harmonic majorant)

$$\limsup_{m_k\to\infty} [\max |s_{m_k}(z)|, \text{ for } |z| = R_0]^{1/m_k} = R_0^{\sigma}, \tag{8}$$

$$\limsup_{n_k\to\infty} [\max |r_{n_k}(z)|, \text{ for } |z| = r_0]^{1/n_k} = r_0^{\tau}. \tag{9}$$

By virtue of (6) and (8) we have

$$\limsup_{m_k\to\infty}\left[\,|c_{\mu_k}|\cdot R_0^{\mu_k},\ \mu_k \leqq m_k\right]^{1/m_k} \leqq R_0^{\sigma}, \tag{10}$$

which implies (4); by virtue of (7) and (9) we have

$$\limsup_{n_k\to\infty}\left[\,|c_{\nu_k}|\cdot r_0^{\nu_k},\ \nu_k > n_k\right]^{1/n_k} \leqq r_0^{\tau}, \tag{11}$$

which implies (5). Theorem 1 is established. The first member of (4) is less than unity so long as we have $\limsup m_k/\mu_k < 1/\sigma$, and the first member of (5) is less than unity so long as we have $\liminf n_k/\nu_k > 1/\tau$.

A further description of the tapering-off of the moduli of the coefficients can be elaborated as follows. Under the conditions of Theorem 1, there exists a sequence c_{p_k} with $\lim_{p_k\to\infty} |c_{p_k}|^{1/p_k}=1$; if necessary we change the notation of m_k, n_k, p_k so that we have also $m_1<n_1<p_1<m_2<n_2<p_2<\cdots$. It is now more convenient to employ (10) rather than (4); by setting $\mu_k=p_{k-1}$ we find

$$\liminf m_k/p_{k-1} \geqq 1/\sigma;$$

consequently the numbers m_k-p_{k-1} cannot be small relative to p_{k-1}. In a similar manner we find from (11) with $\nu_k=p_k$

$$\limsup n_k/p_k \leqq 1/\tau;$$

consequently the numbers p_k-n_k cannot be small relative to p_k. It will be noticed that with our present notation the moduli of the coefficients do taper off from the $|c_{p_k}|$, at least immediately before and after the gaps, because the second members of (4) and (5) are less than unity for $\mu_k=m_k$ and for $\nu_k=n_k+1$. But we have not shown, nor is it true, that the moduli of the coefficients necessarily taper off monotonically.

As an application of Theorem 1 we prove (compare Bourion, chap. 2, §4) the following theorem:

THEOREM 2. *Let the series* (2) *have the radius of convergence unity, so that* (3) *is satisfied. Let unity be an isolated limit point of the set* $\{|c_n|^{1/n}\}$. *Let one of the following conditions be satisfied:*

(a) *the series has gaps of relative lengths bounded from zero, in the sense that* $c_n=0$ *whenever* $m_k<n\leqq n_k$, *with* $m_k<\lambda n_k$, $\lambda<1$;

(b) *for some* $R_0>1$ *and* σ, $0<\sigma<1$, *and for some sequence* m_k, *equation* (8) *is valid;*

(c) *for some* $r_0<1$ *and* $\tau>1$ *and for some sequence* n_k, *equation* (9) *is valid.*

Then the unit circle is a natural boundary for the series (2).

If the unit circle is not a natural boundary for the series (2), the function $f(z)$ represented is analytic along some arc A of the unit circle, and Ostrowski has shown that the conditions (a), (b), (c) imply overconvergence of the respective sequences $s_{m_k}(z)$, $s_{m_k}(z)$, $s_{n_k}(z)$ across the arc A.

Let us suppose that no limit point of the sequence $\{|c_n|^{1/n}\}$ other than unity lies in some interval $(1, 1-\eta)$, $\eta>0$; we set

$$\begin{aligned} c_n' &= c_n, && \text{if } |c_n|^{1/n} > 1 - \tfrac{1}{2}\eta, \\ c_n' &= 0, && \text{if } |c_n|^{1/n} \leqq 1 - \tfrac{1}{2}\eta, \\ c_n'' &= c_n - c_n', \end{aligned}$$

$$f_1(z) \equiv \sum_0^\infty c_n' z^n, \qquad f_2(z) \equiv \sum_0^\infty c_n'' z^n, \qquad f(z) \equiv f_1(z) + f_2(z).$$

The series defining $f_2(z)$ has a radius of convergence greater than unity; any overconvergent sequence for $f(z)$ is an overconvergent sequence for $f_1(z)$; it follows from Theorem 1 that $f_1(z)$ has no overconvergent sequence. Theorem 2 is established.

A necessary condition that a series (2) with (3) satisfied exhibit overconvergence is therefore that unity be a non-isolated limit point of the set $\{|c_n|^{1/n}\}$.

It is instructive in considering Theorem 1 to compare such an example as $\sum[z(z+1)]^{2^n}$, suggested by Bourion as a special case of Porter's original formulas; the function represented has the lemniscate $|z(z+1)|=1$ as a natural boundary; the Taylor development about the origin is convergent in the circle $|z|<\frac{1}{2}5^{1/2}-\frac{1}{2}=0.6$; the coefficients exhibit the characteristics described in Theorem 1.

HARVARD UNIVERSITY

CORRECTION TO "TOTALLY GEODESIC EINSTEIN SPACES"[1]

AARON FIALKOW

The coordinate system (p. 427) in which $H=x^n$ for some fixed value of y and $f_{n\lambda}=0$ exists if and only if $f^{ij}H_{,i}H_{,j}\neq 0$ for this value of y. Hence Theorem 3.1 is valid only if this inequality holds. The remaining case, namely,

$$f^{ij}H_{,i}H_{,j}=0 \tag{3.15}$$

for every y can arise only if $c=0$, as may be seen by differentiating (3.15) covariantly with respect to k and using (3.7). We note that, in accordance with (3.8) and (3.9), $c=0$ implies that $a=b=0$. To obtain the analogue of Theorem 3.1 for the case in which (3.15) holds, we proceed in a manner analogous to that in H. W. Brinkmann, loc. cit., pp. 131–135 or A. Fialkow, *Conformal geodesics*, Transactions of this Society, vol. 45 (1939), p. 473. By these methods, we find a coordinate system such that $H=x^n$ for a fixed value of y and

$$f^{ns}=0, \qquad f^{nn}=0, \qquad f^{(n-1)n}=1,$$
$$f_{t(n-1)}=0, \qquad f_{(n-1)(n-1)}=0, \qquad f_{(n-1)n}=1,$$

where $s, t=1, 2, \cdots, n-2$. In this coordinate system, the characteristic condition (3.7) becomes $\partial g_{ij}/\partial x^{n-1}=0$. (In the Transactions paper, this last equation appears incorrectly as $\partial g_{st}/\partial x^{n-1}=0$.)

If the f_{ij} are to be the components of the metric tensor of an Einstein space E_n, then, as was shown by Brinkmann, the first fundamental form of E_n may be written as

$$\begin{gathered} f_{st}=h_{st}(x^s, x^n), \qquad f_{sn}=0, \qquad f_{nn}=0, \\ f_{(n-1)n}=1, \qquad f_{s(n-1)}=0, \qquad f_{(n-1)(n-1)}=0, \end{gathered} \tag{3.16}$$

where $h_{st}dx^s dx^t$ with x^n constant is the first fundamental form of an Einstein space E_{n-2} of zero mean curvature, and the components of the tensor h_{st} satisfy certain partial differential equations. According to Brinkmann, the conditions (3.16) are the necessary and sufficient conditions that E_n be conformal to another Einstein space by means of a transformation $d\bar{s}=\sigma ds$ with $\Delta_1\sigma=f^{ij}\sigma_{,i}\sigma_{,j}=0$. We note that the most general solution for H of the form $H=H(x^n, y)$ is given by (3.13). Now this solution $H(x^n, y)$ must involve x^n by the hypothesis

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Received by the editors April 4, 1941.

[1] This Bulletin, vol. 45 (1939), pp. 423–428.

of §3 and must also depend upon y since $g^{\alpha\beta}H_{,\alpha}H_{,\beta}\neq 0$ and, according to (3.1) and (3.15), $g^{\alpha\beta}H_{,\alpha}H_{,\beta}=g^{(n+1)(n+1)}(\partial H/\partial y)^2$. Hence the first fundamental form of the E_{n+1} which contains the ∞^1 isometric E_n's is

$$ds^2 = f_{ij}dx^i dx^j + e[\alpha(y)x^n + \beta(y)]^2 dx^{n+1^2}$$

where the f_{ij} satisfy (3.16) and $\alpha\neq 0$ and $(d\alpha/dy)^2+(d\beta/dy)^2\neq 0$. These remarks show that Theorem 3.1 must be modified in the case where (3.15) holds so that the phrase "$\Delta_1\sigma\neq 0$" is replaced by the phrase "$\Delta_1\sigma=0$." Both cases may be included in the theorem:

THEOREM. *A one-parameter family of isometric E_n's may be imbedded as ∞^1 nonparallel, totally geodesic hypersurfaces of an E_{n+1} if and only if each E_n may be mapped conformally on another Einstein space. If a and b are the mean curvatures of E_{n+1} and E_n respectively, then $nb=(n-1)a$.*

BROOKLYN COLLEGE

SUMMER MEETING AT CHICAGO
CONFERENCE REPORTS

CONFERENCE ON ALGEBRA AND TWENTY-THIRD COLLOQUIUM

The Conference on Algebra at the 1941 Summer Meeting of the Society in Chicago was held in three sessions. The first dealt with abstract algebra, especially with lattice theory, and the lectures were given by Professors John von Neumann and Garrett Birkhoff. The session was preceded by the first Colloquium Lecture of Professor Oystein Ore on the allied topic of Mathematical Relations and Structures. The second session was concerned with topics in linear algebra and the theory of matrices and the speakers were Professors Nathan Jacobson and N. H. McCoy. General arithmetic notions came to the fore in the third and final session, with talks by Professors J. F. Ritt and Oscar Zariski, and Dr. O. F. G. Schilling. There was considerable discussion, both organized and spontaneous, after all the talks.

1. *Abstract algebra and lattices*

A number of different algebraic systems can be subsumed under the notion of a lattice-ordered group (or an *l-group*). Professor Birkhoff's talk dealt with the structural properties of such groups. By definition, an *l*-group is a group which is also a lattice and which has the "homogeneity" property ($x \leqq y$ implies $a+x \leqq a+y$ and $x+a \leqq y+a$). Examples include the additive groups of ordered fields, partially ordered function spaces (Kantorovitch and others), and the lattice of all ideals in an integral domain (Clifford, Lorenzen, Krull).

After discussing the elementary properties of *l*-groups and the variety of possible postulate systems, Birkhoff turned to their structure theory. The study of homomorphisms of *l*-groups leads naturally to a concept of an *l*-ideal. The lattice of all *l*-ideals is distributive. A central result asserts that every *l*-group with a chain condition is either directly decomposable, or has exactly one maximal proper *l*-ideal.

Many of the results known for abelian *l*-groups hold without serious change for non-abelian groups. However, any *l*-group in which there is a chain condition for the elements is necessarily abelian. There is a number of unsolved questions: Does there exist a non-abelian *l*-group which is complete, in the sense that every bounded set of elements has a least upper bound? Is there a theory of *l*-rings? Are there simple (non-abelian) *l*-groups which are not simply ordered?

von Neumann began by a comparison of classical and quantum mechanics. Any two operators (observables) of classical mechanics can be added *and* multiplied together. But in quantum mechanics, the observables, being hermitian operators, can be added but not multiplied together, unless they are permutable. Hence they form a commutative group only, and not a ring. This group is partially ordered if the positive semi-definite operators are called "positive." Unfortunately, this ordering does *not* have the lattice property, again unlike classical mechanics. By assuming the lattice property of classical mechanics, but not the ring property, we get a physical background for the study of complete lattice-ordered abelian groups which are not rings.

These have very special structure. Consider the complete modular lattice M of all subgroups of such a group G. In M, the l-ideals of G form a distributive sublattice D, closed in M with respect to unrestricted unions and intersections. The closed l-ideals form a complemented sublattice B of D (hence a Boolean algebra), which is closed in D under unrestricted intersections, but only under finite unions.

Furthermore, using B and the representation theory of Boolean algebras, due to Stone, by bicompact totally disconnected topological spaces, one can obtain a representation theory for G by real functions on such spaces. In this representation, addition corresponds to addition; however, a function is to be called positive if the set on which it assumes negative values is nowhere dense. At any point, a function may assume either all integral or all real values.

2. *Linear algebras and matrices*

Many of the properties of matrices ordinarily proved only for matrices with elements in a field can actually be stated and proved, in appropriate form, for matrices whose elements lie in more general rings. McCoy's conference lecture was concerned with results of this character, especially for noncommutative rings. One of the accomplishments of the E. H. Moore General Analysis was a definition of a determinant for a hermitian matrix whose elements are quaternions. The same technique can be applied to define the determinants of suitably restricted matrices over much more general noncommutative rings. However, the proofs of some of the appropriate properties of these determinants require new devices.

The determinant can be used to define a characteristic polynomial of a matrix A and to prove the analogue of the Cayley-Hamilton theorem that every matrix satisfies its characteristic equation. The minimal polynomial of an ordinary matrix is replaced by a suitable minimal *ideal*, composed of all polynomials of which the matrix is a root. In terms of the elements of the adjoint matrix one can determine a condition of the usual form that a polynomial belong to this minimal ideal. The theorems of Phillips and Ostrowski about equations satisfied by several matrices can also be generalized to matrices with elements in a ring.

Jacobson's talk dealt with Lie Algebras. Let A be an associative algebra of characteristic p. It is well known that the elements of A form a Lie algebra if one defines $[ab]=ab-ba$. Jacobson pointed out that in addition, the following identities hold

$$(a+b)^p = a^p + b^p + s(a, b), \tag{I}$$

$$[\cdots[[ab]b]\cdots b] = [ab^p], \tag{II}$$

provided $s(a, b)$ is a suitably defined, but complicated, Lie polynomial in a and b. He defined a *restricted* Lie algebra of characteristic p as a Lie algebra with operation $a\rightarrow a^p$ which satisfies (I)–(II) and $(\alpha a)^p = \alpha^p a^p$.

He mentioned, as other examples of restricted Lie algebras, the skew-symmetric elements of A under any anti-automorphism of A, and the "derivations" of *any* algebra (associative or not). A derivation is a linear operator D which satisfies, for all x, y, $(xy)D=(xD)y+x(yD)$, and corresponds, over the real field, to an infinitesimal automorphism.

Conversely, Jacobson showed that any restricted Lie algebra L could be realized by a Lie subalgebra of a suitably constructed associative algebra U_L. (The corresponding result for ordinary Lie algebras was known.) Sharper results were given. The subalgebra consists precisely of those elements of U_L which are skew under a certain *involution* of L. Moreover, if L has a finite basis, then so does U_L; hence L has only a finite number of inequivalent irreducible representations by matrices. If, in addition, every element of L is nilpotent, then U_L is a nilpotent associative algebra.

Finally, Jacobson showed that every derivation of U_L induced a "restricted" derivation of L, in the sense that $x^pD=[\cdots[xD, x]\cdots x]$, *and conversely*, every restricted derivation of L was induced by some derivation of U_L. These results have impelled him to the conclusion that restricted Lie algebras of characteristic p are closer analogues of Lie algebras of characteristic ∞ than ordinary (unrestricted) Lie algebras of characteristic p.

3. *General arithmetic*

The theory of polynomial ideals is concerned essentially with the manifold of solutions of polynomial equations in many variables, but there has been developed a parallel theory for formal differential equations and the corresponding manifolds of solutions. Ritt's lecture summarized some of the recent results in this theory, emphasizing the respects in which this theory diverges from the polynomial case.

Starting with a form F which is a polynomial in the unknowns $y_1, \cdots, y_n$ and their derivatives, with coefficients in some (differential) field K, there is a corresponding manifold M, which consists of all the zeros $a_1, \cdots, a_n$ of F lying in some extension of the coefficient field K. There is a unique decomposition of such a manifold into "irreducible" manifolds, which corresponds to the decomposition of a "perfect" differential ideal into an intersection of prime ideals. However, there is no exact analogue of the ordinary algebraic theorem which gives a decomposition of arbitrary ideals into primary components. The dimension of an irreducible manifold can be appropriately defined, but the dimension of an intersection does not behave properly. Among the irreducible components of the manifold of a form, one is essential, in the sense that it does not occur in the manifolds of other forms. Ritt quoted theorems giving conditions that a given manifold be essential for a form in one variable. These conditions, originally found by analytic methods, have been derived in part by purely algebraic arguments, in a recent paper by H. Levi. The discussion after the lecture concerned some new basis theorems for systems of forms over fields of prime characteristic.

The famous and difficult problem of the reduction of the singularities of an algebraic manifold by birational transformations was the subject of Zariski's conference lecture. An r-dimensional manifold is given by a field of algebraic functions of r independent variables. Any n elements $x_1, \cdots, x_n$ which generate this field determine a projective model of the manifold in the n-dimensional projective space with the (nonhomogeneous) coordinates $x_1, \cdots, x_n$. The problem consists in finding some model of the manifold in which there will be no singular points.

Zariski has published an algebraic solution of this problem, for two-dimensional manifolds. His talk discussed the generalization of this attack to higher dimensions. Two steps are necessary. The first is that of local uniformization of the "places" of the manifold. A place is essentially a valuation of the associated field of algebraic functions. The uniformization theorem asserts that there is a projective model of the manifold on which the "center" of the valuation is a simple point. This step has been carried out for r-dimensional manifolds.

The second step is that of getting this uniformization to hold simultaneously for all places. Zariski discussed the procedures available here, which consist essentially in a suitable application of locally quadratic transformations. He announced that his methods now were sufficiently powerful to solve the problem of reducing the singularities of a three-dimensional manifold, in the case where the field of coefficients is the complex number field.

Schilling's talk dealt with a generalized Hilbert theory for fields with a valuation. The original Hilbert theory concerns the successive stages in the decomposition of a rational prime p into prime ideals in a normal algebraic number field. Alternatively, one may start, not with the field of rational numbers, but with the associated p-adic number field. In the general case the prime p is replaced by a valuation, and one requires that the field F be relatively complete with respect to this valuation (this assumption is more appropriate than the stronger requirement of topological completeness). Any normal extension of such a field F may be obtained in two uniquely determined steps. The first step is an "inertial" extension, in which the valuation is unramified (that is, the corresponding prime ideal is not decomposed). The second step is a completely ramified extension, in which the residue class field is not extended. There is a corresponding decomposition of the Galois group into two parts. The whole theory can be carried through for infinite extensions, and the resulting Galois groups can be explicitly characterized. In case the residue class field is finite, and the order of the group is a power of a prime, the reverse problem can be solved: one obtains necessary and sufficient conditions that a given group can be realized as the Galois group of a normal extension of the given base field. The whole theory is analogous to a generalized "Kummer theory" for function fields.

4. *The Colloquium Lectures*

The object of these lectures is to make a general study of the properties of mathematical relations as they occur in the various mathematical theories. One may illustrate such relations by various examples, for instance, the relations: one element contains another, a point is middle point between two given ones, a set is the closure of another set. One finds that a more general type of relations may be considered as correspondences $A \rightarrow R_A$ between sets, where the set R_A corresponding to a set A may depend not only upon the elements in A, but also upon a certain grouping and ordering of the elements in A, as, for instance, in the case of noncommutative or non-associative multiplication. If an element a in a set S belongs to the subset R_A it is convenient to write this as a relation aRA. Such relations may be combined in various ways and certain normal forms of relations may be introduced. Particularly interesting is the problem of finding automorphisms or endomorphisms of a relation R, namely those one-to-one or many-to-one correspondences α such that any relation aRA is taken into another valid relation $a^{\alpha}RA^{\alpha}$. The determination of all automorphisms is connected with the problem of finding all correspondences commuting with a given correspondence. It should also be mentioned that these problems lead to a study of the so-called monomial groups and to problems connected with a general Galois theory, extending the ordinary Galois theory of equations.

Of particular importance are the so-called binary relations aRb between two elements a and b. As examples, one may think of a greater than b, a equivalent to b, a orthogonal to b, a divides b, and so on. The theory of such binary relations may be conceived of in many ways. The first systematic theory of such binary relations was given by Peirce and by Schroeder. In their formulation one can say that the theory appears as a theory of matrices over a Boolean ring. Another method of representing the binary relations consists in conceiving the relation as a graph, joining two vertices a and b by an edge from a to b whenever aRb. The symmetric relations are those for which aRb imply bRa. They correspond to the symmetric matrices or to graphs in which the edges are considered undirected.

When the relations are considered to be matrices one can also unite their theory with vector spaces over special rings. For the relations one can define various operations: sum and intersection, multiplication and dual multiplication. Thus it becomes possible to introduce a ring and ideal theory of binary relations and certain parts of the ordinary theory of linear algebras carry over to relations. In other ways the theory of relations differs considerably from ordinary matrix theory. Also here one is interested in the automorphisms and endomorphisms of the relations. It turns out that the automorphisms correspond to the permutation matrices which commute with the relation matrix, and this property may be used for the determination of automorphisms.

Of particular interest in the theory of relations are the transitive relations for which

$$aRb,\ bRc \text{ implies } aRc.$$

These relations may be characterized in the ring of all relations in various ways, for instance, by the property $R^2 \subset R$ or also as certain units or as certain sum relations. They are closely associated with the idempotents of the rings of relations. Any transitive relation can be broken up in an equivalence relation and a partial order relation. For the equivalence relations a number of important algebraic problems can be solved completely, and the interest is therefore centered upon the partially ordered sets. These sets have gained an increasingly important position in mathematical theories, and a number of useful properties of these sets are known. Only certain aspects of these theories are discussed in detail in these lectures.

Among these partially ordered sets the lattices or structures are particularly important. In these systems there exists a unique minimal element $a\cup b$ containing any two elements a and b, and a unique maximal element $a\cap b$ contained in both. These elements $a\cup b$ and $a\cap b$ may also be considered to be the result of two algebraic operations with certain simple properties. Recent investigations have shown that many mathematical theories may be formulated in terms of such structures, and the systematic use of these concepts gives a unification and a simplification of the various theories. As special instances one may mention the set theory of Stone (Harvard), projective geometry by Garrett Birkhoff (Harvard) and von Neumann (Institute for Advanced Study), euclidean and non-euclidean geometry by Menger (Notre Dame), algebraic theories by the lecturer. Among the many other recent contributions to lattice theory, one can mention papers by Dilworth (Yale) and Ward (California Institute of Technology) on multiplication in such systems, by MacLane (Harvard) on certain types of lattices applicable to chains in algebraic systems, by Glivenko on metric lattices, by Jacotin-Dubreils on equivalence relations. In these lectures only certain specific problems, mainly of an algebraic nature, can be discussed with some completeness.

GARRETT BIRKHOFF
SAUNDERS MACLANE
OYSTEIN ORE

CONFERENCE ON THE THEORY OF INTEGRATION

The sessions of this conference were held concurrently with those of the Conference on Algebra. The following paragraphs give a relatively inadequate summary of the lectures, which provided a very illuminating introduction to the researches of the various speakers, especially in their relation to the historical development of the theory of integration.

The first session was devoted to theories of integration in abstract spaces, and both speakers were concerned with extreme generalizations of the notion of the upper and lower integrals of Darboux. Professor Bochner, in his lecture entitled *Integration and partial ordering*, discussed the problem of extending the domain of a positive and finitely additive operator M on V to Y, when V, V' and Y are partially ordered vector spaces with $V \subset V'$, and Y is also assumed to be a complete lattice. For an arbitrary element ϕ of V' the extension $M\phi$ satisfies the inequalities

$$M_*\phi = \sup_{f \leqq \phi} Mf \leqq M\phi \leqq \inf_{f \geqq \phi} Mf = M^*\phi,$$

where f ranges over V. The largest subset of V' on which the extension of M is uniquely determined is the set $\bar{V}$ on which $M^*\phi = M_*\phi$. The space $\bar{V}$ is complete in the Riemann-Darboux sense, that is, if θ in V' and ϕ_n and ψ_n in $\bar{V}$ are such that

$$\phi_n \leqq \phi_{n+1} \leqq \theta \leqq \psi_{n+1} \leqq \psi_n, \quad \sup M\phi_n = \inf M\psi_n,$$

then θ belongs to $\bar{V}$. In certain special cases, when the elements of V' are point functions, and the space $\bar{V}$ contains sufficiently many step-functions for purposes of approximating to $M\phi$ for arbitrary ϕ in $\bar{V}$, the operation $M\phi$ may appropriately be considered as an integral $\int\phi(x)d\mu(x)$. Bochner showed how the theory may be applied to a generalization of the classical moment problem, that is, to secure a representation of a completely monotone function $f(x)$ with values in Y in the form

$$f(x) = \int_0^\infty e^{-xt}d\mu(t).$$

Among other applications mentioned was the spectral resolution of a bounded hermitian operator in Hilbert space. The discussion of Bochner's paper was initiated by Dr. R. S. Phillips.

Professor Price, in his lecture entitled *Integration and convex operators*, began with an historical discussion of the use of the upper and lower sums of Darboux, leading up to the generalization of this notion in the "integral range" of Garrett Birkhoff. Later developments in this direction by Price himself required the use of a generalized convex operator. In the present discussion the speaker showed that the notion of a convex operator is entirely unnecessary. In contrast to the approach of Bochner outlined above, Price assumes a "measure function" T_σ given in advance, in terms of which an integral operator $M(f) = \int f dT$ is constructed. No partial ordering of the domain or the range of M is used. The integral range is constructed as follows. Let V be a vector space, S a class of elements σ, and f a function of σ whose values are subsets of V. Let T_σ, for each σ in S, be an additive transformation of V into V, let Δ denote a finite subset $(\sigma_1, \cdots, \sigma_n)$ of S, and let D be a class of such subsets Δ which constitutes a directed set in the sense that it is partially ordered and has the composition property. The integral range of f associated with Δ is defined by the formula

$$I(f, \Delta) = \sum_{\Delta' \geqq \Delta} \left[T_{\sigma_1'} f(\sigma_1') + \cdots + T_{\sigma_n'} f(\sigma_n') \right],$$

where $\Delta' = (\sigma_1', \cdots, \sigma_n')$, $\sum$ denotes the logical sum, and $+$ denotes the addition of complexes. This integral range obviously has the following properties: (1) $\Delta_2 > \Delta_1$ implies $I(f, \Delta_2) \subseteq I(f, \Delta_1)$; (2) $I(f_1 + f_2, \Delta) \subseteq I(f_1, \Delta) + I(f_2, \Delta)$. No topological properties of the space V enter into the definition of $I(f, \Delta)$. However, the method of defining a class of integrable functions and an integral $\int f dT$ associated with the class of integral ranges $[I(f, \Delta)]$ depends on the topology in V. If V is a Banach space, we may say that f is integrable in case the lower bound of the diameters of its integral ranges is zero, and $\int f dT$ is then the unique element common to the closures of the integral ranges of f. The integral has the usual properties when suitable restrictions on the basis are made. The speaker indicated the solution of this topological problem of integration also in the case when the space V is a space (L) of Fréchet. Professor Nelson Dunford initiated the discussion of this paper.

Professor Hildebrandt began his lecture on *The general integral of E. H. Moore* by pointing out the three-fold character of Moore's investigations into theories of integrals. Moore first interested himself in improper integrals of the Harnack type and then in his two forms of general analysis. The central feature of both forms is the introduction of a completely unrestricted range $\mathfrak{P}$ on which is defined the class $\mathfrak{M}$ of complex- (or real-) valued functions to be discussed. The first form of general analysis, a group of postulational theories, was an attempt to correlate certain theories of linear equations, including the Fredholm theory. In one formulation of this analysis two classes $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of the type mentioned above are considered, and the integral is a bilinear, continuous operator, continuity being defined with respect to a relative uniform convergence topology. In the case $\mathfrak{M}_1 = \mathfrak{M}_2$, the operator J may be taken to be properly positive and hermitian and is thus the forerunner of the basis of a general Hilbert space.

To illustrate how the second form was developed Hildebrandt discussed the Schmidt theory of the system of equations

$$\sum_{i=1}^{\infty} x_i a_{ij} = y_j, \qquad j = 1, 2, \cdots,$$

and then indicated the extension to the case when the variables i and j have general ranges. For functions μ_1, μ_2 of a class of "modular functions" an integral $J\bar{\mu}_1\mu_2$ is defined which is a bilinear, properly positive and hermitian operator. If $J\bar{\mu}_1\mu_2$ is taken to be the inner product $(\bar{\mu}_1, \mu_2)$, the space of modular functions is seen to be a generalized Hilbert space (with no restriction on the dimension number). Among the important features of Moore's general analysis from the point of view of present day mathematics are his use of a completely unrestricted domain $\mathfrak{P}$ for his modular functions, his conception of an integral as a bilinear operator, and his exploitation of the Moore-Smith limit, which has led to the modern use of directed sets. In discussing the paper Professor Barnard, among other things, developed the modular space determined by functions $f(x-y)$ of a type discussed in the previous lecture by Bochner.

Professor Jeffery began his discussion of *Non-absolutely convergent integrals* with a brief historical summary of the early work of Denjoy and Perron, recalling that it was not until 1924 that Alexandroff and Looman showed the equivalence of the Perron integral with the special Denjoy integral. The problem of formulating an integral of Perron type equivalent to the general Denjoy integral was solved by Ridder in 1933. The speaker gave Ridder's definition and an outline of the equivalence proof. He next discussed the approximately continuous integral of Burkill and also his "Cesàro-Perron" integral of the first order, of all positive integral orders, and finally of all real orders, with a brief description of the application of Burkill's work to Fourier series. For a list of Burkill's papers on these subjects the reader is referred to volume 11 of the Journal of the London Mathematical Society.

In conclusion Jeffery discussed an approach to the problem of non-absolutely convergent integrals which was originated by himself and M. S. Macphail. A function $F(e)$ of the measurable sets e on the interval (a, b) is absolutely-additive relative to the closed set E if it is completely-additive over E and if for the intervals (α_i, β_i) complementary to E, $\sum F(\alpha_i, \beta_i) = F[\sum(\alpha_i, \beta_i)]$. $F(e)$ is generalized absolutely-additive on (a, b) if this interval can be covered by a denumerable sequence of closed sets relative to each of which $F(e)$ is absolutely-additive. Let $f(x)$ be measurable and let $s_n(x)$ be a sequence of summable functions tending to $f(x)$ such that $\int_a^x s_n\,dx$ tends to a limiting function $F(x)$. There is thus associated with $F(x)$ a function $F(e) = \lim \int_e s_n dx$, provided this limit exists. If it turns out that $F(e)$ is generalized absolutely-additive on (a, b), then $F(x)$ is the non-absolutely convergent integral of $f(x)$. This integral is equivalent to the general Denjoy integral. The approach lends itself to a simple and brief treatment which quickly brings out all the main facts about non-absolutely convergent integrals. It is also easily adapted to integrals of the Stieltjes type, and to a formulation of a theory of non-absolutely convergent integrals in abstract space. The discussion of Jeffery's lecture was led by Professor McShane.

In introducing his talk on *Measure in statistical mechanics* Professor Wiener considered a measure-conserving mapping T of the interval (0, 1) into itself and remarked that the set Σ of points P for which $a_i \leqq T^iP < b_i$ $(i=0, \pm 1, \pm 2, \cdots, \pm n)$ is measurable. As a consequence, then, of the ergodic theorem, it follows that $\lim_{N\to\infty}$ (Number of values of k in $(1, N)$ for which T^kP is in $\Sigma)/N = m\Sigma$ and for almost all values of P *this will be true for all such sets* Σ *simultaneously*. A sequence $\{x_k\}$ $(k=0, \pm 1, \pm 2, \cdots)$ is called by Wiener a *time series*, and it is termed *regular* if the number of values of k in $(-N, +N)$ for which $a_i \leqq x_{k-i} < b_i$ $(i=0, \pm 1, \pm 2, \cdots, \pm n)$, when divided by $2N$, tends to a limit for every selection of the a_i, b_i. It was indicated above how the time series T^iP $(i=0, \pm 1, \cdots)$ is regular. Next Wiener showed how every regular time series generates a measure of time series together with a transformation under which this measure is invariant, or in other words establishes a situation under which ergodic theory may be applied. It may be shown that almost all time series are regular and generate the same measure among time series. Thus with respect to this measure, translation in time is metrically transitive. It can be shown that any measure-preserving transformation may be broken up into metrically transitive components. One of the greatest needs of ergodic theory is the technique of producing measure-preserving transformations with desired spectrum characteristics. It is generally desired that these transformations be mutually transitive. One of the easiest ways of producing such transformations is the explicit formation of regular time series. Wiener's theory of Brownian motion was in essence nothing but a proof that almost all Brownian motions generate the continuous analogue of a regular time series. Another interesting application of time series is in the theory of almost periodic functions. Wiener next gave a particularly interesting example due to himself and Kurt Mahler to show the existence of time series with non-absolutely continuous spectra. It is known that the metrically transitive transformation generated by this example is not even mixing in the weak sense, that is, the distribution of $x_k, \cdots, x_{k+j}$ is not even on the average independent of that of $x_{k+l}, \cdots, x_{k+l+j}$, where $l > j$. On the other hand, let $\{p_i\}$ be an increasing sequence of primes. Represent a number N in the form

$A_0+\sum_1^N A_k p_1 \cdots p_k$ $(A_k<p_{k+1})$ and let $x_{\pm n}=\prod_0^N \omega_k{}^{A_k}(n>0)$, where ω_k is a primitive p_{k+1}th root of unity. Then the time series x_n will be regular and the translation operator will be metrically transitive with respect to the measure thus generated. It will in addition be weakly mixing but is not strongly mixing. Wiener concluded his address by conjecturing that a slight generalization of such methods may be used to derive strongly mixing transformations which will be such that the spectrum (in the sense of his generalized harmonic analysis) of some sequence $f(T^N P)$ will be non-absolutely continuous. This would indicate that not all mixing transformations are isomorphic. The discussion was initiated by Professor von Neumann.

The talk on *Product integrals* by Professor Garrett Birkhoff began with a brief discussion of the solution, in terms of a product integral, of a system of ordinary linear differential equations, as developed by Volterra and Schlesinger. Here the matrix product replaces the ordinary sum in the definition of the integral. Birkhoff has generalized this notion, defining a product integral for functions whose values are in a complete normed vector ring with a unit. Thus, let $V(t)$ be defined on the interval (a, b), and let Π be a partition $(\Delta_1, \Delta_2, \cdots, \Delta_n)$ of this interval. Then the Riemann product integral is defined by the equation

$$\widehat{\int_a^b} V(t)dt = \lim_{\Pi}\left[V(t_1)\Delta_1 \circ V(t_2)\Delta_2 \circ \cdots \circ V(t_n)\Delta_n\right],$$

where t_k is in Δ_k, provided the limit exists in the Moore-Smith sense. The ring operation $\circ$ must be distinguished from the "scalar multiplication" by Δ_k. It should be noted that when the operation $\circ$ is interpreted as ordinary addition, the process of product integration just defined reduces to integration in the ordinary sense. Birkhoff indicated that while the notion of product integral may clearly be applied to linear operators on any Banach space, it may also be so interpreted as to apply to non-linear operators. In this case two norms are needed in the space of operators. The distributive law which holds in any ring may be replaced by approximate distributivity (that is, differentiability) near the unity element. In the finite-dimensional case, this permits the integrands to have values in any Lie group; the matrix-valued integrands of Volterra and Schlesinger are thus included as a very special case. In the infinite-dimensional case, the values can be in any "analytical group" (see Birkhoff's paper in Journal of Mathematics and Physics, vol. 16 (1937), pp. 104–132). An extension of the notion of product integral indicated above to one of

Lebesgue type was discussed by the speaker. Certain properties have been secured, but many other problems remain to be elucidated. The connection between product integrals and the fundamental group in the simplest case of a surface was exhibited by Professor Rainich, whose remarks were communicated in a letter read to the conference.

The lecture on *Applications of the theory of integration to the study of linear operators* by Professor Dunford stated some of the unsolved problems as well as the known facts about one type of representation of linear maps between Lebesgue spaces, and also gave an outline of the method of Dunford and Pettis for attacking these problems. If $\psi = U\phi$ is a bounded linear map of $L^p(T)$ onto $L^q(S)$, where S and T are finite intervals of the real axis, it has a representation of the form

$$\psi(s) = \frac{d}{ds} \int_T H(s, t)\phi(t)dt. \tag{1}$$

In case $p=1$ and the map U is weakly compact (that is, U takes the unit sphere into a set whose closure is weakly compact), it has the simpler representation

$$\psi(s) = \int_T K(s, t)\phi(t)dt. \tag{2}$$

However, in case $p=q=1$ there are maps of the form (2) which are not weakly compact, and a topological characterization of the integral operator (2) between L spaces is unknown. One of the results stated by Dunford in this connection asserts that (2) (with $p=1\leqq q$) takes weakly compact sets into compact sets. This shows that the product of two weakly compact operators in L is a compact operator and makes it clear why certain aspects of the Fredholm theory hold for linear equations involving a weakly compact operator in L.

A fundamental class of problems concerning the operators (1) and (2) involves the characterizing in terms of the kernels H and K of those operators U which are bounded, weakly compact, compact, lattice convergence-preserving, and so on. These problems have for the most part been solved in case $p=1\leqq q$ even in the case of abstract S and T, but in all other cases very little is known. One method of investigation which has proved fruitful in the case $p=1$ consists of the following three steps: first, represent the operator U as an abstract integral $U\phi=\int_T\phi(t)x(t)dt$; second, find the relations between the set $J=[(1/|e|)\int_e x(t)dt]$ and the set of values $x(T)$ of the kernel; third, express the abstract integral as a numerical integral, that is, $\int_e x(t)dt =\int_e K(s, t)dt$ where $x(t)=K(\cdot, t)$. For bounded operators U on L to

X the first problem is solvable with a Gelfand integral in case X is the adjoint of a separable space and with a Bochner integral (for arbitrary X) in case U is weakly compact. The principal result under the second problem is that for measurable Pettis integrable functions there is a null set $E_0 \subset T$ such that $x(T-E_0)$ and J have the same closed convex hull. The representation desired in the third step always holds for Bochner integrable functions and at least for those Gelfand integrable functions arising in the representation of linear operations on $L(T)$ to $L^p(S)$. Clearly, it is the second step above which allows us to state the topological properties of the operator (at least those which are invariant under the operation of convex closure) in terms of the kernel. A partial list of those papers treating the problems mentioned above is contained in a paper by Dunford and Pettis appearing in the Transactions of this Society, vol. 47 (1940), pp. 323–392. Dunford's paper was discussed by Professor J. W. Calkin.

H. H. GOLDSTINE
L. M. GRAVES

THE ANNUAL MEETING OF THE SOCIETY

The forty-eighth Annual Meeting of the American Mathematical Society was held at Lehigh University, Bethlehem, Pennsylvania, from Monday, December 29 to Wednesday, December 31, 1941, in conjunction with meetings of the Mathematical Association of America, the Association for Symbolic Logic, and the National Council of Teachers of Mathematics.

There were two general and seven sectional sessions of the Society at which three addresses and seventy-nine research papers (forty-four in person) were given. All sessions were held in Packard Laboratory.

Arrangements for the meetings were made by a committee of which Dean Tomlinson Fort was chairman and Professors Everett Pitcher, G. E. Raynor, and W. M. Smith were the local members. Through the generosity of Lehigh University, rooms were available in the University dormitories.

The attendance was about three hundred fifty including the following two hundred fifty-six members of the Society:

C. R. Adams, R. B. Adams, R. P. Agnew, Leon Alaoglu, C. B. Allendoerfer, H. A. Arnold, H. P. Atkins, Frank Ayres, H. M. Bacon, N. H. Ball, I. A. Barnett, F. S. Beale, A. A. Bennett, Stefan Bergman, A. H. Black, W. A. Blankinship, Salomon Bochner, H. F. Bohnenblust, J. W. Bower, C. B. Boyer, J. W. Bradshaw, C. C. Bramble, R. W. Brink, H. W. Brinkmann, F. L. Brooks, R. H. Bruck, N. R. Bryan. S. S. Cairns, W. D. Cairns, P. A. Caris, M. E. Carlen, I. S. Carroll, W. F. Cheney, Claude Chevalley, D. E. Christie, Alonzo Church, Randolph Church, R. F. Clippinger I. S. Cohen, L. W. Cohen, Nancy Cole, J. B. Coleman, Esther Comegys, R. H. Cook, Richard Courant, A. P. Cowgill, H. B. Curry, E. H. Cutler, D. R. Davis, J. E. Davis, L. J. Deck, F. F. Decker, C. H. Denbow, R. P. Dilworth, L. L. Dines, Jesse Douglas, T. L. Downs, Arnold Dresden, J. C. Durand, J. J. Eachus, Samuel Eilenberg, W. E. Ferguson, F. A. Ficken, N. J. Fine, W. W. Flexner, M. M. Flood, L. R. Ford, Tomlinson Fort, R. M. Foster, J. S. Frame, R. E. Gaskell, H. M. Gehman, H. P. Geiringer, F. J. Gerst, B. P. Gill, A. M. Gleason, J. S. Gold, Michael Goldberg, H. H. Goldstine, Cornelius Gouwens, H. S. Grant, C. H. Graves, L. J. Green, Lewis Greenwald, F. L. Griffin, Theodore Hailperin, D. W. Hall, P. R. Halmos, H. H. Hartzler, G. E. Hay, Alan Hazeltine, E. R. Hedrick, A. E. Heins, M. H. Heins, Edward Helly, Erik Hemmingsen, Coleman Herpel, J. G. Herriot, H. C. Hicks, E. H. C. Hildebrandt, D. L. Holl, T. R. Hollcroft, Witold Hurewicz, W. A. Hurwitz, E. D. Jenkins, Fritz John, Evan Johnson, H. A. Jordan, Mark Kac, G. K. Kalisch, William Karush, J. L. Kelley, A. J. Kempner, R. B. Kershner, S. C. Kleene, R. B. Kleinschmidt, J. R. Kline, P. A. Knedler, T. L. Koehler, H. L. Krall, W. C. Krathwohl, H. N. Laden, K. W. Lamson, O. E. Lancaster, R. E. Langer, V. V. Latshaw, Solomon Lefschetz, Joseph Lehner, Norman Levinson, F. W. Light, M. I. Logsdon, E. N. Lorenz, C. I. Lubin, R. R. R. Luckey, J. J. McCarthy, N. H. McCoy, J. C. C. McKinsey, E. J. McShane, C. C. MacDuffee, G. W. Mackey, Saunders MacLane, H. M. MacNeille, H. F. MacNeish, P. T. Maker, V. S. Mallory, W. T. Martin, A. E. Meder, H. L. Meyer, E. R. C.

Miles, F. H. Miller, H. C. Miller, Deane Montgomery, Richard Morris, Marston Morse, E. J. Moulton, F. D. Murnaghan, C. A. Nelson, C. O. Oakley, Oystein Ore, E. G. Olds, F. W. Owens, H. B. Owens, J. C. Oxtoby, S. T. Parker, B. C. Patterson, G. W. Patterson, E. K. Paxton, E. W. Pehrson, J. W. Peters, C. R. Phelps, H. B. Phillips, R. S. Phillips, Everett Pitcher, J. C. Polley, Willy Prager, G. B. Price, A. L. Putnam, H. A. Rademacher, Tibor Radó, G. Y. Rainich, J. F. Randolph, C. H. Rawlins, G. E. Raynor, Maxwell Reade, O. H. Rechard, Haim Reingold, Moses Richardson, R. G. D. Richardson, R. F. Rinehart, E. K. Ritter, M. S. Robertson, R. E. Root, J. T. Rorer, Barkley Rosser, S. G. Roth, Hans Samelson, S. T. Sanders, L. J. Savage, S. A. Schelkunoff, I. J. Schoenberg, Abraham Schwartz, C. H. W. Sedgewick, I. E. Segal, Abraham Seidenberg, R. W. Shephard, Seymour Sherman, C. A. Shook, M. E. Sinclair, C. H. Sisam, L. L. Smail, D. M. Smiley, M. F. Smiley, P. A. Smith, T. L. Smith, W. M. Smith, Virgil Snyder, I. S. Sokolnikoff, R. H. Sorgenfrey, V. E. Spencer, G. W. Starcher, E. P. Starke, C. N. Stokes, R. W. Stokes, R. R. Stoll, A. H. Stone, Alvin Sugar, J. L. Synge, Otto Szász, J. D. Tamarkin, Alfred Tarski, R. M. Thrall, J. I. Tracey, W. R. Transue, A. W. Tucker, J. W. Tukey, J. L. Vanderslice, Oswald Veblen, R. W. Wagner, T. C. G. Wagner, R. J. Walker, P. R. Wallace, Henry Wallman, J. L. Walsh, André Weil, Alexander Weinstein, B. A. Welch, M. E. Wells, G. W. Whitehead, A. L. Whiteman E. A. Whitman, P. M. Whitman, Norbert Wiener, Audrey Wishard, W. D. Wray, Oscar Zariski.

The meetings of the Society opened Monday afternoon with two sections, Analysis and Algebra, at which Professor Tibor Radó and Dr. R. M. Thrall presided. Tuesday morning there were three sections, Analysis, Topology and Geometry, and Applied Mathematics, with Professors Norbert Wiener, L. W. Cohen, and L. R. Ford presiding.

Tuesday afternoon was devoted to a Symposium on Applied Mathematics, Professor F. D. Murnaghan presiding. Professor L. V. Bewley spoke on *The mathematical theory of traveling waves* and Professor I. S. Sokolnikoff on *Some new methods of solution of two-dimensional problems in elasticity.* The discussion leaders were Professors Alan Hazeltine, Ernst Weber, D. L. Holl, and J. L. Synge.

The annual business meeting and election of officers was held on Wednesday morning, President Marston Morse presiding. This was followed by the award of the Frank Nelson Cole Prize in the Theory of Numbers to Professor Claude Chevalley for his paper *La théorie du corps de classes*, published in Annals of Mathematics, (2), vol. 41 (1940), pp. 394–418. Professor Oscar Zariski then gave an address on *Normal varieties and birational correspondences.*

Wednesday afternoon there were two sections, Mathematical Logic, a joint session with the Association for Symbolic Logic, at which Professor A. A. Bennett presided; and Analysis and Algebra, at which Dr. Jesse Douglas presided.

Sessions of the Mathematical Association of America were held on Thursday morning and afternoon, those of the Association for Symbolic Logic on Wednesday afternoon, and those of the National Council of Teachers of Mathematics on Wednesday and Thursday. There was a Pi Mu Epsilon luncheon on Thursday.

On Tuesday afternoon, the mathematicians and guests were entertained at a tea by the ladies of the Department of Mathematics of Lehigh University.

The attendance at the annual dinner held in the Hotel Bethlehem was two hundred sixty-two. The toastmaster, Professor W. H. Smith, introduced President C. C. Williams of Lehigh University who gave a brief address of welcome. This was followed by "In a Persian Garden" based on the Rubaiyat of Omar Khayyam sung by a quartet with piano accompaniment. Professor Virgil Snyder then read resolutions, prepared by a special committee appointed by the Council, in appreciation of the excellent services of Professor R. C. Archibald as Librarian of the Society for twenty-one years. These resolutions were adopted unanimously by a rising vote. They had been beautifully embossed on parchment for presentation to Professor Archibald. Professor J. R. Kline, Secretary of the Society, spoke of the work of the Society, its role in the present emergency and of plans for the future.

The program continued with a pleasing and well executed mathematical skit written by Dean Tomlinson Fort and presented by a cast of nineteen characters, revealing how Confusion has been dethroned by Mathematics. Following a cello solo by Professor K. W. Lamson, Professor I. S. Sokolnikoff presented resolutions expressing the appreciation of the mathematicians and guests to the President of Lehigh University, the members of the local committee and all who had assisted them for the excellent arrangements, pleasing entertainment, and cordial hospitality. At midnight, 1942 was welcomed with songs and New Year greetings.

Since the meeting, the local committee has reported that after all expenses of the meeting were paid, a small sum remained which was donated to the Red Cross.

At the meeting of the Board of Trustees at 6:00 P.M., December 29, 1941, in Lamberton Hall of Lehigh University, there was no quorum present. An adjourned meeting, therefore, was held on January 2, 1942, at 1:00 P.M. in the Faculty Club of Columbia University, New York City. There were two meetings of the Council, one being a joint session with the Board of Governors of the Mathematical Association of America. The joint session was held at 6:00 P.M. and the regular meeting of the Council at 8:20 P.M. on December 30, 1941 in Lamberton Hall.

At the joint meeting of the Council and Board of Governors, President Marston Morse presented the report of a subcommittee of the War Preparedness Committee (T. Y. Thomas, Chairman) which has been studying the problem of the Supply and Demand for Mathematicians in those institutions which confer the doctorate in mathematics. This study showed that in these institutions during the past year there has been a decrease of 23% in the number of graduate students and an increase of about 2% in the number of teachers of mathematics. It was reported also that there were indications of an increased demand for mathematical instruction of undergraduates. On the basis of this report, the Board of Governors and Council voted that steps should be taken with the proper authorities to have mathematicians employed as teachers in accredited colleges and universities and certain qualified graduate students recommended for individual occupational deferment.

The Secretary announced the election of the following thirty-nine persons to membership in the Society:

Mr. Gaynor Jefferson Adams, State Highway Department, Linden, Ala.;
Mr. Neal Russell Amundson, University of Minnesota;
Mr. Charles Cornelius Andersen, Jr., Chicago, Ill.;
Mr. Henry John Barten, Baltimore, Md.;
Mr. John Keppler Baumgart, Cumberland College, Williamsburg, Ky.;
Dr. Stefan Bergman, Brown University;
Mr. Robert George Blake, Hernando High School, Brooksville, Fla.;
Professor Harold E. Bowie, American International College, Springfield, Mass.;
Professor Mary L. Brady, College of Mount St. Vincent, New York, N. Y.;
Mr. Joseph Francis Brennan, Pacific Gas and Electric Company, San Francisco, Calif.;
Professor Myrtle Cyrena Brown, North Texas State Teachers College, Denton, Tex.;
Professor Irving Wingate Burr, Purdue University;
Mr. Edward Caleb Coker, Jr., Clemson College, Clemson, S.C.;
Mr. Douglas Richard Crosby, Royal Canadian Air Force, Rivers, Manitoba, Canada;
Dr. John A. Daum, Agricultural and Mechanical College of Texas;
Professor James E. Davis, Central Y.M.C.A. College, Chicago, Ill.;
Mr. James Riley Ellis, Booker T. Washington High School, Tulsa, Okla.;
Mr. Sidney Fernbach, Frankford Arsenal, Philadelphia, Pa.;
Mr. Marvin George Harrison, Curtiss Propeller, Caldwell, N.J.;
Mr. William R. Harvey, Bell Telephone Laboratories, New York, N.Y.;
Professor John Henry Hett, Manhattan College;
Mr. Franklin Ferguson Hopper, New York Public Library;
Mr. Walbert C. Kalinowski, St. John's University, Collegeville, Minn.;
Professor John Joseph McCarthy, St. John's University, Brooklyn, N.Y.;
Professor Russle Hindman MacCullough, Defiance College, Defiance, Ohio;
Mr. Joshua Matz, Yeshiva College, New York, N.Y.;
Dr. Harlan C. Miller, Winthrop College, Rock Hill, S.C.;
Mr. Hugh Jordan Miser, Ohio State University;

Professor Ernest Nagel, Department of Philosophy, Columbia University;
Mr. George R. Rich, Tennessee Valley Authority, Knoxville, Tenn.;
Sister Mary De Pazzi Rochford, Briar Cliff College, Sioux City, Iowa;
Professor Paul Anthony Samuelson, Massachusetts Institute of Technology;
Mr. Harry J. Sternberg, Stuyvesant High School, Brooklyn, N.Y.;
Miss Helen Farnam Story, St. Petersburg Junior College, St. Petersburg, Fla.;
Sister Mary Placide Thomas, Mt. Saint Agnes Junior College, Baltimore, Md.;
Dr. Peter Thullen, Departamento Actuarial del Instituto N. de Previsiòn, Quito, Ecuador;
Mr. Marvin Lewis Vest, West Virginia University;
Mr. Thomas Charles Gordon Wagner, University of Maryland;
Dr. Alexander Weinstein, University of Toronto.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

Brown University: Mr. Marshall Evans Munroe, Professor Willy Prager.
Bryn Mawr College: Miss Josephine Margaret Mitchell.
University of California at Los Angeles: Messrs. Paul Arnold Clement and Bill C. Moore.
University of Chicago: Mr. Roy Dubisch, Miss Anne Louise Lewis, Mr. J. Ernest Wilkins, Jr.
College of the City of New York: Mr. Harvey Cohn.
Columbia University: Miss Louise Comer, Mr. Leonard Gillman, Professor Jacques Hadamard, Messrs. Meyer Karlin and Ernst Gabor Straus.
Cornell University: Messrs. W. H. Durfee and R. R. R. Luckey.
Duke University: Mr. Bruce Elwyn Meserve.
Equitable Life Insurance Company of Iowa: Mr. Alton O. Groth.
Harvard University: Messrs. Edwin Hewitt, E. N. Lorenz, and A. L. Putnam.
University of Illinois: Mr. J. C. Bell, Miss Frances E. Ewing, Messrs. W. A. Ferguson, H. F. Gingerich, and N. M. Ginsberg, Miss Corinne Hatcan, Mr. F. F. Helton, Misses Janie C. Lapsley and Naomi M. Livesay, Messrs G. S. Mapes, E. B. Shanks, M. D Springer, O. P. Staderman, and J. C. Stewart.
Indiana University: Miss Barbara Howe.
Institute for Advanced Study: Dr. Felix Adler, Professors Subrahmanyan Chandrasekhar and Kenneth Stewart Cole, Drs. Hans Samelson and Arthur Harold Stone.
Iowa State College: Mr. Ralph H. Tripp.
The State University of Iowa: Messrs. Louis Garfin and Edwin Halfar.
The, Johns Hopkins University: Messrs. Richard Bellman, Seymore Abraham Fenichel, and Leslie G. Peck.
University of Kentucky: Mr. Paul W. Derthick.
Lehigh University: Mr. John Alden Pond.
Massachusetts Institute of Technology: Messrs. B. G. Farley, W. S. Loud, and O. G. Owens.
University of Michigan: Messrs. Wade Ellis, Albert A. Grau, and Jack Irwin Northam.
University of Minnesota: Messrs. William S. H. Crawford and William D. Munro.
Northwestern University: Mr. Herbert J. Greenberg.
University of Pennsylvania: Mr. Erik Hemmingsen.

Stanford University: Mr. Paul Charles Rosenbloom.
Swarthmore College: Professor André Weil.
Syracuse University: Mr. Robert C. Simpson, Jr.
University of Toronto: Mr. Wallace Charles Gordon Fraser.
University of Virginia: Messrs. Richard R. Bernard, William Aubrey Blankinship, and Mariano Garcia, Jr.
University of Washington: Mr. Arthur Henry Jerbert, Miss Dorothy Jeanne Morrow.
Wellesley College: Miss Alberta Schuettler.
Wesleyan University: Mr. Myron Edward White.
Yale University: Miss Martina Elizabeth Doyle, Mr. David McCray Merriell, Miss Mary Kent Peabody.

The following appointments by President Marston Morse were reported: as representative of the Society at the inauguration of Henry Elisha Allen as President of Keuka College (Keuka Park, N.Y.) on November 7, 1941, Professor W. B. Carver; as tellers for the election at the 1941 Annual Meeting, Professors E. R. Lorch and L. L. Smail; as new members of the Committees to Select Hour Speakers: for Annual and Summer Meetings, Professor M. H. Stone (committee now consists of Professors J. R. Kline, chairman, M. H. Ingraham, and M. H. Stone); for Eastern Sectional Meetings, Professor Salomon Bochner (committee now consists of Professors T. R. Hollcroft, chairman, C. R. Adams, and Salomon Bochner); for Western Sectional Meetings, Professor T. H. Hildebrandt (committee now consists of Professors W. L. Ayres, chairman, T. H. Hildebrandt, and W. J. Trjitzinsky); for Far Western Sectional Meetings, Professor G. C. Evans (committee now consists of Dean T. M. Putnam, chairman, Professors G. C. Evans, and T. Y. Thomas); as representative of the Society on the American Year Book for a period of three years, 1942–1944, Professor J. F. Ritt; as a new member of the Committee on Places of Meetings for a period of three years, 1942–1944, Professor P. R. Rider.

The Secretary reported that the ordinary membership in the Society is now 2,453, including 228 nominees of institutional members and 76 life members. There are also 87 institutional members. The total attendance of members at all meetings in 1941 was 1,574; the number of papers read was 474; the number of invited addresses was 38; the number of members attending at least one meeting was 949.

At the annual election which closed on December 31, and at which 492 votes were cast, the following officers were elected:

Vice Presidents, Professors C. C. MacDuffee and J. D. Tamarkin.

Associate Secretaries, Professors W. L. Ayres and M. H. Ingraham, Dean T. M. Putnam.

Librarian, Professor Arnold Dresden.

Members of the Editorial Committee of the Bulletin, Professor A. A. Albert, Dean Tomlinson Fort.

Member of the Editorial Committee of the Transactions, Professor Oscar Zariski.

Member of the Editorial Committee of the Colloquium Publications, Professor M. H. Stone.

Members of the Editorial Committee of Mathematical Reviews, Professors O. E. Neugebauer, J. D. Tamarkin, and Oswald Veblen.

Members-at-large of the Council, Professors G. A. Bliss, Nelson Dunford, W. T. Martin, Gabor Szegö, and S. S. Wilks.

The Council adopted the following resolutions concerning the work of Professor R. C. Archibald as Librarian of the Society:

> The members of the American Mathematical Society take this opportunity to express their appreciation of the services of Raymond Clare Archibald on the occasion of his retirement from the post of its librarian.
>
> During the twenty-one years in which he held this office he has greatly increased the size and usefulness of the library until it has become an important source of reference to all mathematicians. His conduct of this work has been marked by wholehearted devotion, extraordinary ability, and laudable economy.
>
> The Society presents this memorial, attested by the signatures of its officers with assurances of the gratitude and affection of its members, with the hope that the Society may continue to receive the benefit of his advice and of his scholarly work for many years to come.

In an appendix to this report are excerpts from the Report of the Treasurer for the fiscal year 1941 as verified by the Auditors. A copy of the complete report will be sent, on request, to any member of the Society. The Board of Trustees adopted a budget for 1942 showing authorized expenditures of $49,050. This includes an appropriation of $1,000 for Mathematical Reviews from the Marion Reilly Fund.

The Librarian reported that the Library of the Society now contains 9,983 volumes of which 7,159 are bound volumes. Twenty-one years ago when Professor Archibald assumed the librarianship the corresponding numbers were 5,862 and 4,360. There are now over 150 periodicals on our exchange list corresponding to the earlier 123.

The American Journal of Mathematics, which is published jointly by The Johns Hopkins University and the Society, and to which the Society contributes an annual subvention of $2,500, printed 888 pages during 1941.

It was reported that a temporary editorial board consisting of Professors J. L. Walsh (chairman), A. A. Albert, and F. D. Murnaghan had been appointed in connection with the new series of books entitled *Mathematical Surveys.*

It was announced that copies of the recent report of Professor W. L. Hart (Chairman, Subcommittee on Education for Service) entitled *Mathematical Education for Defense* had been distributed in October to approximately 3,700 superintendents of schools throughout the country.

Times and places of meetings during 1942 were set as follows: Fall Far Western Meeting at the University of California at Los Angeles on November 28 and the 1942 Annual Meeting in New York City on December 28–30.

Certain invitations to give hour addresses were announced: Dr. R. P. Boas and Professor George Pólya for the April, 1942, meeting in New York City; Professor Salomon Bochner for the October, 1942, meeting in New York City.

Professor A. D. Michal was appointed Acting Associate Secretary for the far west during the period of Dean Putnam's illness.

Two books were accepted for publication in the Colloquium Series: *Algebraic Topology* by Professor Solomon Lefschetz and *Analytic Topology* by Professor G. T. Whyburn.

The Secretary announced that the Board of Trustees, upon recommendation of the Council, had adopted temporary regulations concerning the dues of those who are serving as enlisted men in the armed forces of the United States or Canada. The following provisions are included: (1) enlisted men who were members in good standing at the time of their induction may have their dues fixed at $1.00 per year during the period of their enlistment; (2) if they were subscribers to the Transactions or Mathematical Reviews at the time of their induction, they may continue to subscribe at the rate of $1.00 per year for the Transactions and $1.00 per year for Mathematical Reviews during the period of their enlistment. It is hoped that these men may thus be enabled to continue their contacts with mathematics while in the service. Full details of these regulations were mailed to members of the Society on January 1.

The Secretary announced that February 15 had been set as the final date for the receipt of orders for microfilm reading machines in connection with three-year subscriptions to Mathematical Reviews.

The Executive Committee of Mathematical Reviews reported that the subscription list of that journal, as of December 1, 1941, was 1,400.

Dean G. D. Birkhoff was elected Chairman of the Emergency Committee for the International Congress of Mathematicians, to replace Professor W. C. Graustein, deceased. It was announced that the Rockefeller Foundation had extended its appropriation for the International Congress to December 31, 1946.

President Marston Morse appointed as the Committee on Nomination of Officers and Members of the Council for 1943, Professors F. D. Murnaghan (chairman), A. A. Albert, H. A. Rademacher, W. M. Whyburn, and R. L. Wilder.

Titles and cross references to the abstracts of papers read at the regular sessions follow below. The papers were read as follows: Papers 1–5 in the section for Analysis on Monday afternoon; papers 6–12 in the section for Algebra on Monday afternoon; papers 13–18 in the section for Analysis on Tuesday morning; papers 19–25 in the section for Topology and Geometry on Tuesday morning; papers 26–33 in the section for Applied Mathematics on Tuesday morning; papers 34–39 in the section for Mathematical Logic on Wednesday afternoon; papers 40–44 in the section for Analysis and Algebra on Wednesday afternoon; and papers 45–79, whose abstract numbers are followed by the letter *t*, were read by title. Dr. Mickle was introduced by Professor Tibor Radó, Dr. Bergman by Professor T. R. Hollcroft, Dr. Miller by Professor R. L. Moore, Dr. Theilheimer by Professor R. E. von Mises, Professor Prager by Dean R. G. D. Richardson, Dr. Weinstein by Professor J. L. Synge. Paper 3 was presented by Professor W. T. Martin, paper 4 by Dr. G. E. Reves, paper 7 by Professor Smiley, paper 8 by Dr. Dilworth, paper 11 by Professor Barnett, paper 12 by Professor MacLane, paper 18 by Dr. Reade, paper 21 by Professor Weil, paper 41 by Professor MacLane, paper 43 by Dr. Phillips, and paper 44 by Dr. Thrall.

1. E. J. Mickle: *Associated double integral variation problems.* (Abstract 48-1-44.)

2. P. T. Maker: *The Cauchy theorem for functions on closed sets.* (Abstracts 48-1-43.)

3. Salomon Bochner and W. T. Martin: *A class of removable singularities in several complex variables.* (Abstract 48-1-32).

4. G. E. Reves and Otto Szász: *Some theorems on double trigonometric series.* (Abstract 48-1-53.)

5. Stefan Bergman: *On operators in the theory of partial differential equations and their application.* (Abstract 48-1-31.)

6. R. H. Bruck: *Isotopy of algebras with a principal unit.* (Abstract 48-1-8.)

7. M. F. Smiley and W. R. Transue: *Metric lattices as singular metric spaces* (Abstract 48-1-20.)

8. Marshall Hall and R. P. Dilworth: *The imbedding problem for modular lattices.* (Abstract 48-1-12.)

9. C. C. MacDuffee: *On the composition of algebraic forms.* (Abstract 48-1-15.)

10. Tomlinson Fort: *Generalizations of the Bernoulli polynomials and numbers and corresponding summation formulas.* (Abstract 47-11-456.)

11. I. A. Barnett and C. W. Mendel: *A property of the traces of a square matrix.* (Abstract 48-1-5.)

12. Saunders MacLane and O. F. G. Schilling: *Groups of algebras over an algebraic number field.* (Abstract 48-1–16.)

13. R. B. Kershner: *The continuity of functions of many variables.* (Abstract 48-1-39.)

14. Mark Kac: *On convergence of certain series of functions.* (Abstract 48-1-38.)

15. W. R. Transue: *Contributions to the theory of subharmonic functions.* (Abstract 48-1-59.)

16. J. W. Bradshaw: *On a certain class of continued fractions.* (Abstract 48-1-33.)

17. Otto Szász: *On a theorem of Hardy and Littlewood.* (Abstract 48-1-56.)

18. E. F. Beckenbach and Maxwell Reade: *Mean-values and harmonic polynomials.* (Abstract 48-1-30.)

19. J. L. Vanderslice: *Invariant theory of vector pencil fields.* (Abstract 48-1-79.)

20. J. W. Peters: *The euclidean geometry of the n-dimensional simplex.* (Abstract 48-1-78.)

21. André Weil and C. B. Allendoerfer: *A general proof of the Gauss-Bonnet theorem.* (Abstract 48-1-80.)

22. W. W. Flexner: *Noncommutative chains.* II. Preliminary report. (Abstract 48-1-90.)

23. G. W. Whitehead: *Homotopy groups of spheres and their rotation groups.* (Abstract 48-1-101.)

24. Harlan C. Miller: *On irreducible continua.* (Abstract 48-1-95.)

25. J. L. Kelley: *Symmetric product spaces.* Preliminary report. (Abstract 48-3-139.)

26. P. M. Whitman: *Note on a certain representation of lattice elements.* Preliminary report. (Abstract 48-1-24.)

27. Feodor Theilheimer: *The potential of curvilinear distributions.* (Abstract 48-1-72.)

28. J. L. Synge: *On the theory of the airfoil of finite span.* (Abstract 48-1-71.)

29. A. E. Heins: *On the transformation theory of the solution of partial differential equations.* II. Preliminary report. (Abstract 48-1-65.)

30. Willy Prager: *Fundamental theorems of a new mathematical theory of plasticity.* (Abstract 48-1-70.)

31. Alexander Weinstein: *On the flexural center and the center of twist.* (Abstract 48-1-74.)

32. R. E. Gaskell: *On longitudinal vibrations of a bar.* (Abstract 48-1-64.)

33. Henry Wallman: *On the reduction in harmonic distortion due to high frequency pre-emphasis.* Preliminary report. (Abstract 48-3-123.)

34. Alonzo Church: *On sense and denotation.* (Abstract 48-1-83.)

35. S. C. Kleene: *On the interpretation of intuitionistic number theory.* (Abstract 48-1-8 5.)

36. G. D. W. Berry: *On formalizing semantics.* (Abstract 48-1-82.)

37. Nelson Goodman: *Sequences.* (Abstract 48-1-84.)

38. Barkley Rosser: *The Burali-Forti paradox.* (Abstract 48-1-86.)

39. E. C. Berkeley: *Application of symbolic logic to punch card operations.* (Abstract 48-1-81-*t*.)

40. I. S. Cohen: *A generalization of Macauley's theorem on unmixed ideals.* (Abstract 48-1-9.)

41. Saunders MacLane and Samuel Eilenberg: *A theorem on group extensions.* (Abstract 48-1-94.)

42. G. Y. Rainich: *Factorization of polynomials, in a ring, with application to partial differential equations.* Preliminary report. (Abstract 48-1-51.)

43. Salomon Bochner and R. S. Phillips: *Absolutely convergent Fourier series in noncommutative normed rings.* (Abstract 48-1-6.)

44. R. M. Thrall and C. J. Nesbitt: *On the modular representations of the symmetric group.* (Abstract 48-1-21.)

45. Reinhold Baer: *Inverses and zero-divisors.* (Abstract 48-1-4-*t*.)

46. E. F. Beckenbach: *Painlevé's theorem and the analytic prolongation of a minimal surface.* (Abstract 48-1-29-*t*.)

47. R. H. Bruck: *Certain numerical invariants of polyadics.* (Abstract 48-1-7-*t*.)

48. Leonard Carlitz: *q-Bernoulli numbers and polynomials.* (Abstract 48-1-26-*t*.)

49. N. A. Court: *On the theory of the tetrahedron.* (Abstract 48-1-77-*t*.)

50. L. L. Dines: *On the mapping of n quadratic forms.* (Abstract 48-3-104-*t*.)

51. J. L. Doob: *Topics in the theory of Markoff chains.* (Abstract 48-1-34-*t*.)

52. H. J. Ettlinger: *The theory of the Riesz integral.* (Abstract 48-1-35-*t*.)

53. Abe Gelbart: *On functions of two complex variables with bounded real parts.* (Abstract 48-1-36-*t*.)

54. D. W. Hall: *On a theorem of E. E. Bets.* (Abstract 48-1-91-*t*.)

55. F. B. Hildebrand: *Note on the integro-differential equation of a problem in the theory of plane stress.* (Abstract 48-1-66-*t*.)

56. F. B. Jones: *A certain non-metric Moore space.* (Abstract 48-3-138-*t*.)

57. Irving Kaplansky and O. F. G. Schilling: *Some remarks on relatively complete fields.* (Abstract 48-1-14-*t*.)

58. H. N. Laden: *An interpolation polynomial involving derivatives of a prescribed function.* (Abstract 48-1-40-*t*.)

59. Joseph Lehner: *The Ramanujan identities and congruences for powers of eleven.* Preliminary report. (Abstract 48-1-27-*t*.)

60. E. J. McShane: *On Perron integration.* (Abstract 48-1-41-*t*.)

61. E. J. McShane: *The derivative of the indefinite Lebesgue integral.* (Abstract 48-1-42-*t*.)

62. S. B. Myers: *An existence theorem for a self-adjoint system of second-order, linear, homogeneous differential equations.* (Abstract 48-1-46-*t*.)

63. N. M. Oboukhoff: *The historical development of total differential as the principal part of the increment of a function of several variables.* (Abstract 48-1-47-*t*.)

64. J. F. Paydon and H. S. Wall: *An extension of the Stieltjes continued fraction theory.* (Abstract 48-1-49-*t*.)

65. Samuel Perlis: *Normal bases of cyclic fields of prime-power degree.* (Abstract 48-1-18-*t*.)

66. Maxwell Reade: *Some remarks on subharmonic functions.* Preliminary report. (Abstract 48-1-52-*t*.)

67. W. H. Roever: *Geometric statement of a fundamental theorem for four-dimensional orthographic axonometry.* (Abstract 48-3-133-*t*.)

68. O. F. G. Schilling: *Normal extensions of relatively complete fields.* (Abstract 48-1-19-*t*.)

69. H. M. Schwartz: *On sequences of Stieltjes integrals.* (Abstract 48-1-55-*t*.)

70. A. R. Schweitzer: *On the genesis of the algebra of logic in the foundations of geometry.* (Abstract 48-1-87-*t*.)

71. M. F. Smiley: *Elementary similarity transformations and the rational canonical form of a matrix.* (Abstract 48-3-107-*t*.)

72. R. H. Sorgenfrey: *Some theorems on co-terminal arcs.* (Abstract 48-1-98-*t*.)

73. E. W. Titt: *A method for integrating the linear hyperbolic equation in three independent variables.* (Abstract 48-1-58-*t*.)

74. W. J. Trjitzinsky: *Analytic theory of parametric linear partial differential equations.* (Abstract 48-1-60-*t*.)

75. S. M. Ulam: *A geometrical approach to the theory of representations of topological groups*. Preliminary report. (Abstract 48-1-61-*t*.)

76. A. D. Wallace: *Chains and structure of continua.* (Abstract 48-3-140-*t*.)

77. T. L. Wade and R. H. Bruck: *The characteristic function of a 2p-tensor*. (Abstract 48-1-22-*t*.)

78. P. A. White: *R-regular convergence spaces.* (Abstract 48-1-100-*t*.)

79. L. R. Wilcox: *Extensions of semi-modular lattices*. III. (Abstract 48-1-25-*t*.)

T. R. HOLLCROFT,
Associate Secretary

APPENDIX

EXCERPTS FROM REPORT OF TREASURER[1]

December 15, 1941

TO THE BOARD OF TRUSTEES OF THE
AMERICAN MATHEMATICAL SOCIETY

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1941.

Surplus account shows a balance of $9,048.08 as compared with $6,966.39 at the beginning of the year. Two of the facts which contributed to this unanticipated increase may be mentioned here: (1) A considerable gain in the membership of the Society resulted in additional revenue from Initiation Fees and Dues; (2) Income from Investments exceeded the estimate by more than $1,000.

The market value on November 29, 1941 of the securities in the Pool was $7,613.09 less than book value. This is to be compared with the total amount, $4,201.96, held in accounts Reserve for Investment Losses and Profit and Loss on Sale of Securities. The market value on November 29, 1941 of securities in General and Restricted Funds was $963.00 less than book value. Profit on the sale of securities for General and Restricted Funds amounting to $295.73 is being carried as a reserve against losses to these funds.

Income earned by General and Restricted Fund Investments during the year amounted to $1,547.08. This represents a return of 1½% on amounts in savings banks, and approximately 4% on securities. Investment Income earned by the Pool amounted to $7,317.85, representing a return of approximately 4¼%. Total Investment Income from all sources was thus $8,864.93, corresponding to a yield of approximately 4¼%. This income and yield may be compared with $6,909.41 and approximately 4% for the fiscal year 1940.

The operating account for Mathematical Reviews continues to show receipts in excess of disbursements. However, this excess is due principally to advance subscriptions.

The Endowment Fund Principal, unchanged during the year, remains on the books at $60,110.68, representing a depreciation of $10,395.39 from a former value of $70,506.07.

During the year final settlement of the Estate of Marion Reilly brought the Society an additional sum of $6,738.78. A gift of $1,000 from Dr. Robert Henderson was also received. As directed by the Trustees, the gifts have been added to the Principal Funds in the Pool.

Respectfully submitted,
BENNINGTON P. GILL, *Treasurer*

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[1] The report of the Treasurer for fiscal 1940 will be published in September 1942 with the biennial list of members.

BALANCE SHEET

Assets

	November 30, 1941	December 1, 1940
GENERAL AND RESTRICTED FUND ASSETS:		
Cash	$ 19,380.05	$ 14,624.30
Investments	28,814.24	21,393.87
	$ 48,194.29	$ 36,018.17
ENDOWMENT AND OTHER NONEXPENDABLE FUND ASSETS:		
Cash	$ 1,156.70	$ 659.28
Investments	160,395.09	152,549.20
	$161,551.79	$153,208.48
TOTAL ASSETS	$209,746.08	$189,226.65

Liabilities

	November 30, 1941	December 1, 1940
GENERAL FUNDS:		
Sinking Fund	$ 420.09	$ 290.80
Profit on Sales of Securities	295.73	
Surplus	9,048.08	6,966.39
	$ 9,763.90	$ 7,257.19
RESTRICTED FUNDS:		
Bôcher Fund Income	$ 597.13	$ 540.96
Brown Fund Income	101.76	54.47
Cole Fund Income	659.39	560.41
Henderson Fund Income	21.63	
Moore Fund Income	1,583.06	1,483.73
Reilly Fund Income	2,650.05	2,280.84
Reilly Fund Principal	3,968.13	4,280.16
Bulletin Reprinting Fund	685.23	457.61
Transactions Reprinting Fund	103.63	103.63
Colloquium	5,187.14	4,718.95
Mathematical Reviews	16,924,45	8,426.11
Congress	5,943.44	5,853.67
War Preparedness	5.35	
	$ 38,430.39	$ 28,760.98
	$ 48,194.29	$ 36,018.17
ENDOWMENT AND OTHER NONEXPENDABLE FUNDS (POOL):		
Endowment Fund Principal	$ 60,110.68	$ 60,110.68
Bôcher Fund Principal	1,188.00	1,188.00
Brown Fund Principal	1,000.00	1,000.00
Cole Fund Principal	2,093.13	2,093.13
Henderson Fund Principal	1,000.00	
Moore Fund Principal	2,100.62	2,100.62
Reilly Fund Principal	19,553.32	12,502.07
Life Membership Reserve	5,191.30	5,507.90
Life Subscription Reserve	112.78	114.09
Colloquium	5,000.00	5,000.00
Mathematical Reviews	60,000.00	60,000.00
Reserve for Investment Losses	2,728.24	1,926.26
Profit on Sales of Securities	1,473.72	1,665.73
	$161,551.79	153,208.48
TOTAL LIABILITIES	$209,746.08	$189,226.65

SUMMARY STATEMENT OF INCOME AND EXPENDITURES
1940–1941

	1941		1940	
	Receipts	*Disbursements*	*Receipts*	*Disbursements*
GENERAL RECEIPTS				
Dues	$14,970.19		$14,380.96	
Dues—Contributing Memberships	1,040.00		1,114.92	
Dues—Institutional Memberships	6,468.31		6,705.00	
Initiation Fees	515.25		584.75	
Investment Income	4,147.27		3,191.46	
Miscellaneous	343.15		695.95	
GENERAL DISBURSEMENTS				
Secretaries		$ 6,091.29		$ 5,800.00
Treasurer		750.00		650.00
Officers Traveling		888.93		749.54
Library		824.50		629.82
Committee Expenses		101.04		211.24
Office Furniture and Fixtures		51.70		31.00
Gibbs Lecture				19.88
Emergency		277.29		287.86
Total	$27,484.17	$ 8,984.75	$26,673.04	$ 8,379.34
Excess of General Receipts	$18,499.42		$18,293.70	
PUBLICATION				
Bulletin	$ 1,755.47	$11,037.14	$ 1,631.65	$12,405.21
Bulletin Reprinting	227.62		145.85	
Transactions	4,816.11	8,968.10	5,411.24	8,521.05
Transactions Reprinting				116.80
Colloquium	3,437.74	2,969.55	3,121.11	2,727.54
Mathematical Reviews	21,467.56[1]	12,969.22	20,302.06[2]	14,356.77
Semicentennial Publications	129.29		141.30	
American Journal		2,500.00		2,500.00
Total	$31,833.79	$38,444.01	$30,753.22	$40,627.37
Excess of Cost of Publication		$ 6,610.22		$ 9,874.15
MISCELLANEOUS				
Profit on Sales of Securities	$ 103.72		$ 1,578.49	
Adjustment for Depreciation of Value of Investments			87.24	$24,804.97
Special Funds	9,526.51[3]	$ 1,000.00	1,366.89	487.10
Total	$ 9,630.23	$ 1,000.00	$ 3,032.62	$25,292.07
Difference	$ 8,630.23			$22,259.45
Net change in assets	$20,519.43			$13,839.90
ASSETS BEGINNING OF YEAR	$189,226.65		$203,066.55	
ASSETS END OF YEAR	$209,746.08		$189,226.65	

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[1] Includes $6,809.37 receipts for subscriptions for 1942–1944 and $1,000 appropriation from Reilly Fund Income.

[2] Exclusive of $1,000 appropriation from Surplus.

[3] Includes $7,738.78 principal of gifts.

JAMES WATERMAN GLOVER—IN MEMORIAM

James Waterman Glover, emeritus professor of mathematics at the University of Michigan, died in Ann Arbor, Michigan, on July 15, 1941. In his passing, actuarial and statistical science has lost one of its most distinguished pioneers.

Professor Glover was born at Clio, Michigan, on July 24, 1868. He was graduated from the University of Michigan with the class of 1892, where he came under the influence of F. N. Cole, and received his doctor's degree from Harvard University in 1895, being the first one to do his thesis work under the supervision of Maxime Bôcher; his thesis subject was *Properties of Solutions of the Partial Differential Equation* $\Delta y+ky=0$. He then returned to the University of Michigan as a member of the Department of Mathematics, holding the various ranks, Assistant Professor in 1903, Junior Professor in 1906, and Professor in 1911, until he retired in 1938. He was chairman of the Department of Mathematics from 1927 to 1934. During the years 1930 to 1932 he had leave of absence, and served as president of the Teachers' Insurance and Annuity Association of America. He was a member of the Council of the American Mathematical Society from 1927–1929.

Beginning in 1902 with a course in actuarial theory, Professor Glover gradually developed courses in the mathematics of finance, life insurance, and statistics, at the University of Michigan. As a consequence of the training afforded by these courses, he was able to place hundreds of students in responsible positions with life insurance companies and state departments. At the time of his retirement about two hundred of his former students, many of them executives of major insurance companies, gave a dinner in Chicago honoring him. At this meeting a fellowship in insurance, bearing his name, was established.

In his administrative capacity he not only built up the work in actuarial mathematics and mathematical statistics at the University, but was also responsible for greatly strengthening the work of the Department in other branches by the addition of promising young men.

His activities outside of the University were numerous. He served as consulting expert for the Canadian Royal Commission on Insurance in 1896, for the Wisconsin Legislative Investigation Committee in 1906, for the Wisconsin Joint Committee on Banks and Insurance in 1907, and for numerous insurance companies. From 1910 to 1929 he served the U. S. Census Bureau as Expert Special Agent and during this time the United States Life Tables were prepared under his supervision.

While Professor Glover was a forceful person so far as being able to accomplish something he thought ought to be done, he was an extremely pleasant person with whom to be associated. One could oppose something he proposed to do, feeling certain it would not affect a personal friendship. He was fond of hunting and fishing and was a delightful companion on an outing. His accomplishments are a lasting monument to his memory.

Following is a list of his publications of mathematical interest:

1. *On groups whose orders are products of three prime factors* (with F. N. Cole), American Journal of Mathematics, vol. 15 (1893), pp. 191–220.

2. *Derivation of the United States mortality table by osculatory interpolation*, Quarterly Publication of the American Statistical Association, (n.s.), vol. 12 (1910), pp. 85–109.

3. *A general formula for the valuation of securities*, American Mathematical Monthly, vol. 22 (1915), pp. 82–88.

4. *Some formulas connected with the calculation of annual dividends on the contribution plan*, Record of the American Institute of Actuaries, vol. 5 (1916), pp. 37–43.

5. *United States Life Tables*: 1910, Government Printing Office, Washington, 1916, 65 pp., 25 life tables.

6. *United States Life Tables*, 1890, 1901, 1910 *and* 1901–1910, Government Printing Office, Washington, 1921, 496 pp., 22 diagrams, 52 graphs.

7. *Tables of Compound Interest Functions and Logarithms of Compound Interest Functions* (with H. C. Carver), Ann Arbor, 1921, 77 pp.

8. *Tables of Applied Mathematics in Finance, Insurance, Statistics*, Ann Arbor, 1923, 13+676 pp.

9. *Tables of Statistical Data for the Use of Students of Mathematical and Economic Statistics* (with H. C. Carver), Ann Arbor, 1923, 53 pp.

10. *Interpolation, summation and graduation*, chap. 3, pp. 34–61, *Handbook of Mathematical Statistics*, Houghton Mifflin, 1924.

11. *Quadrature formulas when ordinates are not equidistant*, Proceedings of the International Congress of Mathematicians, Toronto, 1924, vol. 2.

12. *Report on statistical teaching in American colleges and universities*, Journal of the American Statistical Association, vol. 21 (1926).

13. *An Introduction to the Mathematics of Life Insurance* (with W. O. Menge), New York, Macmillan, 1935, 9+190 pp.

PETER FIELD

BOOK REVIEWS

Tables of Probability Functions. Vol. I. New York, Work Projects Administration. 1941. 28+302 pp. $2.00.

The earliest systematic table of the error function or probability integral was prepared by James Burgess and published in the Transactions of the Royal Society of Edinburgh, volume 39, Part II, 1898, pp. 257–321. This was followed by that compiled by W. F. Sheppard under the auspices of the British Association for the Advancement of Science, published by the Cambridge University Press in 1939, and by various other minor tables.

The present Tables extend the range of all existing ones and provide a smaller tabular interval. The usual control methods employed by the Agency to make them accurate have been used.

Both functions have been calculated to fifteen decimal places at intervals of 0.0001 in the range between 0 and 1 and at intervals of 0.001 in the range from 1 to 5.6. A short supplementary table (Table II) is included giving the values of $H(x)$ and its derivative to eight significant figures for x ranging from 4 to 10 at intervals of 0.01. The procedures for direct and inverse interpolation are explained and the degree of approximation attained are emphasized. The method employed for the actual computation is outlined together with the checks for control of accuracy.

The pages are 9 by 4 inches, both $H'(x)$ and $H(x)$ appearing in contiguous columns. There are 51 lines on a page in blocks of five, the last entry on a page being repeated at the top of the following one.

VIRGIL SNYDER

Guide to Tables in the Theory of Numbers. By Derrick Henry Lehmer. (Bulletin of the National Research Council, Number 105. Division of Physical Sciences, Committee on Mathematical Tables and Aids to Computation, Report 1: Report of the Subcommittee on Section F: Theory of Numbers.) Washington, D. C., National Research Council, 1941. 14+177 pp. $2.50.

In a foreword Professor R. C. Archibald, Chairman of the Committee, states that "In broad outline it exhibits the general plan for all Reports in the series. In adopting this plan the Committee desires to make clear that the Reports are being prepared primarily for scholars and others active in scientific work throughout the world."

Directions for the use of the Report are given in the Introduction.

This also contains an explanation of the practical point of view adopted in deciding what constituted a table in the theory of numbers and what tables were worthy of inclusion. The omissions include old obscure tables which have been superseded by more extensive and more easily available ones, and short tables in which every entry may be easily computed.

The rest of the Report is in three parts:

I. Descriptive Survey.
II. Bibliography.
III. Errata.

In Part I, which is about 80 pages in length, topics in the theory of numbers are classified under 17 headings **a**–**q**, with subheadings indicated by subscripts. Under each heading there is a description of what tables are listed in this topic and what they contain. The pages are numbered at the bottom and the topic described on a page is indicated by the appropriate letter of classification at the top.

With each table mentioned in Part I there appears the author's name followed by a number which refers to the complete bibliographic reference of Part II. Here the material is arranged alphabetically by authors. Following each reference a letter (with or without a subscript) in square brackets indicates the nature of the tables contained in the work referred to. In addition, libraries in which the work may be found are given in the coding used by the *Union List of Serials*. For this purpose 37 representative libraries were selected, the list and key to the code being given following the foreword of the book.

Thus it will be seen that if either the subject matter or the author's name is known the location of the table is a simple matter. This feature in itself makes the book of great value, but even this is overshadowed by Part III which collects for the first time the list of errors which have been discovered in the tables. The authority for corrections and a reference to the source, if published, are usually given after the errors. Tables in which errors have been found are indicated in Part II by an asterisk.

This report is indeed a guide to tables in the theory of numbers and it is, moreover, one which can be followed with ease.

R. D. JAMES

Waves. A Mathematical Account of the Common Types of Wave Motion. By C. A. Coulson. Edinburgh, Oliver and Boyd, 1941. 156 pp. $1.50.

This book will be welcomed by those who are interested in an elementary introduction to the subject of wave motion; it will also interest the experts in some one type of physical wave motion as a book for ready reference concerning other types of physical wave motion. In common with other books belonging to "University Mathematical Texts" this book is remarkably informative for its size. In the first chapter the wave equation and its principal solutions are introduced. From there on are treated in succession waves on strings, waves in membranes, longitudinal waves in bars and springs, waves in liquids, sound waves, and electric waves. The last chapter contains some general considerations. In each chapter the equations for the particular type of waves and the boundary conditions are derived; the methods of solution are illustrated by well-chosen examples. The presentation is clear and straightforward. Each chapter is followed by problems.

In the words of the author, "The object of this book is to consider from an elementary standpoint as many different types of wave motion as possible. In almost every case the fundamental problem is the same, since it consists in solving the standard equation of wave motion; the various applications differ chiefly in the conditions imposed on these solutions. For this reason it is desirable that the subject of waves should be treated as one whole, rather than in several distinct parts; the present tendency is in this direction." If one is to criticize the book in this connection, it is, perhaps, in order to suggest that the announced purpose could be served still better by giving the impedance concept the place it rightly deserves in wave theory. The original wave equation usually consists of two first order equations connecting the force and the velocity (or displacement). The solution will consist of a wave of force and a wave of velocity. By placing emphasis on this "two-wave" aspect, greater uniformity in treatment of reflection can be attained. It is hoped that the author will consider this point of view in the next edition.

SERGEI A. SCHELKUNOFF

Lectures in Topology. The University of Michigan Conference of 1940. Edited by R. L. Wilder and W. L. Ayres. Ann Arbor, University of Michigan Press, 1941. 316 pp. $3.00.

This volume is a collection of the papers presented at the University of Michigan Conference on Topology in June 1940. The scope of the book is indicated by the following list of titles of the longer papers:

Solomon Lefschetz, Abstract complexes; R. L. Wilder, Uniform local connectedness; N. E. Steenrod, Regular cycles of compact metric spaces; Samuel Eilenberg, Extension and classification of continuous mapping; Hassler Whitney, On the topology of differentiable manifolds; S. S. Cairns, Triangulated manifolds and differentiable manifolds; P. A. Smith, Periodic and nearly periodic transformations; Leo Zippin, Transformation groups; Saunders MacLane and V. W. Adkisson, Extensions of homeomorphisms on the sphere; O. G. Harrold, Jr., The role of local separating points in certain problems of continuum structure; L. W. Cohen, Uniformity in topological space; E. W. Chittenden, On the reduction of topological functions. There are also short accounts of nine other papers.

As can be seen from this list, practically every phase of modern topology is touched upon in this collection. Many of the papers are of a discursive nature, with most of the proofs omitted, and so the total amount of ground covered is quite extensive. We heartily recommend this book to any worker in topology as an excellent source of information on the present status of this subject.

R. J. WALKER

An Introduction to Linear Transformations in Hilbert Space. By F. J. Murray. (Annals of Mathematics Studies, no. 4.) Princeton University Press, 1941. 135 pp. $1.75.

The purpose of this book, according to the author, is "to present the most elementary course possible on this subject" and at the same time "to emphasize those notions which seem to be proper to linear spaces." Despite the assertion that these aims are not antagonistic, the exposition would be pretty tough going for the average graduate student. Although the reader is not assumed, except in an isolated section, to know about Lebesgue integration, and although the proof of such a comparatively elementary fact as that a continuous image of a compact set is compact is given in detail (p. 48), many parts of the book assume a great deal more sophistication.

The discussion is almost entirely unmotivated: the beginner might like to know *why* one studies spectral families, or the adjoints of operators. Even to one familiar with the theory it requires proof that von Neumann's definition of T^* is equivalent to the easier one usually given for bounded transformations; T^* is defined as the negative of the transformation whose graph is the orthogonal complement of the graph of T.

Concerning the author's choice of the order of the material, it is questionable whether or not it is pedagogically advisable to aim the discussion at unbounded operators from the very beginning, particularly since the main theorem (the spectral representation of self adjoint operators) is first proved for the bounded case. It would seem better to the reviewer to expound all the easy theory of bounded operators first, and thus prepared to call attention to the delicate considerations necessary to study the unbounded case. Also, in Chapter II we find practically the same proof used twice, once to establish the Riesz theorem on the representation of a bounded linear functional by an inner product (Theorem IV), and once to prove the possibility of projection on any closed linear manifold in Hilbert space (Theorem VI). The extremely elementary derivation, due to Riesz (Acta Szeged, vol. 7 (1934–1935), p. 37), of the former from the latter, could have been used here to good advantage.

The book contains many minor slips and typographical errors which may cause serious confusion. Thus in the statement of the axioms for a linear space (p. 4) the assumption $1f=f$ is omitted, and on p. 34, Theorem I, which is stated for an arbitrary transformation, is proved by reference to a lemma valid only for the additive case. Regrettable also is the author's reluctance to give to well known theorems their usual names: Schwarz's inequality, Bessel's inequality, Parseval's identity, and the Riesz-Fischer theorem are all indiscriminately referred to as Theorem n of Chapter m.

In Chapter VII (footnotes, pp. 67–68) there are some pretty examples of spectral families, and the Hellinger integral is treated courageously from the modern point of view and not reduced by means of weak convergence to the classical numerical case. The last two chapters are an excellent idea, carried out unfortunately too rarely: they contain quick sketches of further developments and applications, and references to the literature. On the whole the book is a compact and unified presentation of a well defined part of Hilbert space theory, and as such may appeal to the reader interested in learning only that part of the theory which even a non-specialist often needs.

PAUL R. HALMOS

NOTES

Brown University is continuing during the summer (June 15–August 29, 1942) and the next academic year the Program of Advanced Instruction and Research in Mechanics already in operation for a year. In the summer, 80 participants will be accepted (of which 20 will be engaged entirely in research) and in the academic year 40. Since this program is supported by the United States Office of Education, the Carnegie Corporation of New York, and the Rockefeller Foundation, no fees will be charged. The faculty for the summer consists of Stefan Bergman, Léon Brillouin, Willy Feller, Richard von Mises, Willy Prager, S. A. Schelkunoff, I. S. Sokolnikoff, and J. D. Tamarkin, who are offering a dozen courses with opportunities for research in various branches of applied mathematics, chiefly in connection with mechanics. For the next academic year eight courses are scheduled. Several substantial fellowships are available for highly qualified participants. Information in regard to each of these programs may be obtained from the Dean of the Graduate School, Brown University, Providence, R. I.

A feature of the program for the coming summer quarter at the University of Chicago will be a seminar on the calculus of variations. Professor Emeritus G. A. Bliss will return to the University to conduct this seminar. Other members of the summer staff who will participate in the seminar include Professors Tibor Radó of Ohio State University, M. F. Smiley of Lehigh University, and L. M. Graves, M. R. Hestenes, and W. T. Reid. Those who are interested may secure further information by writing to Professor Graves. Professor Radó will also give a course on subharmonic functions, and as part of the defense program, Professor Reid will give a course on exterior ballistics.

Connecticut College is establishing a scholarship fund in honor of the late Professor D. D. Leib.

It is planned to establish an Institute of Applied Mathematics at New York University.

Professor E. A. Milne of the University of Oxford has been awarded a Royal Medal by the Royal Society for his researches on the atmospheres of the earth and the sun, on the internal constitution of the stars, and on the theory of relativity.

During 1941 several Russian mathematicians were honored by awards of Stalin Prizes. These prizes were established in honor of the sixtieth birthday of Joseph Stalin, and are of three classes: Frist Prize 100,000 rubles, Second Prize, 50,000 rubles, Third Prize, 25,000 rubles. First Prizes were awarded to Mr. I. M. Vinogradov for his work *A new method in the analytical theory of numbers*; to Mr. P. A. Gelvich for his works *On dispersion, probability of hits, and mathematical anticipation of the number of hits*, *Theoretical fundamentals for the elaboration of gunnery rules*, and *Firing at rapidly moving targets*; to Mr. P. L. Kapitza for his work *Turbodetander for obtaining low temperatures and its application for obtaining liquid air*; to Mr. N. I. Muskhelishvili for his work *Some basic problems in the mathematical theory of elasticity*. Second Prizes were awarded to Mr. A. I. Alikhanov and Mr. A. I. Alikhanian for their scientific work on radio; to Mr. A. N. Kholmogorov and Mr. A. Y. Khinchin for their works *Asymptotic laws of the theory of probability*, *On analytical methods in the theory of probability*, and *Limit laws of the sums of independent chance quantities*; to Mr. L. S. Pontriagin for his work *Topological groups*; and to Mr. S. L. Sobolev for his works *Some questions in the theory of the expansion of vibrations* and *On the theory of non-linear hyperbolic equations with partial derivatives*.

Professor Theodore von Kármán of California Institute of Technology has been awarded the American Society of Chemical Engineers Medal for 1941.

Dr. W. H. Garrett, for thirty-nine years head of the department of mathematics and astronomy at Baker University, has been awarded the doctor of science degree by Illinois College.

Professor C. J. Blackall of the College of St. Thomas has been appointed to a professorship at De Sales College, Toledo, Ohio.

Dr. B. W. Brewer of the Agricultural and Mechanical College of Texas has been promoted to an assistant professorship.

Associate Professor C. M. Cleveland of the University of Texas has been promoted to a professorship in applied mathematics.

Dr. T. J. Higgins of Purdue University has been appointed to an assistant professorship at Tulane University.

Dr. E. E. Ingalls of Albion College has been promoted to an assistant professorship.

Dr. F. B. Jones of the University of Texas has been promoted to an assistant professorship.

Dean Mary N. Keith of the University of Redlands has retired.

Professor J. R. Kline of the University of Pennsylvania has been appointed Thomas A. Scott Professor of Mathematics.

Assistant Professor E. C. Klipple of the Agricultural and Mechanical College of Texas has been promoted to an associate professorship.

Mr. Neil Little of Purdue University has been promoted to an assistant professorship.

Assistant Professor G. A. Lyle of the U. S. Naval Academy has been promoted to an associate professorship.

Reverend P. H. McGrath of St. Peter's College, Jersey City, New Jersey, has been promoted to a professorship.

Dr. C. A. Messick of Oakland City College, Oakland City, Indiana, has been appointed to a professorship and will be head of the department of mathematics.

Assistant Professor J. E. Powell of Michigan State College has been promoted to an associate professorship.

Dr. S. M. Ulam of the University of Wisconsin has been promoted to an assistant professorship.

Miss Margaret C. Weeber of the Teachers College of Connecticut has been promoted to an assistant professorship.

Dr. H. N. Wright of the College of the City of New York will succeed Dr. F. B. Robinson as president of the College.

Dr. R. B. Kershner of Johns Hopkins University has been made an associate in mathematics.

Dr. H. H. Campaigne of the University of Minnesota is on leave to serve in the U. S. Navy.

Professor W. A. Hurwitz of Cornell University is on leave of absence at the Institute for Advanced Study.

Professor Eugenie M. Morenus of Sweet Briar College is on leave for this academic year.

Professor J. F. Ritt of Columbia University is on leave of absence for the spring term.

Dr. E. N. Shawhan of the University of Minnesota is on leave to do defense work in the Naval Ordnance Laboratory at Washington.

The following appointments to instructorships are announced: Central College, Fayette, Missouri: Dr. P. B. Burcham; Haverford College: Dr. L. C. Green, Dr. André Weil; Illinois Institute of Technology: Dr. Herman Meyer; State University of Iowa: Dr. W. D. Berg; University of Maine: Dr. Esther Comegys; University of Minnesota: Dr. Abraham Spitzbart; Montana State College: Dr. C. B. Smith; Agricultural and Mechanical College of Texas: Mr. C. H. Cunkle, Mr. W. B. Evans, Mr. H. W. Grant, Mr. J. R. Smith.

Mr. I. O. Griffith of the University of Oxford died September 22, 1941.

The death of Professor Tullio Levi-Civita of the University of Rome has been announced. He had been a member of the Society since 1904.

Professor Emile Picard, Secretary of the French Academy of Sciences, died December 12, 1941, in Paris. He was eighty-five years old.

Professor C. S. Atchison of Washington and Jefferson College died November 21, 1941, at the age of fifty-nine years. He had been a member of the Society since 1907.

It is reported that Dr. J. D. Barter lost his life at sea through enemy action.

The death of Mr. Robert Coleman of Wilberforce University has been reported.

Dean Emeritus W. P. Durfee of Hobart College died December 17, 1941, at the age of eighty-six years. He had been a member of the Society since 1891.

Professor Emeritus G. I. Gavett of the University of Washington died December 22, 1941.

Professor E. J. Hirschler of Bluffton College died May 22, 1941, at the age of sixty-five years. He had been a member of the Society since 1926.

Professor U. G. Mitchell of the University of Kansas died January 1, 1942, at the age of sixty-nine years. He had been a member of the Society since 1909.

Professor C. M. Sparrow of the University of Virginia died August 30, 1941, at the age of sixty-one years. He had been a member of the Society since 1909.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

102. A. A. Albert: *Non-associative algebras.* II. *New simple algebras.*

It is first shown that non-associative algebras A with a unity quantity e and order n have the same properties for decomposition in direct sums as do associative algebras. Let G be any multiplicative group of order n of non-singular linear transformations S on A such that $eS=e$, H be a subset containing I, g be a set of $g_{S,T}$ in A defined for every S and T of G and which are not zero divisors. Then construct the crossed extension $E=(A, G, H, g)$. It is an algebra of order nm with e as unity quantity. For separable algebras A conditions are given that E be simple and central simple. It is always simple (central simple) when A is and when $H=[I]$. Then an iterative process results for extending ordinary crossed products of order r^2 to central simple algebras of order r^4 which are necessarily non-associative. Every central simple algebra of order n may be extended by the use of an arbitrary permutation group on n letters and a class of permutation algebras is obtained. Finally, a list of fundamental unsolved problems is given. (Received January 7, 1942.)

103. R. A. Beaumont: *Projections of the prime-power abelian group of order p^m and type $(m-1, 1)$.*

A group H is the projection of a group G if there is a $(1-1)$ correspondence between the set of subgroups of G and the set of subgroups of H which preserves the partial ordering of the subgroups. Since R. Baer has given necessary and sufficient conditions that a group H be a projection of a group G which is the direct product of cyclic groups of order p, in the study of the projections of the prime-power abelian group G of order p^m and type $(m-1, 1)$, we may take $m>2$. It is shown that if $p>2$, the only group H, essentially different from G, which is a projection of G is the non-abelian group of order p^m containing an element of order p^{m-1}. If $p=2$ and $m>3$, the only group H, essentially different from G, which is a projection of G is the non-abelian group $\{U_1, U_2\}$ where U_1 and U_2 are subject to the sole defining relations: $U_1^{2^{m-1}}=U_2^2=1$, $U_2U_1U_2=U^{1+2^{m-1}}$. If $p=2$ and $m=3$, a group H is a projection of G if and only if H is isomorphic to G. (Received January 16, 1942.)

104. L. L. Dines: *On the mapping of n quadratic forms.*

A transformation $x_i=Q_i(z)$, $(i=1, 2, \cdots, n)$, in which each $Q_i(z)$ is a real quadratic form in the real variables $z^1, z^2, \cdots, z^m$, maps the m-dimensional z-space into a set of points $\mathfrak{M}$ in the n-dimensional x-space. The present paper considers properties of this map $\mathfrak{M}$, and on the basis of these properties determines necessary and sufficient conditions for the existence of linear combinations $\sum \lambda_i Q_i(x)$ which are positive definite, and also conditions for the existence of such combinations which are semi-definite. The conditions are, for general n, naturally not so simple as those obtained for the case $n=2$ (this Bulletin, vol. 47 (1941), pp. 494–498). But they appear to be simpler as well as more comprehensive than those obtained by Finsler (Commentarii Mathematici Helvetici, vol. 9 (1937), pp. 188–192), and by Hestenes and McShane (Transactions of this Society, vol. 47 (1940), pp. 501–512) with which they may be compared. The paper will appear in this Bulletin. (Received December 17, 1941.)

105. H. B. Mann: *Proof of the conjecture on the density of sums of sets of positive integers.*

Let $A(n)$ denote the number of positive integers less than or equal to n in the set A. If $A+B=C$ then it is proved in this paper that: $C(n)/n \geqq \min\ (A(x)/x+B(x)/x)$ for $x \leqq n$. The conjecture $\gamma \geqq \alpha_1+\alpha_2+\cdots+\alpha_n$ is an immediate consequence of this result. Let $n_1, n_2, \cdots, n_r, \cdots$ be the gaps in C. Let B_r^+ be the numbers of $\overline{B}$ that are not of the form n_r-a with a in A or equal to 0. The theorem is equivalent to the following statement: If for any r, $rn_i > in_r$ and $rn_i > (B_i^+(n_i)+1)n_r$ for $i=1, 2, \cdots, r-1$, then $B_r^+(n_r) \geqq r-1$. Let $n_r - n_i = d_i$ then there always exist values i, j for which $n_r - d_i = a+b$ (a in A or equal to 0, b in B or equal to 0). Among all the b's found in this way the smallest is chosen and denoted by e_1. Let B^1 contain all numbers e_1+d_s and N^1 all numbers n_t that satisfy an equation $a+e_1+d_s=n_t$. The numbers of B^1 are added to the numbers in B and the process repeated. Thus sets $B^1, B^2, \cdots, B^s$ are formed and sets $N^1, N^2, \cdots, N^s$ and it is shown that this construction can be continued until all values n_t have been absorbed into the sets N^j. The sets B^j then contain $r-1$ different numbers of B_r^+. (Received January 15, 1942.)

106. A. R. Schweitzer: *Note on functions which generate an abstract field.*

In a previous report (this Bulletin, vol. 27 (1921), p. 249) the author stated that the following functions each generate, under suitable postulational assumptions, an abstract field: $(x+y)y$, $(1+x)y$, $(x-y)y$, $(1-x)y$. The former two functions were obtained as special instances of the function $(x+y)z$ which, it was stated, also generates a field. In this note a simple a priori proof is given by exhibiting chains of definitions of functions leading to the functions $x+y$, xy, in each instance. These chains are essentially as follows: I. $(x+y)y$: 0, 1, $x+1$, $-x$, $-x+1$, $x-1$, $-(x+y)$, $x+y$, xy. II. $(1+x)y$: 0, -1, $-(1+x)$, $-xy$, $-x$, xy, x/y, $x+y$. III. $(x-y)y$: 0, -1, $-(1+x)$, $-x$, $x-1$, $-x+1$, $x+1$, $x+y$, xy. IV. $(1-x)y$: 0, 1, $1-x$, xy, x/y, $x-y$, $x+y$. Reference is made to a report by the author, this Bulletin, vol. 26 (1920), p. 441. (Received December 29, 1941.)

107. M. F. Smiley: *Elementary similarity transformations and the rational canonical form of a matrix.*

If A is a square matrix with elements in a field F and E is an F-elementary transformation matrix, then $B=EAE^{-1}$ is said to be obtained from A by an elementary similarity transformation. A process is described involving a finite number of such transformations which replaces A by the matrix $\operatorname{diag}\{B^{(1)}, \cdots, B^{(t)}\}$ where the $B^{(i)}$ $(i=1, \cdots, t)$ are matrices in rational canonical form and the characteristic matrix of each $B^{(i)}$ has just one non-trivial invariant factor. The reduction of A to rational canonical form is then easily obtained. Examples are given to support the contention that this process simplifies the computation of the rational canonical form of A. (Received December 15, 1941.)

ANALYSIS

108. G. E. Forsythe and A. C. Schaeffer: *A remark on Toeplitz matrices.*

A doubly infinite matrix (a_{mn}) is said to be regular if for every sequence $\{x_n\}$ with limit x' the corresponding sums $y_m = \sum_n a_{mn} x_n$ are defined for all m and have the limit x'. An apparently more general definition of regularity is that the sums defining y_m exist for all sufficiently large m, depending on $\{x_n\}$, and have the limit x'. Tamarkin (this Bulletin, vol. 41 (1935), pp. 241–243) has obtained necessary and sufficient conditions for the second type of regularity. This result is obtained by elementary methods and related topics are discussed. (Received January 23, 1942.)

109. H. L. Garabedian: *Hausdorff integral transformations.*

This paper involves a study of the integral transformation $v(x) = \int_0^x u(y)d\phi(y/x)$, defining a method of summation $(H, \phi(x))$, where $u(x)$ is bounded and continuous, $x \geqq 0$, and where $\phi(x)$ is either a Hausdorff mass function or satisfies the conditions: (i) $\phi(x)$ is of bounded variation on the interval $0 \leqq x \leqq 1$, (ii) $\phi(x)$ is continuous on the interval $(0, 1)$ except possibly at $x = 1$, (iii) $\phi(0) = 0$, (iv) $\phi(1) = 1$. It is proved that the transformation is regular when and only when $\phi(x)$ is a Hausdorff mass function, and sufficient conditions involving the Silverman-Schmidt integral equations are obtained in order that $(H, \phi_1(x)) \supset (H, \phi_2(x))$, in the case that $\phi_1(x)$ and $\phi_2(x)$ satisfy the conditions stated above. These results are extensions of those obtained by Silverman (Transactions of this Society, vol. 26 (1924), pp. 101–112). (Received January 10, 1942.)

110. A. M. Gelbart: *Functions of two variables with bounded real parts in domains not equivalent to the bicylinder.*

Let $f(z_1, z_2)$ be regular in the interior of a finite four-dimensional domain $\mathfrak{M}^4$, bounded by certain analytic hypersurfaces, and in general not equivalent to the bicylinder, and let $f(z_1, z_2)$ have a bounded real part in $\mathfrak{M}^4$. These domains were first considered by Bergman, and are termed by him, domains with distinguished boundary surfaces. An upper bound for $|f(z_1, z_2)|$ is obtained in terms of only max $Re\, f(z_1, z_2)$ in $\mathfrak{M}^4$, $f(0, 0)$ and the domain considered. From a formula for $\partial^{m+n} f(z_1, z_2)/\partial z_1^m \partial z_2^n$ in $\mathfrak{M}^4$, previously obtained by the author (Transactions of this Society, vol. 49 (1941), pp. 199–210), an upper bound is also obtained for $|\partial^{m+n} f(z_1, z_2)/\partial z_1^m \partial z_2^n|$, again in terms of only max $Re\, f(z_1, z_2)$ in $\mathfrak{M}^4$, $f(0, 0)$ and the domain. These results depend upon the establishment of a form of the Schwarz lemma in $\mathfrak{M}^4$ for two variables. (Received January 29, 1942.)

111. H. J. Greenberg and H. S. Wall: *Hausdorff means included between* $(C, 0)$ *and* $(C, 1)$.

It is shown that if $\phi(u)$ is any function of bounded variation on the interval $0 \leqq u \leqq +\infty$ such that $\phi(+\infty) - \phi(0) = 1$, then the function $\alpha(z) = \int_0^\infty d\phi(u)/(1+zu)$ is a regular moment function; and that when $\phi(u)$ is further restricted to be monotone then $(C, 0) \subset [H, \alpha(n)] \subset (C, 1)$. Conditions under which $[H, \alpha(n)]$ is equivalent to $(C, 0)$ or to $(C, 1)$ are obtained which are analogous to the conditions found by Scott and Wall (abstract 47-3-144) for the special case where $\phi(u)=1$ for $u \geqq 1$, $\phi(0)=0$, namely $(C, 0) \approx [H, \alpha(n)]$ if and only if $\phi(+0)-\phi(0)>0$, and $[H, \alpha(n)] \approx (C, 1)$ if and only if $\int_0^\infty d\phi(u)/u < \infty$. Certain transformations of moment sequences, for example, the Hausdorff transformation, are discussed. (Received January 14, 1942).

112. Walter Leighton and W. J. Thron: *On the convergence of continued fractions.*

In the $z=x+iy$ plane let new coordinate axes x', y' be obtained by rotating the original axes through an angle β. It is shown that if the elements a_n of the continued fraction $1+K(a_n/1)$ are complex numbers and lie in a closed bounded region in the interior of one of the parabolic regions $y'^2 \leqq \cos^2\beta/2(x'+(1/4)\ \cos^2\beta/2), -\pi<\beta<\pi$, the continued fraction converges. Further it is established that the value of this continued fraction lies in the half-plane defined by the relation $(x-1/2)\ \cos\ \beta/2+y\ \sin\ \beta/2 \leqq 0$. (Received January 8, 1942.)

113. Walter Leighton and W. J. Thron: *On value regions of continued fractions.*

If the elements $a_n=\rho e^{i\varphi}$ of a continued fraction $1+K(a_n/1)$ lie in a parabolic region $\rho \leqq 2d(1-d)/(1-\cos\ \theta)$ $(1/2<d<1)$ and if the a_n are bounded in absolute value, the continued fraction is known to converge. It is shown, that the value $z=Re^{i\vartheta}$ of this continued fraction lies in the region $R \geqq 2d(1-d)/(1-2d+\cos\ \theta)$, $-\beta<\theta<\beta$; $\beta=\text{arc cos}\ (1-2d)$. Every value in this region is taken on by at least one continued fraction $1+K(a_n/1)$ with elements in the described parabola. (Received December 12, 1941.)

114. A. N. Lowan, Gertrude Blanch, and William Horenstein: *Inversion of the q-series associated with Jacobi elliptic functions.*

In the computation of the Jacobi elliptic functions, sn, cn, and dn with the aid of the theta functions, it was found necessary to invert the expression $\epsilon=(1/2)(1-k'^{1/2})/(1+k'^{1/2})=(1/2)\theta_2(0, q^4)/\theta_3(0, q^4)$, and thus obtain q as a power series of ϵ. Weierstrass had given the first four terms of this expansion (*Werke*, II (1895), p. 276); Milne-Thomson found two additional terms (Journal of the London Mathematical Society, vol. 5 (1930), pp 148–149). The authors have found the first fourteen terms in the desired expansion. This makes it possible to compute q in terms of ϵ with an accuracy varying between seven places for $\epsilon=0.4$ to eighteen places or better for $\epsilon \leqq 0.25$. These results were obtained in the course of work by the Mathematical Tables Project, Work Projects Administration for the City of New York, conducted under the sponsorship of the National Bureau of Standards. (Received December 17, 1941.)

115. A. N. Lowan and Abraham Hillman: *A short table of the zeros of the equation* $f(x)=J_0(x)\,Y_0(kx)-J_0(kx)\,Y_0(x)=0$.

The Mathematical Tables Project conducted by the Work Projects Administration of New York City, under the sponsorship of the National Bureau of Standards, has computed a short table of the first five zeros of the above equation for $k=1$ (0.5) 4.0. The zeros were first computed with the aid of the method of McMahon, Annals of Mathematics, vol. 9 (1894–1895), pp. 23–30. Each zero was recomputed by inverse interpolation from the values of $f(x)$ for about six arguments at intervals of 0.01 in the neighborhood of the zero in question. As was anticipated from the fact that McMahon's formulae are based on the asymptotic expansions of Bessel functions, the accuracy of the zeros computed by the latter method increases with the order of the zeros. To illustrate, for $k=3$, the first zero is correct to only two decimal places whereas the fifth zero is correct to six places. When each one of the zeros has been computed for a number of k's at sufficiently small intervals, it is expected that the zero in question for any value of k within the given range will be easily obtainable by interpolation. (Received December 6, 1941.)

116. K. L. Nielsen: *Some properties of functions satisfying partial differential equations of elliptic type.*

The author considers the totality, T, of particular solutions of a partial differential equation, $L(u)=\Delta u+a_1\partial u/\partial x+a_2\partial u/\partial y+a_3 u=0$, $a_k=a_k(x, y)$, $[k=1, 2, 3]$. Using results of operators transforming analytic functions into solutions of $L(u)=0$; [see Bergman, Comptes Rendus de l'Academie des Sciences, Paris, vol. 205 (1937), p. 1360 and Matematicheskii Sbornik, vol. 44 (1937), p. 1169], it is shown that for certain types of L there is a subclass S of functions belonging to T with the property that there exists a denumerable set $u_n(x, y)\in S$ each of which satisfies an ordinary differential equation, $\sum_{\nu=1}^{k}A_\nu(x, y; n)d^\nu u(x, y)/dx^\nu$ (A_ν being algebraic functions of x and y and depending on n in a simple way), and an analogous equation with respect to y. Every $u\in S$, regular in $x^2+y^2\leq\rho^2$, can be expanded in the uniformly convergent series $\sum\alpha_n u_n(x, y)$ in this circle. The singularities of $u\in S$ may be branch points of the type that u can be decomposed into $u=u_1+u_2$, where u_2 is regular at the singular point and u_1 satisfies an ordinary differential equation analogous to the one above. If the function element $u=\sum A_{mn}x^m y^n$, $u\in S$, is given, the author indicates a procedure to determine from A_{mn} whether u has only singularities described above, and for the determination of the distances of these branch points from the origin. (Received January 29, 1942.)

117. Harry Pollard: *The generalized Stieltjes transform.*

In this paper the author studies the extension to the generalized Stieltjes transform (1) $f(x)=\int_0^\infty(x+t)^{-\rho}d\alpha(t)$ of the theory developed by Widder for the case $\rho=1$. Specifically, the following results are obtained: (i) It is established that $f(x)$ is also a Laplace-Stieltjes transform for positive x if and only if $\alpha(t)=o(t^\rho)$ as $t\rightarrow\infty$. (ii) The inversion of (1) is accomplished by means of a linear differential operator. (iii) Necessary and sufficient conditions are obtained for the representation of a function $f(x)$ in the most general form (1). (iv) Conditions are obtained for the representation of $f(x)$ in the form (1) with $\alpha(t)$ of preassigned type. The results (i) and (iii) are new even for $\rho=1$. (Received January 30, 1942.)

118. Raphaël Salem: *On singular monotonic functions of the Cantor type.*

The first part of the paper gives the construction of a singular monotonic function *of the Cantor type* with Fourier-Stieltjes coefficients of order $n^{(-1/2)+\epsilon}$ ($\epsilon>0$ as small as desired) and even $n^{-1/2}\Omega(n)$, $\Omega(n)$ increasing to infinity as slowly as desired. The second part gives some new examples of sets of uniqueness and sets of multiplicity for trigonometrical series and shows that a very simple mapping can transform a set of uniqueness into a set of multiplicity. The third part gives the construction of a continuous monotonic function $F(x)$ of the Cantor type such that $\int_0^{2\pi}\exp(n_p ix)dF$ tends to $F(2\pi)-F(0)$ for a sequence $\{n_p\}$ such that p/n_p tends to zero as slowly as desired. (Received January 27, 1942.)

119. A. C. Schaeffer: *On the oscillation of differential transforms.* III.

It is shown that if in an interval (a, b) all derivatives of a function exist and no derivative changes sign more than a fixed bounded number of times in the interval then the function is analytic in the interval. This answers a question which was raised by Pólya as a generalization of a theorem of S. Bernstein. In the case in which all derivatives of a function $f(x)$ exist for $-\infty<x<\infty$ and the function is of exponential growth, let $g(n)$ be the maximum number of variations in sign of $f^{(n)}(x)$ in any interval of length, say, 1. If $g(n)$ tends to infinity with n, but sufficiently slowly, it can be shown that $f(x)$ is an entire function of order 1. These results are obtained by first showing that there is a function $p(n)$ such that the following statement is true: if $f(x)$ is bounded by 1 in $(-1, 1)$ and its first n derivatives are continuous in this interval, then the inequality $|f'(0)|>p(n)$ implies that $f^{(n)}(x)$ changes sign at least $n-1$ times in the interval. The preceding papers in this series have been written by G. Szegö and Einar Hille. (Received January 28, 1942.)

120. L. L. Silverman and Otto Szász: *On a class of Norlund matrices.*

The definition of a Norlund matrix depends upon a sequence of numbers p_n. The Nörlund matrix is then a triangular matrix, whose elements are the numbers p_{n-k} divided by the sum of the numbers p_n from zero to n. A Nörlund matrix is defined to be of *finite rank* if $p_n\neq 0$ for some value of n, and $p_n=0$ for all greater n. A matrix of finite rank is *simple* if the numbers p_n are all zero or unity. In this paper some general properties of matrices of finite rank, and of simple matrices are obtained. A simple matrix for which *all* the numbers $p_n=1$, when $n<r$, and for which $p_n=0$ when $n\geqq r$ is called a Z-matrix. It is denoted by Z_r. The relative inclusion of corresponding summability methods among themselves and with the arithmetic mean methods is investigated. Among the results obtained are the following: if h is a factor of k then Z_h is included in Z_k; if h is prime to k, then the only sequences evaluated by both definitions are the convergent sequences. A study is also made of the inverse if the transformation which is a linear combination of the identity and Z_k. (Received January 22, 1942.)

121. Wolfgang Wasow: *On boundary layer problems in the theory of ordinary differential equations.*

Given a differential equation involving a parameter ρ in such a way that when ρ tends to infinity a "limiting" differential equation of lower order than the original one is obtained. What happens then to the solution $U(x, \rho)$ of a boundary value problem of the differential equation, if ρ tends to infinity? For ordinary linear differential equations of order n depending linearly on the parameter, and for a rather general class of boundary conditions not involving ρ, this question is answered in the paper by an easily applicable rule. This rule shows that in general $u(x)=\lim_{\rho\to\infty}U(x, \rho)$ exists only if the n boundary conditions are not too unevenly divided between the two end points. If the limit function exists it is a solution of the limiting differential equation but satisfies no longer all the boundary conditions prescribed for $U(x, \rho)$. The rule tells which of the n boundary conditions cease to be satisfied after the passage to the limit.

The proof consists in an asymptotic calculation of $U(x, \rho)$ based on the theory of asymptotic solution of ordinary linear differential equations involving a parameter, as developed by G. D. Birkhoff, Noaillon, Tamarkin, Trjitzinsky and others. (Received January 8, 1942.)

APPLIED MATHEMATICS

122. Stefan Bergman: *Two-dimensional flow around two profiles.*

The study of the influence of tail surfaces on lift and pressure distribution of a wing can be reduced to the investigation of problems in the conformal mapping of doubly connected domains. The use of orthogonal functions enables one to give simple formulas for the lift and the moment in the case of a uniform flow around one and around two profiles. In the first case the lift, L, is found to be the expression $L=4(\pi)^{1/2}\rho V^2(\sum|\phi_\nu(0)|^2)^{1/2}\sin[\alpha+\pi-\arg(\sum^*\psi_\nu(b)\phi_\nu(0))]$, where ρ is the density, $Ve^{-i\alpha}$ the velocity at infinity, $\psi_\nu(z)=\int_0^z\phi_\nu(z)dz$, and $\{\phi_\nu(z)\}$ a complete system of orthonormal polynomials of a domain B. (B is the domain obtained from the exterior of the profile by the transformation $z=1/\zeta$, b is the coordinate of the cusp, and the summation $\sum^*$ is understood in a certain special sense.) Analogous formulas exist for the moment and similar ones in the case of a flow around two profiles. (See also Notes of Lectures on Conformal Mapping, publication of Brown University, chap. XI, §§5–7) (Received January 29, 1942.)

123. Henry Wallman: *On the reduction in harmonic distortion due to high frequency pre-emphasis.* Preliminary report.

It is now common practice in high-fidelity sound broadcasting, in either FM or AM, to employ high frequency pre-emphasis, the object being an increase in signal-to-noise ratio. An additional effect, namely a reduction in harmonic distortion, has been noted experimentally. An analysis of this reduction in distortion is made in this paper, and quantitative evaluations are given for single-tone harmonic distortion of all orders. (Received December 30, 1941.)

124. Alexander Weinstein: *Spherical pendulum and complex integration.*

The following theorem, due to Puiseux (Journal de Mathématiques, 1842) is proved by a simple application of the theory of residues: The increment of the azimuth of a spherical pendulum corresponding to its passage from the lowest level z_1 to the highest level z_2 is greater than $\pi/2$. The boundary of the domain in the complex z-plane to which Cauchy's theorem is applied consists of a cut connecting z_1 with z_2 and of a vertical straight line to the right of z_2. (Received January 26, 1942.)

GEOMETRY

125. P. O. Bell: *The parametric osculating quadrics of a family of curves on a surface.*

In this paper the author investigates the properties of the parametric osculating quadrics of a family of curves on a surface. These quadrics were introduced by Dan Sun (Tôhoku Mathematical Journal, vol. 32 (1930), pp. 81–85). His definition is essentially the following: At three neighboring points P, P_1, P_2 on an asymptotic curve C_u of a surface S construct the tangents to the curves of a one-parameter family on S.

The three tangents determine a quadric whose limit, as P_1, P_2 independently approach P along C_u, is the parametric osculating quadric $Q^{(u)}$ of the one-parameter family of curves of S at P. A quadric $Q^{(v)}$ is similarly defined with respect to the other asymptotic curve C_v. The union curves of an arbitrary congruence Γ', the curves of Darboux and the curves of Segre are all characterized geometrically in association with the quadrics $Q^{(u)}$ and $Q^{(v)}$. This paper will appear in the American Journal of Mathematics. (Received January 29, 1942.)

126. Nathaniel Coburn: *Conformal geometry of unitary space.*

The principal purpose of this paper is the determination of those affinors of a unitary K_n which: (I) transform as affinors under the analytic group of K_n; (II) are invariant under conformal transformations of the fundamental tensor of K_n. An equi-conformal fundamental tensor and connection are introduced, both of which are invariant under (II). It is shown that if the group of K_n is equi-analytic (determinants of (I) are constant), then the components of the equi-conformal connection transform as do the components of the ordinary connection of K_n. By use of the equi-conformal connection, a conformal connection is determined which transforms as does the ordinary connection of K_n under (I) and which is invariant under (II). Necessary and sufficient conditions are given for the existence of $k \leqq n^2$ conformal fundamental tensors. Next, expressions for the conformal curvature affinor in terms of the ordinary connection of K_n are derived. It is shown that: (1) if the unitary space K_n is conformal unitary euclidean, then the conformal curvature affinor vanishes; (2) no unitary space of constant nonvanishing curvature $K_n (n>2)$ is conformal to a unitary euclidean space. (Received January 6, 1942.)

127. Nathaniel Coburn: *Congruences in unitary space.*

The general properties of the congruence affinors of ∞^{n-1} curves, which are imbedded in a unitary space of n dimensions, K_n, are developed. The case in which the ∞^{n-1} congruence curves are either real curves X_1 or unitary curves U_1 is completely characterized. Next, by a study of two systems of Pfaffians, two types of orthogonality are defined: (1) ∞^1 hypersurfaces which are *completely unitary* orthogonal to the congruence curves; (2) ∞^1 hypersurfaces which are *semi-unitary* orthogonal to the congruence curves. It is shown that: (1) the ∞^1 completely unitary orthogonal hypersurfaces are ∞^1 unitary K_{n-1}; (2) the ∞^1 semi-unitary orthogonal hypersurfaces are ∞^1 semi-analytic spaces X_{n-1}. An additional analytical characterization of these two types of hypersurfaces is given. The final section of the paper is devoted to two problems: (1) the characterization in terms of congruence affinors of these two types of hypersurfaces; (2) special properties of congruences associated with each type of unitary orthogonal hypersurface. The desired characterizations are obtained in each case. Further, special properties of these congruences are found. Many of these properties are extensions of congruence theorems in Riemannian space. (Received January 6, 1942.)

128. J. J. DeCicco: *A generalization of the dual-isothermal transformations.*

In this paper is discussed a generalization in the plane of the theorem that the only lineal element transformations which carry every dual-isothermal family of curves into a dual-isothermal family are $U=\phi$, $V=(a_2v+b_2w+c_2)/(a_1v+b_1w+c_1)$, $W=(a_3v+b_3w+c_3)/(a_1v+b_1w+c_1)$, where ϕ, a_k, b_k, c_k are functions of u only. A field element may be defined by (u, v, w, p, q), where (u, v, w) are the hessian coordinates of the lineal elements, $p=w_u$, and $q=w_v$. All field element to lineal element transformations which carry every dual-isothermal family into a dual-isothermal family are $U=\phi$, $V=(a_2v+b_2p+c_2)/(a_1v+b_1p+c_1)$, $W=(a_3v+b_3p+c_3)/(a_1v+b_1p+c_1)$, where ϕ, a_k, b_k, c_k are functions of u, q, and $w-qv$ only. The contact field element transformations preserving the dual-isothermal character are also determined. (In the statement of this result, those transformations which carry every field into a single dual-isothermal field are excluded.) (Received December 4, 1941.)

129. J. J. DeCicco: *Point transformations by which straight lines correspond to circles.*

By means of some of Kasner's theorems (*The generalized Beltrami problem concerning geodesic representation*, Transactions of this Society, vol. 4 (1903), pp. 149–152), the author determines all the point transformations by which all the straight lines of the XY-plane correspond to circles of the xy-plane. These form the eleven-parameter set (not a group) $X=(a_2\{x^2+y^2\}+b_2x+c_2y+d_2)/(a_1\{x^2+y^2\}+b_1x+c_1y+d_1)$, $Y=(a_3\{x^2+y^2\}+b_3x+c_3y+d_3)/(a_1\{x^2+y^2\}+b_1x+c_1y+d_1)$. The family of circles consists of the ∞^2 circles orthogonal to a fixed circle (real or imaginary). Any transformation carrying more than $6\infty^1$ circles into straight lines must belong to our set. Conversely, any correspondence converting more than $6\infty^1$ straight lines into circles must be the inverse of a correspondence of our set. Obviously the set contains both the projective and the Moebius groups. (Received December 4, 1941.)

130. Jesse Douglas: *On the geodesic surfaces of a given curve family.*

Consider a family $\mathcal{F}$ of ∞^4 curves in space defined by differential equations of the form: $y''=F(x, y, z, y', z')$, $z''=G(x, y, z, y', z')$. Definitions: (1) relative to $\mathcal{F}$, a surface S is geodesic at one of its points p if every curve of $\mathcal{F}$ tangent to S at p lies entirely upon S; (2) a (totally) *geodesic surface* $\mathcal{G}$ is one which is geodesic at each of its points. In the general case, geodesic surfaces relative to $\mathcal{F}$ are non-existent. The following are shown to be the sole possibilities in which there is at least one geodesic surface through each curve C of $\mathcal{F}$: ∞^1 surfaces $\mathcal{G}$ through each curve C of $\mathcal{F}$ (*linear* family); exactly *two* surfaces $\mathcal{G}$ through each curve C (*intersectional* family, obtainable by cutting ∞^2 surfaces S with ∞^2 surfaces S'); exactly *one* surface $\mathcal{G}$ through each curve C (*semi-intersectional*). Examples of each of the preceding are easily given. Besides, the theory brings out the following possibilities: three or four surfaces $\mathcal{G}$ through each curve C, but the *effective* existence of these types seems doubtful. If n is a finite integer greater than 4, the type: n surfaces $\mathcal{G}$ through each curve C is definitely impossible. An important application is made to the author's work on the inverse problem of the calculus of variations: every curve family $\mathcal{F}$ of *intersectional* type is *extremal*, and in a highly general manner. (Received January 22, 1942.)

131. Edward Kasner and J. J. DeCicco: *A generalization of the isothermal transformations.*

In this paper is obtained a generalization of Kasner's theorem that the only lineal element transformations which carry every isothermal family into an isothermal family are $X+iY=F(x\pm iy)$, $\Theta=a\theta+h$, where a is a nonzero constant and h is a harmonic function. A field element (flat field) is defined by (x, y, θ, p, q), where (x, y) are the cartesian coordinates of the point, θ is the inclination, $p=\theta_x$, and $q=\theta_y$. There are seven types of field element to lineal element transformations which preserve the isothermal character. In the real domain, there are three types: (I) and (II) $X+iY=F(x\pm iy,\ p\mp iq)$, $\Theta=a\theta+h$, where h is a biharmonic function; and (III) $X=\phi$, $Y=\psi$, $\Theta=h(X,\ Y)$, where ϕ and ψ are arbitrary functions, and h is a harmonic function of $(X,\ Y)$. Within the group of contact field element transformations, there are seven types. In the real domain there are five types: (I) and (II) The Kasner extended group; (III) and (IV) $p-iq=F(X\pm iY,\ x+iy)$, $\Theta=a[\theta-2\int(pdx+qdy)]+h(X,\ Y)$; and (V) $X=\phi$, $Y=\psi$, $\Theta=h(X,\ Y)$. Special cases are that the isogonal, the multiplicative, and the isocline trajectories of an isothermal family are always isothermal. (Received December 4, 1941.)

132. Edward Kasner and Don Mittleman: *A general theorem on the initial curvatures of dynamical trajectories.*

The theorem proved is an extension of Kasner's dynamical theorem which states that: If a particle starts from rest in any positional field of force, the initial curvature of the trajectory is one-third of the curvature of the line of force through the initial position. The generalized result is: If a particle starts from maximum rest in an acceleration field of order n, the initial curvature of the trajectory is $n!(n-1)!/(2n-1)!$ of the curvature of the line of force through the initial position. This paper will appear in the Proceedings of the National Academy of Sciences. (Received January 19, 1942.)

133. W. H. Roever: *Geometric statement of a fundamental theorem for four-dimensional orthographic axonometry.*

The theorem stated below makes possible the construction of three-dimensional models for the "picturization" of four-dimensional space in a manner analogous to that furnished by three-dimensional orthographic axonometry for the picturization of three-dimensional space on the plane. Theorem: Three conjugate diameters and the axis of revolution of an oblate spheroid may be regarded, as far as directions are concerned, as the orthographic projection on the three-dimensional space, in which the spheroid lies, of four mutually perpendicular concurrent axes of four-dimensional space. If these axes be taken as rectangular cartesian axes and the three-dimensional space of the spheroid as the picture space, then the ratios which the halves of the chosen conjugate diameters and the radius of the spheroid's focal circle bear to the spheroid's equatorial radius are the foreshortening ratios, that is, the numbers by which the scales on the axes of the four-dimensional space must be multiplied respectively in order to obtain those on the axonometric axes (that is, the three conjugate diameters and the axis of revolution of the given spheroid). (Received December 8, 1941.)

Logic and Foundations

134. A. R. Schweitzer: *On a class of ordered $(n+1)$-ads relevant to the algebra of logic.* I.

Within the frame of his geometric theory (American Journal of Mathematics, vol. 31, (1909), pp. 365–410, chap. 2, 3) the author generates complete classes of ordered $(n+1)$-ads $(n=1, 2, 3, \cdots)$ of generalized type of constituents in the algebra of logic by means of two equivalent processes: (1) Relatively to the given ordered $(n+1)$-ad $\alpha_1\alpha_2\cdots\alpha_{n+1}$ and the ordered dyads $(\alpha_1\lambda_1), \cdots, (\alpha_{n+1}\lambda_{n+1})$ the α's are replaced by the corresponding λ's either singly or in combination yielding $2^{n+1}(n+1)$-ads. (2) Relatively to α_1, λ_1 the elements $\alpha_2\alpha_3\cdots\alpha_{n+1}$; $\lambda_2\lambda_3\cdots\lambda_{n+1}$ are successively adjoined at the right (left) yielding the complete sets $(\alpha_1\alpha_2; \lambda_1\alpha_2; \alpha_1\lambda_2, \lambda_1\lambda_2)$, and so on. It is then assumed that the preceding $(n+1)$-ads belong to the general abstract space $S_{n+1}(G, H)$ as defined by the author (abstract 46-9-438). When $G=H$, = alternating (symmetric) group on $n+1$ variables, the α_i and λ_i are represented graphically as opposite vertices of regular polyhedra in euclidean $(n+1)$-space or opposite faces of duals of the latter. When $n=2$ the polyhedra are the octahedron and the cube. (Received January 30, 1942.)

135. A. R. Schweitzer: *On a class of ordered $(n+1)$-ads relevant to the algebra of logic.* II.

The author develops a finite algebra of logic in which the complete set of ordered $(n+1)$-ads of generalized constituent type relative to $\alpha_1\alpha_2 \cdots \alpha_{n+1}$ and the ordered dyads $(\alpha_1 \lambda_1), \cdots, (\alpha_{n+1}\lambda_{n+1})$ is expressed as a reflexive formal sum: $\sum(\alpha_1\alpha_2 \cdots \alpha_{n+1}) = \sum(I)$. The corresponding terms are $\sum(\Lambda_1\alpha_2 \cdots \alpha_{n+1})$, $\sum(\alpha_1\Lambda_2 \cdots \alpha_{n+1}), \cdots$, $\sum(\Lambda_1\ \alpha_2 \cdots \alpha_{n+1}), \cdots, \sum(\alpha_1\alpha_2 \cdots \Lambda_{n+1})$, where $\sum(\Lambda_1\alpha_2 \cdots \alpha_{n+1})$ is the sum of all $(n+1)$-ads containing α_1, and so on. Then for $n=2$, for example, $\alpha_1\alpha_2\alpha_3 = \sum(\Lambda_1\Lambda_2\Lambda_3) = \sum(\Lambda_1\alpha_2\alpha_3) \times (\alpha_1\Lambda_2\alpha_3) \times \sum(\alpha_1\alpha_2\Lambda_3)$, and so on. Also, $\sum(\Lambda_1\alpha_2\alpha_3) + \sum(\Lambda_1\alpha_2\alpha_3) = \sum(\alpha_1\alpha_2\alpha_3)$, and so on. Finally it is assumed that the preceding $(n+1)$-ads belong to the abstract relational space $S_{n+1}(G, G)$, where G is an arbitrary substitution group on $n+1$ variables (including the identical group). When G is the symmetric group the symbol $\sum(\Lambda_1\alpha_2 \cdots \alpha_{n+1})$ can be replaced by the more economical symbol $\sum(\Lambda_1)$, and so on, and this case reduces essentially to a previous development by the author (abstract 47-9-430). (Received January 30, 1942.)

STATISTICS AND PROBABILITY

136. Irving Kaplansky: *Note on a common error concerning kurtosis.*

In many text books there is to be found a statement to the effect that a frequency curve with positive (negative) kurtosis falls above (below) the corresponding normal curve in the neighborhood of the mean. In this note examples are given to show that there is actually no such connection between kurtosis and the height of the curve at its mean. (Received January 19, 1942.)

TOPOLOGY

137. Paul Civin: *Two-to-one mappings of three-dimensional sets.*

This paper is concerned with the proof of the non-existence of a continuous mapping defined on the closed three-cell in which each inverse image consists of exactly two points. Corresponding theorems for the arc and two-cell were proved by O. G. Harrold (*The non-existence of a certain type of continuous transformation*, Duke Mathematical Journal, vol. 5 (1939), pp. 789–793) and J. H. Roberts (*Two-to-one transformations*, Duke Mathematical Journal, vol. 6 (1940), pp. 256–262), respectively. (Received January 26, 1942.)

138. F. B. Jones: *A certain non-metric Moore space.*

The purpose of this paper is to give an example of a non-metric Moore space which is nevertheless the sum of a monotone increasing collection of completely separable domains each lying together with its boundary in the next. (Received December 31, 1941.)

139. J. L. Kelley: *Symmetric product spaces.* Preliminary report.

Let X be a metric separable arcwise connected space. If every pair (x, y) and (y, x) of $X \times X$ is identified, the resulting space is the symmetric product space X_2. Denote by X^* the subset of all points (x, x) of X_2. Then (a) the fundamental group of X_2 is always abelian; (b) if X is a polyhedron a necessary and sufficient condition that X be contractible is that X^* be a retract of X_2. (Received December 29, 1941.)

140. A. D. Wallace: *Chains and the structure of continua.*

The author gives a decomposition of compact ($=$bicompact) connected H-spaces into chains and p-chains analogous to that given by Kelley and Moore for separable spaces. It is shown that (i) if p is neither an end point nor a cut point then there is a point conjugate to p, (ii) hence each such point is contained in a unique p-chain, (iii) the meet of p-chains is null or a cut point, (iv) a chain is characterized as a continuum which contains each p-chain meeting it in at least two points, (v) each p-chain is the meet of a collection $|A|$ of continua such that $\overline{S-A}$ contains a finite set of components each containing one point of the p-chain. Examples show that chains and J-sets (Kelley) are distinct; p-chains and simple links and F-sets are equivalent in metric spaces and in general a simple link is a p-chain. (Received December 5, 1941.)

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HUREWICZ, W., and WALLMAN, H. Dimension Theory. (Princeton Mathematical Series, vol. 4.) Princeton University Press; London, Humphrey Milford and Oxford University Press, 1941. 7+165 pp. $3.00.

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MCARTHUR, N., and KEITH A. Intermediate Algebra. London, Methuen, 1942. 10+356 pp. 8s. 6d.

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SCHILPP, P. A., edited by. The Philosophy of Alfred North Whitehead. (The Library of Living Philosophers, vol. 3.) Evanston and Chicago, Illinois, Northwestern University, 1941. 20+745 pp. $4 00.

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WALLMAN, H. See HUREWICZ, W.

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STATISTICAL GENETICS AND EVOLUTION

SEWALL WRIGHT

Introduction. When Darwin developed the theory of evolution by natural selection, practically nothing was known of hereditary differences beyond their existence. Since 1900, a body of knowledge on the mechanism of heredity and on mutation has been built up by experiment that challenges any field in the biological sciences in the extent and precision of its results. The implications for evolution are not, however, immediately obvious. It is necessary to work out the statistical consequences.

Studies in the field of statistical genetics began shortly after the rediscovery of Mendelian heredity in 1900. Those of J. B. S. Haldane [7] and R. A. Fisher [4] have been especially important with respect to the application to evolution. My own approach to the subject came through experimental studies conducted in the U. S. Bureau of Animal Industry on the effects of inbreeding, crossbreeding and selection on populations of guinea pigs [21, 22, 23, 37] and through the attempt to formulate principles applicable to livestock breeding [19, 20, 24, 25, 13, 34]. On moving into the more academic atmosphere of the University of Chicago, I have become more directly concerned with the problem of evolution.

I should note that the deductive approach, to which I shall confine myself here, involves many questions that can only be settled by observation and experimental work on natural populations and that a remarkable resurgence of interest in such work is in progress [2, 9].

Postulates. It will be desirable to begin with a brief review of the more important factors of which account must be taken.

The basic fact of modern genetics is that heredity can be analyzed into separable units, "genes," whose most essential property is that of duplicating themselves with extraordinary precision, irrespective of the characteristics of the organism in whose cells they are carried. We shall restrict consideration to changes in the system of genes and aggregates of genes (chromosomes). There are relatively rare and obscure hereditary changes which must be attributed to other cell components but our knowledge of these does not warrant the elaboration of a statistical theory.

Fortunately the same theory applies to a large extent to gene muta-

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The sixteenth Josiah Willard Gibbs Lecture, delivered at Chicago, Illinois, September 3, 1941, under the auspices of the American Mathematical Society; received by the editors October 20, 1941.

tions and to most classes of grosser chromosomal changes (duplications, deficiencies, inversions, translocations, and so on). It will be assumed here that a given kind of mutation occurs at a constant rate per generation. Observed rates in organisms as remote as corn plants, vinegar flies and man are of the order of 10^{-5} or less per generation. Reverse mutation may occur at measurable rates.

It is simplest to deal with mere pairs of alternative conditions (alleles) but the theory remains seriously inadequate unless capable of extension to multiple alleles.

In general I shall assume that the reproductive cells are haploid (that is, contain just one representative from each set of alleles) and that their union results in diploid individuals (with two such representatives in all cells, until reduction occurs in the formation of the germ cells). This is the usual case but there are species in which other situations prevail (tetraploids, hexaploids, aneuploids, and so on). The group of sex linked genes constitutes an important special case in many otherwise completely diploid organisms (including man). I shall not go far into the extension to these cases.

It is simplest to assume that the members of different series of alleles are distributed at random in the reduction division by which the reproductive cells receive a half sample of the genes of the individuals producing them (that, for example, individual *AaBb* produces germ cells *AB*, *Ab*, *aB* and *ab* in equal numbers). The phenomenon of partial linkage, exhibited by genes carried in the same chromosome should, however, be taken into account. These are the principal postulates as far as the mechanism of heredity is concerned though others are required in special cases.

The relations of genes to observed characteristics are important. In general, any measurable character is affected by genes at many loci and a single gene often has multiple apparently unrelated effects. The effects of genes in combinations are often roughly cumulative but marked exceptions are also very common. Account must be taken of noncumulative effects within series of alleles (dominance) and between series (gene interaction).

The breeding structure of the population is important. The situation in nature is so complex that models must be chosen that are compromises between mathematical simplicity and biological adequacy [**35**]. I shall introduce only the simplest models in the course of the present discussion.

Natural selection is an exceedingly complex affair. Selection may occur at various biological levels—between members of the same brood, between individuals of the same local population, between such populations (as through differential increase and migration) and finally between different species, a subject that carries us outside the field of genetics and which has been discussed mathematically by Lotka [12], Volterra [17] and Nicholson and Bailie [14]. Selection among individuals may relate to the mating activities of one or both sexes, to differences in rate of attainment of maturity, to differential fecundity and to differential mortality at all ages. Selection may act steadily or may vary both in intensity and direction in different regions and at different times. Again I can only deal here with the simplest models.

Gene frequency. In such a complex situation, verbal discussion tends toward a championing of one or another factor. We need a means of considering all factors at once in a quantitative fashion. For this we need a common measure for such diverse factors as mutation, crossbreeding, natural selection and isolation. At first sight these seem to be incommensurables but if we fix attention on their effects on populations, rather than on their own natures, the situation is simplified. Such a measure may be found in the effects on *gene frequency* in each series of alleles.

Because of the complete symmetry of the Mendelian mechanism, gene frequency has no tendency to change in an indefinitely large closed population not subject to mutation or selection. Each homozygote (for example, A_1A_1, A_2A_2 or A_3A_3) produces only one kind of germ cell. Each heterozygote (for example, A_1A_2, A_1A_3, A_2A_3) produces two kinds in *equal* numbers. In a population in which the array of gene frequencies is $(q_1A_1+q_2A_2+\cdots+q_mA_m)$ (letting the q's represent the frequencies, and the A's the genes) the frequencies of genotypes come to equilibrium according to the terms in the expansion of $(q_1A_1+q_2A_2+\cdots+q_mA_m)^2$ in the first generation of random mating after attainment of equality of gene frequencies in the sexes [8]. Under sex linkage [10, 15] and in polyploids [6] equilibrium is not reached at once but is rapidly approached. Inbreeding and assortative mating change the relative frequencies of homozygotes and heterozygotes but not the gene frequencies.

One immediate consequence of this persistence of gene frequencies is that variability tends to persist. But the slightest continuing unbalanced pressure on gene frequency tends to cause cumulative change. It is obvious that recurrent mutation, immigration, selection, and the accidents of sampling in an isolated population of small size are all factors that can bring about such change.

The frequencies of combinations of different series of alleles (for example, A, a; B, b) reach equilibrium in a random breeding population in which gene frequencies are constant only when the genes are combined at random (terms of $[(1-q_A)a+q_AA]^2[(1-q_B)b+q_BB]^2$). Equilibrium is not reached immediately, however. The departure from equilibrium is halved in each generation of random mating in the case of two pairs of alleles in different chromosomes. In general, the departure is reduced by the proportion $\bar{c}$, where $\bar{c}$ is the mean chance of recombination [**18**, **11**, **16**].

Systematic changes of gene frequencies. The rate at which gene frequency changes under recurrent mutation is obvious [**27**]. Let q be the frequency of the gene and u the rate at which it mutates to its alleles as a group and Δq the rate of change of q per generation $\Delta q = -uq$.

If reverse mutation occurs at the rate v per generation, the net rate of change of q is

$$\Delta q = v(1 - q) - uq. \tag{1}$$

In the case of multiple alleles, v is the weighted average for the various alleles of the gene in question and is thus a function of their relative frequencies. It is, however, independent of q.

The effect of crossbreeding is similar if we adopt the simplest model [**27**]. If a population with gene frequency q exchanges the proportion m each generation with a random sample of immigrants from the whole species (gene frequency q_t) the rate of change in gene frequency is

$$\Delta q = - m(q - q_t). \tag{2}$$

In actual cases the immigrants are not likely to be a random sample from the whole species but to come largely from neighboring populations. Effective m is thus, in general, smaller than the apparent amount of immigration and is not necessarily the same for all loci. There may also be selective migration. The simplest model must suffice here. It permits identification of the theories of mutation and immigration by substituting mq_t for v, and $m(1-q_t)$ for u.

The effects of selection have been considered extensively by Haldane [**7**] in terms of the frequency ratio $(q/1-q)$ and by Fisher [**4**]. As there can be no selection pressure without at least two alternatives, any expression for it, applicable to all values of q, must include the factor $q(1-q)$, excluding certain limiting cases. Thus the form $aq(1-q)$ has been used by Fisher as the basis for general discussion. For the present purpose somewhat less general forms are more useful. Consider first the case of a random breeding population of diploid individuals in which the combinations of paired alleles may be assigned constant relative selective values [30].

Genotype	*Frequency* (f)	*Value* (W)
AA	q^2	W_{AA}
AA'	$2q(1-q)$	$W_{AA'}$
$A'A'$	$(1-q)^2$	$W_{A'A'}$

For the frequency of A after a generation

$$q_1 = [W_{AA}q^2 + W_{AA'}q(1-q)]/\overline{W}$$

where

$$\overline{W} = \sum fW = W_{AA}q^2 + 2W_{AA'}q(1-q) + W_{A'A'}(1-q)^2,$$

$$(3)\ \Delta q = q_1 - q = q(1-q)[W_{AA}q + W_{AA'}(1-2q) - W_{A'A'}(1-q)]/\overline{W},$$

$$\Delta q = \frac{q(1-q)}{2\overline{W}}\,\frac{d\overline{W}}{dq}.$$

Selection, however, really applies to the organism as a whole not to single series of alleles. If the population is heterallelic in n pairs of pertinent alleles, the number of possible combinations is 3^n. Each of these has a certain frequency and a certain relative selective value, the latter of which we here assume to be constant. If the three phases in the A series of alleles are combined at random with the combinations of the other series, the average selective values of AA, AA' and $A'A'$ are independent of q_A although functions of the other q's. Thus

$$(4)\qquad \Delta q_A = \frac{q_A(1-q_A)}{2\overline{W}}\,\frac{\partial\overline{W}}{\partial q_A}.$$

We have assumed only pairs of alleles, but as any group of alleles may be treated formally as one, this formula may be applied to multiple allelic series. The selective values W_{AA}, $W_{AA'}$ and $W_{A'A'}$ are then functions of the relative frequencies within the group of alleles of the gene under consideration, but not of q_A.

In previous general discussions (for example [35, 36]) I have restricted myself to this convenient model of selection pressure. As this has given rise to misapprehension [5], it should be emphasized that it applies only under the conditions implied in its derivation.

If there are selective differences between the sexes, as is very likely to be the case, there are departures from random combination within series of alleles. These are, however, unimportant for most purposes unless there is rather strong selection.

Selection itself tends to bring about departures from random combination among different series of alleles. Again the effects are unimportant in most cases, especially if all relative selective differences are slight.

The formula must be written in a more generalized form to include polyploidy [32] and sex linkage. For small selective differences in a $2k$-ploid

$$\Delta q_A = \frac{q_A(1-q_A)}{2k\bar{W}} \frac{\partial \bar{W}}{\partial q_A}. \tag{5}$$

This applies (approximately) under sex linkage if $k=3/4$ and $\bar{W}=(\bar{W}_m\bar{W}_f)^{1/2}$ where $\bar{W}_m$ and $\bar{W}_f$ are the mean selective values in males and females, respectively.

There may be departures from random mating because of a constant tendency toward mating of relatives, giving the following genotypic frequencies within a series of alleles [23, 3].

Genotype	*Frequency*
AA	$(1-F)q^2+Fq$
AA'	$2(1-F)q(1-q)$
$A'A'$	$(1-F)(1-q)^2+F(1-q)$

Random combination between series of alleles is not disturbed appreciably if the selective differences are small or if the inbreeding coefficient F is small, giving the following formula in which $\bar{W}_R$ and $\bar{W}_I$ are mean selective values of the random bred and inbred components of the frequencies relative to the A series [3]

$$\begin{aligned} \Delta q_A &= \frac{q_A(1-q_A)}{\bar{W}}\left[\frac{(1-F)}{2}\frac{\partial \bar{W}_R}{\partial q_A} + F\frac{\partial \bar{W}_I}{\partial q_A}\right] \\ &= \frac{q_A(1-q_A)}{2\bar{W}}\left[\frac{\partial \bar{W}}{\partial q_A} + F\frac{\partial \bar{W}_I}{\partial q_A}\right]. \end{aligned} \tag{6}$$

Inbreeding that leads to subdivision into partially isolated groups is best dealt with by a different mathematical model.

Under assortative mating based on similarity in characteristics there are very great departures from random combination of different series of alleles [20]. Again we may best consider such a mating system as one leading to subdivision of the population into partially isolated groups.

Returning to consideration of random breeding populations, it may easily be seen that if the W's are functions of q_A

$$(7)\qquad \Delta q_A = \frac{q_A(1-q_A)}{2\bar{W}}\left[\frac{\partial \bar{W}}{\partial q_A} - \sum\left(f\frac{\partial W}{\partial q_A}\right)\right].$$

Intra-brood selection is an example of a case in which $\partial\bar{W}/\partial q_A = 0$ but gene frequency nevertheless changes. As in our model case, the process does not necessarily lead to fixation of the most favorable of the genotypes possible from the genes present in the population.

Finally, if the genes under consideration affect the system of mating itself, as is the case of the self-sterility alleles of many plants [**33**] or of genes that determine self fertilization [**5**], the changes in gene frequency can only be found from the composition of the population in successive generations.

We may note here that while, in principle, selection must be considered to apply to the organism as a whole, one may analyze the organism into character complexes which evolve largely independently through changes in largely independent systems of genes, the components of which are distributed at random among the chromosomes. The case which we have chosen as a model of selection pressure (4) and its generalization (5) should apply sufficiently well to most reaction systems in freely interbreeding populations and gives an insight into certain aspects of the effects of selection which cannot be obtained as easily from the more complex special cases.

If a character complex is affected by n loci and m_i alleles at a particular locus, it requires

$$\sum_{i=1}^{n}(m_i - 1)$$

dimensions to represent the system of gene frequencies and

$$\prod_{i=1}^{n}[m_i(m_i + 1)/2]$$

kinds of genotypes are possible. Assuming that the relative selective values of these genotypes are independent of their frequencies, the mean selective values ($\bar{W}$) of possible random breeding populations form a surface relative to this multi-dimensional system of gene frequencies, the gradient of which determines the way in which the population tends to change under the influence of selection. The number of loci that may affect even the simplest characters are known in certain cases to be great and many, if not all such loci are probably represented at all times by multiple alleles. Thus the number $\prod_{i=1}^{n} m_i$ of homozygous types possible from genes actually present in a species and affecting a particular character complex may often run into astronomical figures. Under these conditions it is to be expected that in general the surface $\overline{W}$ for any character complex will have numerous peaks, corresponding not only to different combinations of genes that give the same character [**29**] but also to different harmonious combinations of elementary characters that permit the organism to overcome the same conditions in different ways.

Mutation, immigration and selection may all be occurring simultaneously. The net rate of change of gene frequency may be obtained by simply adding the contributions of these factors (1), (2), (4), if these are small. In our ideal case of a random breeding population of diploid individuals subject to reversible mutation, immigration and constant selective differences between genotypes [**27, 30**]

$$\Delta q = v(1-q) - uq - m(q - q_I) + \frac{q(1-q)}{2\overline{W}} \frac{\partial \overline{W}}{\partial q} \cdot \tag{8}$$

There is equilibrium, stable or unstable, if $\Delta q = 0$. With reversible mutation, there must be at least one gene frequency other than 0 or 1 that is in stable equilibrium. There may also be stable equilibrium as a result of opposing selection pressures alone.

Accidents of sampling. There is another possibility of change of gene frequency to be considered. In a population that is not indefinitely large, gene frequency may be expected to change from generation to generation merely from the accidents of sampling. The composition of a population of N diploid individuals depends on that of $2N$ gametes produced by the preceding generation. If these are a random sample from the array $[(1-q)A' + qA]$, the probability array for values of q in the next generation is $[(1-q)A' + qA]^{2N}$ with the standard deviation $(q(1-q)/2N)^{1/2}$. We will call a random deviation of q of this sort δq in contrast with the systematic deviation Δq produced by mutation, migration or selection

$$\sigma^2_{\delta q} = \frac{q(1-q)}{2N} \cdot \tag{9}$$

It might seem that these random deviations would be negligible in any reasonably large population but in the absence of any systematic pressure toward equilibrium, the squared standard deviation for later generations increases approximately with the number of generations until there is an approach to the limiting value $q(1-q)$ of complete fixation, $[(1-q)A'A' + qAA]$. The exact value for the nth generation is $q(1-q)[1-(1-1/2N)^n]$. The rate of fixation of heterallelic genes approaches $1/2N$ per generation.

The effective value of N should often be much smaller than its apparent value [**27, 35**]. It obviously applies only to individuals that reach maturity. If there is cyclic variation in population size, N is more closely related to the minimum than to the maximum number. It is also reduced if there is excessive variability in the number of mature offspring from different parents.

In a $2k$-ploid population [**32**],

$$\sigma_{\delta q}^2 = \frac{q(1-q)}{2kN} \text{ approximately.} \tag{10}$$

For sex linked genes it is approximately $q(1-q)[2/9N_f+1/9N_m]$ where N_f and N_m are the effective numbers of females and males, respectively, and thus is $2q(1-q)/3N$ if these are equal.

The distribution of gene frequencies in the case of equilibrium. The tendency toward a stable equilibrium in the value of q, found when there are opposing systematic pressures, and the tendency to drift away from this point, due to the sampling variance, should result in a probability distribution which one might expect to find realized by the values taken by the frequency of a particular gene over a long period of generations in the ideal case of a population in which all conditions remain constant. It would also be the distribution of values of q taken by this gene at a given time in an array of completely isolated populations, all of which are subject to the same conditions. Finally, all genes subject to systematic pressures of the same magnitude should exhibit such a distribution at one time in a single population. While these are ideal cases, not likely to be approached in actual cases, it is of primary importance in the genetics of populations to be able to reach conclusions on the nature of such distributions.

The distribution of gene frequencies in the case of equilibrium must satisfy the conditions of stability of the mean

$$(\overline{q + \Delta q + \delta q} = \bar{q})$$

and stability of the variance $(\sigma^2_{(q+\Delta q+\delta q)} = \sigma^2_q)$. The possible values of q must range from 0 to 1. It is convenient to use integration for summation in expanding these expressions. Let $\phi(q)$ be the ordinate of the required distribution. The formula of the distribution may be derived as follows [**30, 31**]

$$\int_0^1 (q + \Delta q + \delta q)\phi(q)dq = \int_0^1 q\phi(q)dq,$$

$$\int_0^1 (q + \Delta q + \delta q - \bar{q})^2 \phi(q) dq = \int_0^1 (q - \bar{q})^2 \phi(q) dq.$$

Since the mean value of δq is 0 and since δq is not correlated with $(q+\Delta q)$, these reduce to the following, omitting a term involving $(\Delta q)^2$, negligible if Δq is small

$$\int_0^1 \Delta q \phi(q) dq = 0,$$

$$2\int_0^1 (q - \bar{q}) \Delta q \phi(q) dq + \int_0^1 \sigma_{\delta q}^2 \phi(q) dq = 0.$$

Let $\int \Delta q \phi(q) dq = \chi(q)$ and integrate the first term of the preceding equation by parts

$$\chi(1) - \chi(0) = 0,$$

$$\int_0^1 \chi(q) dq - [\bar{q}\chi(0) + (1 - \bar{q})\chi(1)] - (1/2)\int_0^1 \sigma_{\delta q}^2 \phi(q) dq = 0.$$

It may be found by trial that both of these conditions are satisfied by the following equation if $\phi(0)$ and $\phi(1)$ are finite. Note that $\sigma_{\delta q}^2 = 0$ if $q = 0$ or if $q = 1$, since there can be no sampling variance unless there are alternatives

$$\chi(q) - \chi(1) = (1/2)\sigma_{\delta q}^2 \phi(q),$$

$$d \log [\chi(q) - \chi(1)] = \frac{d\chi(q)}{\chi(q) - \chi(1)} = \frac{2\Delta q dq}{\sigma_{\delta q}^2},$$

$$[\chi(q) - \chi(1)] = \frac{C}{2} e^{2\int (\Delta q/\sigma_{\delta q}^2) dq},$$

$$\phi(q) = (C/\sigma_{\delta q}^2) e^{2\int (\Delta q/\sigma_{\delta q}^2) dq}, \tag{11}$$

where C is a constant such that $\int_0^1 \phi(q) dq = 1$.

The frequency of a particular value of q is approximately $f(q) = \phi(q)/2N$. The amount of exchange between the subterminal and terminal classes is approximately half the frequency of the former from consideration of the Poisson distributions of the classes that are close to fixation [27].

For the model case in which $\Delta q = v(1 - q) - uq - m(q - q_t) + (q(1-q)/2\bar{W})(\partial \bar{W}/\partial q)$ and $\sigma_{\delta q}^2 = q(1-q)/2N$ equation (11) reduces to the following

$$\phi(q) = C\bar{W}^{2N}q^{4N(mq_t+v)-1}(1-q)^{4N[m(1-q_t)+u]-1}, \tag{12}$$

$$\begin{aligned} f(0) &= f(1/2N)/4N[mq_t+v], \\ f(1) &= f(1-1/2N)/4N[m(1-q_t)+u]. \end{aligned} \tag{13}$$

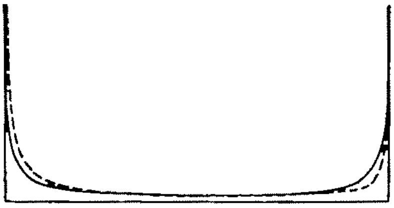

FIG. 1

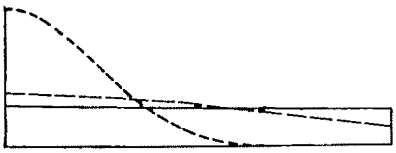

FIG. 2

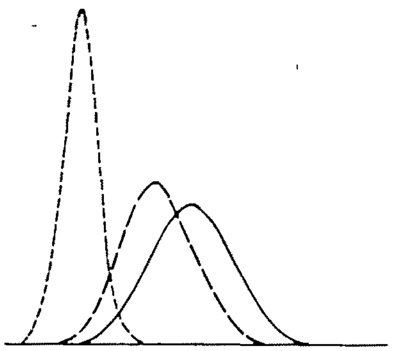

FIG. 3

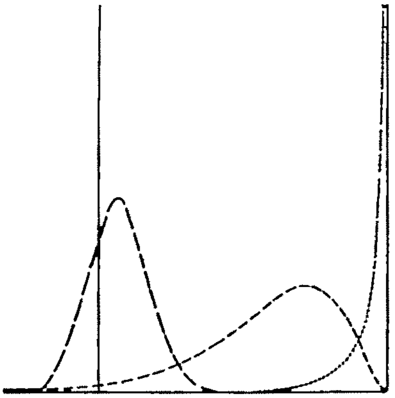

FIG. 4

Figures 1 to 3. Some of the forms taken by the distribution of frequencies of a completely recessive gene. Mutation rates are assumed equal in both directions ($u=v$). $N=1/40v$ in Figure 1, $10/40v$ in Figure 2 and $100/40v$ in Figure 3. In each case the solid line represents the least selection ($t=-v/5$), the long dashes represent selection 10 times as severe (omitted in Figure 1 since practically indistinguishable from the preceding) and the short dashes represent selection 100 times as severe $\phi(q)=Ce^{2Ntq^2}q^{4Nv-1}(1-q)^{4Nv-1}$.

Figure 4. Frequency distribution of a semidominant gene in subgroups of a large population in which the varying conditions of selection among subgroups has lead to a mean gene frequency, $q_t=.25$. The subgroups represented are assumed to be of the same size ($N=1000$) and subject to the same selection pressure ($W_{AA}=1$, $W_{AA'}=.9975$, $W_{A'A'}=.995$) but to different degrees of isolation (long dashes: $m=.01$, short dashes: $m=.001$, dots $m=.0001$), $\phi(q)=Ce^{10q}q^{1000m-1}(1-q)^{3000m-1}$.

This brings the effects of reversible mutation, crossbreeding, selection and size of population into a single formula. Figures 1 to 4 show the forms taken by this distribution in certain special cases. The U-shaped distributions in small populations (Figure 1) may be compared with the I-shaped ones in large populations (Figure 3). The figures bring out the relatively slight effects of selection in small populations.

The joint frequency distribution for multiple pairs of alleles may be written as follows for the model case that we have been considering [30]

$$(14)\quad \phi(q_1, q_2, \cdots, q_n) = C\overline{W}^{2N}\prod_{i=1}^{n} q_i^{4N(m_i q_{Ii}+v_i)-1}(1 - q_i)^{4N[m_i(1-q_{Ii})+u_i]-1}.$$

This applies to $2k$-ploids if $4Nk$ is substituted for $4N$ in the exponents of q_i and $(1-q_i)$. In the case of sex linkage and equal numbers of the sexes, $3N$ is to be substituted for $4N$ in these exponents. As the exponent of $\overline{W}$ is not affected in these cases, the formula applies to joint distributions including different degrees of ploidy (aneuploids) and both autosomal and sex-linked genes.

Figure 5 illustrates the frequencies along two diagonals of the joint distribution for two pairs of alleles which act cumulatively on the same character of which the midgrade is optimum. There are two peak frequencies corresponding to approximate fixation of two different genotypes that give the midgrade of the character. In cases involving large numbers of genes there may be an indefinitely large number of peak frequencies.

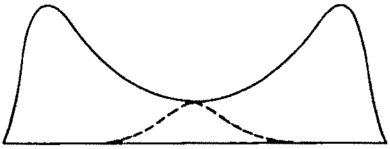

FIG. 5

The frequencies along the diagonals of the joint distribution for two series of alleles with equal and additive effects on a character on which adverse selection acts according to the square of the deviation from the mean ($W_{AAbb}=W_{AaBb}=W_{aaBB}=1$, $W_{AABb}=W_{AaBB}=W_{Aabb}=W_{aaBb}=1-s$, $W_{AABB}=W_{aabb}=1-4s$). The solid line shows the frequencies along the line connecting the two favorable types, $AAbb$ and $aaBB$. The dashes show the frequencies along the line connecting the extreme types $AABB$ and $aabb$. In the case shown, $u_a=v_a=u_b=v_b$, $N=1/2v_a$, $s=5v_a$.

The distribution of gene frequencies under irreversible mutation. It is also important to determine the form taken by the distribution of gene frequencies when fixation of one of the alleles is an irreversible process. The distribution curve should reach constancy of form, but all class frequencies (except that in which fixation is occurring) should fall off at a uniform rate K. The conditions may be expressed as follows [31]

$$\begin{aligned} \overline{q + \Delta q + \delta q} &= (1 - K)\bar{q} + K, \\ \sigma^2_{(q+\Delta q+\delta q)} &= (1 - K)\sigma^2_q + K(1 - \bar{q})^2, \\ K &= (1/2)f(1 - 1/2N). \end{aligned}$$

It can be found by trial that with mutation at rate v, no selection, and $\sigma^2_{\delta q} = q(1-q)/2N$ those conditions are satisfied by the following equation, with decay at rate $K = v$ per generation [31]

$$f(q) = 2vq^{4Nv-1}. \tag{15}$$

I have not been able to obtain a general solution comparable to equation (11) but formulae have been obtained for an important class of cases by another method [27, 30, 31]. Random breeding diploid populations with frequency array $[(1-q)A' + qA]$ are distributed in the following generation according to the expression $[(1-q-\Delta q)A' + (q+\Delta q)A]^{2N}$. Letting $p = 1-q$, the contribution to the frequency $f(q_0)$ of populations characterized by gene frequency q_0, is thus $[(2N)!/(2Np_0)!(2Nq_0)!]\,(p-\Delta q)^{2Np_0}(q+\Delta q)^{2Nq_0}f(q)$. The condition that this frequency be reconstructed after a generation except for a reduction by the amount K can be represented sufficiently accurately as follows:

$$(1 - K)\phi(q_0) = \frac{\Gamma(2N)}{p_0 q_0 \Gamma(2Np_0)\Gamma(2Nq_0)} \int_0^1 (p - \Delta q)^{2Np_0}(q + \Delta q)^{2Nq_0}\phi(q)dq.$$

If $K = 0$, $\Delta q = 0$, this equation is satisfied by $\phi(q) = Cq^{-1} + D(1-q)^{-1}$ for any values of C and D. For irreversible mutation at rate v, it yields equation (15) and for migration and reversible mutation, but no selection, the same result as obtained from equation (12) [27, 31].

Selection pressure gives more difficulty. An important case that can be solved is that for very rare mutations ($4Nv$ negligibly small) subject to selection pressure of the fairly general form $\Delta q = (s + tq)q(1-q)$. Here K may be taken as 0

$$2Nf(q_0) = A \int_0^1 p^{2Np_0}q^{2Nq_0}[1 - q(s + tq)]^{2Np_0}[1 + p(s + tq)]^{2Nq_0}\phi(q)dq,$$

where

$$A = \frac{\Gamma(2N)}{p_0 q_0 \Gamma(2Np_0)\Gamma(2Nq_0)}.$$

The following approximate relations may be used

$$[(1 - q(s + tq)]^{2Np_0} = e^{-2Np_0q(s+tq)}[1 - Np_0q^2(s + tq)^2],$$
$$[(1 + p(s + tq)]^{2Nq_0} = e^{2Nq_0p(s+tq)}[1 - Nq_0p^2(s + tq)^2].$$

Let $\phi(q) = e^{2Nsq+Ntq^2}(C_0 + C_1q + C_2q^2 + \cdots)/q(1-q)$ and use the approximate relation

$$\frac{\Gamma(2N)}{\Gamma(2Np_0)\Gamma(2Nq_0)}\int_0^1 p^{2Np_0-1}q^{2Nq_0-1+x}dq$$
$$= q_0^x + [q_0^{x-1} - q_0^x][x(x - 1)/4N].$$

It may be found that $C_m = [(4N^2s^2 + 2Nt)C_{m-2} + 8N^2stC_{m-3} + 4N^2t^2C_{m-4}]/m(m+1)$ ignoring terms in which the exponent of N is less than the sum of those of s and t. After further reduction

$$(16)\quad f(q) = [Ce^{4Nsq+2Ntq^2} + Dqe^{2Nsq+Ntq^2}\psi(2Nsq, 2Ntq^2)]/q(1 - q)$$

where C and D are any constants and $\psi(a, b)$ is as follows

$$\begin{aligned}\psi(a, b) = 1 &+ \frac{a^2}{3!} + \frac{a^4}{5!} + \frac{a^6}{7!} + \frac{a^8}{9!} + \cdots\\ &+ b(1 + a)\left[\frac{1}{3!} + \frac{2a^2}{5!} + \frac{3a^4}{7!} + \frac{4a^6}{9!} + \cdots\right]\\ &+ b^2\left[\frac{7}{5!} + \frac{2a}{5!} + \frac{69a^2}{7!} + \frac{6a^3}{7!} + \frac{282a^4}{9!} + \frac{12a^5}{9!} + \cdots\right]\\ &+ b^3\left[\frac{27}{7!} + \frac{27a}{7!} + \frac{348a^2}{9!} + \frac{204a^3}{9!} + \cdots\right]\\ &+ b^4\left[\frac{321}{9!} + \frac{132a}{9!} + \cdots\right]\\ &+ b^5\left[\frac{2265}{11!} + \cdots\right] + \cdots.\end{aligned}$$

The special cases of most interest are those of irreversible mutation in one direction or the other and of equilibrium. Consider a population in which $2Ns$ and $2Nt$ have given values but N is indefinitely large and hence s and t are indefinitely small. If mutation is occurring from the class $q = 0$ at an exceedingly minute rate v with irreversible fixation in the class $q = 1$, the frequency of the subterminal class, $q = 1/2N$, must be approximately $4Nvf(0)$ while that of the other subterminal class ($q = 1 - 1/2N$) must be so much smaller as to be negligible. The following are sufficiently accurate, letting $f(0) = 1$

$$f(1/2N) = 2N\left[C + \frac{D}{2N}\right] = 4Nv,$$

$$f(1 - 1/2N) = 2N\left[Ce^{4Ns+2Nt} + De^{2Ns+Nt}\psi(2Ns,\, 2Nt)\right] = 0.$$

As $\psi(2Ns, 2Nt)$ is of the order of e^{2Ns+Nt}, $D/2N$ is negligible compared with C

$$C = 2v,$$

$$D = -\frac{2ve^{2Ns+Nt}}{\psi(2Ns,\, 2Nt)},$$

$$(17)\ f(q) = \frac{2v}{q(1-q)}\left[e^{4Nsq+2Ntq^2} - qe^{2Ns(1+q)+Nt(1+q^2)}\,\frac{\psi(2Nsq,\, 2Ntq^2)}{\psi(2Ns,\, 2Nt)}\right].$$

In the subterminal regions selection is practically inoperative, $\Delta q = s/2N$, approximately for $q = 1/2N$; and $\Delta q = (s+t)/2N$ for $q = (1-1/2N)$. The formula of the curve in the neighborhood of $q = 1/2N$ is therefore approximately $2v/q$. Thus with given v but different values of N, $f(1/2N)(=4Nv)$ always falls very nearly on the same smooth curve. The relation between the selection pressure and the sampling variance (which measures the additive effect of sampling) is constant for given values of $2Ns$, $2Nt$ and q, $(\Delta q/\sigma^2_{\delta q}) = (2Ns+2Ntq)$. Thus the smoothed probability curve, $f(q)$, should be the same throughout with given $2Ns$ and $2Nt$ irrespective of the values of N and of s, t separately. Thus equation (17) is the general formula for the case of irreversible mutation from the class at $q = 0$.

The determination of $\psi(a, b)$ in specific cases is a rather formidable task since it involves two variables and converges slowly. For the case of semidominance however, $t = 0$ and $\psi(2Nsq, 0)$ reduces to $(e^{2Nsq} - e^{-2Nsq})/4Nsq$

$$(18)\qquad f(q) = \frac{2v}{q(1-q)}\,\frac{(1 - e^{-4Ns(1-q)})}{(1 - e^{-4Ns})}.$$

This agrees with a result obtained by Fisher [4] by a different method, involving a transformation of scale ($\theta = \cos^{-1}(1-2q)$) designed to make sampling variance uniform, and expression of the conditions in the form of a differential equation. The chance of fixation of a mutation in this case is given by the ratio of the subterminal classes $(f(1-1/2N)/f(1/2N))$ and is thus $2s/(1-e^{-4Ns})$ where s is the selection favoring the heterozygote. This is practically constant $(2s)$ in large populations. There is a small chance of fixation of even unfavorable mutations $(2s/(e^{4Ns}-1)$. Figure 6 illustrates the distribution curve for various intensities of selection (but with the selection favoring heterozygotes represented by $(1/2)s$ instead of s).

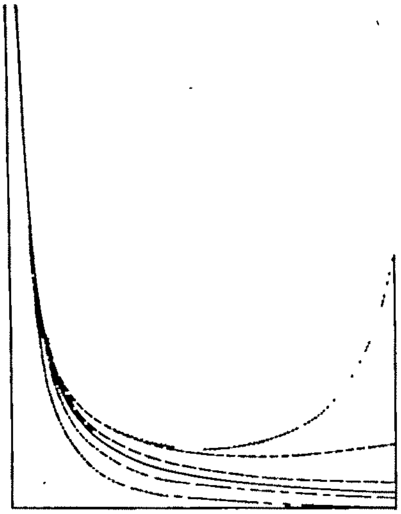

FIG. 6

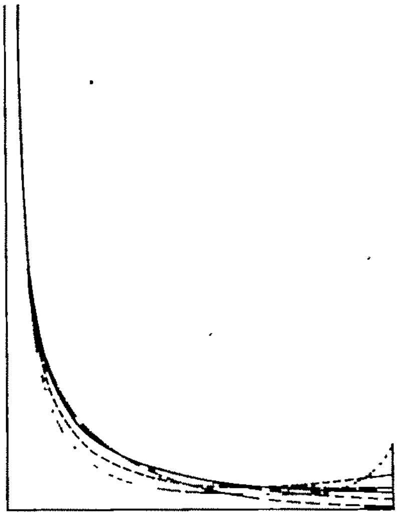

FIG. 7

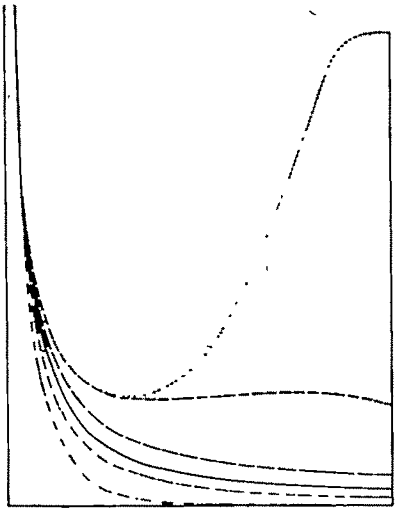

FIG. 8

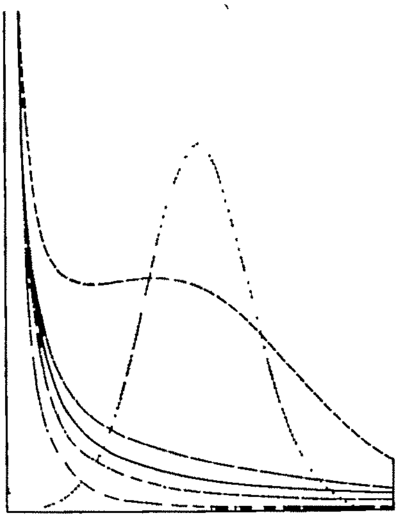

FIG. 9

Figures 6 to 9. Distributions of gene frequencies under irreversible mutation at indefinitely low rates (v). Six cases are shown in each figure. Dash and two dots, $s=-4/2N$; dash and dot, $s=-1/2N$; solid line, $s=0$ ($f(q)=2v/q$ in all figures); long

In the case of recessive mutations, $s=0$ and $\Delta q = tq^2(1-q)$. The law of the series for $\psi(0,\, b)$ is that $E_m = (E_{m-1}+E_{m-2})/2m(2m-1)$ where E_m is the coefficient of b^m. The distribution curve for a number of values of $2Nt$ are shown in Figure 7 (but with s instead of t as the selection coefficient). The chance of fixation was determined empirically from the ratio of subterminal frequencies up to $2Nt=12$. For larger values (up to $2Nt=64$) it was more convenient to calculate it from the ratio of the flux $(2N\, \Delta q\, f(q))$ in the region of maximum Δq, $(q=2/3)$ to the proportion of mutations $(2Nv)$ (usable only for large $\Delta q/\sigma^2_{\delta q}$). For values of t ranging from $4/2N$ to $64/2N$ the average chance of fixation came out $1.1(t/2N)^{1/2}$, apparently approaching $(t/2N)^{1/2}$ and thus a function of N even in large populations contrary to the case of semidominance.

In the case of dominant mutations, $\Delta q = (s-sq)q(1-q)$ if mutations are taken as occurring from the class $q=0$. It is more convenient to assume that they are occurring from the class $q=1$ $(\Delta q = -sq^2(1-q))$ since this merely requires evaluation of $\psi(0,\ -2Nsq^2)$ instead of the two-dimensional series $\psi(2Nsq,\ -2Nsq^2)$. From considerations analogous to those discussed above, it appears that for $\Delta q = (s+tq)q(1-q)$ but irreversible mutation from the class at $q=1$, $C=0$, $D=2ve^{-2Ns-Nt}/\psi(2Ns,\, 2Nt)$ to a sufficient approximation,

$$(19)\qquad f(q) = \frac{2v}{q(1-q)}\left[qe^{-2Ns(1-q)-Nt(1-q^2)}\,\frac{\psi(2Nsq,\ 2Ntq^2)}{\psi(2Ns,\ 2Nt)}\right].$$

---

dashes, $s=1/2N$, short dashes, $s=4/2N$; dots, $s=16/2N$. The ordinate at $q=1/2N$ is the same for all curves $(f(q)=4Nv)$ and far above the range of the figures (except in the case $s=16/2N$, Figure 9, in which all ordinates are greatly reduced).

Figure 6. The case of a semidominant mutation A' $(W_{AA}=1,\ W_{AA'}=1+s/2,\ W_{A'A'}=1+s)$. The probabilities (P) of fixation of a mutation are as follows: For $s=-4/2N$, $P=0.075/2N$; for $s=-1/2N$, $P=0.58/2N$; for $s=0$, $P=1/2N$; for $s=1/2N$, $P=1.6/2N$; for $s=4/2N$, $P=4.1/2N$; for $s=16/2N$, $P=16/2N$; for large s, $P=s$.

Figure 7. The case of a recessive mutation a, $(W_{AA}=W_{Aa}=1,\ W_{aa}=1+s)$. For $s=-4/2N$, $P=0.12/2N$; for $s=-1/2N$, $P=0.70/2N$; for $s=0$, $P=1/2N$; for $s=1/2N$, $P=1.3/2N$; for $s=4/2N$, $P=2.3/2N$; for $s=16/2N$, $P=4.3/2N$; for large s, $P=(s/2N)^{1/2}$.

Figure 8. The case of a dominant mutation A $(W_{aa}=1,\ W_{Aa}=W_{AA}=1+s)$. For $s=-4/2N$, $P=0.042/2N$; for $s=-1/2N$, $P=0.49/2N$; for $s=0$, $P=1/2N$; for $s=1/2N$, $P=1.9/2N$; for $s=4/2N$, $P=6.6/2N$; for $s=16/2N$, $P=31/2N$; for large s, $P=2s$.

Figure 9. The case of a mutation A', selected only in heterozygotes $(W_{AA}=W_{A'A'}=1,\ W_{AA'}=1+s)$. For $s=-1/4N$, $P=0.23/2N$; for $s=-1/2N$, $P=0.71/2N$; for $s=0$, $P=1/2N$; for $s=1/2N$, $P=1.4/2N$; for $s=4/2N$, $P=3.2/2N$; for large s, P approaches 0 and all loci tend to become heterallelic.

Figure 8 shows the form taken by the distribution in special cases (with s as the selection favoring the dominant mutation and mutation taken as occurring from the class $q=0$). The chance of fixation of favorable dominant mutation with large N is approximately $2s$ which is the same as for favorable semidominant mutations provided that s is the selection favoring the heterozygotes in both cases. This is to be expected since homozygotes are relatively rare until mutation has passed through the point of maximum selection pressure.

Figure 9 shows the distribution curve in the case in which there is no selective difference between the homozygotes but selection favors or opposes the heterozygotes. In this case in which $\Delta q = s(1-2q)q(1-q)$, formula (17) was used for values of $2Ns$ up to 4. For large values of $2Ns$, there is so little fixation that the distribution under equilibrium may be used.

By combining the formulae for irreversible mutation in each direction in such a ratio that the amounts of fixation are equal, we obtain a distribution identical with that of reversible mutation occurring at rates at which $4Nv$ and $4Nu$ are negligible. The result agrees with the limiting value obtained from equation (12)

$$f(q) = \frac{C}{q(1-q)} \left[e^{4Nsq+2Ntq^2}\right]. \tag{20}$$

The evolutionary process. I can go only briefly into the implications for evolution. We must distinguish two processes (a) the transformation of a single population until it has become so different that a new species or higher category must be recognized and (b) the cleavage of species.

Consider first the possibilities of transformation in a very large, closed, freely interbreeding population, living under conditions that are the same on the average for thousands of generations [28]. In such a population, random changes in gene frequency are negligible. Gene frequencies can change only according to the systematic pressures of mutation and selection, a process which must stop when all Δq's become zero, unless there is a flow of untried mutations that are favorable from the first. We have a theory of the stability of species in spite of variability due to continually occurring mutations and in spite of continuous action of selection.

A stable state of this sort may be far from being the most adaptive of the systems possible from the genes actually present. Consider here the situation with respect to characters to which our model selection pressure applies. As noted, many distinct peak values are to be expected in the surface of selective values, $\bar{W}$, relative to the multi-dimensional system of gene frequencies. In a species located at a particular point in the system, each gene frequency will change until all Δq's are zero. These changes will be such that the mean selective value of the populations changes approximately by the amount $\Delta\bar{W}=\Sigma(\Delta q\partial\bar{W}/\partial q)$, the species moving up the steepest gradient in the surface $\bar{W}$ except as affected by mutation pressures [29]. It stops when $\Delta\bar{W}=0$, a point in the neighborhood of the peak toward which selection has been directed, but not at the highest point because of the mutation terms in the Δq's. In general there will be other peaks on the surface $\bar{W}$ that are higher but the species cannot reach them.

Perhaps, however, we have been too hasty in assuming that all Δq's would ever reach zero simultaneously. It is probable that the potential alleles at each locus form an indefinitely extended series in which any one allele can give rise to certain others, these to ones at two removes from the first, and so on. A continual flow of untried favorable mutations may keep the population in a state of flux. In general, however, it would seem probable that the rate of the movement toward the equilibrium point indicated by the existing genes would be of a higher order of magnitude than the rate of elevation of this peak by the occurrence of mutations of this very unusual sort.

In a population in approximate equilibrium, the variability due to the balance between mutation and selection is not likely to be great. If, for example, $\Delta q=sq(1-q)-uq$, $q=1-u/s$ at equilibrium. As mutation rates are typically of the order 10^{-5} or less, q is close to 1 if the gene in question has an appreciable advantage. Loci in which there are opposing selection pressures would contribute more to variability. This may occur where a heterozygote combines favorable effects of two genes (Δq of form $q(1-q)[s_2-(s_1+s_2)q]$ with stable equilibrium at $q=s_2/(s_1+s_2)$ if both s_1 and s_2 are positive). It may also occur where different homozygotes have advantages in different ecological niches occupied by the species in such a way that the net selective values are related inversely to the frequencies (a case to which our model applies only approximately). With only one or the same few alleles maintained at high frequencies by the population the chance for the occurrence of fixation of untried favorable mutations at several removes from those present is small. The extreme improbability of such mutations justifies use of the formulae for irreversible mutation at very low rates ($4Nv$ much less than 1) in spite of the fact that we are considering large populations (17), (19) (Figures 6 to 9). There is the possibility of an indefinitely continuing evolutionary process but the rate is restricted by the low probability of the necessary mutations and the incomplete utilization of the potentialities for adaptation provided by the genes actually present (cf. however Fisher [4] and critique [26]).

Conditions are, however, continually changing. Selection of increased severity but unchanged direction merely carries the location of the system of gene frequencies closer to the peak and increases somewhat the chance for a novel favorable mutation to reach fixation. With secular changes in the *direction* of selection, on the other hand, peaks in the surface $\bar{W}$ may become depressed and low places elevated. In species which are sufficiently labile to avoid extinction, the system of gene frequencies is kept continually on the move. It is likely to be shuffled into regions of $\bar{W}$ that are in general the higher ones. This process is undoubtedly of great importance for evolutionary change.

In sufficiently small populations, the random divergences of gene frequencies from their equilibrium values become important. In very small populations, these tend to bring about approximate fixation of some random (and therefore, in general, non-adaptive) combination of genes (Figure 1). Moreover, while selection pressure is less effective in small populations than in large ones, mutation pressure remains the same. Random mutations are more likely to be degenerative than adaptive. Long continued reduction in the size of a population is thus likely to lead to extinction. On the other hand, the less extreme random variations found in populations of intermediate size ($4Nv$ or $4Ns$ of the order 1 to 10) (cf. Figures 2, 3, 5) act somewhat like changes in the direction of the selection. The system of gene frequencies is kept continually on the move and this gives a trial and error process which at times may lead to adaptive combinations which would not have been reached by direct selection. The rate of change of this sort is slow under the required conditions.

Consider next a large population that is divided into many small partially isolated races. These may differ in size and degree of isolation and in the direction and severity of the selection to which they are subjected. The conditions are present for an extensive testing by trial and error of a relatively large number of alleles at each locus and of different combinations of these.

Local differences in direction of selection are effective if the selection coefficient s (writing selection pressure in the form $\Delta q = sq(1-q)$) exceeds the crossbreeding coefficient m (cf. Figure 4). While these differences are primarily of merely local significance, there is the possibility of acquirement of an adaptation that turns out to be of general value which then may spread throughout the species. Moreover, if different alleles come to be characteristic of different races, the store of variability of the species as a whole is increased. This is also true of each local race as a consequence of crossbreeding.

In races in which $4Nm$ is of the order 1 to 10 there is considerable random differentiation without the approach to fixation in equally small completely isolated groups. Such changes may occur with considerable rapidity in this case and while non-adaptive are not necessarily anti-adaptive to an appreciable extent. It is merely that the location of the system of gene frequencies on the surface of selective values in our model case is not constrained to move up the *steepest* gradient but may move up gradients that are not the steepest and occasionally even down hill. Among the many local races exploring the neighborhood of a peak in the surface of selective values, one or more may reach a gradient leading to a higher peak (cf. Figure 10).

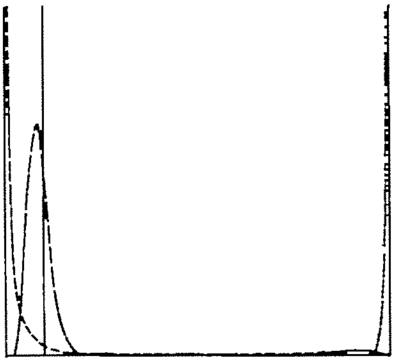

FIG. 10

The frequencies along the diagonal (0, 0) to (1, 1) of the joint distribution of frequencies of recessive genes, *a* and *b* in subgroups of the same size ($N = 1000$) but different degrees of isolation (long dashes: $m = .01$; short dashes: $m = .001$; dots: $m = .0001$) in a large population in which the conditions are everywhere the same (each recessive with selective disadvantage of .001 relative to the type $A-B-$ but the double recessive with an advantage .01 over type). With mutation rates $u_a = v_a = u_b = v_b = 10^{-5}$ there would be two positions of stable equilibrium, one at about (0.1, 0.1) which is assumed to hold for the major portion of the species for historical reasons, and one at about (.999, .999) which is more adaptive

$$\phi(q_a, q_b) = C[1 - .002(q_a^2 + q_b^2) + .012q_a^2q_b^2]^{2000}(q_aq_b)^{400m-1}[(1 - q_a)(1 - q_b)]^{3600m-1}.$$

Local populations for which $m = .01$ vary only slightly from the frequencies characteristic of the species. Most of those for which $m = .001$ show a close approach to fixation of the type genes ($AABB$) but occasionally there is an approach to fixation of the more adaptive double recessive. Those with $m = .0001$ are largely homallelic in *aabb*. In the long run such superior subgroups might be expected to pull the whole species to this position by intergroup selection. The scales are not the same in this figure.

By expansion of numbers and excess migration such races tend to bring the species as a whole under control of this peak. Intergroup selection of this sort, with respect to racial differentiation that has jointly adaptive and non-adaptive aspects, seems to provide the most effective mechanism for testing many alleles at each locus and many combinations of these and is thus the most effective mechanism for a continuing evolutionary process [26, 27, 28, 29, 35, 36].

It should be emphasized that we are not concerned here with local races as incipient species. As long as isolation is incomplete the races are bound together by crossbreeding and thus are carried along by the evolution of the species as a whole although subject to the minor kaleidoscopic changes in character which according to this theory play a major role in the evolution of the whole.

The cleavage of species depends on virtually complete isolation of portions of the species from each other. Even if there are no significant character differences at the time of separation and even if conditions remain substantially the same for the two portions, the process described above will insure that they drift apart. Each continues to be adapted to the conditions but in somewhat different ways. They may be expected to move to increasingly more remote peaks on the surface $\bar{W}$. In the course of time genic and chromosomal differences may be expected to accumulate that prevent crossing and so clinch the specific distinction. Before this point is reached, the occasional occurrence of hybridization may transfer blocks of genes from one species to the other or lead to the origin of a completely hybrid species, presenting a mechanism of reticulate evolution, analogous to that described above but on a coarser scale (cf. [1]).

Under certain conditions the multiplication and diversification of species may be a very rapid process. These include a relaxation of the general selection pressure on the species permitting great increase in numbers and great variability; the opportunity for the occupation of widely distinct ecological niches associated with almost complete isolation of the groups seizing these opportunities and with subdivision of these groups into partially isolated local populations. A species that is the first of its general kind to reach unoccupied territory finds most at least of these conditions realized. This is also the case with a species that by any means acquires an adaptation of first rate general significance which gives its subgroups an advantage over species already established in various ecological niches, that more than compensates for the initial lack of special adaptations for these niches.

The most general conclusion that can be drawn from the attempt to develop a mathematical theory of the simultaneous effects of all statistical processes that affect the genetic composition of populations is that in general the most favorable conditions for evolutionary advance are found when these are balanced against each other in certain ways, rather than where any one completely dominates the situation. Finally it may be said that the more detailed knowledge of heredity and mutation that is now available confirms Darwin's general contention that evolution is a process of statistical transformation of populations.

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UNIVERSITY OF CHICAGO

MATHEMATICAL PROBLEMS CONNECTED WITH THE BENDING AND BUCKLING OF ELASTIC PLATES

J. J. STOKER

1. Introduction. As part of his Gibbs Lecture of last year Theodore von Kármán [5] considered briefly a number of nonlinear problems concerning thin elastic plates. This address will include discussion of linear as well as of nonlinear problems concerning elastic plates, with special stress on a particular aspect of the nonlinear problems. However, it is not my intention to give a more or less complete survey of the literature on the subject; my purpose is rather to present in some detail a few specific leading ideas and points of view and to call attention to a number of unsolved problems.

The mathematical problems connected with thin elastic plates belong with the many problems from mechanics which have interested eminent mathematicians and from which mathematical ideas of fundamental importance have originated. These problems are of at least equal interest in engineering. In fact, a considerable part of the recent interest in the subject of thin plates can be attributed to the need for numerical solutions of problems which arise in the design of the thin-walled structures used in aircraft. The latter problems are in the main nonlinear boundary value problems of an involved type about which almost nothing of a general nature is known, and, aside from the technical difficulties encountered in attempting to obtain numerical solutions, such problems present questions of a purely mathematical nature which require elucidation. The latter part of this address is largely concerned with the mathematical problems which arise in this connection.

The first attempts to formulate mathematically the problem of flexure of thin elastic plates were probably inspired by the experimental researches of Chladni in 1787 on the modes of vibration of thin elastic plates. During the next twenty years a number of unsuccessful attacks were made upon the problem by James Bernoulli the Younger and others. In 1809 the French Academy offered a prize for a theory of the vibrating plate, which, after some controversy, was awarded to Sophie Germain in 1815. Lagrange, who was a member of the prize committee, corrected the theory of Sophie Germain and derived the partial differential equation as we know it now. In

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An address delivered before the New York meeting of the Society on February 21 1941, by invitation of the Program Committee; received by the editors November 6, 1941.

1822 Cauchy, prompted by his membership on a committee to consider a paper of Navier on plates, derived for the first time the general linear theory of elasticity substantially in its present form. It is noteworthy that a relatively complicated special problem in elasticity had been formulated, in the main correctly, before the general linear theory of elasticity was developed. One is tempted to enlarge upon the fact that mathematicians of those days felt no distinction between pure and applied mathematics and often found inspiration in quite special problems taken from mechanics.

2. The linear bending problem. The next outstanding historical event in the theory of plates occurred in 1850 with the publication of a famous paper by Kirchoff. In order to explain the significance of the work of Kirchoff it is necessary to formulate briefly the problem of the bending of a uniform thin plate subjected to forces perpendicular to the plane of the plate. Assuming that the thickness h of the plate is small in comparison with its other dimensions and that the deflection under load is small in comparison with h, it is shown that all stresses in the plate can be calculated in terms of the deflection $w(x, y)$ of the "middle surface" of the plate (Figure 1). The function w is shown to satisfy the linear differential equation:

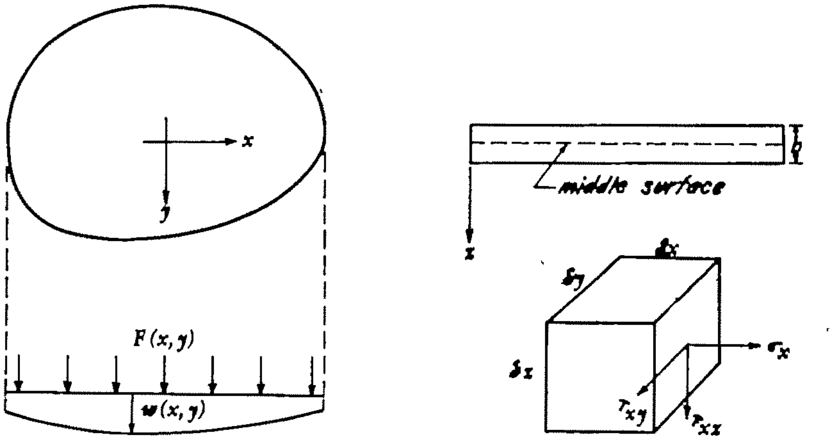

Fig. 1

$$N\Delta\Delta w = F(x, y), \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1}$$

in which F is the force per unit area applied at right angles to the plane of the plate and N is a constant depending only upon the thickness h and the material of the plate. Once a solution $w(x, y)$ of (1) has been obtained the most important quantities from the practical point of view, that is, the corresponding stresses, can be calculated. More specifically, the normal stress σ_x and the shear stresses τ_{xy} and τ_{xz} (on an element perpendicular to the x-axis), for example, are determined from w through the formulas

$$\sigma_x = \alpha_1\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right), \tag{2a}$$

$$\tau_{xy} = \alpha_2\left(\frac{\partial^2 w}{\partial x \partial y}\right), \tag{2b}$$

$$\tau_{xz} = \alpha_3\,\frac{\partial}{\partial x}\,(\Delta w), \tag{2c}$$

where α_1, α_2, α_3 depend only upon the thickness of the plate and upon the distance z from the middle surface to the point where the stresses are to be found. The essential point here is that the α_i are independent of x, y, and w. The quantity ν in (2a) is a material constant called Poisson's ratio.

To solve the problem posed by Chladni's experiments would require the solution of the linear eigenvalue problem associated with (1) (that is, that belonging with the biharmonic equation) under the homogeneous boundary conditions resulting from the assumption that no forces are applied at the edges of the plate. This would mean, one would suppose, that all three of the expressions on the right side of (2) should vanish at an edge perpendicular to the x-axis. But the differential equation is of order four, and consequently only two independent boundary conditions can be imposed. This puzzling fact had not escaped the notice of the early workers in the field, but it remained for Kirchoff [9] to settle the question. Kirchoff started from the usual assumptions of the theory, deduced the expression for the potential of the deflected plate, and found by the calculus of variations not only the differential equation (1) but also the correct boundary conditions. At a free edge he found that condition (2a) equated to zero continues to hold but that conditions (2b) and (2c) must be replaced by the single relation (for an element of the boundary perpendicular to the x-axis)

$$q_x = \frac{\partial^3 w}{\partial x^3} + (2 - \nu)\,\frac{\partial^3 w}{\partial x \partial y^2} = 0. \tag{2d}$$

This was probably the first striking illustration of the power inherent in the calculus of variations to furnish the correct and appropriate boundary conditions for a differential equation. Kirchoff had found what are now called the natural boundary conditions. The idea of natural boundary conditions, when raised to the level of a general concept, has proved to be a fruitful one which has been extensively exploited, for example, in the work of R. Courant and his pupils.

Finally, Kelvin and Tait [8] showed in 1876 how the result of Kirchoff could be reconciled physically with the mechanics of the situation. They observed that conditions (2a, b, c) could all be prescribed for a *thick* plate and that plausible physical arguments could be advanced indicating that conditions (2b) and (2c) coalesce into (2d) when the thickness of the plate is made small. This aspect of the problem has never been given a rigorous mathematical formulation. It is in fact an example of a "boundary layer" problem—a type of problem to be discussed later in another connection.

A vast amount of work has been done in solving the special linear boundary value problems arising from (1) for plates of various shapes and for various prescribed loading functions $F(x, y)$ and conditions at the boundaries. Many of the cases arising in practice have never been solved in closed form or by series, and resort has been taken to various methods of approximation. One of the most famous of these is the method of Ritz [14]. It is interesting to note that Ritz himself applied his method first to a number of plate problems. Among the problems solved by Ritz is included the first and only solution of the Chladni problem, which had remained unsolved for more than a century. The work of Ritz represents one of the early successes of what are now called direct methods of the calculus of variations. However, the method of finite differences has perhaps proved even more serviceable in obtaining approximate numerical solutions of the linear bending problems [cf. **10**].

Although the linear bending problem can be considered as solved in its essentials from the point of view of the mathematician, since the relevant existence and uniqueness theorems are available, there remain nevertheless some unsolved special problems of mathematical interest. For example, there is the curious problem of the rectangular plate resting along its edges on rigid supports and subjected to a constant downward distributed pressure F. It is easily shown that a *downward* concentrated force is necessary at the corners of the plate in order to maintain the edges in contact with the supports; if such forces are not provided, experiments show that the corners will lift up, the plate remaining in contact with the supports only in the central portions of the sides. This presents a boundary value problem of unusual type: The boundary conditions are not given a priori in the form of equalities, since the position of the points at which the edge of the plate rises from the supports is not known in advance. The condition $\sigma_x = 0$ holds everywhere on a boundary line $x = \text{const.}$, but there are two alternatives for the other boundary condition, that is, either $w = 0$ with $q_x \leq 0$ or $w < 0$ with $q_x = 0$. In spite of this, the problem probably has a unique solution.

3. **The von Kármán equations.** The linear theory of bending described above was sufficient for most practical purposes until rather recently. With increased use of very thin metal plates, particularly in the structures used in aircraft, it has become of practical importance to obtain solutions based on a theory which would permit deflections much greater than the thickness of the plate. The experiments indicate that the linear theory fails to mirror the physical facts if deflections as little as one-fourth of the thickness of the plate occur. We owe to von Kármán [6] an extension of the theory in which the squares of the slopes of the deflected middle surface are not neglected, as is the case with the linear theory. At the same time, the hypothesis of the linear theory that the stresses in the middle surface are zero is given up. The resulting equations of von Kármán are the following pair of fourth order nonlinear differential equations for two functions, the deflection $w(x, y)$ and a function $\phi(x, y)$ from which the stresses in the middle surface, often called the "membrane" stresses, are derived:

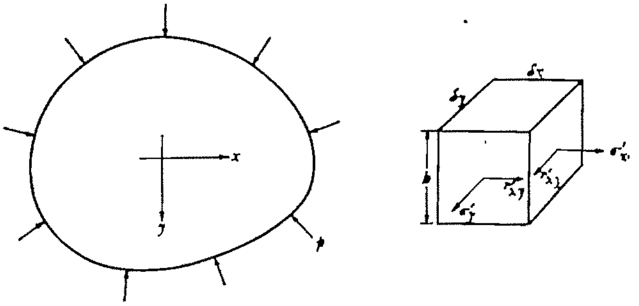

FIG. 2

$$(3)\qquad N\Delta\Delta w = F(x, y) + h\left\{\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right\},$$

$$(4)\qquad \Delta\Delta\phi = E\left\{\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right\}.$$

The quantity h is the thickness of the plate and E is a material constant. The stresses σ_x', τ_{xy}', σ_y', in the middle surface of the plate (Figure 2) are obtained from ϕ by the formulas

$$(5)\qquad \sigma_x' = \frac{\partial^2\phi}{\partial y^2}, \qquad \tau_{xy}' = -\frac{\partial^2\phi}{\partial x\partial y}, \qquad \sigma_y' = \frac{\partial^2\phi}{\partial x^2}.$$

The boundary conditions for w are linear; they remain the same as for the linear bending problem. At a free edge, for example, the Kirchoff conditions are to be satisfied. Boundary conditions for ϕ are obtained from (5) in accordance with whatever "horizontal" forces p are assumed at the boundary of the plate (Figure 2), the term "horizontal" forces being used to distinguish such forces from the "vertical" forces F. It is easily shown that this is equivalent to prescribing the values of ϕ and its normal derivative $d\phi/dn$ at the boundary of the plate.

4. **Buckling problems.** A special class of problems is obtained from (3) and (4) by assuming $F\equiv 0$; this we shall do from now on. In other words, there are assumed to be no lateral forces to cause bending. In addition we take always homogeneous boundary conditions for w. The problems that arise under these circumstances we refer to as buckling problems.

For the sake of simplicity we assume also that the horizontal forces at the boundary are normal *compressive* forces which depend linearly on a factor λ. This is equivalent to taking for ϕ the boundary conditions

$$(5)_1\qquad \phi = \lambda\Phi, \qquad d\phi/dn = \lambda\Psi,$$

where Φ and Ψ are given functions defined at the boundary of the plate and λ is a factor proportional to the applied pressure. Under these circumstances, it is clear that $w\equiv 0$ is always a solution of (3), since w is assumed to satisfy homogeneous boundary conditions. This is also the *unique* solution for w when λ is small enough (in other words, when the applied compressive forces are small enough). However, there is always a critical value λ_c of λ at which the plane state becomes instable and the plate bends, or buckles, in the engineering terminology. Mathematically this means that a bifurcation of the solutions (cf. Figure 4) takes place for $\lambda = \lambda_c$ and solutions appear for which w is not identically zero. In this behavior we are reminded of linear eigenvalue problems, though the theory of such problems is naturally not applicable here. Nevertheless it can be shown that the *lowest* value λ_c of λ for which a bifurcation takes place (or, for which buckling just begins) can always be found from the linear eigenvalue problem which results when the right-hand side of (4) is set equal to zero. This implies that the solution $\phi = \lambda\phi_0$ of (4) does not depend upon w, and hence

$$(6)\qquad N\Delta\Delta w = \lambda h\left\{\frac{\partial^2\phi_0}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2\phi_0}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2\phi_0}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right\}$$

together with linear homogeneous boundary conditions for w then presents a linear eigenvalue problem from which the lowest value of λ for a bifurcation of the solution can be determined.[1]

As in the case of the linear bending problem, a considerable literature has grown up around the eigenvalue problems posed by (6) and many special cases have been treated. Many difficult special cases of practical importance were solved, for example, by Timoshenko [15]; the method used by him bears strong resemblance to the Ritz method. The solutions of these problems have, in general, the properties one is accustomed to associate with linear eigenvalue problems: they possess a discrete spectrum and the eigenfunctions form a complete system in terms of which "arbitrary" functions can be developed.

The problems of existence and uniqueness of the solutions of the eigenvalue problems associated with (6) have been settled in the main. These questions have, however, been reconsidered in recent years by A. Weinstein [17, 18] from a new point of view. Weinstein has shown that certain classes of plate problems (vibration problems as well as buckling problems) are equivalent to an appropriately chosen *sequence* of second order membrane problems. An interesting by-product of Weinstein's work is that his method furnishes for the eigenvalues *lower* bounds which can be calculated numerically.

If a plate is proportioned in such a way that it collapses when compressed by loads only slightly greater than that at which buckling just begins, the linear theory of buckling is evidently sufficient for practical purposes. But very thin metal plates of the type used in various modern structures do not fail even when the applied pressure is many times greater than that at which buckling begins. As a consequence, it is a problem of practical as well as mathematical interest to pursue the solutions of equations (3) and (4) into the nonlinear range for high values of the ratio λ/λ_c. In fact, it is highly desirable to investigate if possible the asymptotic behavior of the solutions as λ/λ_c tends to infinity.

The most important case from the practical point of view would be that of the rectangular plate. It is clear that the numerical solution of the boundary value problem posed by equations (3) and (4) in this case under prescribed boundary conditions and for all values of λ is one of very great difficulty. Some approximate solutions (for example, see [11]) have been given by the energy and perturbation methods, the latter being, roughly speaking, a solution in the form of a power series development with respect to $\lambda - \lambda_s$. It is difficult to estimate the accuracy of such solutions, but it seems certain that they are accurate only for rather low values of the ratio λ/λ_s.

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[1] This approach to equation (6) is not the historical one; equation (6) was derived by G. H. Bryan [1] before the von Kármán equations were known.

5. **The buckled plate as a "boundary layer" problem.** In view of the above remarks, it would seem highly desirable to investigate rigorously and in detail, even if only in a quite special case, the nature of the solutions of (3) and (4), with $F \equiv 0$, in their dependence on λ for an unlimited range of values of the ratio λ/λ_s. One would then be in a better position to judge the degree of accuracy to be expected in applying approximation methods, such as the perturbation method, to other cases not easily solved explicitly. K. Friedrichs and the author have found and solved such a special case [**2**, **3**, **4**].

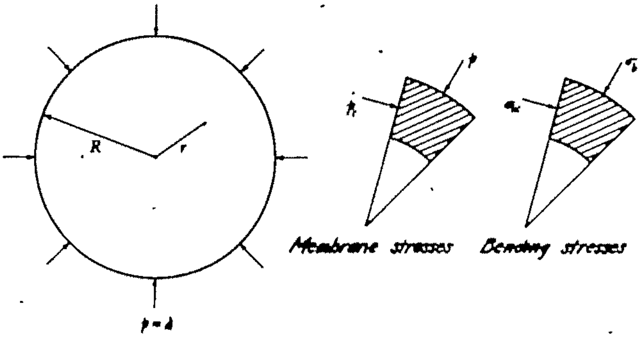

FIG. 3

The special problem in question is that of the circular plate with constant radial pressure applied at the boundary and with radial symmetry assumed (Figure 3). In this case all quantities depend only upon the distance r from the center of the plate and the von Kármán equations become ordinary differential equations.[3] Upon introduction of the quantities $p = (1/r)(d\phi/dr)$ — the radial membrane stress — and $q = -(R/r)(dw/dr)$ in place of ϕ and w, respectively, the von Kármán equations reduce to the following pair of nonlinear differential equations of the second order:

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[3] Solutions of the von Kármán equations for the problem of the bending of the circular plate under vertical forces with radial symmetry assumed have been given by Stewart Way [16]. Way found that the von Kármán equations in this case (in spite of their nonlinearity) can be solved conveniently by power series in r

$$\eta^2 Gq + pq = 0, \tag{7}$$

$$Gp = q^2/2, \tag{8}$$

in which G is the linear operator $R^2 r^{-3} d/dr(r^3 d/dr)$. The quantity η^2 occurring in (7) is given by $\eta^2 = \gamma h/R$, h and R being the thickness and radius of the plate and γ a material constant. At the center $r=0$ the following regularity conditions resulting from the assumed symmetry are imposed:

$$dq/dr = dp/dr = 0 \qquad \text{for } r = 0. \tag{9}$$

At the boundary $r=R$ we assume

$$R\,dq/dr + (1+\nu)q = 0, \tag{10}$$

$$p = \lambda > 0. \tag{11}$$

Condition (10) states that the radial bending stress σ_r is zero at the edge, while the important quantity λ in (11) is the prescribed radial pressure at the edge.

Just as in the general case, the only solution of the boundary value problem formulated in equations (7) to (11) is $q=0$, $p(r)=\text{const.}=\lambda$, if λ is small enough. The lowest value λ_0 of λ for which a different solution appears can be calculated from the linear eigenvalue problem arising from (7) and the homogeneous boundary conditions for q, with $p=\lambda$ in (7) as parameter to be determined. The investigation for $\lambda > \lambda_0$ requires the solution of the nonlinear boundary value problem as formulated above.

The most striking feature of the numerical solutions in their dependence on λ is that p and q (and, in fact, all quantities) appear to tend with increasing λ to become constant in an increasingly large portion of the interior of the plate and to change rapidly in a narrow strip, or "boundary layer," near its edge. In addition, p tends to a *negative* constant in the interior of the plate; physically this means that the compressive stress applied at the edge of the plate (compressive stresses are assumed positive here) results in tension at the center due to buckling and consequent stretch of the plate. To substantiate these statements mathematically it is of course necessary to formulate and solve the limit problem for λ tending to infinity.

There are two distinct limit processes involved, the simpler to deal with being that concerning the limit state in the interior of the plate. One obtains the formulation of this problem from the original differential equations (7) and (8) by observing that to allow λ to tend to infinity for a plate of fixed radius and thickness is mathematically equivalent to allowing the quantity $\eta^2=\gamma h/R$ in (7) to tend to zero, λ being held fixed. The result is the limit system:

$$
Gp = q^2/2, \qquad pq = 0. \tag{12}
$$

The only solution of these equations which satisfies the regularity conditions at $r=0$ is $q\equiv 0$, $p\equiv\text{const.}$ However, the quantity p cannot be set equal to the positive constant prescribed at the boundary, as the value of p in the interior of the plate seems certainly negative in the limit. The limit problem for the interior of the plate is thus not self-contained, since it fails to provide the value to be assigned to p in the limit.

The limit solution for the interior of the plate can be completed only after the limit problem connected with the boundary layer phenomena has been formulated and solved. It should be pointed out that the occurrence of such boundary layer phenomena is not confined to the special problem under consideration here.[3] Phenomena of the same nature have been noted in various problems concerning thin curved shells (see, for example, [13]). Boundary layer problems occur also in certain nonlinear vibration problems, for example in those concerning what are called relaxation oscillations. All such problems have the same general aspect: Each is a boundary value problem associated with a system of differential equations containing a parameter λ. A limit solution is desired for a value of λ which causes the order of the system of differential equations to degenerate. The consequence is that some of the original boundary conditions are lost, with the result that the convergence as regards λ becomes nonuniform at the boundary.

Not all such problems are nonlinear. The problem of Kirchoff-Kelvin-Tait discussed earlier ought to be formulated as a boundary layer problem. The starting point would be the thick plate problem and the limit problem that obtained upon allowing the thickness to approach zero in the differential equations. The differential equations would degenerate and some boundary condition would be lost at the edge. It should be possible to find the nature of the limit state in the boundary layer, at least in some special cases.

No general theory for boundary layer problems exists, but the general scheme of attack can be illustrated by means of the case of the buckled circular plate. The solution of the problem in this case, as in others, requires the introduction of a new independent variable in place of r which depends upon the parameter λ in such a way that the width of the boundary layer, as measured in the new scale, does not shrink to zero when λ tends to infinity. At the same time new dependent variables, replacing p and q and also depending upon λ, are introduced in such a way that the transformed differential equations resulting from (7) and (8) do not degenerate when λ tends to infinity. In short, it is necessary to introduce new variables which have the effect of making the convergence uniform. Limit boundary conditions must also be obtained. The result is in the case of the circular plate a new nonlinear boundary value problem which can be solved to yield limit values for all quantities, including those of most practical importance—the maximum stresses. The maxima of both bending and membrane stresses occur, by the way, in the boundary layer in the limit. Once limit values for all quantities in the boundary layer have been found, we are provided with the particular quantity needed to complete the solution of the interior limit problem—this is, quite naturally, the limit value of p at the inner edge of the boundary layer.

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[3] The first and most famous boundary layer problem is the one discovered and treated by L. Prandtl [12] in studying the flow of a viscous fluid around an obstacle.

It may be mentioned that the occurrence of the boundary layer phenomena in the case of the circular plate was first observed by solving the original differential equations for a series of increasing values of λ. Only afterwards were rigorous proofs of the above statements worked out. I should like at this point to call attention to some remarks made by von Kármán in his Gibbs Lecture relative to the need and value, even to those whose interest is mainly a practical one, of existence, uniqueness, and convergence proofs in cases where it is not obvious that the physical phenomena have received a correct mathematical formulation.

Proofs that the limit situation in the buckled circular plate is as described above can be carried out through a confrontation of the boundary value problem with a minimum problem of the following type: The absolute minimum of the functional V_λ (essentially the potential energy) given by

$$V_\lambda[q] = D[q] - \lambda^2 H[q] + K[q] \tag{13}$$

is to be found, where D, H, and K are integrals, the first two quadratic in q and dq/dr, the third of fourth degree in dq/dr. The minimum of V_λ is sought in the class of all continuous functions q with L^2-integrable derivatives for which the integrals are finite. It is shown, by direct methods, that the minimum of V_λ is taken on for a certain such function (for each λ, including $\lambda=\infty$) and that each such function possesses a continuous second derivative and furnishes a solution (though by no means all solutions) of the boundary value problem. The relation between the solutions of the minimum and boundary value problems is indicated schematically in Figure 4 which shows the

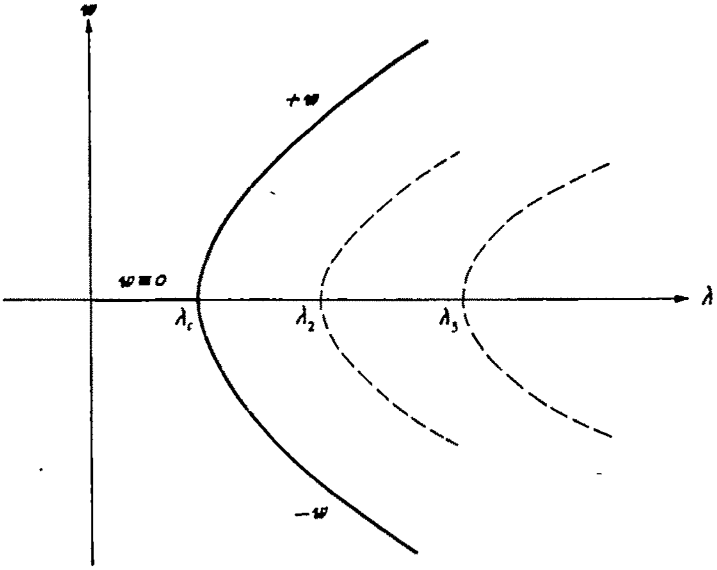

FIG. 4

deflection w of the center of the plate in its dependence on λ. The heavy curve indicates the type of solution of the minimum problem. (That the solution $-w$ always occurs with $+w$ is readily seen.) However, the boundary value problem has many other solutions, indicated by the dotted curves, in addition to $w\equiv 0$, which is always a solution. The solutions of the boundary value problem which do not vanish identically and which are not solutions of the minimum problem begin at values of λ corresponding to the higher eigenvalues of the linearized buckling problem.

6. **Some unsolved problems.** Quite a number of unsolved questions can be raised at once, even for the special case of the buckled circular plate. For example, it is possible that the solution of the minimum problem eventually becomes instable for a certain value of λ and that a new bifurcation arises if the restriction of radial symmetry is given up so that the plate can buckle in waves relative to the original buckled state. Experiments on buckling of rectangular plates indicate

that such a "second buckling," or buckling of higher order, may well occur. Also, it would be of interest to characterize as minimum problems those solutions of the boundary value problem which begin at the higher eigenvalues of the linearized problem. As is well known, a similar question is solved in the case of linear eigenvalue problems by imposing orthogonality conditions, or by the formulation of a minimum-maximum problem. One could also inquire about the limit state in the case of the circular plate when forces perpendicular to the plane of the plate are applied, with various prescribed boundary conditions. It is known, for example, that the boundary layer effect for the membrane stress does not occur if no compressive stresses arise at the boundary of the plate.

From the practical point of view it would be of high interest to carry the solutions through for the case of the buckled rectangular plate. Even the asymptotic boundary layer solution alone would be of value. The problems with two independent variables seem, however, to be of great difficulty.[4]

One could go on to give a lengthy list of unsolved problems concerning the flexure of plates, some of which have never even been formulated mathematically. For instance, a theory of thin plates analogous to the elastica theory for thin rods ought to be worked out. In such a theory the assumptions of small deflections and slopes of the middle surface would be given up. To complete the analogy with the elastica theory, it should be assumed that the middle surface is unstretched, that is, that it remains a developable surface. In this case the right-hand side of the equation corresponding to (4) would be zero, since the right side of (4) is essentially the Gauss curvature of the middle surface. One would have to pay for this simplification by using the exact expressions for the principal curvatures of the middle surface.

There remain unsolved problems of a purely mathematical character among the classical linear bending problems. The solutions of such problems are in general analytic functions with certain singularities, the character of which is known for a variety, though not all, of the commoner physical situations. The singularity at the corner of a clamped rectangular plate ought to be determined. The numerical solution of the previously mentioned problem of the rectangular plate resting on rigid supports but with the corners not held down would be greatly facilitated if the singularity at the point of detachment of the plate from its support could be explicitly determined. The best tools for attacking these problems are probably to be found in complex function theory.

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[4] Some years ago von Kármán, Sechler, and Donnell [7] gave what can be regarded as a rough approximation to the asymptotic solution for the rectangular plate based on physical assumptions which correspond in a general way to what was found later in the boundary layer theory for the circular plate.

A theory with some claim to generality for boundary layer problems would be of high interest. Such a theory should be designed to answer a number of questions. In the first place, one does not know a priori which of the boundary conditions will be lost in a specific problem. This question is coupled with another: In some cases, as in that of the circular plate, it is the boundary layer problem which is self-contained, while the interior limit problem can be solved completely only after the boundary layer problem has been solved. In the case of the Prandtl boundary layer problem the circumstances are reversed; here the solution of the interior limit problem furnishes a quantity which is needed to complete the solution of the boundary layer problem. It is admittedly not easy to see how a general theory capable of answering such questions should be constructed; probably it is necessary to begin, in the time-honored way, by collecting experience through actual solution of a variety of carefully chosen special problems. Some physical problems likely to be of interest in this connection have already been mentioned—the Kirchoff-Kelvin-Tait problem, for example.

The purpose of the addresses on applied mathematics, of which this is the first, is to stimulate interest in applied mathematics not only for the furtherance of practical ends but also for the advancement of mathematics as a whole. A striking illustration of the beneficial effects on pure mathematics of the study of the applications is certainly provided by the topic of this address. One realizes, then, how great the loss to mathematics in solidity, richness, and variety would be if applied mathematics generally were to be neglected.

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New York University

ARITHMETIC OF ORDINALS WITH APPLICATIONS TO THE THEORY OF ORDERED ABELIAN GROUPS

PHILIP W. CARRUTH

1. Introduction.[1] The operations of addition and multiplication of ordinals do not behave as well as one might desire. For example, the commutative laws are not valid, and the distributive law is valid on only one side. Consequently, we make definitions of a sum and product that do not have such defects.

A binary operation, $\alpha\oplus\beta$, on ordinals is termed a ***natural sum*** if $\alpha\oplus\beta$ is a well-determined ordinal for any two ordinals, α and β, such that:

(1) $\alpha\oplus\beta=\beta\oplus\alpha$,
(2) $(\alpha\oplus\beta)\oplus\delta=\alpha\oplus(\beta\oplus\delta)$,
(3) $\alpha\oplus 0=\alpha$,
(4) $\delta\oplus\alpha>\delta\oplus\beta$ if and only if $\alpha>\beta$,

where δ is any ordinal.

Throughout this paper, $\sigma(\alpha, \beta)$ will denote the natural sum defined by Hessenberg.[2] It is the unique natural sum satisfying the condition that $\omega^\alpha\cdot m+\omega^\beta n=\sigma(\omega^\alpha m, \omega^\beta n)$, where α and β are any two ordinals such that $\alpha\geqq\beta$, and where m and n are any two positive integers. $\sigma(\alpha, \beta)$ shall be shown to be the "smallest" natural sum, and it shall be shown to be the best bound for the order type of the join of two well-ordered subsets, of respective order types α and β, of an ordered set.

A binary operation, $\alpha\otimes\beta$, on ordinals is termed a ***natural product*** if $\alpha\otimes\beta$ is a well-determined ordinal for any two ordinals, α and β, such that:

(1) $\alpha\otimes\beta=\beta\otimes\alpha$,
(2) $(\alpha\otimes\beta)\otimes\delta=\alpha\otimes(\beta\otimes\delta)$,
(3) $\alpha\otimes 1=\alpha$,
(4) $\alpha\otimes\delta>\beta\otimes\delta$ if and only if $\alpha>\beta$,
(5) $\sigma(\alpha\otimes\beta, \alpha\otimes\delta)=\alpha\otimes\sigma(\beta, \delta)$,
(6) $\omega^\alpha\otimes\omega^\beta=\omega^\gamma$,

where δ is any ordinal, and where $\gamma=\gamma(\alpha, \beta)$ is a suitable ordinal.

Presented to the Society, September 5, 1941; received by the editors June 6, 1941. The writer is indebted to Professor Reinhold Baer for his advice in the preparation of this paper.

[1] The writer presupposes familiarity with the material on ordinals found in F. Hausdorff's *Mengenlehre*.

[2] G. Hessenberg, *Grundbegriffe der Mengenlehre*, Abhandlungen der Fries'schen Schule, (n. s.), 1.4, Göttingen, 1906, no. 75.

It may be noted that the ordinal γ above is a natural sum of α and β. If we impose the condition that this ordinal be equal to $\sigma(\alpha, \beta)$, we obtain the unique natural product defined by Hausdorff.[3] This particular natural product will be denoted by $\pi(\alpha, \beta)$ throughout this paper. We shall show that $\pi(\alpha, \beta)$ is the "smallest" natural product and that it is the best bound for the order type of a certain rectangular array of ordered elements that has "α rows" and "β columns." Another application of Hausdorff's natural product will be given in determining a bound for the order type of the semi-group generated by a well-ordered set of positive elements in an ordered Abelian group.[4]

It is known that every ordinal α may be represented in the form, $\alpha=\sum_{i=1}^{n}\omega^{\alpha_i}a_i$, where $\omega>n$, $a_1, a_2, \cdots, a_n\geqq 0$ and $\alpha\geqq\alpha_1>\alpha_2>\cdots>\alpha_n\geqq 0$. Henceforth in this paper, when an ordinal is written in summation form, it will be assumed that the summation satisfies the above requirements.

2. **Natural sums and products of ordinals.** We first prove the following theorem.

THEOREM 1. *Let* $\tau\oplus\rho$ *be any natural sum.*

I. *Let* T *and* R *be well-ordered subsets, of respective order types* τ *and* ρ, *of an ordered set of elements. Then the set of elements in the join of* R *and* T, *that is,* $R+T$, *is well-ordered and has order type less than or equal to* $\tau\oplus\rho$.

II. *There exists a well-ordered subset* R, *of order type* ρ, *of an ordered set, and a well-ordered subset* T, *of order type* τ, *of the same ordered set, such that the order type of* $R+T$ *is exactly* $\sigma(\rho, \tau)$.

III. $\sigma(\rho, \tau)\leqq\tau\oplus\rho$.

IV. $\tau+\rho\leqq\tau\oplus\rho$.[5]

PROOF OF I. I is true for $\tau=0$ and for all ρ. Assume it to be true for all ρ and for all τ less than α. It is true for $\rho=0$ and $\tau=\alpha$. Assume it to be true for all ρ less than β and $\tau=\alpha$. We shall now prove I to be true for $\rho=\beta$ and $\tau=\alpha$.

It is easily verified that $R+T$ is well-ordered for $\rho=\beta$ and $\tau=\alpha$. Suppose the order type of $R+T$ is greater than $\beta\oplus\alpha$. Then $R+T$ has a segment Z of order type $\beta\oplus\alpha$. By our induction hypothesis, it is

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[3] F. Hausdorff, ***Mengenlehre***, 3rd edition, p. 70.

[4] For a definition of an ordered Abelian group, see S. MacLane, *The uniqueness of the power series representation of certain fields with valuations*, Annals of Mathematics, (2), vol. 39 (1938), p. 371.

[5] $\sigma(\tau, \rho)$ may be larger than the maximum $\{\tau+\rho, \rho+\tau\}$, for example, let $\tau=\rho=\omega+1$. Then $\tau+\rho=\rho+\tau=\omega 2+1$, and $\sigma(\rho, \tau)=\omega 2+2$.

seen that the order type of the subset of elements of T that are in Z must be α, since $\alpha \oplus \beta > \mu \oplus \beta$ for $\alpha > \mu$. Likewise the order type of the subset of elements of R that are in Z must be β. The elements of R that are in Z form a segment of R, and the elements of T that are in Z form a segment of T. Hence the set $R+T$ is the same as the segment Z of $R+T$. This gives us a contradiction, proving I.

PROOF OF II. Let $\rho = \sum_{i=1}^{n} \omega^{\alpha_i} a_i$, and $\tau = \sum_{i=1}^{n} \omega^{\alpha_i} b_i$. Then $\sigma(\rho, \tau)$ is equal to $\sum_{i=1}^{n} \omega^{\alpha_i}(a_i + b_i)$. Let R be a set of elements of order type ρ. Then R is the join of sets R_i, $1 \leqq i \leqq n$, where R_i has order type $\omega^{\alpha_i} a_i$, and where each element in R_i is smaller than every element in R_{i+1}.

Let T be a set of elements of order type τ. Then T is the join of sets T_i, $1 \leqq i \leqq n$, where T_i has order type $\omega^{\alpha_i} b_i$, and where each element in T_i is smaller than every element in T_{i+1}.

By letting each element in T_i be greater than every element in R_i and smaller than every element in R_{i+1}, we form a set $R+T$ of order type $\sigma(\rho, \tau)$.

III is an immediate consequence of I and II, and IV is an immediate consequence of I.

An ordered set of elements, $A = \{a_{\alpha\beta}\}$, $0 \leqq \alpha < \tau$, $0 \leqq \beta < \mu$, where τ and μ are any two ordinals, is an *ordered* τ, μ*-block* if $a_{\alpha\beta} < a_{\alpha\gamma}$, $a_{\beta\alpha} < a_{\gamma\alpha}$ for $\beta < \gamma$. An ordered τ, μ-block A is said to be *less than* an ordered ρ, λ-block B, that is, $A < B$, if each element in A is considered to be less than every element in B. Two ordered τ, μ-blocks, $A = \{a_{\alpha\beta}\}$ and $B = \{b_{\alpha\beta}\}$, are *congruent* if the following two conditions are equivalent: (1) $a_{\alpha\beta} < a_{\rho\lambda}$, (2) $b_{\alpha\beta} < b_{\rho\lambda}$.

THEOREM 2. *Let* $\tau \otimes \mu$ *be any natural product.*

I. *Let* $A = \{a_{\alpha\beta}\}$ *be an ordered* τ, μ*-block. Then* A *is well-ordered and has order type less than or equal to* $\tau \otimes \mu$.

II. *There exists an ordered* τ, μ*-block of order type exactly* $\pi(\tau, \mu)$.

III. $\pi(\tau, \mu) \leqq \tau \otimes \mu$.

IV. $\tau\mu \leqq \tau \otimes \mu$.[6]

PROOF OF I. Suppose that A is not well-ordered. Then there exists a subset B of A, $a_{\alpha_1\beta_1} > a_{\alpha_2\beta_2} > \cdots$, of order type ω^*. Since the α_i are well-ordered, we may choose a chain C, $a_{\delta_1\gamma_1} > a_{\delta_2\gamma_2} > \cdots$, of order type ω^*, from the set B such that $\delta_1 \leqq \delta_2 \leqq \cdots$. Since the γ_i are well-ordered, there exists a γ_i, say γ_k, such that $\gamma_k \geqq \gamma_1$. But then $a_{\delta_k\gamma_k} \geqq a_{\delta_1\gamma_1}$, which is a contradiction.

The set A has order type less than or equal to $\tau \otimes \mu$ for $\tau = 1$ and all μ. Assume this to be true for all μ and for all τ less than δ. It is true for $\mu=1$ and $\tau=\delta$. Assume it to be true for all μ less than γ and $\tau=\delta$. We shall now prove A has order type less than or equal to $\tau\otimes\mu$ for $\mu=\gamma$ and $\tau=\delta$.

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[6] $\pi(\rho, \tau)$ may be larger than the maximum $\{\tau\rho, \rho\tau\}$, for example, let $\tau = \rho = \omega + 1$. Then $\tau\rho = \rho\tau = \omega^2 + \omega + 1$, and $\pi(\rho, \tau) = \omega^2 + \omega 2 + 1$.

Let $\delta=\omega^{\alpha_1}m_1+\cdots+\omega^{\alpha_r}m_r$ and $\gamma=\omega^{\beta_1}n_1+\cdots+\omega^{\beta_s}n_s$.

Case I. $m_1>1$ or $r>1$.

Let $\rho=\omega^{\alpha_1}m_1+\cdots+\omega^{\alpha_r}(m_r-1)$, where $m_r-1\geqq 0$. By hypothesis, $\delta>\rho>0$ and $\delta>\omega^{\alpha_r}$.

Let T_1 be the set $\{a_{\alpha\beta}\}$, where $0\leqq\alpha<\rho$ and $0\leqq\beta<\gamma$. Let T_2 be the set $\{a_{\alpha\beta}\}$, where $\rho\leqq\alpha<\delta$ and $0\leqq\beta<\gamma$. Then A is the set T_1+T_2. By our induction hypotheses, the order type of T_1 is less than or equal to $\rho\otimes\gamma$. Likewise, the order type of T_2 is less than or equal to $\omega^{\alpha_r}\otimes\gamma$. Theorem 1 implies that the order type of A is less than or equal to $\sigma(\rho\otimes\gamma,\omega^{\alpha_r}\otimes\gamma)=\gamma\otimes\sigma(\rho,\omega^{\alpha_r})=\gamma\otimes\delta$.

Because of the symmetry, A has order type less than or equal to $\tau\otimes\mu$ for $\mu=\gamma$ and $\tau=\delta$, if $n_1>1$ or $s>1$.

Case II. $m_1=r=n_1=s=1$, that is, $\delta=\omega^{\alpha_1}$ and $\gamma=\omega^{\beta_1}$.

Suppose that the order type of A is greater than $\delta\otimes\gamma$. Then A has a segment Z of order type $\delta\otimes\gamma$. Let $A=Z+Y$. Let $a_{\lambda\epsilon}$ be the smallest element in Y. Then $\delta>\lambda$ and $\gamma>\epsilon$.

Let T_1 be the set $\{a_{\alpha\beta}\}$, where $0\leqq\alpha<\delta$ and $0\leqq\beta<\epsilon$. Let T_2 be the set $\{a_{\alpha\beta}\}$, where $0\leqq\alpha<\lambda$ and where $\epsilon\leqq\beta<\gamma$. Let $W=T_1+T_2$.

$$\begin{array}{llll|lll}
a_{00}, & a_{01}, & \cdots, & & a_{0\epsilon}, & a_{0\epsilon+1}, & \cdots \\
a_{10}, & a_{11}, & \cdots, & & \cdot\,, & \cdot\,, & \cdots \\
\cdot\,, & \cdot\,, & \cdots, & & \cdot\,, & \cdot\,, & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdot\,, & \cdot\,, & \cdots, & & a_{\lambda\epsilon}, & \cdot\,, & \cdots \\
\cdot\,, & \cdot\,, & \cdots, & & \cdot\,, & \cdot\,, & \cdots
\end{array}$$

The set at the left of the dotted line is T_1, and that in the upper right-hand corner is T_2.

W certainly contains Z. Therefore, according to our supposition, the order type of W is greater than or equal to $\delta\otimes\gamma$.

By our induction hypothesis, the order type of T_1 is less than or equal to $\delta\otimes\epsilon$, and the order type of T_2 is less than or equal to $\gamma\otimes\lambda$. Theorem 1 implies that the order type of W is less than or equal to $\sigma(\delta\otimes\epsilon,\gamma\otimes\lambda)$. According to (6) of the definition of a natural product, $\delta\otimes\gamma=\omega^{\eta}$, where η is some ordinal. Likewise, by (4) of this definition, it is seen that $\delta\otimes\epsilon<\omega^{\eta}$ and that $\gamma\otimes\lambda<\omega^{\eta}$. The definition of $\sigma(\alpha,\beta)$ implies that $\sigma(\delta\otimes\epsilon,\gamma\otimes\lambda)<\omega^{\eta}=\delta\otimes\gamma$. This then is a contradiction which completes the proof of I.

PROOF OF II. Let $\tau=\omega^{\alpha_1}a_1+\cdots+\omega^{\alpha_r}a_r$ and $\mu=\omega^{\beta_1}b_1+\cdots+\omega^{\beta_s}b_s$.

Then $\pi(\tau, \mu)=\sum_{i,j}\omega^{\sigma(\alpha_i,\beta_j)}a_ib_j=\sum_k\omega^{\gamma_k}c_k$ where $c_k=\sum^k a_ib_j$, where $\sum^k$ is over the finite number of pairs such that $\sigma(\alpha_i, \beta_j)=\gamma_k$.

Any ordered τ, μ-block, $A=\{a_{\alpha\beta}\}$, may be broken up into ordered $\omega^{\alpha_i}a_i$, $\omega^{\beta_j}b_j$-blocks, $D(i, j)=\{a_{\alpha\beta}\}$, where $\omega^{\alpha_1}a_1+\cdots+\omega^{\alpha_{i-1}}a_{i-1}\leqq\alpha<\omega^{\alpha_1}a_1+\cdots+\omega^{\alpha_i}a_i$, and $\omega^{\beta_1}b_1+\cdots+\omega^{\beta_{j-1}}b_{j-1}\leqq\beta<\omega^{\beta_1}b_1+\cdots+\omega^{\beta_j}b_j$.

$$A=\begin{array}{c|c|c|c|c|}
 & \omega^{\beta_1}b_1 & \omega^{\beta_2}b_2 & \cdots & \omega^{\beta_s}b_s \\ \hline
\omega^{\alpha_1}a_1 & D(1, 1) & D(1, 2) & \cdots & D(1, s) \\ \hline
\omega^{\alpha_2}a_2 & D(2, 1) & D(2, 2) & \cdots & D(2, s) \\ \hline
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & & & & \\
\cdot & & & & \\ \hline
\omega^{\alpha_r}a_r & D(r, 1) & D(r, 2) & \cdots & D(r, s) \\ \hline
\end{array}$$

Suppose that we could order our blocks $D(i, j)$ in such a way that each $D(i, j)$ would have order type exactly $\pi(\omega^{\alpha_i}a_i, \omega^{\beta_j}b_j)=\omega^{\sigma(\alpha_i,\beta_j)}a_ib_j$. By then letting $D(i, j)$ be less than $D(k, m)$ if $\sigma(\alpha_i, \beta_j)>\sigma(\alpha_k, \beta_m)$ or if $\sigma(\alpha_i, \beta_j)=\sigma(\alpha_k, \beta_m)$ and $i>k$, we would have an ordered τ, μ-block of order type exactly $\pi(\tau, \mu)$. Thus it is seen that it suffices to prove II for the special case that $\tau=\omega^\alpha a$ and $\mu=\omega^\beta b$.

Hence we now assume that $\tau=\omega^\alpha a$ and that $\mu=\omega^\beta b$. Let $\alpha=\omega^{\gamma_1}f_1+\cdots+\omega^{\gamma_t}f_t$ and $\beta=\omega^{\delta_1}d_1+\cdots+\omega^{\delta_v}d_v$. We first assume that $t=0$. In this case $\alpha=0$ and $\tau=a$. If we let $a_{\lambda\nu}<a_{\theta\phi}$ for $\lambda<\theta$, it is seen that an ordered τ, μ-block is formed of order type $\mu\tau=\pi(\mu, \tau)$. Assume that II is true for all integers t less than k and for all integers v. We shall now prove II to be true when $t=k$.

$$\omega^\alpha=\omega^{\omega^{\gamma_1}f_1+\cdots+\omega^{\gamma_{k-1}}f_{k-1}}\cdot\omega^{\omega^{\gamma_k}f_k}.$$

Let

$$\phi=\omega^{\omega^{\gamma_1}f_1+\cdots+\omega^{\gamma_{k-1}}f_{k-1}}.$$

Then

$$\omega^\alpha=\phi\cdot\omega^{\omega^{\gamma_k}f_k}=\pi(\phi, \omega^{\omega^{\gamma_k}f_k}),$$

$$\omega^\beta=\omega^{\omega^{\delta_1}d_1+\cdots+\omega^{\delta_e}d_e}\cdot\omega^{\omega^{\delta_{e+1}}d_{e+1}+\cdots+\omega^{\delta_v}d_v}$$

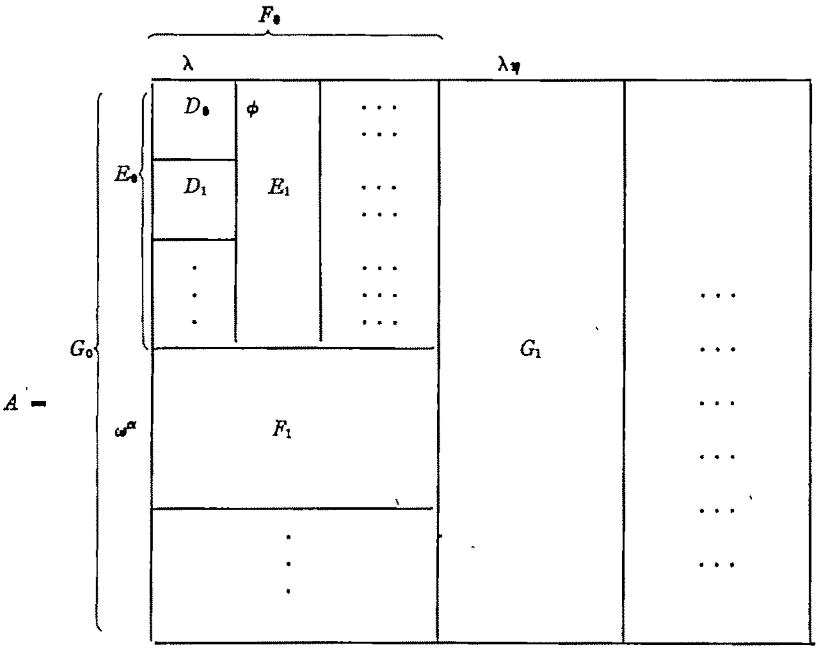
where $\delta_e \geqq \gamma_k > \delta_{e+1}$, $0 \leqq e \leqq v$. Let

$$\lambda = \omega^{\omega^{\delta_1}d_1+\cdots+\omega^{\delta_e}d_e}, \qquad \eta = \omega^{\omega^{\delta_{e+1}}d_{e+1}+\cdots+\omega^{\delta_v}d_v}.$$

If $e=0$, let $\lambda=1$. If $e=v$, let $\eta=1$.

$$\begin{aligned}\pi(\omega^\alpha a, \omega^\beta b) &= \pi(\phi\cdot\omega^{\omega^{\gamma_k}f_k}a, \lambda\eta b)\\ &= \pi(\pi(\phi, \omega^{\omega^{\gamma_k}f_k}), \pi(\lambda, \eta))ab\\ &= \pi(\pi(\phi, \lambda), \pi(\omega^{\omega^{\gamma_k}f_k}, \eta))ab\\ &= \pi(\pi(\phi, \lambda), \omega^{\omega^{\gamma_k}f_k}\eta)ab\\ &= \pi(\phi, \lambda)\omega^{\omega^{\gamma_k}f_k}\eta ab.\end{aligned}$$

DIAGRAM



By our induction hypothesis, there exists an ordered ϕ, λ-block D_0 of order type $\pi(\phi, \lambda)$. We may construct congruent ordered ϕ, λ-blocks D_α, $0 \leqq \alpha < \omega^{\omega^{\gamma_k}f_k}$, as in the Diagram, such that $D_\alpha < D_\beta$ for $\alpha < \beta$.

Let E_0 be the block of all the D_α. E_0 is an ordered ω^α, λ-block. The order type of E_0 is $\pi(\phi, \lambda)\cdot\omega^{\omega^{\gamma_k}f_k}$. We may now construct congruent ordered ω^α, λ-blocks E_β, $0 \leqq \beta < \eta$, as in the Diagram, such that $E_\beta < E_\delta$, for $\beta < \delta$.

Let F_0 be the block of all the E_β. Then F_0 is an ordered ω^α, $\lambda\eta$-block. The order type of F_0 is

$$\pi(\phi, \lambda) \cdot \omega^{\omega^{\gamma_k/k}} \cdot \eta.$$

Now we construct congruent ordered ω^α, $\lambda\eta$-blocks F_i, $0 \leqq i < a$, such that $F_i < F_j$, for $i < j$.

Let G_0 be the block of all the F_i. Then G_0 is an ordered $\omega^\alpha a$, $\lambda\eta$-block. The order type of G_0 is $\pi(\phi, \lambda)\omega^{\omega^{\gamma_k/k}} \cdot \eta a$. We construct congruent ordered $\omega^\alpha a$, $\lambda\eta$-blocks G_i, $0 \leqq i < b$, as in the Diagram, such that $G_i < G_j$, for $i < j$.

Let A be the block of all the G_i. A is an ordered $\omega^\alpha a$, $\omega^\beta b$-block. The order type of A is

$$\pi(\phi, \lambda)\omega^{\omega^{\gamma_k/k}} \cdot \eta \cdot ab = \pi(\omega^\alpha a, \omega^\beta b) = \pi(\tau, \mu).$$

III is an immediate consequence of I and II, and IV is an immediate consequence of I.

3. **Well-ordered subsets of ordered Abelian groups.** We prove the following theorem.

THEOREM 3. *Let G be an ordered Abelian group. Let T be a well-ordered (according to size) set of non-negative elements in G, containing* 0, *of order type $\beta = \tau + n$, where τ is a limit ordinal or* 0, *and n is an integer greater than or equal to* 0. *Let M be the set of all finite sums of elements in T. Then*:

I. *M is well-ordered.*

II. *M has order type less than or equal to $\pi(\omega^\beta, \tau^\tau)$.*[7]

PROOF OF I.

Case 1. There exists an integer N such that each element in M is the sum of N elements in T.

Let M_i be the set of sums of at most i elements in T. Then M_i is well-ordered if $i = 1$. Assume that M_{k-1} is well-ordered. Suppose that M_k is not well-ordered. Then there exists a chain of elements in M_k, $a_1 > a_2 > \cdots$, of order type ω^*. Let $a_i = c_i + d_i$, where c_i is in M_{k-1}, and d_i is in $M_1 = T$. Then $c_1 + d_1 > c_2 + d_2 > \cdots$. The c_i are well-ordered. Likewise the d_i are well-ordered. Hence this chain can not exist, that is, M_k is well-ordered.

Case 2. There exists no integer N such that each element in M is the sum of N elements in T.

Let T be the set $\{t_\alpha\}$, where $t_\alpha < t_\beta$ for $\alpha < \beta$. Let D_α be the subgroup of G that is generated by t_α and all g in G such that $0 \leqq g < t_\alpha$. Then

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[7] If β is an integer, then a better bound of the order type of M is $\pi(\omega^{\beta-1}, \tau^\tau) = \omega^{\beta-1}$.

$0=D_0\leqq D_1\leqq D_2\leqq\cdots$. We pick out the well-ordered set R of all the different D_α. $R=\{R_0, R_1, \cdots\}$; $0=R_0<R_1<\cdots\leqq G$.

Let $M(\alpha)$ be the set of all elements in M such that each may be expressed as a finite sum of elements in T, all of which are in R_α.

$M(0)$ consists of the single element 0 and so is well-ordered. Assume that $M(\alpha)$ is well-ordered for all $\alpha<\gamma$. Suppose that $M(\gamma)$ is not well-ordered. Then there exists a chain in $M(\gamma)$, $a_1>a_2>\cdots$, of order type ω^*.

(a) Only a finite number of the a_i are in $M(\gamma)$ but not in an $M(\alpha)$, $\alpha<\gamma$.

In this case, we may throw out this finite number of a_i and then we have a chain, $c_1>c_2>\cdots$, of order type ω^*, where each c_i is in an $M(\alpha)$, $\alpha<\gamma$. Let c_1 be in $M(\rho)$, $\rho<\gamma$. By our induction hypothesis, there exists a c_i, say c_k, not in $M(\rho)$. But then $c_k>c_1$, a contradiction.

(b) An infinite number of the a_i are in $M(\gamma)$ but not in any $M(\alpha)$, $\alpha<\gamma$.

In this case, we may choose a chain, $c_1>c_2>\cdots$, of order type ω^*, where each c_i is in $M(\gamma)$ but not in any $M(\alpha)$, $\alpha<\gamma$.

Let $c_i=f_i+g_i$, where f_i is in an $M(\alpha)$, $\alpha<\gamma$, and g_i is a sum of terms in T, each of which is in R_γ but not in an R_α, $\alpha<\gamma$. We saw in (a) that the f_i must be well-ordered. Hence the g_i must not be well-ordered, that is, we can choose a chain from the g_i: $g_{\alpha_1}>g_{\alpha_2}>\cdots$, of order type ω^*. Each g_{α_i} is a sum of terms in T, all of which are in R_γ but not in any R_α, $\alpha<\gamma$.

Let t be the smallest positive element in T that is in R_γ but not in an R_α, $\alpha<\gamma$. There exists an integer n such that nt is greater than g_{α_1}. According to Case 1, there must exist a g_{α_i}, say g_{α_j}, that is the sum of more than n positive elements in T. But then $g_{\alpha_j}>nt>g_{\alpha_1}$, a contradiction.

Before we proceed, we shall prove this lemma.

LEMMA 1. *Let G be an ordered Abelian group. Let T be a subset of non-negative elements in G, containing 0, that is well-ordered (according to size) of order type β. Let H_k be the set $\{\sum_{i=1}^{r}a_in_i\}$, where the n_i are non-negative integers such that $\sum_{i=1}^{r}n_i\leqq k$, the a_i are in T, and k is any positive integer. Then H_k is well-ordered and has order type less than or equal to $\phi_k(\beta)=\pi_{k-1}(\pi_{k-2}(\cdots(\pi_1(\beta,\beta),\beta),\cdots),\beta)$, where the subscripts merely denote the number of π's.*

PROOF. H_k is well-ordered, by Case 1 of Theorem 3. The order type of H_1 is β. Assume H_{k-1} has order type less than or equal to $\phi_{k-1}(\beta)$. Let the elements in H_{k-1} be $b_0, b_1, \cdots$, where $b_\alpha<b_\delta$ for $\alpha<\delta$.

The set $R=\{b_\alpha+a_\delta\}$, all b_α in H_{k-1}, and all elements a_δ in T, certainly contains H_k. Theorem 2 implies that the order type of H_k is less than or equal to $\pi(\phi_{k-1}(\beta), \beta)=\phi_k(\beta)$.

We now continue with the proof of Theorem 3.

Proof of II. Let M^α be the set of all elements in M that are sums of elements t_γ, $0\leqq\gamma<\alpha$, in T. Note that $M^\beta=M$. Let $\alpha=\tau_\alpha+n_\alpha$, where τ_α is a limit ordinal or 0, and n_α is an integer greater than or equal to 0.

M^1 has order type 1. Assume that M^α, $\alpha<\delta$, has order type less than or equal to $\pi(\omega^\alpha, \tau_\alpha^{\tau_\alpha})$.

Case 1. $n_\delta>0$.

$M^{\delta-1}$ has order type less than or equal to $\pi(\omega^{\delta-1}, \tau_\delta^{\tau_\delta})$. M^δ is certainly contained in the set $\{a_\alpha+nt_{\delta-1}\}$, for all elements a_α in $M^{\delta-1}$, and $0\leqq n<\infty$. Theorem 2 implies that the order type of M^δ is less than or equal to $\pi(\pi(\omega^{\delta-1}, \tau_\delta^{\tau_\delta}), \omega)=\pi(\omega^\delta, \tau_\delta^{\tau_\delta})$.

Case 2. $n_\delta=0$, that is, $\tau_\delta=\delta$.

Suppose that M^δ has order type greater than $\pi(\omega^\delta, \delta^\delta)$. Then $M^\delta=S+R$, where S is the segment of M^δ of order type $\pi(\omega^\delta, \delta^\delta)$.

Let $b=t_{\alpha_1}m_1+\cdots+t_{\alpha_s}m_s$, $\alpha_1<\alpha_2<\cdots<\alpha_s<\delta$, be the smallest element in R, where the m_i are positive integers. Let $\alpha_s=\mu+m$, where μ is a limit ordinal or zero and m is an integer greater than or equal to 0. Let n be an integer such that nt_{α_s} is greater than b.

Let H_n be the set of all elements $\sum_{i=1}^{r}t_{\beta_i}n_i$ in M^δ, such that $r\leqq n$, $\sum_{i=1}^{r}n_i\leqq n$, and each $\beta_i<\delta$. By Lemma 1, it is seen that the order type of H_n is less than or equal to $\phi_n(\delta)=\pi_{n-1}(\pi_{n-2}(\cdots(\pi_1(\delta, \delta), \delta)\cdots), \delta)$.

Let $\delta=\mathrm{w}^{\delta_1}e_1+\cdots+\omega^{\delta_q}e_q$ and $\delta_1=\omega^{\rho_1}v_1+\cdots+\omega^{\rho_s}v_s$. Then $\delta<\omega^{\omega^{\rho_1}(v_1+1)}$ and $\phi_n(\delta)<\omega^{\omega^{\rho_1}(v_1+1)n}$.

Let x be any element in S. Then $x=t_{\lambda_1}r_1+\cdots+t_{\lambda_v}r_v+t_{\lambda_{v+1}}r_{v+1}+\cdots+t_{\lambda_s}r_s$ where $\lambda_1<\cdots<\lambda_v<\alpha_s\leqq\lambda_{v+1}<\cdots<\lambda_s<\delta$. Let $f=t_{\lambda_1}r_1+\cdots+t_{\lambda_v}r_v$, and $g=t_{\lambda_{v+1}}r_{v+1}+\cdots+t_{\lambda_s}r_s$. Then f is in M^{α_s}. Since x is less than b, it follows that g is less than b. Hence $r_{v+1}+\cdots+r_s\leqq n$. And so g is in H_n. Therefore the set S is contained in the set $\{b_\alpha+c_\rho\}$, for all elements b_α in H_n, and for all elements c_ρ in M^{α_s}. Theorem 2 and our induction hypothesis imply that S has order type less than or equal to $\pi(\pi(\omega^{\alpha_s}, \mu^\mu), \omega^{\omega^{\rho_1}(v_1+1)n})\leqq\pi(\pi(\omega^\delta, \delta^\mu), \delta^{v_1n+n})$.

$$(1)\quad \pi(\pi(\omega^\delta, \delta^\mu), \delta^{v_1n+n}) = \pi(\omega^\delta, \pi(\delta^\mu, \delta^j)), \text{ where } j = v_1n + n \text{ is an integer,}$$

$\pi(\delta^\mu, \delta^j)\leqq\pi((\omega^{\delta_1}(e_1+1))^\mu, (\omega^{\delta_1}(e_1+1))^j)$. $\delta_1>0$, since δ is a limit ordinal. Hence $(\omega^{\delta_1}(e_1+1))^\mu=\omega^{\delta_1\mu}$, since μ is a limit ordinal. $(\omega^{\delta_1}(e_1+1))^j=\omega^{\delta_1j}(e_1+1)$,

$$(2)\qquad \pi(\delta^\mu, \delta^j) \leqq \pi(\omega^{\delta_1\mu}, \omega^{\delta_1j}(e_1 + 1)) = \omega^{\sigma(\delta_1\mu, \delta_1j)}(e_1 + 1).$$

$\delta_1\omega=\omega^{\rho_1+1}$, and $\delta_1 j=\omega^{\rho_1}v_1 j+\omega^{\rho_2}v_2+\cdots+\omega^{\rho_s}v_s<\omega^{\rho_1}(v_1 j+1)$;

$$\sigma(\delta_1\mu,\ \delta_1 j) < \sigma(\delta_1\mu,\omega^{\rho_1}(v_1 j+1)) < \delta_1\mu + \omega^{\rho_1+1} = \delta_1\mu + \delta_1\omega.$$

By (2), $\pi(\delta^\mu,\ \delta^j)<\omega^{\delta_1\mu+\delta_1\omega}=(\omega^{\delta_1})^{(\mu+\omega)}\leqq\delta^{\mu+\omega}\leqq\delta^\delta$.

Hence by (1), the order type of S is less than $\pi(\omega^\delta,\ \delta^\delta)$. This is a contradiction since S was the segment of M^δ of order type $\pi(\omega^\delta,\ \delta^\delta)$.

UNIVERSITY OF ILLINOIS

A CHARACTERIZATION OF ABSOLUTE NEIGHBORHOOD RETRACTS

RALPH H. FOX

By an *absolute neighborhood retract* (ANR) I mean a separable metrizable space which is a neighborhood retract of every separable metrizable space which contains it and in which it is closed. This generalization of Borsuk's original definition[1] was given by Kuratowski[2] for the purpose of enlarging the class of absolute neighborhood retracts to include certain spaces which are not compact. The space originally designated by Borsuk as absolute neighborhood retracts (or $\mathfrak{R}$-sets) will now be referred to as compact absolute neighborhood retracts. Many of the properties of compact ANR-sets hold equally for the more general ANR-sets.[3]

The Hilbert parallelotope Q, that is, the product of the closed unit interval $[0, 1]$ with itself a countable number of times is a "universal" compact ANR in the sense that[4] every compact ANR is homeomorphic to a neighborhood retract of Q. The classical theory of Borsuk makes good use of the imbedding of compact ANR-sets in Q. The problem solved here is that of finding a "universal" ANR.

Received by the editors June 28, 1941.

[1] Fundamenta Mathematicae, vol. 19 (1932), pp. 220–242.

[2] Fundamenta Mathematicae, vol. 24 (1935), p. 270, Footnote 1.

[3] Ibid., pp. 272, 276, and 277, and Footnote 1, p. 279 and Footnote 3. Note that Theorem 12, Fundamenta Mathematicae, vol. 19 (1932), p. 229, is not true for general ANR-sets. In fact let $A=\sum S_n$ where S_n is the plane circle of radius 2^{-n} and center $(3\cdot 2^{-n}, 0)$; let $f(x, y)=(x, |y|)$ for $(x, y)\in A$ and let

$$f_n(x, y)=\begin{cases}(x,\ |y|), & \text{for } (x, y)\in A-S_n,\\ (x,\ y), & \text{for } (x, y)\in S_n.\end{cases}$$

Then $f_n\to f$ in A^A; f can be extended to the half-plane $\{x>0\}$, but none of the maps f_n can. A is an ANR-set. Theorem 16, Fundamenta Mathematicae, vol. 19 (1932), p. 230, is also false for general ANR-sets.

[4] Fundamenta Mathematicae, vol. 19 (1932), p. 223.

Strictly speaking, the problem as just stated has no solution; there is no single "universal" ANR, but rather a whole class of ANR-sets which together serve in the "universal" capacity. Such a class of ANR-sets is the collection of subsets of the Hilbert parallelotope $Q\times[0, 1]$ which contains the open subset[5] $Q\times(0, 1]$ of $Q\times[0, 1]$.

THEOREM 1. *For a separable metrizable space X the following three conditions are equivalent:*

(1) *X is an* ANR-*set;*

(2) *There is a homeomorphism f of X into Q such that $f(X)\times[0]$ is a neighborhood retract of $f(X)\times[0]+Q\times(0, 1]$;*

(3) *$f(X)\times[0]$ is a neighborhood retract of $f(X)\times[0]+Q\times(0, 1]$ for every homeomorphism f of X into Q.*

(1)→(3): If f is a homeomorphism of an ANR-set X into Q then $f(X)\times[0]$ is an ANR-set. Since Q is compact, so that

$$\overline{f(X)\times[0]}\subset Q\times[0],$$

it follows that $f(X)\times[0]$ is closed in $f(X)\times[0]+Q\times(0, 1]$. Hence $f(X)\times[0]$ is a neighborhood retract of $f(X)\times[0]+Q\times(0, 1]$.

(3)→(2): Since X is separable and metrizable a homeomorphism f exists by Urysohn's theorem.[6]

(2)→(1): Let M be a separable metrizable space containing X in which X is closed and let f be a homeomorphism of X into Q. By Tietze's theorem[7] there exists a continuous function g defined on M with values in Q such that $g(x)=f(x)$ for every $x\in X$. Let M be metrized, with metric d, and let $\rho(x)=\min\{1, d(x, X)\}$ for every $x\in M$. Let $h(x)=(g(x), \rho(x))$, so that h is a continuous function defined on M with values in $f(X)\times[0]+Q\times(0, 1]$ which has the property $h(M-X)\subset Q\times(0, 1]$. Let V be a neighborhood of $f(X)\times[0]$ in $f(X)\times[0]+Q\times(0, 1]$ and let $U=h^{-1}(V)$ so that U is a neighborhood of X in M. If r is a retraction of V onto $f(X)\times[0]$ then the mapping[8] $f^{-1}\pi rh|U$, where π denotes the projection of $Q\times[0]$ onto Q, is a retraction of U onto X.

Kuratowski also gave an analogous generalization of the notion of absolute retract.[9] According to the extended definition a separable metrizable space is an *absolute retract* (AR) if it is a retract of every containing separable metrizable space in which it is closed.

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[5] The symbol (0, 1] denotes the half-open interval $0<t\leq 1$.

[6] Alexandroff and Hopf, *Topologie*, p. 81.

[7] Ibid., p. 73.

[8] If $B\subset B'$ and e is a function defined on B' then the notation $d=e|B$ means that d is the function defined on B such that $d(x)=e(x)$ for every $x\in B$.

THEOREM 1′. *For a separable metrizable space X the following three conditions are equivalent*:

(1′) *X is an* AR;

(2′) *There is a homeomorphism f of X into Q such that $f(X)\times[0]$ is a retract of $f(X)\times[0]+Q\times(0, 1]$;*

(3′) *$f(X)\times[0]$ is a retract of $f(X)\times[0]+Q\times(0, 1]$ for every homeomorphism f of X into Q.*

The proof of this theorem is an obvious modification of the preceding proof.

COROLLARY. *If C denotes the open n-cell $0<x_i<1$ $(i=1, \cdots, n)$ and D denotes the closed n-cell $0\leqq x_i\leqq 1$ $(i=1, \cdots, n)$ then any set E such that $C\subset E\subset D$ is an* AR.

By condition (2′) and a retraction of $Q\times[0, 1]$ onto $D\times[0, 1]$ it is sufficient to show that $E\times[0]$ is a retract of $E\times[0]+D\times(0, 1]$. This can be done by projecting from the point $(1/2, \cdots, 1/2, -1)$ of Euclidean $(n+1)$-space.

It may be worth noting that conditions (2) and (2′) make possible a simpler proof of the Borsuk-Kuratowski[9] theorem(s):

If W is a closed subset of a normal space Z and X is an AR-*set* (ANR-*set*) *then every continuous map of W into X can be extended to Z (to a neighborhood of W in Z).*

In fact conditions (2) and (2′) replace a theorem of Kuratowski[10] which involves infinite polyhedra.

THEOREM 2. *An* ANR *is locally contractible.*[11] *An* AR *is also contractible.*

Using (2) we can suppose that our ANR-set Y is contained in $Q\times[0]$ and that there is a retraction r of an open neighborhood V of Y in $Y+Q\times(0, 1]$ onto Y. But V is the intersection of $Y+Q\times(0, 1]$ with an open set V' of $Q\times[0, 1]$. Let $y\in Y$ and let S_ϵ denote the ϵ-sphere in $Q\times[0, 1]$ about the point y. Since r is continuous there is a $\delta>0$ such that the intersection T_δ of the δ-sphere S_δ and $Y+Q\times(0, 1]$ is contained in V', hence in V, and $r(T_\delta)\subset S_\epsilon$. Let u_t denote a contraction of S_δ to a point $p\in S_\delta\cdot(Q\times(0, 1])$ which moves points rectilinearly, so that $u_t(x)\in Q\times(0, 1]$ for every $0<t\leqq 1$ and

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[9] Fundamenta Mathematicae, vol. 24 (1935), p. 275.

[10] Fundamenta Mathematicae, vol. 24 (1935), p. 266, Theorem 2.

[11] But not uniformly. See the example in Footnote 3. This theorem was proved by Borsuk, Fundamenta Mathematicae, vol. 19 (1932), p. 237 for compact ANR-sets.

$y \in Y \cdot S_\delta$. Then $ru_t | Y \cdot S_\delta$ contracts $Y \cdot S_\delta$ in $Y \cdot S_\epsilon$. The second statement is a consequence of Theorem 3′.

THEOREM 3. *A separable metrizable space X is an* ANR *if and only if for every separable metrizable space M containing X (in which X need not be closed!) there is a neighborhood U of X and a continuous function h defined on $X \times [0] + U \times (0, 1]$ with values in X such that*[8] *$h | X \times [0, 1]$ is a deformation.*[12]

Suppose X is an ANR and M a separable metrizable space containing X. We may assume that $M \subset Q$. By (2) and (3) there is an open neighborhood V' of $X \times [0] + Q \times (0, 1]$ and a retraction r of $V = V' \cdot (X \times [0] + Q \times (0, 1])$ onto $X \times [0]$. Let $\lambda(x) = d(x \times [0], Q \times [0, 1] - V')$ for every $x \in M$ and let $U = \pi(V' \cdot (Q \times [0]))$ where, as before, π denotes the projection of $Q \times [0]$ onto Q. Define for every $(x, t) \in X \times [0] + U \times (0, 1]$,

$$\begin{aligned} h(x, t) &= \pi r(x, t), && \text{when } t \leqq \lambda(x), \\ &= \pi r(x, \lambda(x)), && \text{when } t \geqq \lambda(x). \end{aligned}$$

Since λ is continuous and $\lambda(x) > 0$ when $x \in U$ it follows that h is continuous.

Conversely, let U be a neighborhood of X in $M = Q$ and let h be a continuous function defined on $X \times [0] + U \times (0, 1]$ with values in X such that $h(x, 0) = x$ for every $x \in X$. Then h is a retraction of $X \times [0] + U \times (0, 1]$ onto $X \times [0]$. Furthermore $X \times [0] + U \times (0, 1]$ is a neighborhood of $X \times [0]$ in $X \times [0] + Q \times (0, 1]$.

THEOREM 3′. *A separable metrizable space X is an* AR *if and only if for any separable metrizable space M containing X there is a continuous function h defined on $X \times [0] + M \times (0, 1]$ with values in X such that*[8] *$h | X \times [0, 1]$ is a contraction.*[12]

Let X be an AR and M a separable metrizable space containing X; we may assume that $M \subset Q$. Let r be a retraction of $X \times [0] + Q \times (0, 1]$ onto $X \times [0]$. Let $p \in Q$ and let

$$h(x, t) = \pi r(tp + (1 - t)x, t)$$

for every $(x, t) \in X \times [0] + M \times (0, 1]$, where π is the projection of $Q \times [0]$ onto Q. Then h maps $X \times [0] + M \times (0, 1]$ continuously into X and $h | X \times [0, 1]$ is a contraction of X.

The converse is proved as in Theorem 3.

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[12] A *deformation* of X is a continuous mapping h of $X \times [0, 1]$ into X such that $h(x, 0) = x$ for every $x \in X$. If $h(X, 1)$ is a point then h is called a *contraction* of X.

If X is locally compact the deformation $h|X\times[0, 1]$ of Theorems 3 and 3′ can be chosen in advance of M. For then there exists[13] a compact set M^* and a homeomorphism g of X into M^* such that $M^*-g(X)$ is a point. (We can suppose X not compact so that $M^*\neq g(X)$.) Let $M^*\subset Q$. The homeomorphism g can be extended[13] to a continuous mapping g^* of $\overline{X}$ into M^* by defining $g^*(\overline{X}-X)=M^*-g(X)$. The mapping g^* of $\overline{X}$ into Q can be extended, by Tietze's theorem, to a mapping k of M into Q. In the case of Theorem 3 let h be the mapping of $X\times[0]+U\times(0, 1]$ into X defined by

$$h(x, t) = g^{-1}\pi r(k(x), \min\{t, \lambda(x)\}),$$

where $U=g^{-1}\pi(V'\cdot(Q\times[0]))$. In the case of Theorem 3′ let h be the mapping of $X\times[0]+M\times(0, 1]$ into X defined by

$$h(x, t) = g^{-1}\pi r(tp + (1-t)k(x), t).$$

In both cases $h|X\times[0, 1]$ is independent of M.

If X is not locally compact it may not be possible to pick a deformation $h|X\times[0, 1]$ satisfying the conditions of Theorems 3 or 3′ for all M. An example is the AR-set $\{0\leqq x\leqq 1;\ y=0\}+\sum_{n=1}^{\infty}\{x=1/n;\ 0\leqq y\leqq 1\}$.

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[13] Alexandroff and Hopf, *Topologie*, I, p. 93.

RESOLUTION OF BOUNDARY PROBLEMS BY THE USE OF A GENERALIZED CONVOLUTION

R. C. F. BARTELS AND R. V. CHURCHILL

1. The Laplace transformation of the convolution. The generalized convolution $F^*(t)$ of $F(t, t')$ is defined as follows:

$$F^*(t) = \int_0^t F(t - t', t')dt'.$$

In case $F(t, t') = F_1(t)F_2(t')$, the function $F^*(t)$ is the ordinary convolution $F_1 * F_2$, or Faltung,[1] of the two functions F_1 and F_2.

Let $L\{F^*(t)\}$ denote the Laplace transform of F^* with respect to t,

$$L\{F^*(t)\} = \int_0^\infty e^{-st}F^*(t)dt,$$

and let $\bar{f}(s)$ denote the iterated transform of $F(t, t')$,

$$\bar{f}(s) = \int_0^\infty e^{-st'}dt' \int_0^\infty e^{-st}F(t, t')dt. \tag{1}$$

It will be seen that

$$L\{F^*(t)\} = \bar{f}(s), \tag{2}$$

which, in terms of the inverse Laplace transformation, implies that

$$L^{-1}\{\bar{f}(s)\} = F^*(t).$$

THEOREM. *Let $F(t, t')$ be an integrable function of t and t' in every finite rectangle $0 \leqq t \leqq T$, $0 \leqq t' \leqq T'$ and, for some real α, let $e^{-\alpha(t+t')}|F(t, t')|$ be bounded for all t and t' $(t \geqq 0,\ t' \geqq 0)$. Then if $R(s) > \alpha$, the integral $L\{F^*(t)\}$ is absolutely convergent and satisfies the equation* (2).

Under the conditions stated, the iterated integral in (1) exists if $R(s) > \alpha$ and is equal to the absolutely convergent double integral

$$\iint e^{-s(t+t')}F(t, t')d(t, t'),$$

over the quadrant $t \geqq 0$, $t' \geqq 0$. However, the latter is equal to

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Presented to the Society, April 12, 1940, under the title *An extension of Duhamel's theorem*; received by the editors March 28, 1941.

[1] G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Berlin, 1937, p. 155 ff.

$$\lim_{k\to\infty}\iint_{T_k} e^{-s(t+t')}F(t, t')d(t, t'), \tag{3}$$

where T_k is the triangle bounded by the lines $t'=0$, $t=0$ and $t+t'=k$. After the substitutions $\tau=t+t'$, $\tau'=t'$ have been made, the integral in (3) is transformed into the integral over the triangle bounded by the lines $\tau'=0$, $\tau=\tau'$, and $\tau=k$ in the $\tau\tau'$-plane. Hence (3) can be written

$$\lim_{k\to\infty}\int_0^k e^{-s\tau}d\tau\int_0^\tau F(\tau-\tau', \tau')d\tau',$$

which is $L\{F^*(t)\}$. Therefore the equality in (2) holds.

The convergence of the integral $L\{|F^*(t)|\}$, that is, the absolute convergence of the Laplace integral in (2), follows immediately from the absolute convergence of the limit in (3).

2. The Duhamel integral formula. Let Λ and λ denote the differential operators defined as follows:

$$\Lambda\{U\} = C_0U + \sum_{i=1}^{3} C_i\frac{\partial}{\partial x_i}\left(K_i\frac{\partial U}{\partial x_i}\right),$$
$$\lambda\{U\} = c_0U + \sum_{i=1}^{3} c_i\frac{\partial U}{\partial x_i},$$

where the coefficients of U and its derivatives are functions of x_1, x_2, and x_3 only. Then, if P denotes an arbitrary interior point (x_1, x_2, x_3) of a region R, and Q any point on the boundary S of R, a general boundary value problem for the temperature $U(P, t)$ in the region R can be written

$$\text{(A)}\qquad \begin{aligned} \frac{\partial}{\partial t}U(P, t) &= \Lambda\{U\} + F(P, t), && t > 0,\\ \lambda\{U(Q, t)\} &= G(Q, t), && t > 0,\\ U(P, 0) &= H(P), \end{aligned}$$

where F, G, and H are prescribed functions. In the second of these equations it is understood that $\lambda\{U(Q, t)\}$ represents the limit of $\lambda\{U(P, t)\}$ as P approaches the point Q on S in a prescribed manner. Similarly, in the last equation $U(P, 0)$ is written for $U(P, +0)$.

Applying the Laplace transformation with respect to t to the first two equations in problem (A) gives, in view of the last condition,

$$
(\mathrm{A}') \qquad
\begin{aligned}
su(P, s) - H(P) &= \Delta\{u(P, s)\} + f(P, s),\\
\lambda\{u(Q, s)\} &= g(Q, s),
\end{aligned}
$$

where u, f, and g are the transforms of U, F, and G, respectively.

Let the temperature function $V(P, t, t')$, depending on the *fixed* parameter t', be the solution of problem (A) when the source function F and the surface-temperature function G have at each point of R and S, respectively, the ***constant*** values $F(P, t')$ and $G(Q, t')$. Then

$$
(\mathrm{B}) \qquad
\begin{aligned}
\frac{\partial}{\partial t} V(P, t, t') &= \Delta\{V(P, t, t')\} + F(P, t'),\\
\lambda\{V(Q, t, t')\} &= G(Q, t'),\\
V(P, 0, t') &= H(P).
\end{aligned}
$$

Also let $v(P, s, t')$ represent the Laplace transformation of $V(P, t, t')$ with respect to t, and $\bar{v}(P, s)$ the iterated transform of $V(P, t, t')$ with respect to t and t'. Then applying the transformation with respect to t, it follows from (B) that

$$
\begin{aligned}
sv(P, s, t') - H(P) &= \Delta\{v(P, s, t')\} + \frac{1}{s} F(P, t'),\\
\lambda\{v(Q, s, t')\} &= \frac{1}{s} G(Q, t'),
\end{aligned}
$$

and consequently, on applying the transformation with respect to t' to these equations, that

$$
(\mathrm{B}') \qquad
\begin{aligned}
s\bar{v}(P, s) - \frac{1}{s} H(P) &= \Delta\{\bar{v}(P, s)\} + \frac{1}{s} f(P, s),\\
\lambda\{\bar{v}(Q, s)\} &= \frac{1}{s} g(Q, s).
\end{aligned}
$$

Upon multiplying each member of the equations in (B′) by s, it is at once evident that the problems (A′) and (B′) are equivalent and that the function $s\bar{v}(P, s)$ is a solution of (A′). Thus, assuming that the solution of (A′) is unique, it follows that

$$
(4) \qquad u(P, s) = s\bar{v}(P, s).
$$

According to equation (2),

$$
\bar{v}(P, s) = L\{V^*(P, t)\} = L\left\{\int_0^t V(P, t - \tau, \tau)\,d\tau\right\}.
$$

Moreover, since $V^*(P, 0) = 0$,

$$L\left\{\frac{\partial V^*}{\partial t}\right\} = s\bar{v}(P, s).$$

Hence it follows from equation (4), on performing the inverse transformation, that

$$U(P, t) = \frac{\partial}{\partial t}\int_0^t V(P, t - \tau, \tau)d\tau. \tag{5}$$

The solution of the problem (A) *with variable source and surface conditions is therefore given by formula* (5) *in terms of the solution of the problem* (B) *with constant source and surface conditions.*

Sufficient conditions can, of course, be given in order to justify all of the steps in the formal derivation of (5). However, more relaxed conditions can be obtained in particular cases by verifying directly that the function given by (5) is a solution of problem (A). When the latter procedure is applied to the above general case, it is seen that the foregoing result in italics is true, provided that V, $\partial V/\partial x$, $\partial^2 V/\partial x_i^2$, and their derivatives with respect to t are continuous functions of P, t, and t' interior to R when $t \geqq 0$ and $t' \geqq 0$, and that the function $(\partial/\partial t)\int_0^t \lambda\{V(P, t-\tau, \tau)\}d\tau$ is continuous with respect to P at the points Q when $t > 0$.

The relation (5) is known in the theory of heat conduction as Duhamel's integral formula.[2] It has been shown above that this formula applies to a very general temperature problem. It is clear that the procedure can be applied in case of discontinuous media, and to other boundary value problems as well. It is, of course, applicable to problems with partial differential equations of higher order than the second.

3. **Resolution of temperature problems.** It will now be shown that the general problem (A) can be further resolved into still simpler problems which are of two basic types. It should first be observed that the solution of problem (B) can be written

$$V(P, t, t') = V_1(P, t') + V_2(P, t, t') + W(P, t, t'), \tag{6}$$

where V_1, V_2, and W are solutions of the problems:

$$\text{(a)} \qquad \Lambda\{V_1(P, t')\} = 0, \qquad \lambda\{V_1(Q, t')\} = G(Q, t');$$

---

[2] J. M. C. Duhamel, ***Mémoir sur la méthode générale relative au mouvement de la chaleur dans les corps solides plongers dans des milieux dont la température varie avec le temps***, Journal de l'École Polytechnique, vol. 14 (1830), pp. 20–77; also see H. S. Carslaw, *Conduction of Heat*, 1921, pp. 16–19.

$$\text{(b)}\qquad \begin{gathered}\frac{\partial}{\partial t}V_2(P,t,t') = \Delta\{V_2(P,t,t')\},\\ \lambda\{V_2(Q,t,t')\} = 0,\quad V_2(P,0,t') = H(P) - V_1(P,t');\end{gathered}$$

and

$$\begin{gathered}\frac{\partial}{\partial t}W(P,t,t') = \Delta\{W(P,t,t')\} + F(P,t'),\\ \lambda\{W(Q,t,t')\} = 0,\qquad W(P,0,t') = 0.\end{gathered}$$

In terms of the transform $w(P, s, t')$ of $W(P, t, t')$, the last of the foregoing problems becomes

$$sw(P,s,t') - \frac{1}{s}F(P,t') = \Delta\{w(P,s,t')\},\qquad \lambda\{w(Q,s,t')\} = 0.$$

On multiplying each member of these two equations by s, it is immediately evident that the function $sw(P, s, t')$ is a solution of the transform of the following problem:

$$\text{(c)}\qquad \begin{gathered}\frac{\partial}{\partial t}V_3(P,t,t') = \Delta\{V_3(P,t,t')\},\\ \lambda\{V_3(Q,t,t')\} = 0,\qquad V_3(P,0,t') = F(P,t').\end{gathered}$$

Therefore, if $v_3(P, s, t')$ denotes the transform of $V_3(P, t, t')$, then $w = v_3/s$ and, consequently,

$$\text{(7)}\qquad W(P,t,t') = \int_0^t V_3(P,\tau,t')d\tau.$$

According to equations (5), (6), and (7), the solution of problem (A) is given in terms of the solutions V_1, V_2, and V_3 of the problems (a), (b), and (c), respectively, as follows:

$$U(P,t) = \frac{\partial}{\partial t}\int_0^t \Big[V_1(P,\tau) + V_2(P,t-\tau,\tau) + \int_0^{t-\tau} V_3(P,\tau',\tau)d\tau'\Big]d\tau,$$

which can be written

$$\text{(8)}\qquad \begin{aligned}U(P,t) = V_1(P,t) &+ \frac{\partial}{\partial t}\int_0^t V_2(P,t-\tau,\tau)d\tau\\ &+ \int_0^t V_3(P,t-\tau,\tau)d\tau.\end{aligned}$$

Problems (b) and (c) are variable state problems of the same type in which the initial condition alone is nonhomogeneous. On the other hand, (a) is a steady state problem. *The general temperature problem* (A) *is thus resolved by formula* (8) *into the solution of problems of two basic types, one of steady state, the other, of variable state.*

4. **Resolution of vibration problems.** A general form of the problem of displacements in elastic media with variable boundary conditions, including forced vibrations of membranes, shafts, and strings, can be written

$$\text{(C)}\qquad \begin{gathered} \frac{\partial^2}{\partial t^2}\Phi(P,t)+b\frac{\partial}{\partial t}\Phi(P,t)=\Lambda\{\Phi(P,t)\}+F(P,t),\\ \lambda\{\Phi(Q,t)\}=G(Q,t),\\ \Phi(P,0)=H(P),\qquad \frac{\partial}{\partial t}\Phi(P,0)=I(P). \end{gathered}$$

As before, P denotes a point in a region R, and Q a point on the boundary of R; F, G, H, I, and b are prescribed functions, b being a function of P alone. The resolution of this problem can, as in the case of the foregoing temperature problem, be easily obtained by formal application of the Laplace transformation. The problem corresponding to (C) in $\phi(P, s)$, the transform of $\Phi(P, t)$, is

$$\text{(C}'\text{)}\qquad \begin{gathered} (s^2+bs)\phi(P,s)-(s+b)H(P)-I(P)=\Lambda\{\phi(P,s)\}+f(P,s),\\ \lambda\{\phi(Q,s)\}=g(Q,s). \end{gathered}$$

Let the function $\Theta(P, t, t')$, depending upon the fixed parameter t', be the solution of problem (C) when F and G have at each point the constant values $F(P, t')$ and $G(Q, t')$, respectively, that is,

$$\text{(D)}\qquad \begin{gathered} \frac{\partial^2}{\partial t^2}\Theta(P,t,t')+b\frac{\partial}{\partial t}\Theta(P,t,t')=\Lambda\{\Theta(P,t,t')\}+F(P,t'),\\ \lambda\{\Theta(Q,t,t')\}=G(Q,t'),\\ \Theta(P,0,t')=H(P),\qquad \frac{\partial}{\partial t}\Theta(P,0,t')=I(P). \end{gathered}$$

If $\bar{\theta}(P, s)$ is the iterated transform of $\Theta(P, t, t')$, it follows from (D) that $s\bar{\theta}$ satisfies the equations (C'). Therefore, $\phi(P, s) = s\bar{\theta}(P, s)$ and, according to (2), we have the formula

$$\text{(9)}\qquad \Phi(P,t)=\frac{\partial}{\partial t}\int_0^t\Theta(P,t-\tau,\tau)d\tau.$$

Repeating the procedure of the preceding section, it is seen that

$$\Theta(P, t, t') = \Theta_1(P, t') + \Theta_2(P, t, t') + \int_0^t \Theta_3(P, \tau, t')d\tau,$$

where Θ_1, Θ_2, and Θ_3 are the solutions of the following problems, respectively,

$$\Lambda\{\Theta_1(P, t')\} = 0, \qquad \lambda\{\Theta_1(Q, t')\} = G(Q, t'); \tag{d}$$

$$\begin{gathered}
\frac{\partial^2}{\partial t^2}\Theta_2(P, t, t') + b\frac{\partial}{\partial t}\Theta_2(P, t, t') = \Lambda\{\Theta_2(P, t, t')\}, \\
\lambda\{\Theta_2(Q, t, t')\} = 0, \qquad \Theta_2(P, 0, t') = H(P) - \Theta_1(P, t'), \\
\frac{\partial}{\partial t}\Theta_2(P, 0, t') = I(P);
\end{gathered} \tag{e}$$

$$\begin{gathered}
\frac{\partial^2}{\partial t^2}\Theta_3(P, t, t') + b\frac{\partial}{\partial t}\Theta_3(P, t, t') = \Lambda\{\Theta_3(P, t, t')\}, \\
\lambda\{\Theta_3(Q, t, t')\} = 0, \qquad \Theta_3(P, 0, t') = 0, \\
\frac{\partial}{\partial t}\Theta_3(P, 0, t') = F(P, t').
\end{gathered} \tag{f}$$

Therefore, in view of (9),

$$\begin{aligned}
\Phi(P, t) = \Theta_1(P, t) &+ \frac{\partial}{\partial t}\int_0^t \Theta_2(P, t - \tau, \tau)d\tau \\
&+ \int_0^t \Theta_3(P, t - \tau, \tau)d\tau.
\end{aligned} \tag{10}$$

Thus the solution of the general problem (C) *is resolved into the solution of the two basic types* (d), *and* (e). Problem (f) is a special case of (e). The latter can, of course, be still further resolved into two simpler problems, one being of the same form as (f) with the second initial condition nonhomogeneous, and the other having the nonhomogeneity occurring in the first initial condition.

THE UNIVERSITY OF MICHIGAN

ROOTS OF CERTAIN CLASSES OF POLYNOMIALS

LOUIS WEISNER

It is well known[1] that if the roots of the polynomials $\phi(z)$ and $F(z)$ are real, so are the roots of the polynomial $\phi(D)F(z)$, where $D=d/dz$. This result has been applied to certain types of entire functions and trigonometric integrals.[2] The following example illustrates the method employed. If

$$f(z) = \sum_{k=0}^{n} c_k z^k \tag{1}$$

is a polynomial whose roots $\lambda_1, \cdots, \lambda_n$ lie on the unit circle, then the roots of the polynomials

$$F_p(z) = c_n \prod_{k=1}^{n} \left[(1 + z/p)^p - \lambda_k(1 - z/p)^p\right], \qquad p = 1, 2, \cdots,$$

lie on the axis of pure imaginaries. Therefore, if the roots of the polynomial $\phi(z)$ lie on the axis of pure imaginaries, so do the roots of the polynomials[3] $\phi(D)F_p(z)$, $p=1, 2, \cdots$. Now the sequence of polynomials $\{F_p(z)\}$ converges uniformly in every finite region to the function

$$F(z) = e^{-nz}f(e^{2z}) = \sum_{k=0}^{n} c_k e^{(2k-n)z},$$

and the sequence $\{\phi(D)F_p(z)\}$ converges likewise to

$$\phi(D)F(z) = \sum_{k=0}^{n} c_k \phi(2k - n)e^{(2k-n)z}.$$

The roots of $\phi(D)F(z)$ therefore lie on the axis of pure imaginaries. Removing the innocuous factor e^{-nz}, and replacing e^{2z} by z, the following theorem results: *If the roots of $f(z)$ lie on the unit circle, and the roots of $\phi(z)$ lie on the axis of pure imaginaries, then the roots of the polynomial*

Presented to the Society, September 5, 1941; received by the editors June 6, 1941.

[1] Ch. Hermite, Nouvelles Annales de Mathématiques, vol. 5 (1866), p. 479; Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis* II, p. 47, Problem 62.

[2] See G. Pólya, *Über trigonometrische Integrale mit nur reellen Nullstellen*, Journal für die reine und angewandte Mathematik, vol. 158 (1927), pp. 6–18.

[3] Replacing z by iz it follows from the above theorem of Hermite that if the roots of $\phi(z)$ and $F(z)$ lie on the axis of pure imaginaries, so do the roots of $\phi(D)F(z)$.

$$g(z) = \sum_{k=0}^{n} c_k \phi(2k - n) z^k \tag{2}$$

lie on the unit circle.

This theorem provides us with a large class of polynomials whose coefficients are explicitly given and whose roots lie on the unit circle. As the theorem itself concerns polynomials, while the proof is transcendental, an elementary proof is desirable. An elementary proof, with extensions, follows.

From the polynomials $f(z)$ and $\phi(z)$, where $f(z)$ has the explicit form (1), construct the polynomial $g(z)$ defined by (2). Setting

$$\psi(z) = \phi(2z - n),$$

we have

$$g(z) = \sum_{k=0}^{n} c_k \psi(k) z^k.$$

If $\alpha_1, \cdots, \alpha_m$ are the roots of $\psi(z)$, so that

$$\psi(z) = a \prod_{k=1}^{m} (z - \alpha_k), \qquad a \neq 0,$$

then

$$g(z) = a \prod_{k=1}^{m} (zD - \alpha_k) \cdot f(z) = \psi(zD) f(z). \tag{3}$$

Now suppose that the roots of $\phi(z)$ lie on the axis of pure imaginaries and the roots of $f(z)$ in the ring $r_1 \leqq |z| \leqq r_2$, where $0 \leqq r_1 \leqq r_2$. From the relation between $\phi(z)$ and $\psi(z)$ it is evident that the roots of $\psi(z)$ lie on the line $\Re z = n/2$. Let α be a root of $\psi(z)$ and ζ a root of the equation

$$zf'(z) - \alpha f(z) = 0. \tag{4}$$

If ζ is also a root of $f'(z)$, ζ is a multiple root of $f(z)$ and hence lies in the ring $r_1 \leqq |z| \leqq r_2$. In the contrary case the centroid of $f(z)$ relative to ζ is different from ζ. This centroid is[4] $\zeta_1 = \zeta - nf(\zeta)/f'(\zeta) = (1 - n/\alpha)\zeta$, by (4). From $\Re \alpha = n/2$, we infer that $|1 - n/\alpha| = 1$; therefore $|\zeta_1| = |\zeta|$. If ζ is outside the ring $r_1 \leqq |z| \leqq r_2$, the circle with center at the origin which passes through ζ and ζ_1 fails to separate the roots of $f(z)$, in violation of a theorem of Laguerre.[5] Therefore all the roots of (4) lie in the ring $r_1 \leq |z| \leq r_2$. Applying this result repeatedly it follows from (3) that the roots of $g(z)$ lie in the ring.

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[4] E. Laguerre, *Oeuvres*, vol. 1, p. 56.

[5] Laguerre, loc. cit., p. 57, p. 134; Pólya and Szegö, loc. cit., p. 57, Problem 106.

THEOREM 1. *If the roots of $\phi(z)$ lie on the axis of pure imaginaries, and the roots of $f(z)$ in the ring $r_1 \leq |z| \leq r_2$, then the roots of $g(z)$ lie in this ring.*

COROLLARY 1. *If the roots of $\phi(z)$ lie on the axis of pure imaginaries, and the roots of $f(z)$ on the circle $|z| = r$, then the roots of $g(z)$ lie on this circle.*

COROLLARY 2. *If the roots of $\phi(z)$ lie on the axis of pure imaginaries, the roots of the polynomial*

$$\sum_{k=0}^{n} \phi(2k - n) z^k$$

lie on the unit circle.

To prove the first corollary, take $r_1 = r_2 = r$. To prove the second, take $f(z) = 1 + z + \cdots + z^n$.

We suppose now that the roots of $f(z)$ lie in the circular region $|z| \leq r$, and that the roots of $\phi(z)$ lie in the half-plane $\Re z \leq 0$. We reëxamine (4), retaining our previous notation. If $\alpha = 0$, the roots of (4) lie in the region $|z| \leq r$ by the Gauss-Lucas theorem; hence we need only consider the case $\alpha \neq 0$. We may also suppose that ζ is not a root of $f'(z)$.

We now have $\Re\alpha \leq n/2$; hence $|1 - n/\alpha| \geq 1$, and $|\zeta_1| \geq |\zeta|$. If $|\zeta| > r$, then $|\zeta_1| > r$, and a circle may be drawn through ζ and ζ_1 which fails to separate the roots of $f(z)$. As this result contradicts Laguerre's theorem, the region $|z| \leq r$ includes all the roots of (4) and consequently all the roots of $g(z)$.

THEOREM 2. *If the roots of $\phi(z)$ lie in the half-plane $\Re z \leq 0$, and the roots of $f(z)$ lie in the circular region $|z| \leq r$, then the roots of $g(z)$ lie in this circular region.*

Theorem 2 may be extended to the case in which $\phi(z)$ is the limit of a sequence of polynomials whose roots lie in the half-plane $\Re z \leq 0$ and which converges uniformly in every finite region.[4] Likewise Theorem 1 and its corollaries are valid if $\phi(z)$ is the limit of a sequence of polynomials whose roots lie on the axis of pure imaginaries and which converges uniformly in every finite region.

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[4] Entire functions which are limits of such sequences of polynomials are characterized by Eduard Benz, *Über lineare, verschiebungstreue Funktionaloperationen und die Nullstellen von ganzen Funktionen*, Commentarii Mathematici Helvetici, vol. 7 (1935), p. 246.

HUNTER COLLEGE

GENERALIZED LAPLACE INTEGRALS

R. P. BOAS, JR.

We consider the linear space $\mathfrak{H}(c)$ whose elements are functions $f(z)$ $[z=x+iy]$ which are analytic for $x>c$ and satisfy

$$(1) \qquad \int_{-\infty}^{\infty} |f(x+iy)|^2 dy \leq M, \qquad x > c,$$

where the finite number M depends on the function in question. It is well known that an element $f(z)$ of $\mathfrak{H}(c)$ has boundary values $f(c+iy)$ almost everywhere on $x=c$, and that $\mathfrak{H}(c)$ is a Hilbert space if the norm of $f(z)$ is defined by

$$\|f(z)\|^2 = \int_{-\infty}^{\infty} |f(c+iy)|^2 dy.$$

Furthermore, it is known [5, p. 8] that if $f(z) \in \mathfrak{H}(c)$, then $f(z)$ is representable as a Laplace integral for $x>c$, in the sense that there is a unique function[1] $\phi(t)$ with $e^{-ct}\phi(t) \in L^2(0, \infty)$ such that

$$(2) \qquad \lim_{T\to\infty} \left\| f(z) - \int_0^T e^{-zt}\phi(t)dt \right\| = 0;$$

we shall express (2) by writing

$$(3) \qquad f(z) = \int_0^\infty e^{-zt}\phi(t)dt, \qquad x > c.$$

It is easily verified that the integral in (3) converges in the ordinary sense for $x>c$. A Laplace integral may be regarded as a generalized power series; the object of this note is to generalize the integral representation (3) by replacing e^{-zt} by a kernel $g(z, t)$ which is in some sense "nearly" e^{-zt}, just as power series $\sum a_n z^n$ have been generalized[2] by replacing the functions z^n by functions $g_n(z)$.

Presented to the Society, September 5, 1941; received by the editors May 24, 1941.

[1] Unique, that is, up to sets of measure zero.

[2] For a bibliography of this problem, see [1].

We suppose that $g(s, t)$ is, for each fixed t in $0<t<\infty$, an analytic function of s in $x>c$; and that for each fixed s in $x>c$, $g(s, t)\in L^2(0, R)$ for every $R>0$.

THEOREM 1. *If for each positive* T

$$e^{-ct}\phi(t) \in L^2(0, T) \textit{ implies } \int_0^T g(s, t)\phi(t)dt \in \mathfrak{H}(c); \tag{4}$$

and if there is a number λ, $0<\lambda<1$, *such that*

$$\left\| \int_R^S a(t)\left[e^{-st} - g(s, t)\right]dt \right\| \leqq \lambda \left\| \int_R^S a(t)e^{-st}dt \right\| \tag{5}$$

for all $S>R>0$ *and all functions* $a(t)\in L^2(R, S)$, *then there exists for each* $f(s)\in\mathfrak{H}(c)$ *a unique* $\psi(t)$ *with* $e^{-ct}\psi(t)\in L^2(0, \infty)$ *such that*

$$f(s) = \int_0^\infty g(s, t)\psi(t)dt, \qquad x > c, \tag{6}$$

where the integral is a mean-square limit for $x\geqq c$, *and also converges in the ordinary sense for* $x>c$.

In Theorems 2, 3 and 5 we shall replace the conditions of Theorem 1 by more convenient conditions; in Theorem 6 the theory will be applied to the generalized Laplace integrals recently discussed by Meijer [4] and Greenwood [7], namely

$$f(s) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty K_\nu(st)(st)^{1/2}\phi(t)dt,$$

where $K_\nu(s)$ is the usual notation for a Bessel function of imaginary argument [6, p. 78], and $-\frac{1}{2}<\Re(\nu)<\frac{1}{2}$ (if $\nu=\pm\frac{1}{2}$, we have (3) again).

Theorem 1 is easily proved by the method of successive approximations used for a similar purpose by Paley and Wiener [5, p. 100].

Let $f(s)\in\mathfrak{H}(c)$. Then there is a function $\phi(t)$ with $e^{-ct}\phi(t)\in L^2(0, \infty)$ such that

$$f(s) = \int_0^\infty e^{-st}\phi(t)dt, \qquad x > c;$$

here, and throughout the proof, integrals over $(0, \infty)$ are taken as mean-square limits, as in (2). Then the integral

$$\int_0^\infty g(s, t)\phi(t)dt, \qquad x > c,$$

exists; for, by (5),

$$\left\| \int_R^S \phi(t)[e^{-st} - g(s, t)]dt \right\| \leqq \lambda \left\| \int_R^S \phi(t)e^{-st}dt \right\| \to 0$$

as R and $S \to \infty$.

We now define inductively sequences $\{\phi_n(t)\}$ and $\{f_n(s)\}$ by setting $\phi_0(t) = \phi(t)$;

$$f_1(s) = \int_0^\infty \phi(t)[e^{-st} - g(s, t)]dt = \int_0^\infty \phi_1(t)e^{-st}dt;$$

and generally

$$f_n(s) = \int_0^\infty \phi_n(t)e^{-st}dt,$$

$$f_{n+1}(s) = \int_0^\infty \phi_n(t)[e^{-st} - g(s, t)]dt.$$

Thus

$$f(s) - f_{n+1}(s) = \int_0^\infty \sum_{k=0}^n \phi_k(t)g(s, t)dt;$$

and by (5)

$$\text{(7)} \qquad \begin{aligned} \|f_{n+1}\| &\leqq \lambda \left\| \int_0^\infty \phi_n(t)e^{-st}dt \right\| = \lambda\|f_n\| \\ &\leqq \cdots \leqq \lambda^{n+1}\|f\| \to 0, \qquad n \to \infty. \end{aligned}$$

Consequently

$$\text{(8)} \qquad \left\| f(s) - \int_0^\infty \sum_{k=0}^n \phi_k(t)g(s, t)dt \right\| \to 0, \qquad n \to \infty.$$

Now

$$\left\| \int_0^\infty \phi_n(t)e^{-st}dt \right\| = \|f_n\| \leqq \lambda^{n+1}\|f\|,$$

and hence

$$\left\| \int_0^\infty \sum_m^n \phi_k(t)e^{-st}dt \right\| \leqq \sum_m^n \lambda^{k+1}\|f\| \to 0, \qquad m, n \to \infty.$$

Therefore, since $\mathfrak{H}(c)$ is complete, there exists $F(s) \in \mathfrak{H}(c)$ such that

$$\text{(9)} \qquad \left\| F(s) - \int_0^\infty \sum_{k=0}^n \phi_k(t)e^{-st}dt \right\| \to 0, \qquad n \to \infty,$$

and

$$F(s) = \int_0^\infty \psi(t)e^{-st}dt, \qquad e^{-ct}\psi(t) \in L^2(0, \infty).$$

If we set

$$\psi_n(t) = \psi(t) - \sum_{k=0}^{n} \phi_k(t) \tag{10}$$

we then have

$$\left\| \int_0^\infty \psi_n(t) e^{-st} dt \right\| \to 0;$$

then by (5)

$$\left\| \int_0^\infty g(s, t)\psi_n(t) dt \right\| \leqq (1 + \lambda) \left\| \int_0^\infty e^{-st}\psi_n(t) dt \right\| \to 0.$$

By (10) and (8) we thus have

$$f(s) = \int_0^\infty \psi(t) g(s, t) dt,$$

and the existence of the representation (6) is established. That (6) converges in the ordinary sense for $x > c$ follows easily from the fact that functions of $\mathfrak{H}(c)$ are represented by their Cauchy integrals [**2**, p. 338].

To show that the representation is unique, we have only to show that

$$\int_0^\infty \omega(t) g(s, t) dt \equiv 0, \qquad x > c, \tag{11}$$

implies $\omega(t) = 0$ almost everywhere. Now if (11) is true, for every positive T we have, by (5),

$$\begin{aligned} \left\| \int_0^T \omega(t) e^{-st} dt \right\| &\leqq \left\| \int_0^T \omega(t) [e^{-st} - g(s, t)] dt \right\| + \left\| \int_0^T \omega(t) g(s, t) dt \right\| \\ &\leqq \lambda \left\| \int_0^T \omega(t) e^{-st} dt \right\| + \left\| \int_0^T \omega(t) g(s, t) dt \right\|. \end{aligned}$$

The limit of the last term on the right is zero, and since $\lambda < 1$ it follows that

$$\int_0^\infty \omega(t) e^{-st} dt = 0, \qquad x > c.$$

Since the Laplace representation (3) is unique, $\omega(t) = 0$ almost everywhere.

THEOREM 2. *If for* $x > c$

$$g(s, t) = e^{-st}\{1 + h(s, t)\} \tag{12}$$

with

$$k(z, t) = \int_0^\infty e^{-zu}\omega(t, u)du, \qquad x > c,$$

$$|\omega(t, u)| \leqq \omega(u),$$

$$h(z) = \int_0^\infty e^{-zu}\omega(u)du, \qquad x > c,$$

and

$$h(c) < 1, \tag{13}$$

then the function $g(z, t)$ *has properties* (4) *and* (5), *and consequently the representation* (6) *is possible and unique.*

If $a(t) \in L^2(R, S)$, we set $a^*(t) = a(t)$ in (R, S) and $a^*(t) = 0$ outside (R, S), and use the Parseval theorem for Laplace transforms; with $z = c + iy$,

$$\int_{-\infty}^{\infty}\left|\int_R^S e^{-zt}a(t)k(z, t)\right|^2 dy$$

$$= \int_{-\infty}^{\infty}\left|\int_0^\infty e^{-zt}a^*(t)dt\int_0^\infty e^{-zu}\omega(t, u)du\right|^2 dy$$

$$= \int_{-\infty}^{\infty}\left|\int_0^\infty e^{-zv}dv\int_0^v a^*(t)\omega(t, v - t)dt\right|^2 dy$$

$$= \frac{1}{2\pi}\int_0^\infty\left|e^{-cv}\int_0^v a^*(t)\omega(t, v - t)dt\right|^2 dv$$

$$\leqq \frac{1}{2\pi}\int_0^\infty\left|e^{-cv}\int_0^v |a^*(t)|\,\omega(v - t)dt\right|^2 dv$$

$$= \int_{-\infty}^{\infty}\left|\int_0^\infty e^{-zv}dv\int_0^v |a^*(t)|\,\omega(v - t)dt\right|^2 dy$$

$$= \int_{-\infty}^{\infty}\left|\int_0^\infty e^{-zt}|a^*(t)|\,dt\int_0^\infty e^{-zu}\omega(u)du\right|^2 dy$$

$$= \int_{-\infty}^{\infty}\left|h(z)\int_0^\infty e^{-zt}|a^*(t)|\,dt\right|^2 dy$$

$$\leqq [h(c)]^2\int_{-\infty}^{\infty}\left|\int_0^\infty e^{-zt}|a^*(t)|\,dt\right|^2 dy$$

$$= [h(c)]^2\int_{-\infty}^{\infty}\left|\int_R^S e^{-zt}a(t)dt\right|^2 dy.$$

Since $h(c)<1$, this gives us (5). But it also implies

$$\left\{\int_{-\infty}^{\infty}\left|\int_0^T \phi(t)g(z,t)dt\right|^2 dy\right\}^{1/2}$$
$$\leqq \{1+h(c)\}\left\{\int_{-\infty}^{\infty}\left|\int_0^T \phi(t)e^{-zt}dt\right|^2 dy\right\}^{1/2}$$
$$=\frac{1}{2\pi}\{1+h(c)\}\left\{\int_0^T |\phi(t)|^2 e^{-2xt}dt\right\}^{1/2},$$

and hence (4).

For example, (13) is satisfied, with $c=1$, if $h(z,t)=z/(z^2+t^2)$ or $1/(z+t)$.

THEOREM 3. *If* $h(z,t)$ *is defined by* (12) *and satisfies*

$$(14)\qquad |h(z,t)| \leqq |k(z)|\, l(t), \qquad x>c,$$

with $\|k(z)\|(\int_0^\infty |l(t)|^2dt)^{1/2}=\mu<(2\pi)^{-1/2}$, *then* $g(z,t)$ *has properties* (4) *and* (5) *and the representation* (6) *is possible.*

In fact, we have

$$\left|\int_R^S a(t)e^{-zt}h(z,t)dt\right| \leqq |k(z)|\int_R^S e^{-xt}l(t)\,|a(t)|\,dt,$$

$$\left\|\int_R^S a(t)e^{-zt}h(z,t)dt\right\|$$
$$\leqq \|k(z)\|\left\{\int_R^S |l(t)|^2dt\right\}^{1/2}\left\{\int_R^S e^{-2ct}|a(t)|^2dt\right\}^{1/2}$$
$$\leqq (2\pi)^{1/2}\mu\left\|\int_R^S a(t)e^{-zt}dt\right\|.$$

This verifies (5); and (4) follows because

$$\left\|\int_0^T a(t)g(z,t)dt\right\| = \left\|\int_0^T a(t)e^{-zt}[1+h(z,t)]dt\right\|$$
$$\leqq \left\|\int_0^T a(t)e^{-zt}dt\right\| + \left\|\int_0^T a(t)e^{-zt}h(z,t)dt\right\|$$
$$\leqq [1+(2\pi)^{1/2}\mu]\left\|\int_0^\infty a(t)e^{-zt}dt\right\|.$$

THEOREM 4. *If $g(s, t)=g^*(s, t)$ for $t \geqq s>0$; if the representation* (6) *is possible and unique for $x>c^*$, with $g(s, t)$, for every $f(s)$ of $\mathfrak{H}(c^*)$; and possible and unique for $x>c$, with $g^*(s, t)$, for every $f(s)$ of $\mathfrak{H}(c)$ (where $c<c^*$), then the representation is also possible and unique for $x>c$, with $g(t, s)$, for every $f(s)$ of $\mathfrak{H}(c)$.*

We have

$$f(s) = \int_0^\infty g(s, t)\psi(t)dt, \qquad x > c^*. \tag{15}$$

Define

$$F(s) = f(s) - \int_0^s g(s, t)\psi(t)dt = \int_s^\infty g(s, t)\psi(t)dt = \int_s^\infty g^*(s, t)\psi(t)dt.$$

The last integral defines $F(s)$ for $x>c$; and since the representation of $F(s)$ in terms of $g^*(s, t)$ is unique for $x>c$, the integral

$$\int_s^\infty g(s, t)\psi(t)dt$$

must represent $F(s)$ for $x>c$; hence (15) represents $f(s)$ for $x>c$ also.

THEOREM 5. *Theorem 3 remains true if* (14) *is satisfied with $k(s)\in\mathfrak{H}(c)$ and $l(t)\in L^2(0, \infty)$.*

If $k(s)\in\mathfrak{H}(c)$ we have

$$\lim_{x\to\infty} \int_{-\infty}^\infty | k(x + iy) |^2 dy = 0; \tag{16}$$

for, if $x>c$,

$$k(x + iy) = \int_0^\infty e^{-xt-iyt}\phi(t)dt, \qquad e^{-ct}\phi(t) \in L^2(0, \infty);$$

and so

$$\int_{-\infty}^\infty | k(x + iy) |^2 dy = \frac{1}{2\pi}\int_0^\infty e^{-2xt} | \phi(t) |^2 dt \to 0, \qquad x \to \infty.$$

Choose s so large that

$$\| k(s) \| \left(\int_s^\infty | l(t) |^2 dt\right)^{1/2} = \mu < (2\pi)^{-1/2}. \tag{17}$$

Then define $h^*(s, t)=h(s, t)$ for $t>s$, $h^*(s, t)=0$ for $0\leqq t\leqq s$; and

$g^*(s, t) = e^{-st}[1+h^*(s, t)]$. The function $g^*(s, t)$ satisfies the conditions of Theorem 3, with $l(t)$ replaced by the function $l^*(t)$ equal to $l(t)$ in (s, ∞) and to zero in $(0, s)$. On the other hand, if b is so large that

$$\left\{\int_{-\infty}^{\infty} |k(b+iy)|^2 dy\right\}^{1/2} \left\{\int_0^{\infty} |l(t)|^2 dt\right\}^{1/2} < (2\pi)^{-1/2}$$

(such a b exists because of (16)), the function $g(s, t)$ satisfies the conditions of Theorem 3 with c replaced by b. Thus the representation (6) is possible and unique with $g(s, t)$ for $x > b$, and with $g^*(s, t)$ for $x > c$. Theorem 5 now follows from Theorem 4.

Let $K_\nu(s)$ have its usual meaning in the theory of Bessel functions [6, p. 78].

THEOREM 6.[3] *If* $-\frac{1}{2} < \Re(\nu) < \frac{1}{2}$, *then for every* $f(s) \in \mathfrak{H}(0)$ *there is a unique* $\psi(t) \in L^2(0, \infty)$ *such that*

$$f(s) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} K_\nu(st)(st)^{1/2}\psi(t)\,dt, \qquad x > 0, \tag{18}$$

where the integral is a mean-square limit, as in (2), *for* $x \geq 0$, *and converges in the ordinary sense for* $x > 0$.

We shall show that the function $(2st/\pi)^{1/2}K_\nu(st)$ satisfies the hypotheses of Theorem 5, with $c = 0$. The following inequalities for $K_\nu(s)$ are valid for $\Re(s) > 0$ [6, p. 219; 3, p. 658].

$$(2s/\pi)^{1/2}e^{s}K_\nu(s) = R_0(s), \tag{19}$$

$$(2s/\pi)^{1/2}e^{s}K_\nu(s) = 1 + R_1(s)/(2s), \tag{20}$$

where

$$|R_j(s)| \leq \left|\frac{\cos \nu\pi}{\cos \Re(\nu\pi)}\right|, \qquad j = 0, 1.$$

If now $h(s, t)$ is defined by

$$(2st/\pi)^{1/2}K_\nu(st) = e^{-st}[1 + h(s, t)],$$

from (20) and (19) we have, with $A(\nu)$ depending only on ν,

$$|h(s, t)| \leq \frac{A(\nu)}{|st|}, \qquad |h(s, t)| \leq A(\nu), \qquad x > 0,\ 0 < t < \infty.$$

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[3] A closely related theorem is given by Meijer [4, p. 603]. Meijer also obtains an inversion formula for (18).

Hence, with some $B(\nu)$,

$$| h(z, t) | \leqq \frac{B(\nu)}{| 1 + z | (1 + t)}, \quad x > 0,\ 0 < t < \infty,$$

and the conditions of Theorem 5 are satisfied.

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DUKE UNIVERSITY

CLASSES OF MAXIMUM NUMBERS ASSOCIATED WITH TWO SYMMETRIC EQUATIONS

H. A. SIMMONS

1. **Introduction.** Let $\sum_{i,j}(1/x)$ stand for the elementary symmetric function of the jth order of the i reciprocals $(1/x_p)$ $(p=1, 2, \cdots, i>0)$ with

$$\sum_{i,j}(1/x) = 0 \quad \text{when} \quad i<j \text{ or } j<0,$$
$$= 1 \quad \text{when} \quad j=0$$

($\sum_{i,j}(x)$ having a similar meaning for the x_p themselves).

Here we extend the work of papers I,[1] II,[2] III[3] by obtaining relative to equations (1) and (1.1) below results analogous to those in I, II, III

$$\text{(1)}[4] \qquad \sum_{n,n-1}(1/x) + \sum_{t=1}^{m} a_t[\pi(x)]^{-t} = b/a,$$
$$a = (c+1)b - 1, \quad \pi(x) = x_1x_2\cdots x_n,$$

$$\text{(1.1)} \qquad \sum_{n,n-2}(1/x) + \lambda\sum_{n,n-1}(1/x) + \mu\sum_{n,n}(1/x) = b/a;$$

in (1), b, c, and m are arbitrary positive integers, $n>1$, and the a_t are any non-negative real numbers; in (1.1), a and b are as in (1), $n>2$, λ is a non-negative integer, and μ is a positive integer.

We have not seen previous mention of (1); the case of (1.1) in which $\mu=0$ was treated in II and that in which $\lambda=\mu=1$ was treated in III. Our procedure for (1) does not suffice for the equation that is obtained by adding to the left member of (1.1) the terms

$$\sum_{t=2}^{m} a_t[\pi(x)]^{-t}.$$

The following definitions and notation from I will be frequently

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Presented to the Society, September 5, 1941; received by the editors June 27, 1941.

[1] H. A. Simmons, Transactions of this Society, vol. 34 (1932), pp. 876–907.

[2] Norma Stelford and H. A. Simmons, this Bulletin, vol. 40 (1934), pp. 884–894.

[3] H. A. Simmons and W. E. Block, Duke Mathematical Journal, vol. 2 (1936), pp. 317–340.

[4] In so far as we know, the form of the right member of equation (1) was first used by Tanzo Takenouchi in the Proceedings of the Physico-mathematical Society of Japan, (3), vol. 3 (1921), pp. 78–92.

used here: $x_{1\ldots p}$, $1 \leq p \leq n$, stands for the set $(x_1, x_2, \cdots, x_p)$; $P(x) \equiv P(x_1, x_2, \cdots, x_n)$ stands for a polynomial, not a constant, which is symmetric in the x_i $(i=1, 2, \cdots, n)$ and has at least one positive, and no negative, coefficient; the *Kellogg solution* of equation (e), where e stands for (1) or (1.1), is the solution that is obtained by minimizing the variables $x_1, x_2, \cdots, x_{n-1}$ in this order, one at a time, in (e) among positive integers; an *E-solution* of (e) is any solution of it in which $x_1 \leq x_2 \leq \cdots \leq x_n$ while $x_1, x_2, \cdots, x_n$ are positive integers. When any further definition or notation from I, II, or III is used here, a suitable reference to the appropriate article will be given.

We can now state accurately our purpose here. It is to prove that the Kellogg solution w of equation (e) has the following two properties, which were called *remarkable properties* in III: (i) It contains the largest number that exists in any E-solution of (e) and no other E-solution of (e) contains this number. (ii) If X, with $X \neq w$, is an E-solution of (e), then $P(X) < P(w)$.

The discussion from §2 to the end of this paper is divided into two parts as follows: Part 1 treats (1), §§2 to 6 (inclusive); Part 2, (1.1), §§7 to 12.

This paper involves innovations of notation and procedure of I, II, III. The terms *set* σ and *set* τ, which were important in I, II, III, are not used here; they are not needed because of our use of a new term that is very convenient for present purposes, namely *s-set* (cf. §4). This change is accompanied by new procedure for both (1) and (1.1): in Part 1, we use a new lemma, namely Lemma 4.1; in Part 2, we introduce an upper bound $R(X)$ (cf. §8) for the maximum number that we seek to identify and we show that $R(X)$ is uniquely maximized, with respect to values that $R(x)$ can assume on E-solutions X of (1.1), by the Kellogg solution of (1.1). In so far as we know, our reasoning about $R(X)$ in §11 affords the first strong resemblance of our procedure (for identifying maximum numbers) to that which Curtiss[5] used in solving Kellogg's problem.[6]

PART 1. THE REMARKABLE PROPERTIES OF THE KELLOGG SOLUTION OF (1)

2. The Kellogg solution of (1). This solution is $x=w$ where [I, (23)]

$$(2) \qquad w_p = 1 \quad (p = 1, 2, \cdots, n-2), \qquad w_{n-1} = c + 1,$$

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[5] D. R. Curtiss, American Mathematical Monthly, vol. 29 (1922), pp. 380–387, and note, in particular, his upper bound (10), p. 384.

[6] O. D. Kellogg, ibid., vol. 28 (1921), pp. 300–303.

with w_n defined to be the unique positive solution x_n of the equation that is obtained by substituting in (1) for each x_p ($p=1, 2, \cdots, n-1$) its value w_p from (2).

If $n=2$, only the last equation in (2) is to be retained.

3. **Our transformation.** In considering an E-solution $X \neq w$ of (1), we classify and transform elements as we did in §§15, 17 of I. Thus we define our transformation of X ($X_{1 \ldots n}$) into a new set X' by (t_1) or (t_2) [I, (33) and (52)]:

$$(3) \qquad \begin{aligned} &(t_1)\colon X'_p = X_p (p \neq q_1, {}_1q;\ p \leqq n),\ X'_{q_1} = w_{q_1},\ Q(1/X') = Q(1/X);\\ &(t_2)\colon X'_p = X_p (p \neq q_1, {}_1q;\ p \leqq n),\ X_{{}_1q} = w_{{}_1q},\ Q(1/X') = Q(1/X); \end{aligned}$$

according as (t_1) requires $X'_{{}_1q}$ to be not greater than $w_{{}_1q}$ or greater than $w_{{}_1q}$, respectively, where $Q(1/X)$ is the case $x=X$ of the left member of (1); if (t_1) defines $X'_{{}_1q}$ to be equal to $w_{{}_1q}$, (t_1) and (t_2) are the same transformation.

If $X' \neq w$, our transformation from X' to X'' is obtained from (3) by replacing in (3) X, X', q by X', X'', q', respectively, where $X'_{q'_1}$ ($X'_{{}_1q'}$) is of class A'(B'), and the new transformation is regarded as a transformation (3). Thus we avoid giving here an analogue of (52) of I.

Replacement of (1) by (1.1) in this section gives our transformation for §§11, 12; it will be called (3a).

4. **Important lemmas; s-set.** In the sequel, we use Lemma 4 and Lemma 4.1 below. Lemma 4.1 depends on Lemma 4, which is essentially Lemma 1a of I.

LEMMA 4. *Let $Q(1/x)$ stand for a symmetric polynomial in the n reciprocals $(1/x_p)$ ($p=1, 2, \cdots, n>1$) which is not a mere constant and contains at least one positive, and no negative, coefficient; with i and j equal to distinct positive integers each less than or equal to n, let x_i, x_j, α, β be positive numbers with $\alpha < x_i \leqq x_j$; and suppose that the expression that is obtained by replacing in $Q(1/x)$ the numbers x_i, x_j* by $(x_i-\alpha)$, $(x_j+\beta)$, *respectively, equals $Q(1/x)$, then*

$$(4) \qquad x_i x_j \leqq (x_i-\alpha)(x_j+\beta), \quad x_i^h + x_j^h < (x_i-\alpha)^h + (x_j+\beta)^h,$$

where h is a positive integer. Furthermore, the equality sign holds in (4) *if, and only if, $Q(1/x)$ is a polynomial in $[\pi(x)]^{-1}$.*

In the proof of Lemma 4.1 and in §§6, 12, we use the following definition.

Definition of s-set. *If in a set $X^{(\alpha)}\neq w$ every element of class $B^{(\alpha)}$* [I, p. 898] *is at least as large as every element of class $A^{(\alpha)}$, we call $X^{(\alpha)}$ an s-set (relative to w), s meaning satisfactory in the sense that $X^{(\alpha)}$ can be transformed into w by one or more transformations of type* (3) *every one of which accords with* (4).

Lemma 4.1. *Let k be an integer greater than or equal to* 1; *let $W\equiv W_{1\ldots v}$* $(v>1)$ *be the Kellogg solution of the equation $(x_1x_2\cdots x_v)^{-1}=k^{-1}$; and let $X\equiv X_{1\ldots v}$ be a set of v positive integers with $X_1\leqq X_2\leqq\cdots\leqq X_v$ satisfying the relation*

$$(4.1)\qquad (x_1x_2\cdots x_v)^{-1}\leqq k^{-1},$$

then

$$(4.2)\qquad \sum_{v,s}(1/X)\leqq\sum_{v,s}(1/W),\qquad 1\leqq s<v,$$[7]

with $<$ holding in (4.2) *except when $X=W$.*

Proof. We first consider the case where

$$(4.3)\qquad X_1X_2\cdots X_v=k.$$

Then $X_{1\ldots v}$ is an s-set (relative to W). Therefore $P(X)\leqq P(W)$, [I, Lemma 3], the equality sign holding in this relation if, and only if, $X=W$ or $P(X)$ is a polynomial in the product of all of the v variables $X_1, X_2, \cdots, X_v$. In particular, then, when $P(x)\equiv\sum_{v,r}(x)$, we have

$$(4.4)\qquad \sum_{v,r}(X)\leqq\sum_{v,r}(W),\qquad 1\leqq r<v,$$

the equality sign holding in (4.4) if, and only if, $X=W$ since $r<v$. But, using (4.3), we find

$$(4.5)\quad \sum_{v,r}(1/X)=\frac{\sum_{v,v-r}(X)}{X_1X_2\cdots X_v}=\frac{\sum_{v,v-r}(X)}{k},\quad \sum_{v,r}(1/W)=\frac{\sum_{v,v-r}(W)}{k},$$

while, by (4.4),

$$(4.6)\qquad \sum_{v,v-r}(X)\leqq\sum_{v,v-r}(W),\qquad 1\leqq v-r<v.$$

From (4.5) and (4.6), it follows that in the present case (4.2) holds.

Suppose now that $<$ holds in the case $x=X$ of (4.1), say

$$(4.7)\qquad (X_1X_2\cdots X_v)^{-1}=(k')^{-1},$$

---

[7] The case $s=v$ is excluded merely for convenience in our applications of Lemma 4.1; the case $k=1$ is included for convenience in writing, not for use.

where k' is an integer greater than k. By considering the Kellogg solution of (4.7), one can prove that (4.2) still holds: indeed if the Kellogg solution of (4.7) is U, one finds readily that

$$\sum_{v,s} (1/X) \leqq \sum_{v,s} (1/U) < \sum_{v,s} (1/W), \qquad 1 \leqq s - v.$$

5. **Proof of Property** (i) **for the** w **of** (1). We substitute any E-solution $X \neq w$ of (1) for x in (1) and employ the following equivalent of the resulting equation ($\sum_{i,j}$ standing for $\sum_{i,j}(1/X)$ here as in the sequel)

$$(5) \qquad \sum_{n-1,n-1} + X_n^{-1}\Big(\sum_{n-1,n-2} + a_1 \sum_{n-1,n-1} \Big) + \sum_{i=2}^{m} a_i X_n^{-i} \Big(\sum_{n-1,n-1} \Big)^i = b/a.$$

In order to establish Property (i), it suffices to prove that w has Property (i) when $n=2$ and to prove that the following relations hold

$$(5.1) \qquad \sum_{n-1,n-1} \leqq \sum_{n-1,n-1} (1/w), \quad \sum_{n-1,n-2} < \sum_{n-1,n-2} (1/w), \qquad n > 2.$$

When $n=2$, equation (1) reduces to

$$(5.2) \qquad X_1^{-1} + X_2^{-1}(1 + a_1 X_1^{-1}) + \sum_{i=2}^{m} (a_i X_1^{-i}) X_2^{-i} = ba^{-1}.$$

In this case, $X \neq w$ implies that $X_1 < w_1$. This fact and (5.2) imply that $X_2 < w_2$ [I, Lemma 2].

When $n > 2$, the first relation of (5.1) is true by the definition of Kellogg solution since (cf. (2))

$$\sum_{n-1,n-1} \leqq (c+1)^{-1} = \sum_{n-1,n-1} (1/w),$$

and since $X \neq w$, the second relation of (5.1) follows from Lemma 4.1 (cf. the case of (4.2) in which $X \neq w$ and $(v, s) = (n-1, n-2)$ with $n > 2$).

6. **Proof that the** w **of** (1) **has Property** (ii). Let $X \neq w$ be an E-solution of (1). The discussion of the case $n=2$ in §5 shows that in this case X is an s-set (so that $P(X) < P(w)$). Suppose $n > 2$. Then, by §5, $X_n < w_n$, and by (2) every classified element of $X_{1 \cdots (n-2)}$ is of class A. Therefore, whether X_{n-1} is of class A, class B, or unclassified ($= w_{n-1}$), X is an s-set.

Part 2. The remarkable properties of the Kellogg solution of (1.1)

7. **The Kellogg solution of** (1.1). This solution is $x = w$ where [I, (23)]

$$
(7)\qquad
\begin{aligned}
&w_p = 1, && p = 1, 2, \cdots, n-3,\\
&w_{n-2} = c + 1, \quad w_{n-1} = a\Big[\sum_{n-2,1}(w) + \lambda\Big],\\
&w_n = a\Big[\sum_{n-1,2}(w) + \lambda\sum_{n-1,1}(w) + \mu\Big].
\end{aligned}
$$

If $n=3$, the first set of equations in (7) is, of course, to be omitted.

8. **An upper bound for X_n.** In the sequel X stands for an E-solution of (1.1), arbitrary except as we specify.

If we substitute X for x in (1.1) and solve the resulting equation for X_n, we find after simple algebraic manipulations that

$$
(8)\qquad
\begin{aligned}
X_n = a&\Big[\sum_{n-1,2}(X) + \lambda\sum_{n-1,1}(X) + \mu\Big]\\
&\cdot\Big[bX_1X_2\cdots X_{n-1} - a\Big(\sum_{n-1,1}(X) + \lambda\Big)\Big]^{-1}.
\end{aligned}
$$

Since the X_p ($p=1, 2, \cdots, n-1$) are positive integers, the second factor in the right member of (8) is the reciprocal of a positive integer. Therefore,

$$
(8.1)\qquad X_n \leqq a\Big[\sum_{n-1,2}(X) + \lambda\sum_{n-1,1}(X) + \mu\Big].
$$

9. **An inequality for X_{n-1} when X_n is the maximum number.** In the sequel, the statement that $X \neq w$ *and* X_n *is the maximum number that we seek to identify* (so that $X_n \geqq w_n$) will be referred to as *hypothesis* H, or merely as H. We use H henceforth until a contradiction of it is reached in §11.

Under hypothesis H, we now desire to prove that

$$
(9)\qquad X_{n-1} \leqq w_{n-1}.
$$

Suppose that this is not true, so that (with H holding)

$$
(9.1)\qquad X_{n-1} > w_{n-1}.
$$

We presently contradict (9.1). The case $x=X$ of (1.1) is equivalent to

$$
(9.2)\qquad
\begin{aligned}
&\sum_{n-2,n-2} + (1/X_{n-1})\Big(\sum_{n-2,n-3} + \lambda\sum_{n-2,n-2}\Big)\\
&\quad + (1/X_n)\Big(\sum_{n-1,n-3} + \lambda\sum_{n-1,n-2} + \mu\sum_{n-1,n-1}\Big) = b/a,\\
&\qquad\qquad\qquad a = (c+1)b - 1,
\end{aligned}
$$

with $n>2$. To reach the contradiction, we first establish the following relations

$$\sum_{n-2,n-2} \leqq \sum_{n-2,n-2} (1/w), \qquad n > 2, \tag{9.3}$$

$$\sum_{n-2,n-3} + \lambda \sum_{n-2,n-2} \leqq \sum_{n-2,n-3} (1/w) + \lambda \sum_{n-2,n-2} (1/w), \qquad n > 2. \tag{9.4}$$

Since X is an E-solution not equal to w of (1.1) and $n>2$, $X_1X_2 \cdots X_{n-2} \geqq c+1 = w_1w_2 \cdots w_{n-2}$; therefore, (9.3) is true. Consequently, to prove (9.4), it suffices to show that

$$\sum_{n-2,n-3} \leqq \sum_{n-2,n-3} (1/w), \qquad n > 2. \tag{9.5}$$

When $n=3$, (9.5) states that $1=1$; when $n>3$, (9.5) is a case of Lemma 4.1 in which $X \neq W$ and $(v, s) = (n-2, n-3)$ with $n-3 \geqq 1$; therefore, (9.5) is true.

Next, using (9.1), (9.3), and (9.4), we find that the sum of the terms in the first line of (9.2) is less than U, where

$$U = \sum_{n-2,n-3} (1/w) + (1/w_{n-1}) \left[\sum_{n-2,n-3} (1/w) + \lambda \sum_{n-2,n-2} (1/w) \right].$$

Indeed, if in the first line of (9.2) we should replace X_{n-1} by $X_{n-1}-1$, the resulting expression would not exceed U. Consequently, there exists for (1.1) an E-solution Y in which

$$Y_p = X_p \ (p = 1, 2, \cdots, n-2), \quad Y_{n-1} = X_{n-1} - 1, \quad Y_n > X_n, \tag{9.6}$$

and the inequality in (9.6) contradicts H. Hence, under hypothesis H, (9.1) is false.

10. On the classification of the elements of $X_{1 \ldots (n-1)}$ when H holds. For use in §11, the following statement, S, will presently be proved: *In $X_{1 \ldots (n-1)}$ every element of class B is at least as large as every element of class A.*

To avoid vacuous language in the proof of S,[8] we consider separately the cases $n=3$ and $n>3$.

Case $n=3$. Here $X_{1 \ldots (n-1)} = (X_1, X_2)$, and either $X_p \geqq w_p$ $(p=1, 2)$ or $X_1 > w_1$, $X_2 < w_2$; in both cases S is true.

Case $n>3$. Here, by (7), any classified element of $X_{1 \ldots (n-3)}$ is of class A; X_{n-2} is of class A, class B, or unclassified $(=w_{n-1})$; and by (9) X_{n-1} is either unclassified or of class B. Therefore S is true.

---

[8] If $n=3$, it is vacuous to say that any classified element of $X_{1 \ldots (n-3)}$ is of class A (cf. our discussion of the case $n>3$).

11. Proof of Property (i) **for the** w **of** (1.1). If $X_{1\cdots(n-1)}$ contains no element of class B, $X\neq w$ implies that $X_n<w_n$, which contradicts H.

Suppose that $X_{1\cdots(n-1)}$ contains at least one element of class B and, therefore, at least one element of class A, since the first classified element of X is necessarily of class A. Then, by S, every application of transformation (3a) to X or to an *intermediate set* of X [I, p. 898], which does not change the magnitude of the nth element of a set, accords with (4). Further, the last such transformation in the *exhaustive set* for X [I, p. 898] yields a set $X^{(t)}$ in which $X^{(t)}_n=X_n\geqq w_n$ (cf. H) and

$$X^{(t)}_p \leqq w_p, \qquad p = 1, 2, \cdots, n-1; \tag{11}$$

otherwise $X^{(t)}$ would be a set satisfying (1.1) and having at least one element of class $A^{(t)}$ and no element of class $B^{(t)}$, which is impossible. We reach a contradiction of H as follows. Let $R(X)$ stand for the right member of (8.1), so that $X_n\leqq R(X)$. Certainly $R(X)$ is expressible in the form

$$R(X) = F + G(X_{t1} + X_{1t}) + HX_{t1}X_{1t},$$

in which F, G, H are positive and independent of X_{t1} and X_{1t}, while $X_{t1}\leqq X_{1t}$ [I, p. 898]; therefore, the first transformation (3a) that one uses in passing from X to $X^{(t)}$ is such that $R(X)<R(X')$ (cf. (4)). If $t>1$, our transformation of X' into X'' is such that $R(X')<R(X'')$, and so on. On arriving at $X^{(t)}$, one has

$$X_n \leqq R(X) < R(X') \leqq R(X^{(t)}), \qquad t \geqq 1. \tag{11.1}$$

But by (11) and the fact that $R(X^{(\alpha)})$ depends only on the first $n-1$ elements of $X^{(\alpha)}$, we have

$$R(X^{(t)}) \leqq R(w). \tag{11.2}$$

By (11.1) and (11.2), $X_n<R(w)$; and since $R(w)=w_n$ (cf. (7)), $X_n<w_n$. This contradicts H.

12. Proof that the w **of** (1.1) **has Property** (ii). H negated, we now merely suppose that $X\neq w$. We again avoid vacuous language by treating separately the cases $n=3$ and $n>3$.

If $n=3$, $X_3<w_3$ (cf. §11) and either $X_p\geqq w_p$ ($p=1$, 2) or $X_1>w_1$, $X_2<w_2$; in either case X is an s-set (and $P(X)<P(w)$).

If $n>3$, X_n is of class B, and every classified element of $X_{1\cdots(n-3)}$ is of class A (cf. (7)). Then if one of X_{n-2}, X_{n-1} is unclassified, X is an s-set; the same is true if X_{n-2}, X_{n-1} are of the same class or if

X_{n-2} (X_{n-1}) is of class A (B). Therefore, we only need to consider the case in which X_{n-2} (X_{n-1}) is of class B (A). Then

$$X_1X_2\cdots X_{n-2} \geqq w_1w_2\cdots w_{n-2} = c+1, \qquad n > 3, \tag{12}$$

and $X_{1\cdots(n-2)}$ contains one or more elements of class A (preceding the element X_{n-2}, of class B). Apply transformation (3a) to X, or to X and one or more intermediate sets of X, until a set $X^{(t)}$ is obtained in which $X_{1\cdots(n-2)}^{(t)}$ does not contain both an element of class $A^{(t)}$ and an element of class $B^{(t)}$. Since each transformation that has been applied to this point has increased the $(n-2)$d element of a set and decreased a not larger element (with subscript less than $n-2$) each transformation applied has accorded with (4), so that necessarily

$$X_1^{(t)}X_2^{(t)}\cdots X_{n-2}^{(t)} > c+1$$

(cf. the first two lines below (4)), and

$$X_p^{(t)} \geqq w_p, \qquad p = 1, 2, \cdots, n-2,$$

with $>$ holding for at least one of the indicated values of p (cf. (12)). Further, by hypothesis $X_{n-1} > w_{n-1}$, and by §11 $X_n < w_n$, while no transformation used in arriving at $X^{(t)}$ has changed the value of the $(n-1)$th or nth element of a set. Therefore,

$$X_p^{(t)} \geqq w_p \quad (p = 1, 2, \cdots, n-1), \qquad X_n^{(t)} = X_n < w_n, \tag{12.1}$$

with $>$ holding in (12.1) for at least one of the indicated values of p. If u is a value of p for which $>$ holds in (12.1), then $X_u^{(t)} \leqq X_u$ since transformation (3a) never increases the value of an element of class $A^{(a)}$. Consequently, the classified elements of $X^{(t)}$, like the elements of X, do not decrease as their subscripts increase, and so the $X^{(t)}$ of (12.1) is an s-set.

NORTHWESTERN UNIVERSITY

ON A FAMILY OF FOURIER TRANSFORMS

AUREL WINTNER

A procedure leading to specific properties of the symmetric stable distribution functions results by subjecting certain generalizations of the Bessel functions $J_\kappa(t)$ to a limit process.[1] There arises the question as to the existence of the corresponding distributions in case the limit process involved is omitted. The object of this note is to delimit the conditions under which the answer is affirmative. It turns out that the situation is quite different from that resulting in the limiting case of stability.

Let $L(t; \phi)$ denote the Fourier transform,

$$L(t; \phi) = \int_{-\infty}^{\infty} e^{itx} d\phi(x), \qquad -\infty < t < \infty,$$

of a distribution function $\phi = \phi(x)$, $-\infty < x < \infty$, that is, of a monotone function satisfying the boundary conditions $\phi(-\infty) = 0$ and $\phi(\infty) = 1$. If ϕ' denotes the derivative of ϕ (a derivative which necessarily exists and is finite almost everywhere), the Stieltjes integral $L(t; \phi)$ reduces to

$$L(t; \phi) = \int_{-\infty}^{\infty} e^{itx} \phi'(x) dx$$

if and only if ϕ is absolutely continuous; in which case $\phi'(x)$ is called the density of $\phi(x)$.

For a real or complex number z, let z_+ denote z or 0 according as z is or is not positive. Thus, if $\lambda > 0$ and $\mu > 0$, the even function $L = (1 - |t|^\lambda)_+^\mu$ of the real variable t represents a continuous curve in a (t, L)-plane, this symmetric curve being situated above or on the t-axis according as $|t| < 1$ or $1 \leqq |t| < \infty$. It should be noted for later reference that the curve has at $t = 0$ a cusp with a tangent perpendicular to the t-axis, a corner with two distinct finite slopes or a tangent parallel to the t-axis, according as $\lambda < 1$, $\lambda = 1$ or $\lambda > 1$, while μ is arbitrary.

If $\lambda = 1 = \mu$, the function $(1 - |t|^\lambda)_+^\mu$ is $1 - |t|$ or 0 according as $|t| < 1$ or $|t| \geqq 1$, and represents therefore the Fourier transform $L(t; \phi)$ of an absolutely continuous distribution function $\phi(x)$. This is

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Received by the editors July 10, 1941

[1] A. Wintner, *On a class of Fourier transforms*, American Journal of Mathematics, vol. 58 (1936), pp. 45–90.

seen either by subjecting the kernel of Dirichlet's factor of discontinuity to the process of auto-convolution or by applying a Fourier inversion. In fact,

$$\int_{-\infty}^{\infty}\left(\frac{\sin x/2}{x\pi^{1/2}/2}\right)^2 \cos tx \text{ or } dx = \begin{cases} 1-|t| & \text{if } \quad |t| \leqq 1, \\ 0 \quad \text{if} \quad 1 \leqq |t| < \infty; \end{cases}$$

so that the density $\phi'(x)$ is the square $(\ )^2$ on the left.

Thus, if $(\lambda, \mu) = (1, 1)$, there exists a distribution function $\phi_{\lambda\mu} = \phi_{\lambda\mu}(x)$ such that

$$L(t; \phi_{\lambda\mu}) = \begin{cases} (1-|t|^{\lambda})^{\mu} & \text{if } \quad |t| \leqq 1, \\ 0 \quad \text{if} \quad 1 \leqq |t| < \infty. \end{cases} \tag{1}$$

There arises the question for what additional pairs (λ, μ) does (1) determine a distribution function $\phi_{\lambda\mu}$. Since the derivative

$$\phi_{11}'(x) = (\sin x/2)^2/(x\pi^{1/2}/2)^2, \tag{2}$$

though non-negative, has zeros, no continuity consideration can indicate that there exists a distribution function $\phi_{\lambda\mu}$ for certain pairs (λ, μ) sufficiently close to $(1, 1)$; in fact, the derivative $\phi_{\lambda\mu}'(x)$ might then become negative, precluding a distribution function $\phi_{\lambda\mu}(x)$.

Since the function $(1-|t|^{\lambda})_{+}^{\mu}$ of t is integrable over $(-\infty, \infty)$; it is clear by Fourier inversion that, if there exists at all a distribution function $\phi_{\lambda\mu}$ satisfying (1), then $\phi_{\lambda\mu}$ is absolutely continuous and has a density $\phi_{\lambda\mu}'(x) \geqq 0$ proportional to

$$f_{\lambda\mu}(x) = \int_0^1 (1-t^{\lambda})^{\mu} \cos xt \, dt; \tag{3}$$

the factor of proportionality being $1/\pi$ for arbitrary (λ, μ). Accordingly, the question can be restated by asking, which are those pairs (λ, μ) for which the function (3) satisfies the conditions represented by,

$$f_{\lambda\mu}(x) \geqq 0, \qquad 0 < x < \infty, \tag{4}$$

and

$$\int_0^{\infty} f_{\lambda\mu}(x) dx < \infty. \tag{5}$$

If the unit of length on the x-axis is changed in the proportion $1:a$, the definition of the Fourier transform $L(t; \phi)$ shows that the unit of length on the t-axis must be changed in the proportion $a:1$. Hence, if $a = \mu^{1/\lambda}$, the function (1) must be replaced by $(1-|t|^{\lambda}/\mu)^{\mu}$ if $|t| \leqq \mu^{1/\lambda}$, and 0 if $|t| \geqq \mu^{1/\lambda}$. This function of t tends, as $\mu \to \infty$, to $\exp(-|t|^\lambda)$ for $-\infty < t < \infty$. This means that the limiting case $\phi_{\lambda\infty}$ of $\phi_{\lambda\mu}$ is the symmetric stable distribution function of index λ; a distribution function which is known to exist[3] if and only if λ does not exceed the critical value $\lambda = 2$ belonging to a Gaussian distribution (the condition $f_{\lambda\infty}(x) \geqq 0$ corresponding to (4) being violated for certain values of x if and only if $\lambda > 2$).

It will turn out that the characteristic condition $\lambda \leqq 2$ belonging to $\mu = \infty$ must be replaced by $\lambda \leqq 1$ for *every* finite $\mu \geqq 1$. In other words, if μ is arbitrarily fixed on the range $1 \leqq \mu < \infty$, the distribution function $\phi_{\lambda\mu}$ does or does not exist according as $\lambda \leqq 1$ or $\lambda > 1$. Incidentally, the trivial case (2) happens to be a limiting case, in the sense that (4), where $\lambda \leqq 1 \leqq \mu$, can be refined to

$$f_{\lambda\mu}(x) > 0, \qquad 0 < x < \infty \tag{6}$$

unless $(\lambda, \mu) = (1, 1)$.

It seems to be of interest that there exists a *fixed* lower bound (equal to 1) for the admissible values of μ in the *whole* admissible range of λ ($\leqq 1$). That μ must be limited by *some* lower bound for every *given* admissible λ, is clear from Lévy's theory of indefinitely divisible laws. In fact, if λ is arbitrarily fixed, (1) requires that $L(t; \phi_{\lambda\mu})$ be the μth power of $L(t; \phi_{\lambda 1})$ for every μ, although $L(t; \phi_{\lambda 1})$ vanishes for certain values of t. But this contradicts the stochastic theory just mentioned.

Actually, it is easy to see directly that there exists for every $\lambda > 0$ a sufficiently small $\mu > 0$ such that the corresponding function (3) becomes negative for certain values of x and violates, therefore, (4). In fact, it is clear for reasons of continuity that there must exist for every $\lambda > 0$ such a sufficiently small $\mu > 0$, if it is true that, for every fixed $\lambda > 0$, the function (3) becomes negative for certain values of x in the limiting case $\mu = 0$. But if $\mu = 0$, then (3) reduces for every $\lambda > 0$ to

$$f_{\lambda 0}(x) = \int_0^1 \cos xt\, dt = x^{-1} \sin x,$$

and attains therefore negative values.

For sake of simplicity, it will from now on be assumed that μ has a fixed value not less than 1, while $\lambda > 0$ is arbitrary.

It will be shown that (4) is violated or the necessary and sufficient condition, represented by (4) and (5) together, is satisfied, according as $\lambda > 1$ or $\lambda \leq 1$; that is, according as the function on the right of (1) is or is not differentiable at $t = 0$.

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[3] For sharper results see, loc. cit., where references are given to the investigations of Lévy and Pólya.

First, if $\lambda > 2$, the function on the right of (1) is $1 + o(t^2)$ as $t \to 0$. But this implies[3] that either (4) is violated or $L(t; \phi_{\lambda\mu})$ is independent of t, and the latter case is excluded by (1). Next, if $\lambda = 2$, then (3) reduces to a constant multiple of the classical integral representation of the ratio $J_{\mu+1/2}(x)/x^{\mu+1/2}$; so that (4) is violated in view of the asymptotic formula of the Bessel functions for large real x. But a perfectly similar asymptotic formula, again with a sine function in the numerator, can be proved[4] for (3) even if $\lambda < 2$, provided that $\lambda > 1$. Accordingly, (4) is violated[5] whenever $\lambda > 1$.

It will now be shown that (4) is satisfied whenever $\lambda \leq 1$. To this end, it will be sufficient to prove (6), since the limiting case $(\lambda, \mu) = (1, 1)$ is taken care of by (2).

Partial integration of (3) shows that

$$xf_{\lambda\mu}(x) = \lambda\mu \int_0^1 (1 - t^\lambda)^{\mu-1} t^{\lambda-1} \sin xt \, dt. \tag{7}$$

But, if $F_{\lambda\mu}(t)$ is an abbreviation for $(1 - t^\lambda)_+^{\mu-1} t^{\lambda-1}$, the last integral can be written in the form

$$\int_0^\infty F_{\lambda\mu}(t) \sin xt \, dt = x^{-1} \sum_{n=0}^{\infty} (-1)^n \int_{n\pi/x}^{(n+1)\pi/x} F_{\lambda\mu}(t/x) \sin t \, dt$$

for every $x > 0$. Hence, in order to prove (6), it is sufficient to ascertain that the non-negative function $F_{\lambda\mu}(t/x)$ of t/x is steadily decreasing as long as it does not vanish; in other words, that the function $(1 - t^\lambda)^{\mu-1} t_+^{\lambda-1}$ of t is steadily decreasing on the interval $0 < t < 1$. But both factors $(1 - t^\lambda)^{\mu-1}$, $t^{\lambda-1}$ are positive and non-increasing, and at

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[3] Loc. cit., p. 80.

[4] Cf. A. Wintner, *On the asymptotic formulae of Riemann and of Laplace*, Proceedings of the National Academy of Sciences, vol. 20 (1934), pp. 57–62.

[5] Naturally, the problem is less deep than to necessitate an actual application of the sharp asymptotic fact, referred to before. For instance, if $\mu = 1$, a partial integration transforms (3) into

$$f_{\lambda 1}(x) = \lambda x^{-1} \int_0^1 t^{\lambda-1} \sin xt \, dt = \lambda x^{-1-\lambda} \int_0^x t^{\lambda-1} \sin t \, dt.$$

But this function violates (4) whenever $\lambda > 1$, since, if $\lambda - 1 > 0$, the successive "waves"

$$(-1)^n \int_{n\pi}^{(n+1)\pi} t^{\lambda-1} \sin t \, dt = \int_{n\pi}^{(n+1)\pi} t^{\lambda-1} |\sin t| \, dt, \qquad \text{where } n = 0, 1, 2, \cdots,$$

increase with n. It is easy to transfer this primitive proof from $\mu = 1$ to every $\mu > 1$, if use is made of the second mean-value theorem.

least one of them is decreasing, on the interval $0<t<1$. In fact, $\mu-1\geqq 0$ and $\lambda-1\leqq 0$ by assumption, where at least one of the two signs of equality is excluded by $(\lambda, \mu)\neq(1, 1)$.

This proves (4) for $\lambda\leqq 1\leqq\mu$. It remains to be shown that (5) is satisfied for $\lambda\leqq 1\leqq\mu$. To this end, it is sufficient to verify that

$$f_{\lambda\mu}(x) = O(x^{-1-\lambda}), \qquad \text{as } x\to\infty. \tag{8}$$

Suppose first that $\lambda=1$. Then (7) shows that (8) reduces to

$$\int_0^1 (1-t)^{\mu-1}\sin xt\,dt = O(x^{-1}), \qquad \text{as } x\to\infty.$$

But this estimate is clear from the second mean-value theorem.

In the remaining case, where $\lambda<1$, the estimate (8) can be refined to the asymptotic formula

$$f_{\lambda\mu}(x) \sim \frac{\mu\Gamma(1+\lambda)\sin(\pi\lambda/2)}{x^{1+\lambda}}, \quad \text{as } x\to\infty;\ \lambda<1. \tag{9}$$

In fact, since $0<\lambda<1$ implies the existence of the integral

$$\int_0^\infty t^{\lambda-1}\sin t\,dt = \Gamma(\lambda)\sin(\pi\lambda/2), \tag{10}$$

it is clear from (7) that (9) is equivalent to the assertion that, as $x\to\infty$, the function

$$x^\lambda\int_0^1 (1-t^\lambda)^{\mu-1}t^{\lambda-1}\sin xt\,dt$$

tends to the limit (10). This function can be written in the form

$$\int_0^x (1-t^\lambda/x^\lambda)^{\mu-1}t^{\lambda-1}\sin t\,dt,$$

and tends therefore to the limit (10) if it is legitimate to carry out the process $x\to\infty$ *beneath* the last integral sign. But this is readily justified by the second mean-value theorem.

THE JOHNS HOPKINS UNIVERSITY

EVERYWHERE DENSE SUBGROUPS OF LIE GROUPS

P. A. SMITH

A recent note by Montgomery and Zippin[1] leads one to speculate concerning the nature of everywhere dense proper subgroups of continuous groups. Such subgroups can easily be constructed. Suppose for example that G is a non-countable continuous group which admits a countable subset G_0 filling it densely. The group generated by G_0 is everywhere dense in G but is not identical with G. In the case of Lie groups, it is easy to see that an abelian G admits non-countable subgroups of the sort in question; whether or not a non-abelian G does so, appears to be a more difficult question. We shall, however, show that if G is simple, proper subgroups of G cannot, so to speak, fill G *too* densely.

Let G be a simple[2] Lie group of dimension r with $r>1$, and let U be a canonical nucleus of G—that is, a nucleus which can be covered by an analytic canonical coordinate system. An arbitrary point x of U is contained in the central of at least one closed proper Lie subgroup of G with non-discrete central. In fact, through x there passes a one-parameter subgroup γ; the closure of γ is an abelian Lie subgroup and this subgroup is proper since G is simple and $r>1$.

THEOREM. *Let G be a simple Lie group of dimension r greater than one and let $\mathfrak{g}$ be a proper subgroup filling G densely. There exists at least one proper closed Lie subgroup H of G such that those left- (right-) cosets of H which fail to meet $\mathfrak{g}$ fill G densely. For H one may take any closed proper Lie subgroup of G whose central is non-discrete and contains an arbitrarily chosen point p in $\mathfrak{g}\cap U$, U being any given canonical nucleus of G.*

PROOF. Let U, p, H be chosen and let us consider only the left-cosets of H. It will be sufficient to prove that there exists at least one coset, say aH, which fails to meet $\mathfrak{g}$. For, the cosets obtained by multiplying aH on the left by arbitrary elements of $\mathfrak{g}$ fail to meet $\mathfrak{g}$ and fill G densely.

Received by the editors July 15, 1941.

[1] Deane Montgomery and Leo Zippin, *A theorem on the rotation group of the 2-sphere*, this Bulletin, vol. 46 (1940), pp. 520–521. Our theorem may be regarded as a generalization of the theorem of Montgomery and Zippin and the proofs of the two theorems may be regarded as being the same in principle.

[2] We use *simple* here in the sense of having a simple Lie algebra. A simple group need not be connected.

Let us assume the contrary, namely that *every* coset of H meets $\mathfrak{g}$. Let H^* be the totality of cosets of H and let the elements of H^* be denoted by $e^*=H$, $a^*=aH, \cdots$. Let σ be the mapping $x\rightarrow x^*$ $(x^*=xH)$ of G into H^*. Let H^* be topologized in the usual way by taking as open in H^* every set of the form σA where A is an open subset of G. The space H^* is homogeneously locally euclidean.—Now let x^* be an element of H^* and let x be a representative of the *coset* x^*. Then xpx^{-1} (where p is defined in the theorem) is independent of x. For if y is a second representative of x^*, then $x^{-1}y\subset H$ so that $x^{-1}yp=px^{-1}y$ since p is in the central of H. Hence $xpx^{-1}=ypy^{-1}$. Thus the formula $\tau(x^*)=xpx^{-1}$ defines a mapping τ of H^* into G which, in particular, carries e^* into p. Evidentally τ is continuous. In fact it is easy to see that τ is analytic relative to an arbitrarily chosen analytic canonical coordinate system $x_1, \cdots, x_r$ covering U, and a suitably chosen coordinate system covering a neighborhood of e^*.

The mapping τ carries H^* into a subset of $\mathfrak{g}$. For, by our assumption on the cosets of H, an element y^* of H^* can be written in the form $y^*=gH$ where $g\subset\mathfrak{g}$. Hence we have $\tau(y^*)=gpg^{-1}\subset\mathfrak{g}$.—Moreover, any given neighborhood V^* of e^* contains at least one point x^* such that $\tau(x^*)\neq p$. For otherwise we have $\tau(yH)=p$ for every y in a certain nucleus V of G, that is, for every y in V and h in H we have $yhp(yh)^{-1}=p$ or $ypy^{-1}=p$. But then the one-parameter subgroup of G determined by p would be invariant, contrary to the hypothesis that G is simple.

Let W be a nucleus of G such that $W^{-1}WW\subset U$. It follows from the last two paragraphs that there exists in H^* a point s^* near e^* such that the linear segment e^*s^* is carried by τ into an analytic arc contained in $\mathfrak{g}\cap W$ and consisting of more than a single point. A suitably chosen piece of this arc, when multiplied on the left by the inverse of one of its points, furnishes an analytic 1-cell K contained in $\mathfrak{g}\cap W$ and containing e, the identity of G. Starting with K we shall construct a dimensionally increasing sequence of analytic continua, subsets of $\mathfrak{g}$. In what follows, let it be understood that all functions are real, single-valued and analytic over the domains indicated.

We may suppose that K is defined parametrically, say by $x_i=f_i(t)$ where $-1<t<1$ and $f(0)=e$. The set KK is in $\mathfrak{g}$ and is defined by equations of the form $x_i=g_i(s, t)$ where $-1<s, t<1$. Suppose that $\dim KK>\dim K$; that is, suppose $\dim KK=2$. Then being an analytic locus, KK contains points at which it is locally euclidean 2-dimensional. If b is such a point, then $b^{-1}KK$ (a subset of $\mathfrak{g}$) is locally euclidean at e. Hence $\mathfrak{g}\cap W$ contains a 2-cell K_2 defined say by $x_i=h_i(u, v)$ where $-1<u, v<1$ and $h(0, 0)=e$. We next consider the set K_2K_2 and suppose that its dimension exceeds that of K_2. On continuing in this manner, we finally obtain a k-cell E in $\mathfrak{g}\cap W$ defined say by $x_i = h_i(u_1, \cdots, u_k)$ where $-1<u_i<1$ and $h(0, \cdots, 0)=e$, and such that $\dim EE=\dim E=k$. We assert that E contains subsets E^* and F such that (1) E^* and F are k-cells; (2) $e\subset F\subset E^*$; (3) $FF\subset E^*$.

To prove this, we first note that by the theory of implicit functions, E contains a k-dimensional sub-cell E^* definable, after renaming the coordinates x_i if necessary, by equations

$$(1) \qquad x_i = X_i(x_1, \cdots, x_k), \qquad i = k+1, \cdots, r,$$

where $(x_1, \cdots, x_k)$ ranges over the cube C_δ: $-\delta<x_i<\delta$, and where $X_i(0, \cdots, 0)=e_i=0$. On replacing δ by a smaller number if necessary, it is easy to see that C_δ contains a cube C_μ: $-\mu<x_i<\mu$ $(i=1, \cdots, h)$ such that if F is the k-cell defined by (1) with $(x_1, \cdots, x_k)$ restricted to the cube C_μ, and if q is an arbitrary point of F, then qF, like F, is definable by equations of the form (1):

$$x_i = X_i^q(x_1, \cdots, x_k)$$

where $(x_1, \cdots, x_k)$ ranges over a certain open subset A^q of C_δ. Now EE is the union of k-cells qE $(q\subset E)$, hence is k-dimensional at every point. Being an analytic locus, the points q at which EE is locally euclidean k-dimensional fill it densely. Consider such a point q. The k-cells F and qF intersect at q. But since both are contained in EE which is locally euclidean k-dimensional at q, they coincide identically in the neighborhood of q. Hence the functions X_i and X_i^q are identically equal over an open subset of A^q; hence, by the theory of analytic functions, they are equal over the whole of A^q. Hence $qF\subset E^*$, and this is true for a set of points q filling F densely. By continuity this relation holds for arbitrary q in F. Hence $FF\subset E^*$, proving our assertion.

It is easy to see that on replacing F by a smaller k-cell if necessary, we have also $F^{-1}\subset E^*$. In short F is a k-dimensional local Lie subgroup of G; hence it is an open subset of a k-dimensional linear subspace L of the linear space of the canonical coordinates $x_1, \cdots, x_r$. If $k<r$, there exists in W an element a such that the linear subspace L' determined by $F'=aFa^{-1}$ is different from L; otherwise the Lie subalgebra represented by L is invariant. Since $\mathfrak{g}$ is everywhere dense in G, we may assume, so far as the relation $L\neq L'$ is concerned, that $a\subset\mathfrak{g}$. Then $FF'\subset\mathfrak{g}$. Moreover, it is evident that $\dim FF'>k$. We can now repeat the construction described above starting with a suitably chosen analytic cell of dimension exceeding k in FF'. We obtain finally an analytic r-cell contained in $\mathfrak{g} \cap W$. Hence $\mathfrak{g}$ contains a nucleus of G and hence $\mathfrak{g} = G$, a contradiction which proves the theorem.[3]

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[3] We have proved, incidentally, that if an everywhere dense subgroup $\mathfrak{g}$ of a simple Lie group G_r $(r>1)$ contains an analytic arc, then $\mathfrak{g} = G$.

VECTOR SPACES OVER RINGS

C. J. EVERETT[1]

1. **Introduction.** Let $\mathfrak{M} = u_1K + \cdots + u_mK$ be a vector space (linear form modul [5, p. 111]) over a ring $K = \{0, \alpha, \beta, \cdots; \epsilon \text{ unit element}\}$. By a *submodul* $\mathfrak{N} \leqq \mathfrak{M}$ is meant an "admissible" submodul: $\mathfrak{N}K \leqq \mathfrak{N}$. Elements $v_1, \cdots, v_n$ of a submodul $\mathfrak{N}$ form a *basis* for $\mathfrak{N}$ (notation: $\mathfrak{N} = v_1K + \cdots + v_nK$) in case $\sum v_i\alpha_i = 0$ implies $\alpha_i = 0$, $i = 1, \cdots, n$, and if every element of $\mathfrak{N}$ is expressible in the form $\sum v_i\alpha_i$, $\alpha_i \in K$. The equivalent formulations of the ascending chain condition for submoduls of a vector space, and for right ideals of a ring will be used without further comment [5, §§80, 97].

2. **Basis number, linear transformations.** We remark that the following holds.

(A) *The ascending chain condition is satisfied by the submoduls of a vector space $\mathfrak{M}$ over K if and only if it is satisfied by the right ideals of K.*

An infinite chain of right ideals $\mathfrak{r}_1 < \mathfrak{r}_2 < \cdots$ in K yields an infinite chain of submoduls $u_1\mathfrak{r}_1 < u_1\mathfrak{r}_2 < \cdots$ in $\mathfrak{M}$. The other implication is proved in [5, p. 87].

[By using a lemma due to N. Jacobson (*Theory of Rings*, in publication) Theorem (A) and the corresponding theorem for descending chain condition are easily proved in a unified manner.]

Linear transformations of $\mathfrak{M}$ on $\mathfrak{M}$ are given by $u_j \to u_j' = \sum u_i\alpha_{ij}$. Write $(u_1', \cdots, u_m') = (u_1, \cdots, u_m)A$, $A = (\alpha_{ij})$. Under $u_j \to u_j'$, let $\mathfrak{M}_0 \to 0$. Thus $\mathfrak{M}/\mathfrak{M}_0 \cong \mathfrak{M}A \leqq \mathfrak{M}$. Clearly $\mathfrak{M}_0 = 0$ if and only if $Av = 0$ implies $v = 0$, v an $m \times 1$ matrix over K, and $\mathfrak{M}A = \mathfrak{M}$ if and only if there exists an $m \times m$ matrix R with $AR = I$, the identity matrix.

Possibilities (i) $\mathfrak{M}_0 = 0$ and $\mathfrak{M}A = \mathfrak{M}$; (ii) $\mathfrak{M}_0 > 0$ and $\mathfrak{M}A < \mathfrak{M}$; (iii) $\mathfrak{M}_0 = 0$ and $\mathfrak{M}A < \mathfrak{M}$ are familiar. The possibility of (iv) $\mathfrak{M}_0 > 0$

Presented to the Society, September 5, 1941; received by the editors May 27, 1941.

[1] The results presented here were obtained while the author was Sterling Research Fellow in mathematics, Yale University, 1940–1941. Thanks are due to Professors Oystein Ore, R. P. Dilworth, and the referee for helpful suggestions.

and $\mathfrak{M}A = \mathfrak{M}$ is demonstrated later in (D), thus settling a question raised by van der Waerden [5, p. 115].

Case (iii) *implies an infinite descending chain in* $\mathfrak{M}$, *case* (iv) *an infinite ascending chain in* $\mathfrak{M}$.

(B) *The set* $(v_1, \cdots, v_n) = (u_1, \cdots, u_m)A$, $n<m$, *forms a basis for* $\mathfrak{M} = u_1K + \cdots + u_mK$ *if and only if the* $m \times m$ *matrix* $(A0)$ *has a right inverse*: $(A0)R = I$, *and* $Av = 0$ *implies* $v = 0$, v *an* $n \times 1$ *matrix over* K.

This is an immediate consequence of the basis definition.

(C) *If the right ideals of* K *satisfy the ascending chain condition, every basis of a vector space* $\mathfrak{M} = u_1K + \cdots + u_mK$ *has* m *elements.*

For a matrix $(A0)$ of the type in (B) defines a linear transformation of type (iv) violating the chain condition in K.

Hence with every vector space $\mathfrak{M}$ over a ring K with ascending chain condition for right ideals is associated a unique *basis number* $b(\mathfrak{M})$. K a quasi-field is a trivial special case.

(D) *If* K *is the ring of all infinite matrices over a field, with only a finite number of nonzero elements in each row and each column, then the vector space* $\mathfrak{M} = u_1K + \cdots + u_mK$, $m > 1$, *has a basis of one element*: $\mathfrak{M} = uK$. *Thus there exist, for arbitrary* m, $1 \times m$ *matrices* $(\alpha_1, \cdots, \alpha_m)$, $(\beta_1, \cdots, \beta_m)$ *over* K *such that* $(\alpha_1, \cdots, \alpha_m)'(\beta_1, \cdots, \beta_m) = I$, *the* $m \times m$ *identity matrix, with* $\alpha_i\beta = 0$, $i = 1, \cdots, m$, $\beta \in K$ *implying* $\beta = 0$.[2]

Let δ_i be the vector $(0, 0, \cdots, 0, 1, 0, \cdots)'$ with 1 in the ith position from above. Matric elements of K are defined by their column vectors; let the unit of K be $\epsilon = (\delta_1, \delta_2, \cdots)$ and $\alpha_1 = (0, \delta_1, 0, \delta_2, 0, \delta_3, \cdots)$, $\alpha_2 = (\delta_1, 0, \delta_2, 0, \delta_3, 0, \delta_4, \cdots)$, $\alpha_3 = \alpha_1'$, $\alpha_4 = \alpha_2'$. Let

$$A = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha_3 & \alpha_4 \\ 0 & 0 \end{pmatrix}.$$

Then $AB = I$, and $\alpha_1\beta = \alpha_2\beta = 0$ implies $\beta = 0$, $\beta \in K$. Let

$$A_1 = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix},$$

where I is the $(m-2) \times (m-2)$ identity matrix. It follows from (B) that $u_1, \cdots, u_{m-2}, v$ form a basis for $\mathfrak{M}$, where $(u_1, \cdots, u_{m-2}, v, 0)$

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[2] A' means A transpose.

$=(u_1, \cdots, u_m)A_1$. The induction is obvious, and $\mathfrak{M}$ has a basis of a single element. The theorem follows from (B).

3. **Vector spaces over right principal ideal rings.** We now remark that the following holds:

(E) *If* $\mathfrak{M}=u_1K+\cdots+u_mK$ *is a vector space over a ring* K *in which every right ideal* $\mathfrak{r}>0$ *is of type* ρ_0K, *where* $\rho_0\alpha=0$, $\alpha\in K$ *implies* $\alpha=0$, *then every submodul* $\mathfrak{N}$, $0<\mathfrak{N}\leqq\mathfrak{M}$, *has a basis of* n *elements*, $n\leqq m$.

This is only a trivial modification of the van der Waerden result [5, pp. 88, 121], appropriate since the condition subsequently also appears to be necessary (see (F)).

Lemma 1. *If every submodul* $\mathfrak{N}$, $0<\mathfrak{N}\leqq\mathfrak{M}=u_1K+\cdots+u_mK$ *has a basis of* $n\leqq m$ *elements, and* $\mathfrak{r}$ *is a right ideal of* K, $0<\mathfrak{r}\leqq K$, *then the submodul* $\mathfrak{N}=u_1\mathfrak{r}\cup\cdots\cup u_m\mathfrak{r}$, *consisting of all sums* $\sum u_i\rho_i$, $\rho_i\in\mathfrak{r}$, *has a basis* $u_{11},\cdots,u_{m1}$ *with* $u_1\mathfrak{r}=u_{i1}K$, $i=1,\cdots,m$, *and* u_{i1} *is a basis for* $u_i\mathfrak{r}$.

For $0<u_i\mathfrak{r}=u_{i1}K+\cdots+u_{in_i}K$, $1\leqq n_i\leqq m$, and $\mathfrak{N}=u_1\mathfrak{r}\cup\cdots\cup u_m\mathfrak{r}$ is a submodul for which the u_{ij} together form a basis of $\sum n_i$ elements. The hypothesis of the lemma implies the ascending chain condition in $\mathfrak{M}$, and hence in K (by (A)). Hence by (C) the basis number for $\mathfrak{N}$ is unique and $m\geqq\sum n_i\geqq m$, $n_i=1$, $i=1,\cdots,m$. Thus $u_i\mathfrak{r}=u_{i1}\cdot K$.

(F) *Let* $\mathfrak{M}=u_1K+\cdots+u_mK$ *be a vector space over* K. *Then every submodul* $\mathfrak{N}$, $0<\mathfrak{N}\leqq\mathfrak{M}$, *has a basis of* $n\leqq m$ *elements, if and only if every right ideal* $\mathfrak{r}>0$ *in* K *is of type* ρ_0K, *where* $\rho_0\alpha=0$, $\alpha\in K$, *implies* $\alpha=0$.

For if $\mathfrak{r}>0$ is a right ideal of K, by the lemma, $u_1\mathfrak{r}=u_{11}K$, $u_{11}=u_1\rho_0$, $\rho_0\in\mathfrak{r}$. Then $u_1\mathfrak{r}=u_1\rho_0K$ and $\mathfrak{r}=\rho_0K$. Moreover $\rho_0\alpha=0$ implies $u_{11}\alpha=0$ and $\alpha=0$.

Now suppose $\mathfrak{M}=u_1K+\cdots+u_mK$ is a vector space over a ring K of the type in (F). To every submodul $\mathfrak{N}$, $0<\mathfrak{N}\leqq\mathfrak{M}$, corresponds a unique basis number $b(\mathfrak{N})$. Define $b(0)=0$.

(G) *If* $\mathfrak{M}=u_1K+\cdots+u_mK$ *is a vector space over a ring* K *of the type in* (F), *the basis number* $b(\mathfrak{N})$, $0\leqq\mathfrak{N}\leqq\mathfrak{M}$, *is a positive modular functional* [1, p. 40]:

M1. $b(\mathfrak{A}\cup\mathfrak{B})+b(\mathfrak{A}\cap\mathfrak{B})=b(\mathfrak{A})+b(\mathfrak{B})$,

M2. $\mathfrak{A}\leqq\mathfrak{B}\leqq\mathfrak{M}$ *implies* $b(\mathfrak{A})\leqq b(\mathfrak{B})$.

M2 is clear from (F). A proof of M1 may be made by induction on $b(\mathfrak{A})$. We treat here only the following case:

Let K be a (noncommutative) domain of integrity in which every right ideal is principal.[3] The vector space $\mathfrak{M}=u_1K+\cdots+u_nK$ may then be regarded as imbedded in the vector space $\mathfrak{M}^*=u_1\bar{K}+\cdots+u_n\bar{K}$ where $\bar{K}$ is the quotient quasi-field of K. The existence of $\bar{K}$ follows from theorems developed by Ore [3, p. 466] and a proof by Teichmüller [4] that the least common multiple of nonzero elements in such a K is not zero. The correspondence

$$(\gamma)\ \mathfrak{N}=v_1K+\cdots+v_nK\to\mathfrak{N}^*=v_1\bar{K}+\cdots+v_n\bar{K}$$

is a well-defined correspondence on the lattice L of all K-submoduls of $\mathfrak{M}$ to the entire lattice $\bar{L}$ of $\bar{K}$-submoduls of $\mathfrak{M}^*$, (since $\mathfrak{N}^*$ is independent of the $\mathfrak{N}$-basis). Observe that $b(\mathfrak{N})=b(\mathfrak{N}^*)$ as a submodul of $\mathfrak{M}^*$. For the K-independence of a basis $(v_1,\cdots,v_n)$ of $\mathfrak{N}$ implies the $\bar{K}$-independence of $v_1,\cdots,v_n$: Let $\sum v_i\bar{\alpha}_i=0$, $\bar{\alpha}_i=\alpha_i/\beta_i\in\bar{K}$ (Ore quotient); if μ is the (nonzero) least common multiple of the β_i, $\sum v_i\bar{\alpha}_i\mu=0$, and $\bar{\alpha}_i\mu\in K$ by the Ore theory referred to. Hence $\bar{\alpha}_i\mu=0$, and $\bar{\alpha}_i=0$, $i=1,\cdots,n$.

It is trivial to verify that:

(1) $\mathfrak{A}\geqq\mathfrak{B}$ implies $\mathfrak{A}^*\geqq\mathfrak{B}^*$.

(2) $(\mathfrak{A}\cup\mathfrak{B})^*=\mathfrak{A}^*\cup\mathfrak{B}^*$.

(3) $(\mathfrak{A}\cap\mathfrak{B})^*=\mathfrak{A}^*\cap\mathfrak{B}^*$.

For example, in (2) $(\mathfrak{A}\cup\mathfrak{B})^*\geqq\mathfrak{A}^*\cup\mathfrak{B}^*$ follows from (1). But every element in $(\mathfrak{A}\cup\mathfrak{B})^*$ is a $\bar{K}$-form in a K-basis of $\mathfrak{A}\cup\mathfrak{B}$, hence is in $\mathfrak{A}^*\cup\mathfrak{B}^*$. Since $b(\mathfrak{A}^*)$ is the dimension of $\mathfrak{A}^*$ over $\bar{K}$, it follows that $b(\mathfrak{A})$ is a positive modular functional on L.

We may now apply the theory of such functionals [1, p. 42, Theorem 3.10] to show that $\delta(\mathfrak{A},\mathfrak{B})=b(\mathfrak{A}\cup\mathfrak{B})-b(\mathfrak{A}\cap\mathfrak{B})$ is a quasi-metric on L:

(4) $\delta(\mathfrak{A},\mathfrak{B})\geqq 0$, $\delta(\mathfrak{A},\mathfrak{A})=0$.

(5) $\delta(\mathfrak{A},\mathfrak{B})+\delta(\mathfrak{B},\mathfrak{C})\geqq\delta(\mathfrak{A},\mathfrak{C})$.

The relation $\mathfrak{A}\sim\mathfrak{B}$ defined by $\delta(\mathfrak{A},\mathfrak{B})=0$ is an equivalence relation, and the correspondence $\mathfrak{A}\to[\mathfrak{A}]$, the equivalence class containing $\mathfrak{A}$, is a lattice homomorphism of L onto the metric lattice L' of equivalence classes. For want of a name, we call L' the metric homomorph of L. However, in the correspondence (γ), $\mathfrak{A}^*=\mathfrak{B}^*$ if and only if $\mathfrak{A}\sim\mathfrak{B}$. For, if $\mathfrak{A}\sim\mathfrak{B}$, $b(\mathfrak{A}\cup\mathfrak{B})=b(\mathfrak{A}\cap\mathfrak{B})$, and $\mathfrak{A}^*\cup\mathfrak{B}^*=\mathfrak{A}^*=\mathfrak{B}^*=\mathfrak{A}^*\cap\mathfrak{B}^*$, since all these have the same dimension over $\bar{K}$. Conversely, if $\mathfrak{A}^*=\mathfrak{B}^*$, then $(\mathfrak{A}\cup\mathfrak{B})^*=\mathfrak{A}^*=(\mathfrak{A}\cap\mathfrak{B})^*$, $b(\mathfrak{A}\cup\mathfrak{B})=b(\mathfrak{A}\cap\mathfrak{B})$ and $\mathfrak{A}\sim\mathfrak{B}$.

(H) *If K is a right principal ideal domain of integrity, quotient field*

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[3] For the elementary divisor theory of matrices over such domains, and references to the literature, see [2].

K, then the basis number $b(\mathfrak{N})$ is a positive modular functional on the lattice L of submoduls of $\mathfrak{M}=u_1K+\cdots+u_mK$, and the metric homomorph L' of L is lattice isomorphic with the lattice of submoduls of $\mathfrak{M}^*=u_1\bar{K}+\cdots+u_m\bar{K}$.

4. **Vector spaces over quasi-fields.** We now typify vector spaces over quasi-fields by (I) and (J).

REMARK. *A ring $K=\{0,\alpha,\cdots\}$ with unit ϵ, whose only right ideal $\mathfrak{r}>0$ is K, is a quasi-field.*

Let $\alpha\neq 0$. Then $0<\alpha K=K$, $\alpha\beta=\epsilon$. The right annihilator (right) ideal $\mathfrak{r}$ of α is (0), for $\mathfrak{r}>0$ implies $\mathfrak{r}=K$, and $\alpha\epsilon=\alpha=0$. Hence $\alpha(\beta\alpha-\epsilon)=\alpha\beta\alpha-\alpha=\alpha-\alpha=0$ and $\beta\alpha=\epsilon$.

(I) *Let $\mathfrak{M}=u_1K+\cdots+u_mK$ be a vector space. Then every submodul $\mathfrak{N}$, $0<\mathfrak{N}\leqq\mathfrak{M}$, has a basis of $n\leqq m$ elements, with $\mathfrak{N}<\mathfrak{M}$ implying $n<m$, if and only if K is a quasi-field; that is, the modular functional $b(\mathfrak{N})$ on a vector space over a ring K of the type in* (F) *is sharply positive* [1, p. 41] *if and only if K is a quasi-field.*

These are well known properties of a vector space over a quasi-field. If they hold, then by Lemma 1, the existence of a right ideal $\mathfrak{r}$, $0<\mathfrak{r}<K$ implies $\mathfrak{N}=u_1\mathfrak{r}\cup\cdots\cup u_m\mathfrak{r}<\mathfrak{M}$ with $b(\mathfrak{N})=b(\mathfrak{M})$, contrary to hypothesis. Hence (I) follows from the remark above.

(J) *Let $\mathfrak{M}$ be a vector space over a ring K of the type in* (F). *Then $\mathfrak{M}$ satisfies the descending chain condition if and only if K is a quasi-field.*

For rings of this type, the descending chain condition in $\mathfrak{M}$ and sharp positiveness of $b(\mathfrak{N})$ are equivalent. If $\mathfrak{A}<\mathfrak{B}$ with $b(\mathfrak{A})=b(\mathfrak{B})$, the transformation of $\mathfrak{B}$-basis into $\mathfrak{A}$-basis is of type (iii), on $\mathfrak{B}$.

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UNIVERSITY OF MICHIGAN

PSEUDO-CONFORMAL GEOMETRY: FUNCTIONS OF TWO COMPLEX VARIABLES

EDWARD KASNER AND JOHN DECICCO

1. Introduction. The theory of functions of a *single* complex variable is essentially identical with the conformal geometry of the real (or complex) plane. However, this is *not* the case in the theory of functions of *two* independent complex variables. Any pair of functions of two complex variables induces a correspondence between the points of a real (or complex) four-dimensional space S_4. The infinite group G of all such correspondences is obviously *not* the conformal group of S_4. Poincaré in his fundamental paper in Palermo Rendiconti (1907) has called G the group of *regular* transformations. In an abstract presented to the Society, 1908, Kasner found it more appropriate to term it the *pseudo-conformal group G.*

In a preceding paper, Kasner has given a purely geometric characterization. His main result is that *the pseudo-conformal group G is characterized by the fact that it leaves invariant the pseudo-angle between any curve and any hypersurface at their point of intersection.*[1]

In the present work, we shall find *all* the differential invariants of *first* order between the curves, surfaces, and hypersurfaces at a given point under the pseudo-conformal group. We shall take every combination of any two elements—six possible cases.[2] The number of independent invariants may be 0, 1, or 2.

A general pair of curve elements possesses no invariants. However, in the special case of an isoclinal pair, there is a unique invariant (the angle between them). A similar result is true for two hypersurface elements.

A hypersurface element and a curve element possess only one invariant—the *pseudo-angle.*[1]

To any general surface element S, there is associated a quadric regulus R of curve elements. There are no invariants between a general surface element S and a curve element ϵ which is not on the regulus R of S. On the other hand, if ϵ is in R, then there is a unique

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Presented to the Society, September 5, 1941; received by the editors July 19, 1941.

[1] Kasner, *Conformality in connection with functions of two complex variables*, Transactions of this Society, vol. 48 (1940), pp. 50–62. See also the following paper: Kasner, *Biharmonic functions and certain generalizations*, American Journal of Mathematics, vol. 58 (1936), pp. 377–390.

[2] We shall denote by ϵ a curve element, that is, a lineal element; by S a general surface element; and by π a hypersurface element. The six possible cases are (ϵ, ϵ), (ϵ, π), (π, π), (ϵ, S), (π, S), (S, S).

invariant between e and S. We find similar results by substituting hypersurface element π for curve element e in the preceding statements.

Finally, two general surface elements possess two independent invariants.

2. **The minimal coordinates.** Let $(\epsilon, \eta, \zeta, \xi)$ denote the cartesian coordinates of any point of complex euclidean four-space S_4. In our work, we shall find it very convenient to introduce the *minimal* coordinates (x, y, u, v) defined by

$$(1)\qquad x = \epsilon + i\eta, \qquad y = \zeta + i\xi, \qquad u = \epsilon - i\eta, \qquad v = \zeta - i\xi.$$

The inverse of this correspondence is

$$(2)\quad \epsilon = \frac{1}{2}(x + u),\ \eta = \frac{1}{2i}(x - u),\ \zeta = \frac{1}{2}(y + v),\ \xi = \frac{1}{2i}(y - v).$$

In minimal coordinates, the distance ds between any two nearby points of space is

$$(3)\qquad ds^2 = dxdu + dydv.$$

If θ is the angle between any two directions of space, then

$$(4)\qquad \cos\theta = \frac{dx_1du_2 + dx_2du_1 + dy_1dv_2 + dy_2dv_1}{2[(dx_1du_1 + dy_1dv_1)(dx_2du_2 + dy_2dv_2)]^{1/2}}.$$

3. **The pseudo-conformal group** This is given in minimal coordinates by

$$(5)\quad X = X(x, y), \qquad Y = Y(x, y), \qquad U = U(u, v), \qquad V = V(u, v),$$

where $X_xY_y - X_yY_x \neq 0$ and $U_uV_v - U_vV_u \neq 0$. Our problem is to inaugurate the study of the geometry of this group.[3]

In what follows, we shall omit from consideration the special minimal planes x=const., y=const., and u=const., v=const. Obviously our pseudo-conformal group may be defined as that preserving the $2\infty^2$ special minimal planes.

4. **The pseudo-conformal geometry of differential elements of first order.** In this and the following sections, we shall mainly be interested in the geometry of the curve elements at a fixed point of S_4. Obviously these form a three-dimensional manifold Σ_3.

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[3] We shall study only the *continuous* pseudo-conformal group, not the more extensive *mixed* group obtained by adjoining functions of the conjugate complex variables.

The pseudo-conformal group induces the seven-parameter group G_7 among the ∞^3 curve elements of Σ_4 at a given point, defined as follows

$$\rho X' = ax' + by', \qquad \rho U' = \alpha u' + \beta v',$$
$$\rho Y' = cx' + dy', \qquad \rho V' = \gamma u' + \delta v', \tag{6}$$

where $ad-bc\neq 0$ and $\alpha\delta-\beta\gamma\neq 0$. Note that $(\rho x', \rho y', \rho u', \rho v')$ are proportional to the four differentials (dx, dy, du, dv).

5. **The isoclinal surface elements.** A surface element of Σ_4 is said to be an *isocline* if it is given by two equations of the form

$$x'/x_1' = y'/y_1' \neq u'/u_1' = v'/v_1'. \tag{7}$$

Otherwise a surface element *not* an isocline is called a *general* surface element. There are ∞^2 isoclines among the ∞^4 surface elements of Σ_4.

There is a unique isocline through any given curve element. Therefore two isoclines never possess a common curve element.

By applying (6) to (7), we find that *any isocline becomes an isocline under* G_7. The group of correspondences between the isoclines is the six-parameter set

$$\mu X_1' = ax_1' + by_1', \qquad \nu U_1' = \alpha u_1' + \beta v_1',$$
$$\mu Y_1' = cx_1' + dy_1', \qquad \nu V_1' = \gamma u_1' + \delta v_1'. \tag{8}$$

6. **The invariants of two curve elements of Σ_4.** In the first place, it is observed that the most general transformation of G_7 which will carry the curve element (1, 0, 1, 0) into the curve element $(\rho_1 X_1', \rho_1 Y_1', \rho_1 U_1', \rho_1 V_1')$ is of the form

$$\rho X' = \rho_1 X_1' x' + by', \qquad \rho U' = \rho_1 U_1' u' + \beta v',$$
$$\rho Y' = \rho_1 Y_1' x' + dy', \qquad \rho V' = \rho_1 V_1' u' + \delta v'. \tag{9}$$

Under G_7, any curve element of Σ_4 may be carried into any other curve element of Σ_4. The canonical form of any curve element is (1, 0, 1, 0). There are ∞^4 such transformations.

A pair of curve elements is said to be an *isoclinal pair* if both elements lie in the same isocline. Otherwise it is said to be a *general pair*. The conditions necessary and sufficient for an isoclinal pair are (7).

Any curve element which lies in the same isocline with (1, 0, 1, 0) must be of the form $(x_2', 0, y_2', 0)$. The transform of this under the correspondence (9) is

$$\rho_2 X_2' = \rho_1 X_1' x_2', \qquad \rho_2 U_2' = \rho_1 U_1' u_2',$$
$$\rho_2 Y_2' = \rho_1 Y_1' x_2', \qquad \rho_2 V_2' = \rho_1 V_1' u_2'. \tag{10}$$

These equations immediately guarantee that the transformed element (X_2', Y_2', U_2', V_2') is in the same isocline with (X_1', Y_1', U_1', V_1'). Moreover the expression

$$(11) \qquad X_2'/X_1' \cdot U_1'/U_2' = Y_2'/Y_1' \cdot V_1'/V_2'$$

is invariant. By taking the logarithm of this invariant and then multiplying the result by $1/2i$, it is found by (4) that the resulting invariant represents the angle between the isoclinal pair of curve elements.

Next let us observe that the group which preserves the curve element (1, 0, 1, 0) is

$$(12) \quad \rho X' = \rho_1 x' + by', \ \rho Y' = dy', \ \rho U' = \rho_1 u' + \beta v', \ \rho V' = \delta v'.$$

This group demonstrates that the curve element $(x', 0, u', 0)$ is preserved. Hence we find the following proposition to be true.

THEOREM 1. *An isoclinal pair of curve elements possesses the unique invariant*

$$(13) \qquad \theta = \frac{1}{2i}\log\frac{x_2'}{x_1'}\cdot\frac{u_1'}{u_2'} = \frac{1}{2i}\log\frac{y_2'}{y_1'}\cdot\frac{v_1'}{v_2'}\cdot$$

This actually is the angle θ between the two directions.

The canonical forms of any isoclinal pair of curve elements are (1, 0, 1, 0) and $(x', 0, u', 0)$. There are ∞^4 such transformations.

Next any transformation of the group (12) which carries the curve element (0, 1, 0, 1) into the curve element $(\rho_2 X_2', \rho_2 Y_2', \rho_2 U_2', \rho_2 V_2')$ is of the form

$$(14) \qquad \begin{aligned} \rho X' &= \rho_1 x' + \rho_2 X_2' y', & \rho U' &= \rho_1 u' + \rho_2 U_2' v', \\ \rho Y' &= \rho_2 Y_2' y', & \rho V' &= \rho_2 V_2' v'. \end{aligned}$$

This demonstrates immediately that any general pair may be sent into any other general pair of curve elements.

The group which preserves the general pair of curve elements (1, 0, 1, 0) and (0, 1, 0, 1) is

$$(15) \qquad \rho X' = \rho_1 x', \qquad \rho Y' = \rho_2 y', \qquad \rho U' = \rho_1 u', \qquad \rho V' = \rho_2 v'.$$

THEOREM 2. *Any general pair of curve elements may be converted into any other general pair.*

7. The nonexistence of invariants for an isocline paired with an isocline or a curve element. By (6), it may be proved that any isocline may be carried into the isocline $x' = 0$, $u' = 0$. The subgroup of (6) which preserves this isocline $x'=0$, $u'=0$ is, in curve element coordinates,

$$(16)\qquad \rho X' = ax', \quad \rho Y' = cx' + dy', \quad \rho U' = \alpha u', \quad \rho V' = \gamma u' + \delta v'.$$

Any isocline may be carried into any other isocline. The canonical form of an isocline is $x'=0$, $u'=0$. There are ∞^5 such transformations of G_7 which carry a given isocline into a prescribed isocline.

By (16) it may be proved that any isocline distinct from the isocline $x'=0$, $u'=0$ may be carried into the isocline $y'=0$, $v'=0$. The group which preserves the two isoclines $x'=0$, $u'=0$, and $y'=0$, $v'=0$ is

$$(17)\qquad \rho X' = ax', \quad \rho Y' = dy', \quad \rho U' = \alpha u', \quad \rho V' = \delta v'.$$

THEOREM 3. *Any two distinct isoclines may be carried into any other two distinct isoclines.*

The canonical forms of two distinct isoclines are $x'=0$, $u'=0$; and $y'=0$, $v'=0$. There are ∞^3 such transformations of G_7.

By combining (9) and (16), it is found that any curve element not on the isocline $x'=0$, $u'=0$ may be carried into the curve element (1, 0, 1, 0). The group preserving the isocline $x'=0$, $u'=0$ and the curve element (1, 0, 1, 0) is

$$(18)\qquad \rho X' = \rho_1 x', \quad \rho Y' = dy', \quad \rho U' = \rho_1 u', \quad \rho V' = \delta v'.$$

THEOREM 4. *An isocline γ and a curve element e not in γ can be converted into any other isocline Γ and a curve element E not in Γ.*

The canonical forms of an isocline γ and a curve element e not in γ are $\gamma: u'=0$, $x'=0$; and $e(1, 0, 1, 0)$. There are ∞^2 such transformations of G_7.

It may be proved that any isocline γ and a curve element e in γ can be carried into the isocline $x'=0$, $u'=0$ and the curve element (0, 1, 0, 1). The group which preserves the isocline $x'=0$, $u'=0$ and the curve element (0, 1, 0, 1) is given by (16).

An isocline γ and a curve element e in γ can be carried into any other isocline Γ and a curve element E in Γ.

The canonical forms of an isocline γ and a curve element e in γ are $\gamma: u'=0$, $x'=0$; and $e(0, 1, 0, 1)$. There are ∞^4 such transformations.

8. **The invariants for a hypersurface element paired with a curve element, or hypersurface element, or isocline.** Any hypersurface element of Σ_3 is

$$(19)\qquad kx' + ly' + mu' + nv' = 0.$$

Any hypersurface element may be given by $\pi(\sigma k, \sigma l, \sigma m, \sigma n)$.

Note that k and l, or m and n, cannot both be zero as hypersurface elements which contain the two minimal planar directions $x'=y'=0$, or $u'=v'=0$ are excluded from consideration.

The induced group between the hypersurface elements of Σ_4 is

$$\sigma k = aK + cL, \qquad \sigma m = \alpha M + \gamma N, \qquad\qquad (20)$$
$$\sigma l = bK + dL, \qquad \sigma n = \beta M + \delta N.$$

Let a hypersurface element $\pi(k, l, m, n)$ and a curve element $e(x', y', u', v')$ be given, (e not in π). Upon drawing the isocline γ through e, we find that γ intersects π in the curve element

$$X'/x' = Y'/y' = -\omega(mu' + nv'), \qquad\qquad (21)$$
$$U'/u' = V'/v' = \omega(kx' + ly').$$

By Theorem 1, we know that the angle between e and the above curve element is invariant.

By (12) and (20), we see that the group which preserves the curve element (1, 0, 1, 0) may be written in hypersurface element coordinates as

$$\sigma k = \rho_1 K, \quad \sigma l = bK + dL, \quad \sigma m = \rho_1 M, \quad \sigma n = \beta M + \delta N. \qquad (22)$$

By these equations, it may easily be proved that the hypersurface element $\pi(\sigma k, \sigma l, \sigma m, \sigma n)$ can be sent into the hypersurface element $\Pi(K, 0, M, 0)$. The group which preserves the curve element $e(1, 0, 1, 0)$ and the hypersurface element $\pi(k, 0, m, 0)$ is

$$\sigma k = \rho_1 K, \qquad \sigma l = dL, \qquad \sigma m = \rho_1 M, \qquad \sigma n = \delta N. \qquad (23)$$

THEOREM 5. *A hypersurface element π and a curve element e (e not in π) possess the unique invariant*

$$\theta = \frac{1}{2i}\log\left[-\frac{mu' + nv'}{kx' + ly'}\right]. \qquad (24)$$

This is the angle between e and the unique curve element e^* in π such that e and e^* form an isoclinal pair. This is identical with Kasner's pseudo-angle.

The canonical forms of a hypersurface element π and a curve element e (e not in π) are π: $kx' + mu' = 0$ and $e(1, 0, 1, 0)$. The pseudo-angle is then $(1/2i)\log(-m/k)$. There are ∞^2 such transformations.

Any hypersurface element contains a unique isocline. Thus the isocline in the hypersurface element $\pi(k, l, m, n)$ is

$$kx' + ly' = 0, \qquad mu' + nv' = 0. \qquad (25)$$

A hypersurface element π and a curve element e contained in π but not in the isocline of π can be carried into any other hypersurface element Π and a curve element E contained in Π but not in the isocline of Π.

The canonical forms of a hypersurface element π and a curve element e contained in π but not in the isocline of π are π: $x' = u'$ and $e(1, 0, 1, 0)$. There are ∞^2 such transformations.

A hypersurface element π and a curve element e contained in the isocline of π can be sent into any other hypersurface element Π and a curve element E contained in the isocline of Π.

The canonical forms of a hypersurface element π and a curve element e contained in the isocline of π are $\pi: y' + v' = 0$ and $e(1, 0, 1, 0)$. There are ∞^3 such transformations.

A pair of hypersurface elements are said to be an *isoclinal pair* if they intersect in an isocline. Otherwise they are said to be a *general pair.*

In hypersurface element coordinates, the equations of an isocline are

$$k/k_1 = l/l_1 \neq m/m_1 = n/n_1. \tag{26}$$

Of course, these are the conditions that the hypersurface elements (k_1, l_1, m_1, n_1) and (k, l, m, n) form an isoclinal pair.

THEOREM 6. *An isoclinal pair of hypersurface elements π_1 and π_2 possess the fundamental invariant*

$$\theta = \frac{1}{2i} \log \frac{k_1}{k_2} \cdot \frac{m_2}{m_1} = \frac{1}{2i} \log \frac{l_1}{l_2} \cdot \frac{n_2}{n_1}. \tag{27}$$

This is Kasner's pseudo-angle between π_2 and any curve element of π_1.

The canonical forms of an isoclinal pair of hypersurface elements are $x' + u' = 0$ and $kx' + mu' = 0$. The invariant (27) is then $(1/2i) \log (m/k)$. There are ∞^4 such transformations.

THEOREM 7. *In the other possible pairs of elements, no further invariants are found.*

On examination, we find the following standard forms:

The canonical forms of a general pair of hypersurface elements are $x' + u' = 0$ and $y' + v' = 0$. There are ∞^1 such transformations.

The canonical forms of any isocline γ and a hypersurface element π, γ not in π, are $\gamma: y' = 0$, $v' = 0$, and $\pi: x' + u' = 0$. There are ∞^2 such transformations.

The canonical forms of an isocline γ and a hypersurface element

π, γ in π, are $\gamma: y'=0$, $v'=0$, and $\pi: y'+v'=0$. There are ∞^4 such transformations.

9. **The invariants for a general surface element paired with a curve element, or a hypersurface element, or a general surface element, or isocline.** A surface element, not an isocline, may be given by u' and v' expressed as linear functions of x' and y'; or by x' and y' expressed as linear functions of u' and v'. Because of the symmetry of our group (6), we find that it always can be supposed that a general surface element can be written in the form

$$u' = px' + qy', \qquad v' = rx' + sy'. \tag{28}$$

Since we wish to omit from consideration those general surface elements, which contain a curve element of the minimal planar directions $x'=y'=0$, or $u'=v'=0$, we find that $ps-qr\neq 0$.

The correspondence between the general surface elements is

$$\begin{aligned} (\alpha\delta-\beta\gamma)p &= a\delta P + c\delta Q - a\beta R - c\beta S,\\ (\alpha\delta-\beta\gamma)q &= b\delta P + d\delta Q - b\beta R - d\beta S,\\ (\alpha\delta-\beta\gamma)r &= -a\gamma P - c\gamma Q + a\alpha R + c\alpha S,\\ (\alpha\delta-\beta\gamma)s &= -b\gamma P - d\gamma Q + b\alpha R + d\alpha S. \end{aligned} \tag{29}$$

Let $S(p, q, r, s)$ be a given general surface element. Through each curve element of S, there is a unique isocline. There are, therefore, ∞^1 isoclines which intersect a given surface element S. These isoclines generate a *nondegenerate quadric regulus* R, which is given by

$$\frac{u'}{v'} = \frac{px'+qy'}{rx'+sy'}. \tag{30}$$

Let $e(x', y', u', v')$ be a curve element of the regulus R but not in S. The isocline through e will intersect the general surface element S in the curve element e^* given by

$$X'/x' = Y'/y' = \omega u', \qquad U'/u' = V'/v' = \omega(px'+qy'). \tag{31}$$

By Theorem 1, the angle between e and e^* is invariant.

Since any general pair of curve elements may be carried into any other general pair, it follows that any general surface element may be sent into any other general surface element. In particular, we may carry any general surface element into the general surface element $u'=x'$, $v'=y'$.

Let us find the subgroup of G_7 which leaves the general surface element $u'=x'$, $v'=y'$ invariant. By (29), we must have

$$(32)\quad a\delta - \beta\gamma = a\delta - c\beta = -b\gamma + d\alpha, \quad 0 = b\delta - d\beta = -a\gamma + c\alpha.$$

By these equations, it follows immediately that $\alpha = a$, $\beta = b$, $\gamma = c$, $\delta = d$. *Thus the subset of G_7, which leaves our general surface element $u' = x'$, $v' = y'$ fixed is the three-parameter group in curve element coordinates*

$$(33)\quad \rho X' = ax' + by', \quad \rho U' = au' + bv', \quad \rho Y' = cx' + dy', \quad \rho V' = cu' + dv'.$$

Now there exists a transformation of G_7 which carries our general surface element into $S: u' = x'$, $v' = y'$. Therefore our curve element is carried into a curve element which belongs to the regulus of S. Hence the coordinates of this new curve element satisfy the condition $u'/v' = x'/y'$. Now in (33) take $-c/d = y'/x' = v'/u'$. Obviously this transformation will take our curve element into the curve element $(x', 0, u', 0)$.

THEOREM 8. *Let the curve element $e(x', y', u', v')$ be on the regulus R of the general surface element $S: u' = px' + qy'$, $v' = rx' + sy'$. That is, let $(u'/v') = (px' + qy')/(rx' + sy')$. Then e and S possess the unique invariant*

$$(34)\qquad \theta = \frac{1}{2i}\log\frac{u'}{px' + qy'} = \frac{1}{2i}\log\frac{v'}{rx' + sy'}\,.$$

This is the angle between e and the curve element e^* on S such that e and e^* are an isoclinal pair.

The canonical forms of a curve element e and a general surface element S, e on the regulus R of S but not in S, are $e(x', 0, u', 0)$ and $S: u' = x'$, $v' = y'$. There are ∞^2 such transformations. The invariant (34) is then $\theta = (1/2i)\log(u'/x')$.

Any pair consisting of a general surface element S and a curve element e contained in S can be converted into any other such pair.

The canonical forms of a general surface element S and a curve element e contained in S are $S: u' = x'$, $v' = y'$ and $e(1, 0, 1, 0)$.

Next let us note that the transformation (33) where

$$(35)\qquad \begin{aligned} a &= \frac{\rho_1}{2i}(X_1' - U_1'), & b &= \frac{\rho_1}{2}(X_1' + U_1'),\\ c &= \frac{\rho_1}{2i}(Y_1' - V_1'), & d &= \frac{\rho_1}{2}(Y_1' + V_1'), \end{aligned}$$

will carry the curve element $(i, 1, -i, 1)$, not in the regulus of the general surface element $S: u' = x'$, $v' = y'$, into any curve element $(\rho_1 X_1', \rho_1 Y_1', \rho_1 U_1', \rho_1 V_1')$, not in the regulus of S.

We also note that the only transformation which preserves the general surface element $u'=x'$, $v'=y'$ and the curve element $(i, 1, -i, 1)$ is the identity.

THEOREM 9. *There exist no invariants between any curve element e and a general surface element S such that e is not on the regulus R of S. There exists one and only one transformation of G_7 which will carry any such general pair into another general pair.*

Let (x', y', u', v') be the coordinates of the curve element of contact between the tangent hypersurface $\pi(k, l, m, n)$ and the regulus R of the general surface element S. Then

$$(36)\quad \rho k = ru' - pv',\ \rho l = su' - qv',\ \rho m = rx' + sy',\ \rho n = -px' - qy'.$$

Solving these equations for (x', y', u', v'), we find

$$(37)\quad \sigma x' = -mq - ns,\quad \sigma u' = pl - kq,\quad \sigma y' = pm + rn,\quad \sigma v' = -ks + rl.$$

The angle between this curve element e and the curve element e^* in S such that e and e^* are an isoclinal pair is invariant. The preceding equations in conjunction with Theorem 8 yield the following information.

THEOREM 10. *Let a hypersurface element $\pi(k, l, m, n)$ be tangent to the regulus R of the general surface element $S: u'=px'+qy'$, $v'=rx'+sy'$. The condition necessary and sufficient for this is $(-n/m) = (kq-pl)/(ks-rl)$. Then π and S possess the unique invariant*

$$(38)\qquad \theta = \frac{1}{2i}\log\left[\frac{qk - pl}{n(ps - qr)}\right] = \frac{1}{2i}\log\left[-\frac{sk - rl}{m(ps - qr)}\right].$$

This is the angle between the curve element e of contact between the hypersurface element π and the regulus R, and the curve element e^* on the general surface element S such that e and e^* form an isoclinal pair.

Again by Theorem 9 and equations (37), we deduce the following theorem:

THEOREM 11. *There exist no invariants between a hypersurface element π and a general surface element S with π not tangent to the regulus R of S. There exists one and only one transformation of G_7 which will carry any such general pair into another general pair.*

The canonical forms are $S: u'=x'$, $v'=y'$, and $\pi: ix'+y'-iu'+v'=0$.

Let θ be the angle between any curve element e of the regulus R_1 of the general surface element S_1 and the curve element e^* on S_1 such that e and e^* are an isoclinal pair. Upon setting $\phi=e^{2i\theta}$, we find by Theorem 11 that the regulus R_1 may be given parametrically by

$$u' = \phi(p_1x' + q_1y'), \qquad v' = \phi(r_1x' + s_1y'). \tag{39}$$

Now the general surface element S_2 will intersect R_1 in two curve elements e_1 and e_2, given by

$$\begin{aligned} u' &= \phi(p_1x' + q_1y') = p_2x' + q_2y', \\ v' &= \phi(r_1x' + s_1y') = r_2x' + s_2y'. \end{aligned} \tag{40}$$

The elimination of x' and y' gives a quadratic equation in ϕ. The roots of this quadratic equation are therefore invariants of the two general surface elements S_1 and S_2.

Now we shall show that these two invariants are the only ones. In the first place, we may assume that S_1 is of the form $u'=x'$, $v'=y'$. Then from (40), we find that our invariants are the roots of

$$\phi^2 - (p_2 + s_2)\phi + (p_2s_2 - r_2q_2) = 0. \tag{41}$$

Since the sum and product of these roots are invariant, we know that (p_2+s_2) and $(p_2s_2-r_2q_2)$ are both invariants.

Next we shall show that we can carry our surface $S_2(p_2, q_2, r_2, s_2)$ into $(P_2, 0, 0, S_2)$. In the first place, the two invariants of the preceding paragraph yield the following two conditions

$$P_2 + S_2 = p_2 + s_2, \qquad P_2S_2 = p_2s_2 - r_2q_2. \tag{42}$$

The elimination of s_2 yields the condition

$$p_2^2 - (P_2 + S_2)p_2 + P_2S_2 + r_2q_2 = 0. \tag{43}$$

Let us in the first place assume that our general surface element S_2 is contained in the regulus R_1 of S_1. Then S_2 must be of the form $(p_2, 0, 0, p_2)$ which is the desired form. As a matter of fact, it is seen that our group (33) preserves any general surface element S_2 contained in R_1.

Now let us suppose that our general surface element S_2 is *not* contained in the regulus R_1. Then its transformed surface element $(P_2, 0, 0, S_2)$ is *not* contained in R_1 so that $P_2 \neq S_2$.

The group (33) shows that if (p_2, q_2, r_2, s_2) is carried into $(P_2, 0, 0, S_2)$, then

$$\begin{aligned} (ad - bc)p_2 &= adP_2 - bcS_2, & (ad - bc)r_2 &= -ac(P_2 - S_2), \\ (ad - bc)q_2 &= bd(P_2 - S_2), & (ad - bc)s_2 &= -bcP_2 + adS_2. \end{aligned} \tag{44}$$

By use of the first of equations (42), it may be shown that the last of the preceding equations is equivalent to the first equation. Hence our conditions reduce to

$$(45)\qquad \begin{aligned} ad(p_2 - P_2) &= bc(p_2 - S_2),\\ (ad - bc)q_2 &= bd(P_2 - S_2),\\ (ad - bc)r_2 &= -ac(P_2 - S_2). \end{aligned}$$

Now in the first of these equations, we see that $p_2 - P_2$ and $p_2 - S_2$ cannot both be zero, for otherwise $P_2 = S_2$, which is impossible. Hence there exists a nonzero number λ so that

$$(46)\qquad ad = \lambda(p_2 - S_2),\quad ac = -\lambda r_2,\quad bc = \lambda(p_2 - P_2),\quad bd = \lambda q_2.$$

From these equations, we deduce that since $\lambda \neq 0$, then

$$(47)\qquad \frac{c}{d} = -\frac{r_2}{p_2 - S_2} = \frac{p_2 - P_2}{q_2}.$$

Of course the last equation is true because of (43). Substituting (47) into (46), we see that $a:b:c:d$ are determined except for an arbitrary constant.

THEOREM 12. *Two general surface elements* S_1 *and* S_2 *possess the two differential invariants* ϕ_1 *and* ϕ_2 *which are roots of the equation*

$$(48)\qquad (p_1s_1 - q_1r_1)\phi^2 + (q_1r_2 + q_2r_1 - p_1s_2 - p_2s_1)\phi + (p_2s_2 - q_2r_2) = 0.$$

If we let $\theta = (1/2i)\log\phi$, *then* θ *is actually the angle between the curve element* e_i, *or* e_j, *in which* S_2 *intersects the regulus* R_1 *of* S_1 *and the curve element* e^* *in* S_1 *such that* e_i, *or* e_j, *and* e^* *are an isoclinal pair. There are no more invariants.*

Finally, we may prove the following result.

THEOREM 12*. *There are no invariants between a general surface element* S *and an isocline* γ. *Any such pair may be carried into any other.*

In conclusion, we observe that Kasner's pseudo-angle may quite easily be extended to characterize the group of pseudo-conformal transformations defined by n complex functions of n complex variables. This group is characterized within the group of arbitrary point transformations of a $2n$-dimensional space by the preservation of the pseudo-angle between a curve and a $(2n-1)$-dimensional hypersurface at their common point of intersection.

COLUMBIA UNIVERSITY AND
ILLINOIS INSTITUTE OF TECHNOLOGY

ON INTEGRAL FUNCTIONS OF INTEGRAL OR ZERO ORDER

S. M. SHAH

Let $F(z)$ be an integral function of finite order ρ. We write $F(z) = z^k e^{g(z)} f(z)$ where $g(z)$ is a polynomial of degree $q \leq \rho$ and

$$f(z) = \prod_1^\infty \left\{\left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \cdots + \frac{1}{p}\left(\frac{z}{a_n}\right)^p\right)\right\}$$

is the canonical product of order ρ_1 and genus p. Let $M(r, F) = \max_{|z|=r} |F(z)|$ and $n(r, F-a) = n(r, a)$ be the number of zeros of $F(z) - a$ in $|z| = r$. In an earlier paper[1] I proved the following result.

THEOREM 1. *If $F(z)$ be of integral order ρ and if the genus of the canonical product $f(z)$ be $p = \rho$, then*

$$\liminf_{r \to \infty} \frac{\log M(r, F)}{n(r, F)\phi(r)} = 0 \tag{1}$$

where $\phi(x)$ is any positive continuous increasing function of the real variable x such that

$$\int_a^\infty \frac{dx}{x\phi(x)} \tag{2}$$

is convergent.

In this note I prove a similar result for the canonical products of order ρ and genus $p = \rho - 1$, and discuss whether the result can be extended to integral functions which are not canonical products. The main result is the following.

THEOREM 2. *If $f(z)$ is a canonical product of integral order ρ and genus $p = \rho - 1$ then*

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{n(r, f)\Phi(r)} = 0 \tag{3}$$

where $\Phi(x)$ is any positive increasing function such that

$$\int_a^\infty \frac{dx}{x\Phi(x)} \tag{4}$$

Received by the editors July 20, 1941.

[1] *A Theorem on integral functions of integral order*, Journal of the London Mathematical Society, vol. 15 (1940), pp. 23–31. I shall refer to this paper as (1).

is convergent and

$$(4.1) \qquad \Phi(x)/x^{\alpha}$$

is monotonic for all large x, *say* $x \geqq \Delta > 0$; α *a constant such that* $0 < \alpha < 1$.

LEMMA 1. *For all* $r \geqq r_0(A, \beta)$

$$J = \int_{\Delta_1}^{r} \frac{dx}{x^{\beta}\Phi(x)} < \frac{Ar^{1-\beta}}{\log r},$$

where A *and* Δ_1 *are positive constants, and* β *is a constant such that* $0 < \beta < 1$.

PROOF. From the convergence of the integral in (4), we have $\log x < \Phi(x)$ for all $x \geqq \Delta_1$. Hence for $r \geqq r_0$

$$J = \int_{\Delta_1}^{r^{1/2}} + \int_{r^{1/2}}^{r} \leqq \frac{1}{\Phi(\Delta_1)} \frac{r^{(1-\beta)/2}}{(1-\beta)} + \frac{2}{(1-\beta)} \frac{r^{1-\beta}}{\log r} < \frac{Ar^{1-\beta}}{\log r}.$$

LEMMA 2. *Suppose that the real functions* $\psi(x)$ *and* $\theta(x)$ *satisfy the following conditions*:

(1) $\psi(x)$ *is continuous in* (δ, ∞) *where* $\delta > 0$, *except for isolated points where* $\psi(x)$ *has ordinary left-hand discontinuities.*

(2) $\psi(x)$ *is non-increasing as* $x \geqq \delta$ *increases in any interval between two consecutive discontinuities.*

(3) $\theta(x)$ *is a positive continuous increasing function for* $x \geqq \delta$.

$$(4) \qquad \limsup_{x\to\infty} \psi(x) = \infty, \qquad \limsup_{x\to\infty} \frac{\psi(x)}{\theta(x)} = 0.$$

Then we can find a sequence $\{x_n\}$ *of values of* x *tending to* ∞ *such that the two inequalities*

$$\psi(x) \leqq \psi(x_n), \qquad x_1 \leqq x < x_n,$$

$$\frac{\psi(x)}{\theta(x)} \leqq \frac{\psi(x_n)}{\theta(x_n)}, \qquad x > x_n,$$

are satisfied simultaneously.

The x_n *are points of discontinuity so that* $\psi(x_n) = \psi(x_n + 0)$ *and* x_1 *is the first point of discontinuity in* (δ, ∞).

The proof is similar to that of Lemma 2 of my paper referred to above, and is based on the following lemma of Pólya.[2]

If

$$l_1, l_2, l_3, \cdots, \qquad l_m > 0,$$

$$s_1, s_2, s_3, \cdots, \qquad s_1 > 0;\ s_{m+1} > s_m;\ m = 1, 2, 3, \cdots,$$

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[2] Mathematische Annalen, vol. 88 (1923), p. 170.

are two sequences of positive numbers, of which the second is monotonic and increasing, such that

$$\lim_{n\to\infty} l_n = 0, \qquad \limsup_{n\to\infty} l_n s_n = \infty,$$

then we can find an infinite sequence $\{n\}$ *of the indices* n *such that the two sets of inequalities*

$$l_n > l_\nu, \qquad \nu > n,$$
$$l_n s_n > l_\mu s_\mu, \qquad \mu < n,$$

are satisfied simultaneously.

To prove Theorem 2 we first consider the case when

$$\limsup_{r\to\infty} \frac{n(r, f)\,\Phi(r)}{r^{p+1}} > 0. \tag{5}$$

We have

$$r^{p+1} < An(r)\,\Phi(r) \tag{5.1}$$

for an infinity of values $r = R_n$ tending to ∞ and so

$$\frac{\log M(R_n)}{n(R_n)\,\Phi(R_n)} < \frac{A \log M(R_n)}{R_n^{p+1}} \to 0 \qquad \text{as } n \to \infty.$$

Hence

$$\liminf_{r\to\infty} \frac{\log M(r)}{n(r)\,\Phi(r)} = 0.$$

Suppose secondly that

$$\lim_{r\to\infty} \frac{n(r)\,\Phi(r)}{r^{p+1}} = 0. \tag{5.2}$$

Here $\Phi(x)/x^\alpha$ must be monotonic decreasing, for if not $\Phi(x) \geqq Ax^\alpha$, and so

$$\limsup_{r\to\infty} \frac{n(r)\,\Phi(r)}{r^{p+1}} \geqq A \limsup_{r\to\infty} \frac{n(r)}{r^{p+1-\alpha}} = \infty,$$

contradicting hypothesis (5.2); so $\Phi(x)/x^\alpha$ is monotonic decreasing for $x \geqq \Delta$. We apply Lemma 2 putting

$$\psi(x) = \frac{n(x)\,\Phi(x)}{x^{p+1-\beta}} = n(x)\,\frac{\Phi(x)}{x^\alpha}\,\frac{1}{x^{p+1-\alpha-\beta}},$$

and choosing $\theta(x) = x^\beta$, β a constant such that $0 < \beta < 1 - \alpha$, $\delta = \Delta$. The conditions of Lemma 2 are satisfied, and hence, putting $x_n = R$ we obtain

$$\frac{n(x)\Phi(x)}{x^{p+1-\beta}} \leqq \frac{n(R)\Phi(R)}{R^{p+1-\beta}}, \quad \text{for } \Delta \leqq x_1 \leqq x \leqq R,$$

$$\frac{n(x)\Phi(x)}{x^{p+1}} \leqq \frac{n(R)\Phi(R)}{R^{p+1}}, \qquad \text{for } x > R.$$

Thus for $R > x_1$,

$$\begin{aligned}
\log M(R, f) &< AI(R, f) \\
&= A \int_0^\infty \frac{n(x, f)}{x^{p+1}} \frac{R^{p+1}}{(x+R)}\,dx \\
&\leqq A\left\{A_1 R^p \int_{x_1}^R \frac{n(x)}{x^{p+1}}\,dx + R^{p+1}\int_R^\infty \frac{n(x)}{x^{p+2}}\,dx\right\} \\
&\leqq A\left\{A_1 R^p \frac{n(R)\Phi(R)}{R^{p+1-\beta}} \int_{x_1}^R \frac{dx}{x^\beta \Phi(x)} + n(R)\Phi(R)\int_R^\infty \frac{dx}{x\Phi(x)}\right\} \\
&\leqq A\left\{\frac{A_2 n(R)\Phi(R)}{\log R} + o(n(R)\Phi(R))\right\}.
\end{aligned}$$

Hence

$$\liminf_{r\to\infty} \frac{\log M(r, f)}{n(r, f)\Phi(r)} \leqq A \liminf_{r\to\infty} \frac{I(r, f)}{n(r, f)\Phi(r)} = 0 \tag{6}$$

and this completes the proof of the theorem.

COROLLARY. *If* $F(z) = z^k e^{q(z)} f(z)$ *is of integral order* ρ *and genus* $g = \rho - 1$ *then*

$$\liminf_{r\to\infty} \frac{\log M(r, F)}{n(r, f)\Phi(r)} = 0. \tag{7}$$

We have $g = \rho - 1 = \max(p, q)$. It is easily seen that $p = \rho - 1$, $q \leqq \rho - 1$ and

$$\log M(r, F) < A\left\{r^{\rho-1} + \log r\right\} + \log M(r, f). \tag{8}$$

If

$$\limsup_{r\to\infty} \frac{n(r, f)\Phi(r)}{r^{p+1}} > 0$$

then R_n being defined in (5.1),

$$\frac{\log M(R_n, F)}{n(R_n, F)\Phi(R_n)} < \frac{A\left\{R_n^{\rho-1} + \log R_n\right\}}{n(R_n, F)\Phi(R_n)} + \frac{\log M(R_n, f)}{n(R_n, f)\Phi(R_n)}.$$

Hence

$$\liminf_{r\to\infty} \frac{\log M(r, F)}{n(r, F)\Phi(r)} = 0.$$

If now

$$\lim_{r\to\infty} \frac{n(r, f)\Phi(r)}{r^{p+1}} = 0$$

then for all large r

$$\log M(r, F) < A\left\{r^{\rho-1} + \log r\right\} + AI(r, f) < A_2 I(r, f)$$

and hence from (6) the required result follows.

The condition (4) on $\Phi(x)$ is sufficient but not necessary[3] for (3) and (7) to hold. The condition (4.1) is also not necessary for we can take $\Phi(x)$ to be any function

$$(l_1x)(l_2x)\cdots(l_{k-1}x)(l_kx)^{1+\eta}, \qquad \eta > 0,$$

of the logarithmic comparison scale, and hence any function for which

$$\liminf_{x\to\infty} \frac{\Phi(x)}{(l_1x)(l_2x)\cdots(l_{k-1}x)(l_kx)^{1+\eta}} \geqq A > 0.$$

We can take $\Phi(x)$ to be any positive L function[4] which satisfies (4) but we cannot take $\Phi(x)$ (or $\phi(x)$ in Theorem 1) to be $(l_1x)(l_2x)\cdots(l_kx)$.

Consider for instance

$$f_1(z) = \prod_N^\infty \left\{1 - \frac{z}{a_n}\right\}, \qquad f_2(z) = \prod_N^\infty \left\{\left(1 - \frac{z}{a_n'}\right)\exp\left(\frac{z}{a_n'}\right)\right\},$$

where

$$a_n = -n(l_1n)\cdots(l_kn)(l_{k+1}n)^2,$$
$$a_n' = n(l_1n)\cdots(l_kn)(l_{k+1}n).$$

The functions $f_1(z)$ and $f_2(z)$ are canonical products of order 1. The genus of $f_1(z)$ is 0, and of $f_2(z)$ is 1. For each of them we have

$$\lim_{r\to\infty} \frac{\log M(r)}{n(r)(l_1r)\cdots(l_kr)} = \infty.$$

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[3] Cf. p. 4 of (1).

[4] For definition see G. H. Hardy, *Orders of Infinity*, 1924, p. 17.

In what follows we shall take $\phi(x)$ to be a positive L function satisfying the condition (2).

Suppose now $F(z)$ is of integral order ρ. There are four possibilities:

(1) $\rho_1 < \rho,\ p \leqq \rho_1,\ q = \rho,$ (2) $\rho_1 = p = \rho,\ q \leqq \rho,$

(3) $\rho_1 = q = \rho,\ p = \rho - 1,$ (4) $\rho_1 = \rho,\ q < \rho,\ p = \rho - 1.$

Combining the above results we have in cases (2) and (4)

$$\liminf_{r\to\infty} \frac{\log M(r, F)}{n(r, F)\phi(r)} = 0. \tag{9}$$

In cases (1) and (3), (9) does not hold.[5] For functions of fractional order and zero order[6] it certainly holds. In particular (9) *is true for any canonical product of finite order; it also holds for functions of maximum or minimum type, order ρ.*

It is known that if $F(z)$ is of integral order ρ, then[7]

$$\liminf_{r\to\infty} \frac{\log M(r, F)}{n(r, F - a)} < \infty \tag{10}$$

for every a, except possibly a single exceptional value of a. Since $F(z)$ and $F(z) - a$ belong to the same type, we deduce from (9) that *if $F(z)$ is of maximum or minimum type, order ρ, where ρ is an integer, then*

$$\liminf_{r\to\infty} \frac{\log M(r, F)}{n(r, F - a)\phi(r)} = 0 \tag{11}$$

for every a. If $F(z)$ is of mean type then (11) need not hold for one exceptional value of a. For example, ze^z and

$$e^z \prod_2^\infty \left\{1 + \frac{z}{n(\log n)^2}\right\}$$

are both functions of mean type, order 1. For each of these two functions

$$\lim_{r\to\infty} \frac{\log M(r, F)}{n(r, F - 0)(\log r)^{3/2}} = \infty.$$

MUSLIM UNIVERSITY
ALIGARH, INDIA

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[5] (1), p. 29.

[6] (1), pp. 29–30.

[7] G. Valiron, *Lectures on the General Theory of Integral Functions*, 1923, p. 86.

THE FEBRUARY MEETING IN NEW YORK

The three hundred eighty-fifth meeting of the American Mathematical Society was held at Columbia University on Saturday, February 28, 1942. The attendance was approximately two hundred including the following one hundred forty-seven members of the Society:

R. L. Anderson, Lawrence Annenberg, R. G. Archibald, B. H. Arnold, K. J. Arnold, L. A. Aroian, Fred Assadourian, F. E. Baker, J. D. Bankier, Alfred Basch, P. T. Bateman, Stefan Bergman, Felix Bernstein, D. H. Blackwell, Gertrude Blanch, Samuel Borofsky, C. B. Boyer, A. D. Bradley, A. B. Brown, G. E. Bulloch, J. H. Bushey, J. W. Calkin, J. A. Clarkson, Harvey Cohn, K. S. Cole, L. M. Comer, T. F. Cope, Richard Courant, J. H. Curtiss, M. D. Darkow, Norman Davids, J. L. Doob, Jesse Douglas, Arnold Dresden, Nelson Dunford, Jacques Dutka, J. E. Eaton, M. L. Elveback, Paul Erdös, Aaron Fialkow, D. A. Flanders, A. I. Forsythe, Tomlinson Fort, R. M. Foster, G. A. Foyle, Hans Fried, K. O. Friedrichs, H. P. Geiringer, A. M. Gelbart, B. P. Gill, M. C. Gray, H. J. Greenberg, Bernard Greenspan, Lewis Greenwald, H. M. Griffin, E. J. Gumbel, Jacques Hadamard, D. W. Hall, F. C. Hall, N. A. Hall, M. H. Heins, Edward Helly, E. H. C. Hildebrandt, Banesh Hoffmann, T. R. Hollcroft, E. A. Hoy, E. M. Hull, W. A. Hurwitz, W. H. Ingram, G. K. Kalisch, Edward Kasner, L. S. Kennison, J. R. Kline, E. R. Kolchin, B. O. Koopman, Arthur Korn, J. S. LeCaine, Solomon Lefschetz, Howard Levi, Madeline Levin, Norman Levinson, E. R. Lorch, A. N. Lowan, L. A. MacColl, C. C. MacDuffee, H. F. MacNeish, P. T. Maker, W. T. Martin, Imanuel Marx, Walther Mayer, H. L. Mintzer, E. C. Molina, Deane Montgomery, M. E. Munroe, F. J. Murray, C. A. Nelson, K. L. Nielsen, L. F. Ollmann, Oystein Ore, J. C. Oxtoby, A. S. Peters, R. S. Phillips, Harry Pollard, E. L. Post, Walter Prenowitz, M. H. Protter, H. A. Rademacher, M. S Rees, C. F. Rehberg, H. J. Riblet, Moses Richardson, R. G. D. Richardson, J. F. Ritt, I. F. Ritter, H. E. Robbins, S. L. Robinson, Benjamin Rosenbaum, J. E. Rosenthal, Raphaël Salem, Arthur Sard, I. E. Segal, Seymour Sherman, Max Shiffman, L. L. Silverman, James Singer, M. F. Smiley, A. H. Sprague, Wolfgang Sternberg, J. J. Stoker, A. H. Stone, R. C. Strodt, W. C. Strodt, Alvin Sugar, M. M. Sullivan, J. D Tamarkin, J. W. Tukey, J. L. Vanderslice Wolfgang Wasow, Louis Weisner, M. E. Wells, W. F. Whitmore, Nobert Wiener, John Williamson, Audrey Wishard, H. A. Wood, M. A. Woodbury, Leo Zippin.

Professor J. H. Curtiss presided at the morning session which was devoted to short papers. At the afternoon session, at which Vice President J. D. Tamarkin presided, an address entitled *On a general equation for relaxation oscillations* was given by Professor Norman Levinson.

Titles and cross references to the abstracts of papers read at this meeting follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1–10 were read before the morning session and papers numbered 11–27 were read by title. Paper 6 was read by Professor Silverman and paper 7 by

Professor Kasner. Dr. Mann was introduced by Professor H. A. Rademacher.

1. Wolfgang Wasow: *On boundary layer problems in the theory of ordinary differential equations.* (Abstract 48-3-121.)

2. Harry Pollard: *The generalized Stieltjes transform.* (Abstract 48-3-117.)

3. K. L. Nielson: *Some properties of functions satisfying partial differential equations of elliptic type.* (Abstract 48-3-116.)

4. A. M. Gelbart: *Functions of two variables with bounded real parts in domains not equivalent to the bicylinder.* (Abstract 48-3-110.)

5. Raphaël Salem: *On singular monotonic functions of the Cantor type.* (Abstract 48-3-118.)

6. L. L. Silverman and Otto Szász: *On a class of Nörlund matrices.* (Abstract 48-3-120.)

7. Edward Kasner and J. J. DeCicco: *A generalization of the isothermal transformations.* (Abstract 48-3-131.)

8. E. R. Lorch: *The theory of analytic functions in normed abelian vector rings.* Preliminary report. (Abstract 47-11-469.)

9. Stefan Bergman: *Two-dimensional flow around two profiles.* (Abstract 48-3-122.)

10. H. B. Mann: *Proof of the conjecture on the density of sums of sets of positive integers.* (Abstract 48-3-105.)

11. A. A. Albert: *Non-associative algebras.* II. *New simple algebras.* (Abstract 48-3-102-*t*.)

12. P. O. Bell: *The parametric osculating quadrics of a family of curves on a surface.* (Abstract 48-3-125-*t*.)

13. Paul Civin: *Two-to-one mappings of three-dimensional sets.* (Abstract 48-3-137-*t*.)

14. Nathaniel Coburn: *Conformal geometry of unitary space.* (Abstract 48-3-126-*t*.)

15. Nathaniel Coburn: *Congruences in unitary space.* (Abstract 48-3-127-*t*.)

16. J. J. DeCicco: *A generalization of the dual-isothermal transformations.* (Abstract 48-3-128-*t*.)

17. J. J. DeCicco: *Point transformations by which straight lines correspond to circles.* (Abstract 48-3-129-*t*.)

18. Jesse Douglas: *On the geodesic surfaces of a given curve family.* (Abstract 48-3-130-*t*.)

19. H. L. Garabedian: *Hausdorff integral transformation.* (Abstract 48-3-109-*t*.)

20. H. J. Greenberg and H. S. Wall: *Hausdorff means included between* $(C, 0)$ *and* $(C, 1)$. (Abstract 48-3-111-*t*.)

21. Irving Kaplansky: *Note on a common error concerning kurtosis.* (Abstract 48-3-136-*t*.)

22. Edward Kasner and Don Mittleman: *A general theorem on the initial curvatures of dynamical trajectories.* (Abstract 48-3-132-*t*.)

23. A. N. Lowan, Gertrude Blanch, and William Horenstein: *Inversion of the q-series associated with Jacobi elliptic functions.* (Abstract 48-3-114-*t*.)

24. A. N. Lowan and Abraham Hillman: *A short table of the zeros of the equation* $f(x)=J_0(x)Y_0(kx)-J_0(kx)Y_0(x)=0$. (Abstract 48-3-115-*t*.)

25. A. R. Schweitzer: *Note on functions which generate an abstract field.* (Abstract 48-3-106-*t*.)

26. Alexander Weinstein: *Spherical pendulum and complex integration.* (Abstract 48-3-124-*t*.)

27. Don Mittleman: *Spin in newtonian mechanics.* (Abstract 48-5-210-*t*.)

T. R. HOLLCROFT,
Associate Secretary

BOOK REVIEWS

Research—A National Resource. II. *Industrial Research.* (Report of the National Research Council to the National Resources Planning Board.) Washington, D.C., United States Government Printing Office, 1941. 11+370 pp. $1.00.

This report, based on a canvass of industrial laboratories, gives an account of the nature and extent of present day industrial research in this country. It consists of an organized collection of over twenty studies, prepared by specially qualified men, covering such topics as the development of industrial research, fundamental research in industry, careers in research, research in the aeronautics, petroleum and steel industries, location and extent of research, and research abroad. There are studies on the role of chemistry, of physics, biology, mathematics, and various fields of engineering, in industrial research. The part on mathematics (pp. 268–288), prepared by Dr. T. C. Fry, will be found interesting and valuable to all mathematicians.

Dr. Fry's report, entitled *Industrial mathematics*, has been reprinted as a supplementary issue of the American Mathematical Monthly (vol. 48 (1941), no. 6, part 2), and also in the Bell System Technical Journal (vol. 20 (1941), pp. 255–292). It begins with a discussion of the nature of the industrial mathematician's work, followed by a list of the qualifications that these men should possess, pointing out that the mathematician in industry must generally function as a consultant rather than as a project man. The lack of a center of training for such men is stressed. In recent months steps have been taken to improve this situation. A study of the uses of mathematics in industry and a section on statistics make up the rest of the report. The many interesting examples give a vivid impression of the growing use of mathematics in industry.

R. V. CHURCHILL

The Analysis of Economic Time Series, by Harold T. Davis. (Cowles Commission for Research in Economics, Monograph no. 6.) Bloomington, Indiana, Principia Press, 1941. 14+620 pp. $5.00.

It is unfortunate and a sign of the extreme specialization of science and the scientists that many advances made in related fields are ignored or neglected by the mathematicians. This is not altogether their fault, because nobody could possibly follow the ever growing literature in all phases of science utilizing mathematics. The whole mathematical profession and especially its statistical branch is indebted to Professor H. T. Davis, who has in this monumental volume assembled and lucidly interpreted a great number of the recent developments in the interesting, if somewhat puzzling, field of the study and analysis of economic time series. It also contains many of his own very important contributions. I feel certain that this book is destined to exert a deep influence upon the future development of the field. It also will stimulate the thinking of many who have been working on entirely different problems and who may find some of their difficulties related to the questions discussed in Professor Davis' book.

The first chapter deals with the history of the analysis of time series and gives a very valuable account of the most important writings in the field. It also affords an excellent introduction to the other parts of the book. The mathematical reader may not feel he can even glance through the whole 600 pages of the book. But he will be able to gain a rapid survey of the field and to select the material most valuable to himself by reading carefully the 60 pages of the first chapter.

The second chapter deals with the technique of harmonic analysis; this is the representation of empirical data by Fourier series. It does not on the whole add very much to the classical treatment of the problems. The last section relates the technique of harmonic analysis and more generally the method of decomposing empirical series by orthogonal function to the theory of multiple correlation. It contains some very valuable and original ideas. There are also, as in all other chapters, applications to economic data, which will be of great interest to economists.

The third chapter treats serial correlation. This is the phenomenon of the appearance of leads and lags among correlated variables. A special case is auto-correlation, where the individual items of one time series are correlated with each other. This chapter presents the best exposition of these extremely interesting and difficult problems which have given so much headache to the statistician. A great deal of the work originally done by Norbert Wiener is reproduced but important theorems are added. There are also new calculation techniques. The problem of continuous spectra is treated as well as some questions connected with random variation which have received so much attention lately. Finally, this chapter gives an application of the ideas of lag correlation to supply and demand curves, which leads to a discussion of the so-called cobweb theorem well known in economic statistics.

The fourth chapter brings an extensive discussion of the important problem of the theory of random series, which naturally is of great interest not only to professional statisticians but also to workers outside the field of economic statistics. It has a definite relationship to some of the fundamental assumptions of probability theory, for instance, to the frequency definition of probability by von Mises. Goutereau's method is discussed and Yule's theory of random series is generalized with the help of the auto-correlation function. The theory of accumulated random series which has received some attention from mathematicians recently is presented as well as the influence of moving averages upon random series. The theory of sequences and reversals originally introduced by Bortkiewicz and recently further developed by H. E. Jones receives considerable attention. Finally, an application to the economic problems of stock market action is given.

Chapter 5 deals with the difficult and puzzling problem of degrees of freedom in economic time series, which has troubled statisticians for a long time. It is well known to the economic statistician that additional observations within the same time interval do not in general add materially to our information about economic phenomena we are studying with the help of statistical methods. If, however, the Fisherian theory of degrees of freedom is uncritically applied, it would seem that we are gaining a great deal of information. Professor Davis attacks the question as a problem in inverse probability essentially on the basis of Bayes' theorem. He considers particularly Schuster's, Walker's and Fisher's tests of significance in harmonic analysis and some problems recently discussed by Thurston, Frisch, and Hotelling in the field of factor analysis. Dissatisfied with previous results he finally proposes a very ingenious method which he calls the method of elementary energies.

Chapter 6 deals with the analysis of trends. The secular trend represents the non-periodic movements in economic time series. The discussion adds some new formulae for the correlation of residuals from polynomial trends. It treats the elimination of seasonal variation by the method of link relatives and the variate difference method which utilizes series of successive finite differences of the data. A very interesting section deals with the fitting of the logistic, which is a somewhat difficult problem because of the nature of the curve. Its solution is, however, important for social and economic statistics because of the role of the logistic in population theory and related fields. Hotelling's method of fitting the curve to its differential equation is discussed extensively as well as other approaches. Additional sections deal with the growth of population and of production. Finally Mr.

Leavens' interesting theory of frequency distributions of time series is presented.

Chapter 7 deals with periodogram analysis. This is one particular technique for the search for hidden periodicities. It utilizes the amplitudes of sine-cosine curves fitted to trial products. Davis presents some material on sine-cosine series, random series, smoothed and cumulated random series, and economic applications to various empirical data. A section deals with sunspot numbers and another with galvanometer series. It is a clear and valuable presentation of the technique of periodogram analysis.

Chapter 8 deals with the evidence and explanation of economic cycles. This is the much discussed and still largely unsolved problem of the somewhat irregular appearance of booms and depressions. Professor Davis first discusses and classifies economic business cycle theories and treats in more detail the mathematical theories of Evans, Roos, Frisch, Tinbergen, and Kalecki. He also goes into some problems on statistical hysteresis which are very interesting from a purely mathematical point of view. He discusses the methods proposed by Lotka, Volterra, Jones, and others in this field. The remainder of the chapter is devoted to some original work in the theory of crises which Professor Davis tries to explain by a sort of resonance theory. Chapter 9 deals with the nature of wealth and income and presents first statistical data on the personal distribution of income as well as the classical theory of Pareto's law. Professor Davis also tries to give a new and very interesting general distribution function which seems to fit the empirical income distribution better than Pareto's function or variants proposed by other writers. The chapter deals, however, mainly with economic and social problems.

Chapter 10 is entitled, "The dynamic trends from the point of view of the equation of exchange," and will not be of very great interest to non-economists.

Chapter 11 takes up very courageously the difficult problem of forecasting economic time series, especially as applied to stock prices. It reviews the classical work of Henry Schultz, *The standard error of a forecast from a curve.* Then it presents Professor Davis' own contribution which essentially assumes that it is not possible to forecast in the future beyond a range equivalent to the range of the available data. There are some interesting ideas about moving periodogram analysis and the method of probable error bands.

The last chapter finally deals with interpretation and critique of the material presented, especially from the point of view of business cycles. It also gives a new "economic" interpretation of history and presents some interesting analogies between mathematics and economics.

This book should prove very useful to the statistician as a handbook. He will find a great deal of stimulation and valuable material and procedures even if he is not professionally concerned with economic problems. The pure mathematician can gain a survey of the field. He may be interested especially in some of the unsolved problems of serial correlation and periodogram analysis which evidently require very powerful tools of analysis. One has to be extremely grateful to the author for having accumulated so much material and presented it in such a readable and interesting form.

GERHARD TINTNER

A Survey of Modern Algebra. By Garrett Birkhoff and Saunders MacLane. New York, Macmillan 1941. 11+450 pp. $3.75.

In this book Professors Birkhoff and MacLane have made an important contribution to the pedagogy of algebra. Their emphasis is on the methods and spirit of modern algebra rather than on the subject matter for itself. The word "survey" in the title is quite accurate; for, although many topics are treated, none of them is really completely developed. The most important parts of each theory are included and that is all that can be asked of an introductory textbook.

Because of the authors' emphasis on "method" rather than "fact" the book will not be of much use as a reference work. But there is no dearth of good reference works in algebra, and in the reviewer's opinion the present textbook will prove more useful than another encyclopedic treatise would have been.

A discussion of the topics included will help to indicate the authors' purposes.

The first three chapters of the book are ostensibly devoted to the development of number systems: starting with postulates for the integers, then defining rational numbers in terms of the integers, and next giving an outline of the Dedekind construction of real numbers. Actually much more is happening. Such fundamental concepts as congruence, residue class, isomorphism, and ordered and well ordered sets are introduced and applied in a natural manner to the theory. Also the generalizations from integers to integral domain, and from rational number to field are made at suitable stages of the development. This procedure of starting with properties of a familiar mathematical system and generalizing to an abstract system is typical of the spirit of the book and is used in the development of almost every topic treated.

The traditional theory of equations, which is all too frequently the undergraduate's only contact with algebra, places much emphasis on the solution of polynomial (and sometimes also non-algebraic) equations by approximation methods. This involves considerable attention to the analysis of polynomial functions, including such topics as Descartes' rules of signs, and Sturm's and Budan's theorems for isolation of roots. Such theorems may indeed be nice applications of the calculus but they certainly do not give the student any new general concepts either in algebra or analysis. A result of this procedure is that most beginning graduate students have not even the ghost of an idea as to what algebra is really about, even those parts of algebra that have proved to be of almost universal mathematical interest. (Contrast this with the situation in analysis where the undergraduate is getting acquainted with such basic ideas as continuity, limit, and infinite processes.)

In contrast to the traditional approach, in the authors' treatment (Chapters IV and V) the parts of the theory of polynomials and theory of equations that belong to analysis are omitted. (One major exception to this general policy is the inclusion of several sections on the fundamental theorem of algebra, but these can be skipped without loss of continuity.) Because of these omissions, an undergraduate course in algebra based on this book has time to include numerous more generally useful algebraic topics and methods; that is, the student can be presented with ideas that lead somewhere, rather than being sent up a blind alley.

Chapter VI deals with group theory. Only the most elementary (and therefore the most frequently needed) parts of the theory of abstract groups are included. The importance of groups of transformations and of permutation groups is emphasized throughout. As usual the topics treated are pegs upon which are hung new general ideas: this time the concepts of automorphism, homomorphism, and a general discussion of abstract equivalence and congruence relations.

The authors express the belief that "for many students, the value of algebra lies in its applications to other fields: higher analysis, geometry, physics, and philosophy." This belief is reflected in many places throughout the book; but one of the most striking examples is found in their treatment of matrices and determinants (Chapters VII–X), which is so arranged as to be an excellent foundation for a later study of Hilbert and Banach spaces. Before matrices are mentioned at all the concepts of vector space, linear dependence, dimension (including infinite-dimensional spaces), inner product, euclidean vector space, normal orthogonal base are considered. Then matrices are introduced as arrays which describe linear transformations of a vector space (relative to some basis); and various properties of matrices are defined relative to the central concept of vector space. Linear groups are introduced as automorphisms of vector spaces. The full linear, the affine, the orthogonal, and the euclidean groups are studied. The problem of classifying quadratic forms relative to these groups is considered.

To clarify the two possible interpretations of change of basis the authors introduce the terms *alibi* (the point moves elsewhere) and *alias* (the point is renamed). This distinction pays rich dividends in the later discussions of various kinds of equivalence and canonical forms. While treating groups of transformations it is natural to discuss invariants under these groups. At this stage the authors insert a discussion of the place of canonical forms in invariant theory. In the reviewer's opinion these little general discussions that appear from time to time are one of the most valuable contributions of the book.

The last of these four chapters opens with a discussion on ranks of matrices and a treatment of the elementary properties of determinants. The authors' interest in applications of algebra to other fields is evidenced here by an interpretation of determinants as volumes. The chapter closes with a survey of the theory of similarity of matrices. Their development does not cover the case of a matrix whose minimum equation has repeated roots, but what they do give illustrates the general ideas and includes many classes of matrices important in other fields.

The recent wave of interest in lattice theory and partially ordered sets is reflected in Chapter XI. The first part of this chapter is a brief treatment of Boolean algebras; then lattices are introduced as a generalization. Chapter XII is devoted to transfinite arithmetic.

In Chapters XIII and XIV enough material is included to give the student the flavor of the theory of rings and fields, but for the real substance of these topics he will have to look elsewhere. The treatment of fields goes far enough to serve as a basis for the final chapter which gives Artin's development of the Galois theory of equations. Solvable groups are studied, and the insolvability by radicals of the general quintic equation is proved.

A very important feature of the book is the inclusion of many exercises. Most of these are illustrative in nature, but a goodly number involve useful extensions of the theory as developed in the text.

The authors' preface includes a summary of the purposes and contents of each chapter, and also indicates several levels of courses in which the book is designed to be used as a text. The most elementary of these is a one year undergraduate course based on only high school algebra. Provision is made for a semester course covering the tools used in physical applications. Finally, a semester course on abstract algebra can be carved out by selecting suitable chapters. The reviewer recommends the use of the book as a text in any of these ways.

The format is pleasing and the text seems to be remarkably free from typographical errors and minor slips.

R. M. THRALL

NOTES

The following one hundred doctorates, with mathematics or mathematical physics as a major subject, were conferred during 1941 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and the title of the dissertation are given in each case if available.

J. C. Abbott, Notre Dame, August, *The projective theory of noneuclidean geometry.*

R. L. Anderson, Iowa State, June, minors in statistics and economics, *Serial correlation in the analysis of time series.*

Elizabeth S. Arnold, California (Berkeley), December, *On certain projective properties of the configurations of m general hyperplanes in hyperspace.*

K. J. Arnold, Massachusetts Institute of Technology, June, minor in economics, *On spherical probability distributions.*

J. D. Bankier, Rice, June, *Arithmetical continued fractions.*

C. B. Barker, California (Berkeley), December, *The Lagrange multiplier rule for two dependent and two independent variables.*

J. H. Bell, Wisconsin, March, *Topics related to the factorization of matrices.*

T. J. Benac, Yale, June, *The associativity condition for linear associative algebras.*

W. D. Berg, Iowa, August, *Theorems on certain type-A difference-equation graduations.*

F. C. Biesele, Texas, June, *Substitutes for the commutative law in the theory of semi-groups.*

D. H. Blackwell, Illinois, June, minor in physics, *Some properties of Markhoff chains.*

E. E. Blanche, Illinois, June, major in statistics, minor in economics, *A systematic analysis of frequency distributions by the Edgeworth method.*

E. L. Buell, Massachusetts Institute of Technology, June, minor in physics, *Solution of some problems in plane stress using functions of a complex variable.*

P. B. Burcham, Northwestern, June, *Certain inclusion relations in the domain of Hausdorff methods of summation.*

I. W. Burr, Michigan, June, *Cumulative frequency functions.*

Ethel B. Callahan, Columbia, April, *Some properties of the system of trajectories defined by equations* $\dot{X} = \Phi(X, Y)$, $\dot{Y} = \Psi(X, Y)$.

P. W. Carruth, Illinois, June, minor in astronomy, *Valuations with given residue class field and value group.*

A. B. Carson, Chicago, August, *An analogue of Green's theorem for multiple integral problems in the calculus of variations.*

Mary D. Clement, Chicago, August, *Loci of osculants of asymptotic plane sections of a surface.*

A. C. Cohen, Jr., Michigan, June, *Estimation of parameters in truncated Pearson frequency distributions.*

Esther Comegys, Radcliffe, June, *Sufficient conditions for applicability of Riemannian spaces.*

E. L. Crow, Wisconsin, June, *The expansion problem associated with an ordinary differential equation of the first order which is quadratic in the parameter.*

Benjamin Epstein, Illinois, June, minor in physics, *On a certain class of transforms.*

G. E. Forsythe, Brown, June, *Summability of random variables.*

David Gilbarg, Indiana, June, minor in physics, *On the structure of the group of* $\mathfrak{P}$*-adic* 1*-units.*

F. G. Gravalos, Harvard, June, *The algebraic integrals of Hill's equations.*

P. E. Guenther, Harvard, June, *On a canonical form for the linear homogeneous q-difference equations.*

W. W. Gutzman, Iowa, August, *On the distribution of means of middle items.*

Margaret M. Hansman, Illinois, February, minors in physics and statistics, *A geometric investigation into metabelian groups generated by four elements of order P.*

W. J. Harrington, Cornell, June, *A study of certain functions auxiliary to Brun's method in number theory.*

J. G. Herriot, Brown, June, I. *Cesàro summability of ordinary double Dirichlet series.* II. *Nörlund summability of double Fourier series.*

G. P. Hochschild, Princeton, June, *Semi-simple algebras and generalized derivations.*

F. E. Hohn, Illinois, October, minor in physics, *Curves on Cayley's dianodal surface.*

C. T. Hsu, Columbia, June, *Tests of certain statistical hypotheses concerning bivariate normal populations.*

H. B. Huntington, Princeton, June, major in mathematical physics, *Mechanisms for self-diffusion in a copper lattice.*

L. W. Johnson, Princeton, January, *A linear algebraic theory of complexes.*

R. E. Johnson, Wisconsin, June, *Rings of infinite matrices and polynomial rings.*

H. F. S. Jonah, Purdue, June, minor in mathematical physics, *Congruences connected with the solution of a certain diophantine equation.*

Irving Kaplansky, Harvard, June, *Maximal fields with valuations.*

Lois Kiefer, Illinois, June, minor in astronomy, *A radial analysis of the structure of the Milky Way in Auriga, based on star counts and color excesses.*

E. R. Kolchin, Columbia, September, *On the exponents of differential ideals.*

David Krabill, Ohio State, August, *Matrices whose elements are functions of one variable.*

H. N. Laden, Pennsylvania, June, *An application of the classical orthogonal polynomials to the theory of interpolation.*

J. P. LaSalle, California Institute of Technology, June, minor in physics, *Pseudo-normed linear sets over valued rings.*

S. J. Lawwill, Cincinnati, June, *Concerning harmonic functions especially with reference to the theory of approximation.*

Jeanne S. LeCaine, Radcliffe, June, *On the generalized gamma function and its analogue for q-difference equations.*

H. L. Lee, Duke, June, *Power sums of polynomials in a Galois field.*

Joseph Lehner, Pennsylvania, June, *A partition function connected with the modulus five.*

R. J. Levit, California (Berkeley), May, *Characterisations of fields as systems of single composition.*

C. B. Lindquist, Wisconsin, September, *Rectangular isotropic and anisotropic plates under forces in their planes.*

D. B. Lloyd, Catholic University, February, minors in physics and education, *Some properties of rational quintic equations.*

L. A. Lorch, Cincinnati, June, *Some problems in Borel summability of Fourier series.*

Leonard McFadden, Brown, October, *Absolute Nörlund summability.*

Rhoda Manning, Stanford, June, minor in biochemistry, *On the derivatives of the sections of bounded power series.*

A. E. Marston, California (Berkeley), December, *A linear elliptic partial differential equation whose leading coefficients may vanish at a point.*

F. L. Martin, Chicago, August, *Integral domains in quartic fields.*

Herman Meyer, Chicago, June, *Polynomial approximations to functions defined on abstract spaces.*

E. J. Mickle, Ohio State, June, *Hamiltonian and quasi-Hamiltonian functions associated with double integral variation problems.*

D. D. Miller, Michigan, February, *Extension and reduction theorems for certain types of continuous transformations.*

D. S. Miller, Cornell, September, *Some properties of Carathéodory and Gillespie linear measure.*

G. T. Miller, Purdue, June, minors in statistics and mathematical physics, *Analytic continuation of functions defined by factorial series.*

Harlan C. Miller, Texas, June, *On compact unicoherent continua.*

Philip Newman, Columbia, June, *The geometry of the planar* (2, 2) *connex.*

E. H. Nicholson, Washington University, June, minor in physics, *On the degree of approximation in some convergence theorems concerning derivatives of the mapping function in conformal mapping.*

E. N. Nilson, Harvard, June, I. *Interpolation and approximation to an analytic function by functions analytic and bounded in a region.* II. *Approximation to a harmonic function by functions harmonic and bounded in a region.*

C. G. A. Nordling, Massachusetts Institute of Technology, June, minor in mechanical engineering, *Solution by polynomial approximation of the integral equation for the circulation around an airfoil.*

Sister Jeanette Obrist, Catholic University, February, minors in physics and botany, *A problem arising from the special symmetric correspondence* C_4 *set up by the rational quartic curve with two cusps.*

R. E. O'Connor, Harvard, June, *Representation of integers by power-products of two real numbers.*

J. W. Odle, Michigan, February, *Non-alternating and non-separating transformations modulo a family of sets.*

O. G. Owens, California (Berkeley), May, *An explicit formula for the solution of the ultrahyperbolic equation in four variables.*

A. S. Peters, New York, June, minor in physics, *On a simply supported-free half-plane plate and a clamped infinite-sector plate.*

Anatol Rapoport, Chicago, December, *Construction of non-abelian fields with prescribed arithmetic.*

G. E. Reves, Cincinnati, June, *On the absolute convergence of double Fourier series.*

C. E. Rickart, Michigan, June, *Integration in a convex, linear topological space.*

L. A. Ringenberg, Ohio State, March, *On functions of intervals.*

F. E. Satterthwaite, Iowa, August, *Developments on the theory of Chi-square.*

L. J. Savage, Michigan, June, *The application of vectorial methods to the study of distance spaces.*

Ivor Schilansky, Michigan, June, *On generalized zeta functions and their associated lattice point problems.*

W. M. Scott, Michigan, February, *On matrix algebras over an algebraically closed field.*

D. M. Seward, Duke, June, minor in physics, *Harmonic continuation in space.*

Ernst Snapper, Princeton, January, *Structure of linear sets.*

C. D. Solin, Toronto, June, *Positive ternary quadratic forms of determinant* $\leqq 200$, *and positive ternary quadratic forms in genera of one class.*

R. H. Sorgenfrey, Texas, June, *Concerning triodic continua.*

A. H. Sprague, Princeton, January, *Surfaces whose lines of curvature are nets* R, *and their transformations.*

C. F. Strobel, Illinois, June, minor in physics, *The quadrilinear form* (1, 1, 1, 2).

R. L. Swain, Texas, June, I. *Proper and reductive-transformations.* II. *Continua obtained from sequences of simple chains of point sets.* III. *Distance axioms in Moore spaces.* IV. *Linear metric space.* V. *A space in which there may exist uncountable convergent sequences of points.*

C. J. Thorne, Iowa State, June, minor in physics, *The approximate solution of linear differential equations by the use of functionals.*

L. V. Toralballa, Michigan, June, *The sum of the values of a rational function of* s *variables over the set of all the* n*-partite permutational partitions of a given positive integer.*

W. R. Transue, Lehigh, October, minor in physics, *Contributions to the theory of subharmonic functions.*

J. R. Vatnsdal, Michigan, February, *Minimal variance and its relation to efficient moment tests.*

L. I. Wade, Jr., Duke, June, minor in philosophy, *Certain quantities transcendental over the field* $GF(p^n, x)$.

Muriel Wales, Toronto, June, *Theory of algebraic functions based on the use of cycles.*

Dzung-shu Wei, Iowa, February, *Necessary and sufficient condi-*

tions that regression systems of sums with elements in common be linear.

William Wernick, New York, June, minor in physics, *Complete sets of logical functions.*

R. L. Westhafer, Ohio State, August, *Singular solutions of ordinary differential equations of the first order.*

G. W. Whitehead, Jr., Chicago, June, *Homotopy properties of the real orthogonal groups.*

P. M. Whitman, Harvard, June, *Free lattices.*

W. F. Whitmore, California (Berkeley), May, *Convergence theorems for functions of two complex variables.*

W. D. Wray, Cornell, June, *Some applications of uniformity trials.*

Paul Young, Ohio State, August, *On the approximation of functions by integral means.*

Professor J. L. Gibson of the University of Utah has retired with the title professor emeritus.

Assistant Professor Max Astrachan of Antioch College has been promoted to an associate professorship and made chairman of the department of mathematics.

Dr. Alfred Basch of Holy Cross College has accepted a position at the College of Paterson, Paterson, New Jersey.

Assistant Professor T. C. Benton of Pennsylvania State College has been promoted to an associate professorship.

Associate Professor J. E. Davis of Drexel Institute of Technology has been promoted to a professorship.

Associate Professor M. A. Hill of the University of North Carolina has been promoted to a professorship.

Assistant Professor Nathan Jacobson of the University of North Carolina has been promoted to an associate professorship.

Professor R. D. James of the University of Saskatchewan is at the University of Wisconsin for the second semester of this current academic year.

Dr. A. E. Marston of the University of California has accepted a position at Compton Junior College, Compton, California.

Associate Professor F. C. Ogg of Bowling Green State University has been promoted to a professorship.

Dr. Arthur Rosenthal has been appointed to a lectureship at the University of New Mexico.

Professor M. H. Stone has been appointed chairman of the department of mathematics at Harvard University, succeeding Professor J. L. Walsh.

Assistant Professor Antoni Zygmund of Mount Holyoke College has been appointed visiting assistant professor at the University of Michigan for the second semester of the current academic year.

Mr. C. L. Carroll of the Georgia School of Technology is on leave at the University of North Carolina.

Assistant Professor B. E. Gatewood of Louisiana Polytechnic Institute is on leave and with MacDonnell Aircraft Corporation.

Professor W. R. Hutcherson of Berea College is on leave at Brown University.

Professor H. W. March of the University of Wisconsin is on leave of absence.

Professor F. W. Owens of Pennsylvania State College is on leave of absence.

Professor Mary E. Sinclair of Oberlin College is on leave of absence.

Dean T. M. Putnam and Assistant Professor B. C. Wong of the University of California are on leave of absence.

Professor H. C. Carver, Assistant Professor P. S. Dwyer, Professor C. E. Love, and Associate Professor J. A. Nyswander of the University of Michigan are on leave of absence.

The following appointments to instructorships are announced: Massachusetts Institute of Technology: Dr. J. P. LaSalle; University of Wisconsin: Dr. R. E. Johnson, Dr. R. L. Swain.

Miss Frances Hardcastle of Stocksfield-on-Tyne, England, died December 26, 1941. She had been a member of the Society since 1894.

Professor R. L. Charles of Franklin and Marshall College died December 13, 1941, at the age of fifty-six years.

Professor W. V. N. Garretson of Oklahoma Agricultural and Mechanical College died January 17, 1942, at the age of sixty-five years.

Dr. Robert Henderson died February 16, 1942. Until his retirement he had been vice president of Equitable Life Assurance Society of the United States. Dr. Henderson joined the American Mathematical Society in 1910 and served on its Board of Trustees for many years. A more complete notice will appear later in this Bulletin.

Dr. E. R. van Kampen of Johns Hopkins University died February 11, 1942.

Dr. E. J. Maurus, who was professor of mathematics at the University of Notre Dame from 1897 to 1939, died November 26, 1941, at the age of sixty-nine years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

141. A. A. Albert: *The radical of a non-associative algebra.*

The multiplications of an algebra A generate a corresponding associative polynomial algebra $T(A)$. Every ideal B of A determines a corresponding ideal S of $T(A)$ and it is shown that $T(A-B)$ is equivalent to $T(A-S)$. If H is the radical of $T(A)$ the set AH is an ideal of A and $A-AH$ is either a zero algebra Z, a semi-simple algebra G (direct sum of simple algebras), or a direct sum $Z\oplus G$. If $A-AH=Z$ the algebra A is not homomorphic to a semi-simple algebra. For all other algebras the radical is defined to be the minimal ideal N such that $A-N$ is semi-simple. Then $N=AH$ if $A-AH$ is semi-simple. Otherwise $Z=N-AH$, the quantities of N are the quantities in the cosets which make up Z. If A and A_1 are isotopic algebras with unity quantities, their radicals $N=AH$ and $N_1=A_1H_1$ are isotopes and the difference algebras $A-N$ and A_1-N_1 are isotopes. An example is given of an algebra A with a unity quantity whose radical, with respect to our definition, is a field. (Received February 12, 1942.)

142. B. H. Arnold: *Rings of transformations of certain vector spaces.* Preliminary report.

Eidelheit (Studia Mathematica, vol. 9 (1940), pp. 97–105) showed that the algebraic properties of the ring of all continuous linear transformations of a real Banach space into itself characterizes the Banach space up to an isomorphism. Mackey has extended this result to a more general class of spaces. In this paper the author shows what algebraic properties of the ring correspond to certain properties of the space, such as completeness, reflexivity, reflexivity of the completion, or being the conjugate of some space. Eidelheit's result is extended to the complex case and a set of conditions is given which are necessary and sufficient that an abstract ring be the ring of all continuous linear transformations of some space into itself. (Received March 3, 1942.)

143. Emil Artin and Peter Scherk: *On the sum of two sets of integers.*

Let A, B, respectively, be sets of non-negative integers a, b. Let C be the set of all integers of the form $a+b$. Let $A(x)$, $B(x)$, $C(x)$ denote the number of positive inintegers of the sets less than or equal to x. A few months ago, H. B. Mann succeeded in proving: If $0\subset A$ and $0\subset B$, and if $C(n)<n$, then $C(n)/n\geq\min_{x=1,\cdots,n}(A(x)+B(x))/x$. Applying and simplifying Mann's method, the authors have proved: If $C(n)<n$, then $C(n)-C(n-m)=A(m-1)+B(m-1)+Z_m(n)$, for a suitable $m\not\subset C$

with $0 < m \leq n$; $Z_m(n)$ denotes the number of decompositions of n of a certain type (hence $Z_m(n) \geq 0$). If $0 \subset A$, $0 \subset B$, this result contains various estimates of $C(n)$ including that by Mann. If $0 \subset A$ a result of A. S. Besicovitch's follows (Journal of the London Mathematical Society, vol. 10 (1935), pp. 246–248). If $0 \not\subset A$, $0 \not\subset B$, new estimates of $C(n)$ follow immediately. (Received March 14, 1942.)

144. S. P. Avann: *A numerical condition for modularity of a finite lattice.*

Let $U(a)$ be the number of elements other than a which are covered by elements covering a, an element of a finite lattice L in which the Jordan-Dedekind chain condition holds; let $V(a)$ be the number of elements other than a which cover elements covered by a. Then a necessary and sufficient condition that L be modular is that $U(a) = V(a)$ for every $a \in L$. Finiteness of L is required for sufficiency. Hence L may be tested for modularity by counting coverings or bonds in the Hasse diagram of L. (Received March 14, 1942.)

145. S. P. Avann: *On complete classes of projective quotients in a modular lattice.*

Let L be a finite modular lattice in which Q is a complete class: $[a_1/b_1, a_2/b_2, \cdots, a_r/b_r]$ of projective quotients defined by the equivalence relation of projectivity. Denote the set of numerators a_i by A and the set of denominators b_i by B. Then the following conditions are equivalent: (1) A and B are transposed quotient sublattices of L, (2) the meet a_0 of the a_i is in A and to each $a_i \neq a_0$ belongs a unique denominator b_i in B, (2′) the join b of the b_i is in B and to each $b_i \neq b$ belongs a unique numerator a_i in A, (3) $a_0 \in A$ and $b \in B$, (4) uniqueness of denominators and numerators. If the quotient sublattices of Q are complemented then (1), (5) $a_0 \in A$, (5′) $b \in B$, (6) uniqueness of denominators, (6′) uniqueness of numerators, are equivalent conditions. (Received March 14, 1942.)

146. Reinhold Baer: *Radical ideals.*

An ideal is termed a radical ideal, if it is a two-sided nil ideal modulo which there do not exist nilpotent right-ideals different from 0. Every ring possesses at least one radical ideal; and both the cross-cut and the sum of all the radical ideals are themselves radical ideals which are called the lower and the upper radical, respectively. It is possible that lower and upper radical are different and that neither of them is nilpotent. If every right ideal, not 0, in the quotient ring modulo the lower radical contains a smallest right ideal different from 0, then upper and lower radical are equal; and if this condition is satisfied by every quotient ring, then a finite or transfinite power of the radical ideal is 0. As an application of this theory a generalization of a theorem of C. Hopkins is proved. (Received March 12, 1942.)

147. Richard Brauer and Hsio-Fu Tuan: *Some remarks on simple groups of finite order.* Preliminary report.

Using the theory of group characters, some lemmas on simple groups of finite order are obtained. For instance, a group G of order $3 \cdot p^a \cdot q^b$ (p, q primes) can only be simple if G has one of the orders 60 or 168. It can be shown that the list of simple groups (cf. L. E. Dickson, *Linear Groups*) is complete up to the order 10,000. (Received March 6, 1942.)

148. Leonard Carlitz: *On the reciprocal of certain types of Hurwitz series.*

In the first part of this paper previous results (Duke Mathematical Journal, vol. 9 (1942)) are extended to series $\sum A_m t^{p^{nm}}/F_m$, where the A_m are polynomials of a more general type. In the second part of the paper the case $\sum A_m t^m/G_m$ is treated. (Received March 6, 1942.)

149. R. F. Clippinger: *Matrix products of matrix powers.*

If $A_1, A_2, \cdots, A_m$ are given matrices of constants, the closure of the set of products of arbitrary positive powers of the matrices $\bar{A}_1 = \exp A_1, \bar{A}_2 = \exp A_2, \cdots, \bar{A}_m = \exp A_m$ is identical with the closure of the set of solutions for positive t of the matrix differential equation: $dy/dt = Y(\rho_1(t)A_1 + \rho_2(t)A_2 + \cdots + \rho_m(t)A_m)$, $Y(0) = 1$, where $\rho_1(t), \rho_2(t), \cdots, \rho_m(t)$ are arbitrary, non-negative, measurable, bounded functions of t. (Received March 19, 1942.)

150. R. P. Dilworth: *The partition lattice.*

In a previous paper (abstract 48-1-11) a dependence relation for elements of rank k in a Birkhoff lattice was defined. In this paper it is shown that if this dependence relation is applied to the elements of rank two in a Boolean algebra, the closed subsets form a lattice isomorphic to the partition lattice. (Received March 9, 1942.)

151. F. A. Ficken: *On Rosser's generalization of Euclid's algorithm.* Preliminary report.

Let $X_1, \cdots, X_n$ be a set of vectors in an euclidean space. The set of independent vectors $Y_1, \cdots, Y_m$ is a greatest common factor (G.C.F.) of the X's if each $X_j = \sum p_{ji} Y_i$ and each $Y_k = \sum q_{kr} X_r$, where the p's and q's are integers. In a minimal G.C.F., the lengths of the Y's satisfy certain minimal conditions. Full definitions and a number of important applications may be found in Duke Mathematical Journal, vol. 9 (1942), pp. 59–95. Rosser has given there an algorithm for finding G.C.F.'s and all minimal G.C.F.'s. By examining the termination of the algorithm, he establishes several properties of minimal G.C.F.'s. In the present paper, this examination is conducted by producing explicit lower bounds for $(\sum t_i X_i)^2$, $(i = 1, \cdots, n; n = 2, 3, 4, 5)$. These bounds furnish simplified proofs for the bulk of the results and confirm certain of the conjectures stated in Rosser's paper. (Received March 6, 1942.)

152. J. S. Frame: *Double coset matrices and group characters.*

This paper extends to the characters of the irreducible components of an intransitive permutation group a theorem proved earlier by the author (this Bulletin, vol. 47 (1941), p. 459) for the degrees of the irreducible components of a transitive permutation group. To the element γ of G corresponds the $n \times n$ permutation matrix $R(\gamma)$ with the $n^t \times n^t$ transitive constituents $R^t(\gamma)$ which contain the irreducible representation Γ_s just μ_s^t times. To each transitive constituent corresponds a subgroup H^t of order h^t leaving its first symbol fixed. The matrices V permutable with G break up into submatrices V^{tu} linearly dependent on μ^{tu} double coset matrices V_α^{tu}, each associated with a double coset $H_\alpha^{tu} = H^t \gamma_\alpha H^u / h_\alpha^{tu}$ formed from the given subgroups. With the structure coefficients $c_{\alpha\beta\gamma}^{tuv}$ which appear in the product formula $H_\alpha^{tu} H_\beta^{uv} / h^u = \sum_\gamma c_{\alpha\beta\gamma}^{tuv} H_\gamma^{tv}$, with the constants $h_{\gamma\lambda}^{t}$ which tell how many conjugate elements from the class C lie in the double coset H_γ^{tt}, and with a set of arbitrary parameters a_λ, a matrix

$K^{st}=\sum_{\lambda,\nu}c_{\alpha\beta\nu}k^{t}_{\nu\lambda}a_\lambda$ is formed. Its determinant $|K^{st}|$ will factor into linear factors $\chi_s=\chi_s(\sum a_\lambda C_\lambda)$ which give the characters of the irreducible components Γ_s common to two constituents $R^t(\gamma)$ and $R^s(\gamma)$. The theorem is: $(n^t)^{\mu^{st}}|K^{st}|=\mathrm{P}_{st}\bar{\mathrm{P}}_{st}\prod_s(\chi_s)^{\mu_s}$, where $\mu_s=\mu_s^t\mu_s^s$, $\mu^{st}=\sum\mu_s$, and P_{st} is an algebraic integer. An illustrative example is given. (Received March 5, 1942.)

153. R. D. James: *On Euler's conjecture.*

It was conjectured by Euler that every integer $N\equiv 2 \pmod 4$, $N\geqq 10$, could be represented as a sum of two primes each congruent to 1 (mod 4). However even the weaker statement that every such integer is a sum of two integers which have all their prime factors congruent to 1 (mod 4) has not yet been proved. In a previous paper (Transactions of this Society, vol. 43 (1938) pp. 296–302) the author proved that every sufficiently large integer $N\equiv 2 \pmod 4$ is a sum of two integers which have all except possibly two of their prime factors congruent to 1 (mod 4). This result is improved in the present paper as indicated in the following theorem: Every sufficiently large integer $N\equiv 2 \pmod 4$ is a sum of two integers which either have at most six prime factors all congruent to 1 (mod 4) or else have exactly two prime factors congruent to 3 (mod 4) and at most three prime factors congruent to 1 (mod 4). The method of proof is a general one and applies to any infinite sequence of primes for which $\sum p^{-1}\log p=\mu\log x+O(1)$, where μ is any positive number. (Received March 20, 1942.)

154. R. E. Johnson: *On structures of infinite modules.*

Let Q be a principal ideal ring containing the field P, and Ξ be a P-module which has Q as an operator domain. It is assumed that Ξ has a countable P-basis. Then the problem studied is under what conditions there exist elements $\xi_1, \xi_2, \cdots$ in Ξ such that $\Xi=Q\xi_1+Q\xi_2+\cdots$, this last being a supplementary sum. Necessary and sufficient conditions are found for this to be the case. If $Q/(m)$, m not a unit of Q, is taken as the operator domain of Ξ, then there always exist elements $\xi_1, \xi_2, \cdots$, of Ξ such that $\Xi=Q/(m)\xi_1+Q/(m)\xi_2+\cdots$. Now let $\mathfrak{M}_\omega$ be the matrix algebra of order type ω over P with finitely nonzero columns, and Ξ be the P-module composed of the columns of these matrices. Then an element A of $\mathfrak{M}_\omega$ can be transformed into a reduced form in case $\Xi=P[A]\xi_1+P[A]\xi_2+\cdots$. (Received March 18, 1942.)

155. H. F. S. Jonah: *Congruences connected with the solution of a certain Diophantine equation.*

The paper begins with a function and functional equation used by Maier (Mathematische Annalen, vol. 104 (1931), pp. 745–769) and generalizes both. Then by means of the calculus of residues the author obtains a quadratic functional equation of a function closely related to the $h_n^{(a,b)}(x)$ of Vandiver (Annals of Mathematics, (2), vol. 27 (1926), pp. 171–176). Using this functional equation it is possible to transform the general congruences used by Vandiver in his work on Fermat's last theorem. (Received March 18, 1942.)

156. Joseph Lehner: *On certain irrational modular equations.*

The author obtains by simple group-theoretic considerations the modular equation connecting $\eta(\tau)/\eta(\tau/p)$ and $\eta(p\tau)/\eta(\tau)$, where $\eta(\tau)=\exp(\pi i\tau/12)\cdot\prod_1^\infty(1-\exp(2\pi in\tau))$ and $p=5, 7, 13$. These are the only three primes p for which an equation of degree p exists, this fact being connected with the genus of the subgroup

$\Gamma_0(p)$ of the full modular group. The equations for $p=5, 7$ were obtained by Watson (Crelle, 1937) by more complicated methods; they are important in the theory of Ramanujan's congruences concerning the unrestricted partition function, $p(n)$. (Received February 21, 1942.)

157. F. A. Lewis: *Generators of permutation groups simply isomorphic with $LF(2, p^n)$.*

Generators of any permutation group of the class are obtained by a slight extension of an argument used in a recent paper (this Bulletin, vol. 47 (1941), p. 631). (Received February 20, 1942.)

158. Knox Millsaps: *Abstract polynomials in non-abelian groups.*

After an abstract calculus of finite differences is defined, functional definitions of a monomial and a polynomial for elements of the central subgroup or elements of the group as increments are given. The theorem on the homogeneity of a polynomial is proved for central and arbitrary differences; for central differences the difference being a function of the increment alone implies the difference is a monomial; the independence of the central difference of polynomials and the unique decomposition for the abelian valued case are made to depend on the product of a Vandermonde determinant and a finite product of binomial coefficients; and these last results can be extended to the non-abelian valued case for certain free groups. The theory is essentially a generalization of the work of Van der Lijn on abstract polynomials in abelian groups. (Received March 4, 1942.)

159. Ivan Niven: *On matric polynomials.*

Let B be a matrix with elements from any field, and let P be a scalar polynomial in B with coefficients from the field. Then the index of P cannot exceed the index of B. Furthermore, B is expressible as a polynomial in P if and only if the index of P equals the index of B. A new method is offered for finding those solutions X of $X^m=A$ which are polynomials in A, the elements of A being from an algebraically closed field with characteristic 0 (cf. M. H. Ingraham, *Rational methods in matrix equations*, this Bulletin, vol. 47 (1941), p. 64, Theorem 3). If A is non-derogatory every solution of $X^m=A$ is a polynomial in A, and the equation has no solutions if and only if 0 is a repeated root of the minimal equation of A. (Received March 17, 1942.)

160. Rufus Oldenburger: *Expansions of quadratic forms.*

A quadratic form Q is ordinarily given as a sum $a_{ij}x_ix_j$ of power products of the variables. Although this is a sum of products of linear forms, this is not in general the most economical way of writing Q as such a sum. The minimum τ such that Q is a sum of τ products of linear forms is $r-\sigma$, where r is the rank of Q, and σ is the index of Q introduced elsewhere by the author. This result is valid for fields with characteristic not 2. (Received February 12, 1942.)

161. Rufus Oldenburger: *Repeated linear factors of forms.*

With a form F of degree n there are associated 2-way displays, $A_0, A_1, \cdots, A_n$ of the symmetric matrix A of coefficients of F, used by W. Mayer, Thrall and Chandler, F. L. Hitchcock, and others. The form F has a linear factor L repeated r times if and only if L^{n-r+1} is a covariant directly constructible from A_{n-r+1}. If n is even the coefficients in L^{n-r+1} can be expressed simply as higher-dimensional determinants. (Received February 20, 1942.)

162. Rufus Oldenburger: *The index of a quadratic form for an arbitary field.*

The classical index of a quadratic from Q is defined in terms of the order of elements in the real field. In the present paper a definition of the index of Q is given which is valid for any field, and which reduces in the real case to the index of Q or $-Q$. It is proved that a form Q with rank r and index σ has a splitting $G+H$, where G has index σ and rank 2σ, whereas H has index 0 and rank $r-2\sigma$. A second form Q' with a corresponding splitting $G'+H'$ is equivalent under a nonsingular linear transformation to Q if and only if the rank and index of Q equal the rank and index of Q', and H is equivalent to H'. A form H has index 0 if and only if H is a nonzero form. Thus the problem of the equivalence of quadratic forms reduces to that of the equivalence of nonzero forms. Such forms have been studied by Siegal, Weyl, Dickson, Legendre, and many others. (Received February 13, 1942.)

163. M. A. Sadowsky: *Approximation by rational sequences.*

A certain graphical principle is employed to select the best approximations to a real number. It is shown that this process leads to the consecutive convergents of the continued fraction development. (Received March 20, 1942.)

164. R. D. Schafer: *Alternative algebras over an arbitrary field.*

The results of M. Zorn concerning simple alternative algebras are incomplete for algebras over modular fields of characteristics two or three. This paper presents a unified treatment of such algebras over an arbitrary field. Define a quaternion algebra Q over F as in A. A. Albert's *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939, p. 145. Then an alternative, but not associative, algebra A over F is central simple (simple for all scalar extensions) if and only if A is a Cayley-Dickson algebra $A=Q+gQ$, in which multiplication is defined by $(q_1+gq_2)(q_3+gq_4)=(q_1q_3+\delta q_4\cdot q_2S)+g(q_1S\cdot q_4+q_3q_2)$, where $g^2=\delta\neq 0$ in F and S is the involution $x\leftrightarrow xS=t(x)-x$, the principal function of A being $x^2-t(x)\cdot x+n(x)$. Moreover A is a central division algebra if and only if Q is a division algebra and there exist no λ, μ, ρ, σ in F such that $\delta=\lambda^2+\lambda\mu-\beta\mu^2-\gamma\rho^2-\gamma\rho\sigma+\beta\gamma\sigma^2$. (Received March 20, 1942.)

165. M. F. Smiley and W. R. Transue: *Applications of transitivities of betweenness in lattic theory.*

This paper is an expansion of our note ***Metric lattices as singular metric spaces*** (this Bulletin, abstract 48-1-20) which was to appear in this Bulletin. The properties of betweenness studied in the original note are generalized and applied to the characterization of modular lattices and of arbitrary lattices. The proofs of the extensions of our original results are based in part on certain of the results of the paper *Transitivities of betweenness* by Everett Pitcher and M. F. Smiley (forthcoming in the Transactions of this Society). (Received February 27, 1942.)

166. R. M. Thrall: *On the decomposition of modular tensors.*

Let k be any field, and denote by A any element of the full linear group $G(n, k)$ of n-rowed k-matrices. The mapping $A\to\pi^m(A)=A\times\cdots\times A$ (Kronecker direct product with m factors A) is a representation of $G(n, k)$ as linear transformations on the space V_m of all tensors of rank m. For k of characteristic zero the theory of the decomposition of V_m into invariant subspaces is classical. A key tool of the development of this theory is use of the relationship between V_m and the symmetric group of degree m. For k of characteristic p the classical theory is valid only for $m<p$. In the present paper the decomposition of V_m is obtained for $p\leq m<2p$. Further progress on the decomposition problem will likely have to await the development of the theory of modular representations of the symmetric groups of degree $m\geq 2p$; the measure of difficulty here being the power of p which divides $m!$. (Received March 26, 1942.)

167. Bernard Vinograde: *Split rings and their representation theory.*

Suppose a ring R, with minimum condition on left ideals, can be split into a direct sum, $R=R^*+N$, of a semi-simple ring and the radical. This splitting property of R is equivalent to the existence of a certain set of noncommutative fields in R. Let R have a unit, and let V be a commutative group possessing R in its left operator domain and a finite composition series with respect to R. Then V is a direct sum of modules, each over one of the noncommutative fields. Thus V gives rise to a composite representation module for its homomorphism ring. In the case $V=R$ the resulting representation of R displays the essential structure of R^* and N, and may be considered a natural extension of the Wedderburn theorem for simple rings to the class of split rings with unit and finite composition series. (Received March 20, 1942.)

168. R. W. Wagner: *Expressibility relations for bilinear operations.* Preliminary report.

If xy is a multiplication operation for a vector space over a field, an expression like $(a((xb)(cy)))d$ is also a multiplication for the vector space. The operation x^*y is called simply expressible in the operation xy if x^*y is a finite iteration of the original operation. The operation x^*y is called strongly expressible if it is a linear combination of weakly expressible operations. Necessary and sufficient conditions are found for both kinds of expressibility. These conditions involve the enveloping algebra of the algebra formed by the original multiplication. Mutual expressibility, both simple and strong, is an equivalence relation and differs from isotopy. Reducibility is a property common to mutually expressible algebras. Associativity, commutativity, and the existence of nilpotents are properties which need not be preserved. (Received March 18, 1942.)

ANALYSIS

169. Dorothy L. Bernstein and S. M. Ulam: *On the problem of completely additive measure in classes of sets with a general equivalence relation.*

The problem of finding a necessary and sufficient condition for the existence of a finitely additive measure in a class of sets, with the property that equivalent sets have equal measure, has been solved by Tarski (Fundamenta Mathematicae, vol. 31 (1938), pp. 47–66). This paper considers the existence of a completely additive measure in a class K which is a Borel field over a given sequence $\{A_n\}$ of sets and which has the property that $m(X)=m(Y)$ if X is equivalent to Y. Equivalence is defined in the most general sense: Given a division of the sets of K into disjoint classes $A_1, A_2, A_3, \cdots$, sets X and Y are called equivalent if they belong to the same class A_α. It should be noted that even if no requirement for equivalence is made, it is not always possible to find a measure in a Borel field over a given sequence of sets. A necessary and sufficient condition is obtained which involves the characteristic set of real numbers formed from the sequence $\{A_n\}$ and the existence of a homeomorphism of this set with a set of positive outer Lebesgue measure, in which intervals corresponding to equivalent sets have equal outer measure. (Received March 20, 1942.)

170. Lipman Bers: *A property of harmonic functions.*

Suppose that $u(z)=(x+iy)$ is a non-negative harmonic function defined in $|z|<1$. Consider the upper and lower limits, $L(\theta)$ and $l(\theta)$, of $u(z)$ as z goes toward the point 1 along a simple Jordan curve which possesses at 1 a definite direction forming the angle θ, $|\theta|<\pi/2$, with the positive x-direction. (These limits depend only upon θ.) $L(\theta)$ and $-l(\theta)$ are uniformly continuous and convex functions. If $-\pi/2\leqq\theta_1<\theta<\theta_2\leqq\pi/2$, $l(\theta)\leqq\mu L(\theta_1)+(1-\mu)l(\theta_2)\leqq L(\theta)$ and $l(\theta)\leqq\mu l(\theta_1)+(1-\mu)L(\theta_2)\leqq L(\theta)$ where $\mu=(\theta_2-\theta)/(\theta_2-\theta_1)$. If $u(z)$ possesses at 1 a limit in one direction, that is, if $L(\theta_0)=l(\theta_0)$, $L(\theta)$ and $l(\theta)$ are linear functions in the intervals $-\pi/2\leqq\theta\leqq\theta_0$ and $\theta_0\leqq\theta\leqq\pi/2$. If $u(z)$ possesses at 1 limits in two different directions, $u(z)$ possesses at 1 a limit in any direction and $L(\theta)=l(\theta)$ is a linear function. If $u(z)$ is bounded and its Dirichlet integral finite, $L(\theta)$ and $l(\theta)$ are equal to the upper and lower limits of $\tau^{-1}\{\mu[V(\tau)-V(0)]+(1-\mu)[V(0)-V(-\tau)]\}$, $\tau\rightarrow+0$, where $\mu=(\pi+2\theta)/2\pi$ and $V(\tau)$ is the indefinite integral of $U(\tau)=u(e^{i\tau})$. (Received March 28, 1942.)

171. F. G. Dressel: *The fundamental solution of the parabolic equation.* II.

The existence of a fundamental solution in an unbounded region for the parabolic equation with variable coefficients is proved in this paper. In addition, several properties of this fundamental solution are demonstrated. (Received February 5, 1942.)

172. E. D. Hellinger and H. S. Wall: *Contributions to the analytic theory of continued fractions.*

The theory of (1) $1/(b_1+z)-a_1^2/(b_2+z)-a_2^2/(b_3+z)-\cdots$, as developed by Stieltjes, Hamburger, Hellinger, R. Nevanlinna, and others is extended to the case $a_n\neq0$ real and $\Im(b_n)\geqq0$. The nth approximant lies upon a circle $K_n(z)$ contained in $K_{n-1}(z)$, $(\Im(z)>0)$. *Case* I. For $\Im(z)>0$ the circles have only one point in common and (1) converges; the associated J-form which, in contrast with former extensions of this theory, is not hermitian, has a unique resolvent bounded on an "ellipsoid" E. *Case* II. Otherwise, (1) converges for all z (to a meromorphic function of z) if and only if it converges for $z=0$; the J-form has infinitely many resolvents completely continuous on E. The associated power series of (1) represents $f(z)$ asymptotically in $\epsilon\leqq\arg z\leqq\pi-\epsilon$, $\epsilon>0$, if and only if for every n there exists an M_n such that $f(z)$ lies in $K_n(z)$ for z in the sector and $\Im(z)>M_n$. If for $\Im(z)>0$ the value of an analytic function $f(z)$ is in $K_n(z)$ for every n, then $f(z)=\int_{-\infty}^{+\infty}d\phi(u)/(z+u)$, $\phi(u)$ real, bounded, monotone non-decreasing; and (1) represents a function of this form whenever it converges. The moments $\int_{-\infty}^{+\infty}u^n d\phi(u)$ do not in general exist. (Received March 6, 1942.)

173. Fritz John: *The character of solutions of linear partial differential equations.*

Let $L(u)=0$ denote a linear partial differential equation with analytic coefficients for a function $u(x_1,\cdots,x_n)$ (order N or "type" of equation as well as the number n of independent variables are arbitrary). Characteristic directions with respect to L at a point may be defined in the usual manner. Let there be given an analytic k-

parametric family of r-dimensional manifolds $M_{\alpha_1 \cdots \alpha_k}$ in $x_1 \cdots x_n$-space ($r<n$), such that (a) all $M_{\alpha_1 \cdots \alpha_k}$ have the same boundary, and (b) none of their normals is characteristic. Let $d\omega$ be any analytic element of mass on the $M_{\alpha_1 \cdots \alpha_k}$. The main result of this paper states that $\int u\,d\omega$ over $M_{\alpha_1 \cdots \alpha_k}$ will be an analytic function of $\alpha_1, \cdots, \alpha_k$, whenever u is any solution of class C^N of the equation $L(u)=0$. The analyticity of solutions of linear elliptic equations with analytic coefficients is a special case. It also follows that the values of u cannot be prescribed arbitrarily on any analytic manifold in $x_1 \cdots x_n$-space, which contains a sub-manifold M of lower dimension, which has no characteristic normals. (Received March 12, 1942.)

174. M. L. Kales: *Tauberian theorems.*

Let $T_{m,n} \geqq 0, \sum_{n=0}^{\infty} T_{m,n} = 1$, and $\{A_n\}$ be a sequence of real numbers. Under fairly general conditions it is shown that if $\sum_{n=0}^{\infty} T_{m,n}A_n = O(1)$ as $m \rightarrow \infty$, and $A_m - A_n > -\psi(m, n)$ (where the function $\psi(m, n)$ is defined in the statement of the theorem) then $A_n = O(1)$ as $n \rightarrow \infty$. This result is then applied to prove the following theorem: Let $\lim_{x\rightarrow\infty} g''(x) = \alpha (0 < \alpha \leqq \infty)$. Let $T_{m,n} = (1/\sum_{k=0}^{\infty} \exp\{-g(k)+kg'(m)\}) \exp\{-g(n) + ng'(m)\}$. If $\lim_{m\rightarrow\infty} \sum_{n=0}^{\infty} T_{m,n}A_n = A$, and if $A_n - A_{n-1} > -K \exp\{g(n)-g(n-1) - g'(n-1)\}$, then $\lim_{n\rightarrow\infty} A_n = A$. The proof of this theorem for the case where α is infinite follows immediately from the first theorem. When α is finite the problem can be reduced to one where an application of one of Wiener's general Tauberian theorems is possible. (Received March 18, 1942.)

175. Meyer Karlin: *Characterization of the extremals of a variation problem of higher order in the plane.* Preliminary report.

A variation problem of second order in the plane, $\int \phi(x, y, y', y'')dx = \min$, gives rise to a quadruply infinite system of curves for its extremals. E. Kasner (this Bulletin, vol. 13 (1907), pp. 289–292) found their differential equation to be of the form: $y^{\mathrm{IV}} = Ay'''^2 + By''' + C$, where A, B, C involve x, y, y', y''. With the help of a theorem by A. Hirsch (Mathematische Annalen, vol. 49 (1897), pp. 49–72), the author develops a set of necessary and sufficient conditions that A, B, C must satisfy in order that the curves obeying the above differential equation be identifiable with the totality of extremals connected with a variation problem of second order in the plane. When the conditions are fulfilled, the solution is unique up to an additive arbitrary total derivative with respect to x of a function of x, y, y'. Examples of extremal and nonextremal families of the above quadratic type are given. The paper also contains some necessary conditions for, as well as geometric characterizations of, the ∞^6 and ∞^{2n} curves constituting, respectively, the totality of extremals of variation problems of the third and nth orders in the plane. (Received March 6, 1942.)

176. A. N. Lowan, H. E. Salzer and Abraham Hillman: *Coefficients of differences in the expansion of derivatives in terms of advancing differences.*

The exact values of the coefficients $A_{n,r}$ of the rth differences in Markoff's formula for the nth derivative in terms of advancing differences were computed for $n=1, 2, \cdots, 20$ and $r=n, n+1, n+2, \cdots, n+20$ by the Mathematical Tables Project, Work Projects Administration, New York City. The coefficients are defined by $A_{n,r} = (n/r(r-n)!)\,B_{r-n}^{n}$ where B_{r-n}^{r} is the $(r-n)$th Bernoulli number of the rth order. These coefficients are useful in computing the derivatives of functions from tabulated values. In particular, they constitute a powerful tool for calculating values of functions for complex values of the argument when the function is known to be analytic in the neighborhood of the real axis and when the function has been tabulated for real arguments with sufficient accuracy and to a sufficiently small interval. Coefficients were computed with the aid of the recurrence formula $A_{n,r+1}=(nA_{n-1,r}-rA_{n,r})/(r+1)$ and further checked against the values obtained from the factorial polynomials. The usefulness of the table of coefficients here discussed will be enhanced by a table of $(x+iy)^n$ the preparation of which is now contemplated. (Received February 20, 1942.)

177. G. T. Miller and H. K. Hughes: *Analytic continuation of functions defined by factorial series.*

Given a function of s defined by a factorial series of the form: $\Omega(s)=\sum_{m=0}^{\infty}(-1)^{m(k+1)}g(m)/s\cdot(s+1)\cdots(s+m)$, where $g(w)$ is a single-valued analytic function of the complex variable $w=x+iy$, for $x\geqq -1/2$, and $g(w)$ is such that for all such x and for $|y|$ sufficiently large, $|g(x+iy)/g(x)|<M\exp(\pi(k+(1/2))-\epsilon)|y|$ then an analytic continuation of the function $\Omega(s)$ is given by the sum of two infinite integrals. This result may be considered to be an extension of a result obtained by H. K. Hughes (American Journal of Mathematics, vol. 53 (1931), pp. 771–775). The method of proof is based on the calculus of residues and bears a close analogy to the work of W. B. Ford (*Asymptotic Developments*, University of Michigan Studies, 1936) and to that of C. V. Newsom (American Journal of Mathematics, vol. 60 (1938), pp. 561–572). In case the function $g(w)$ has singularities, certain loop integrals are added to the expression already obtained. (Received March 10, 1942.)

178. Marston Morse and G. M. Ewing: *The non-regular case in the variational theory in large.*

The original treatment both of the absolute minimum theory and of the more general variational theory in the large made use of broken extremals. In the non-regular case these lost their most useful characteristics, making it necessary to treat the variational theory in the large in a new way. This has been done in three papers, the first on functional topology by Morse which makes the necessary alterations in the underlying topological theory, and in two papers by Morse and Ewing dealing with the local analytical aspects of the problem such as upper reducibility and the generalized Euler or homotopy theorem. The classical minimum theory including the Lindeberg theorem is first reviewed and refined, making a more systematic use of the hypothesis of convexity of the integrand than previously. In the second paper the more difficult question of upper-reducibility is taken up. The final theorems are on a general non-regular problem on a Riemannian manifold. (Received March 16, 1942.)

179. Isaac Opatowski: *Confluent hypergeometric functions and Markhoff chains.*

New simple relations between confluent hypergeometric functions are obtained by means of the Laplace transformation. The study is based on the function $[(n+m-1)!]^{-1}t^{n+m-1}e^{-t}\,{}_1F_1(m,\ n+m,\ t)=\gamma_m(t,\ n)$, which has certain advantages of simplicity with respect to the Erdélyi's function $N_{k,m}$ (Mathematische Zeitschrift, vol. 42 (1936), pp. 125–143) for example, $\gamma_m(t,\ n)^*\gamma_p(t,\ q)=\gamma_{m+p}(t,\ n+q)$, where * means the convolution. Then γ_1 is the Pearson incomplete gamma function and $\gamma_{m+1}(t,\ n)$ is the m times repeated integral of $\gamma_1(t,\ n)$ between $t=0$ and $t=t$ (the integration being understood for an arbitrary m as the general derivative of Liouville-Riemann). The function $\gamma_m(t,\ n)$ is used to express the probability function $P_n(t)$

and calculate the moments of the Markoff chain (a generalization of the Poisson distribution): $dP_i/dt = k_iP_{i-1} - k_{i+1}P_i$, where $k_1 = 0$ and the positive constants k_i's have for $i = 1, 2, 3, \cdots$ two different values only. (Received March 20, 1942.)

180. E. J. Pinney: *Calculus of variations in abstract spaces and related topics*. I.

The calculus of variations problem considered is that of minimizing the integral $I(y) = \int f(x, y(x), y_u(x))dV(x)$, where $y(x)$ is a function on an n-dimensional arithmetic space R to a linear topological space T which is required to contain a convex open set properly, $y_u(x)$ denotes the set of linear topological derivatives, $f(x, h, h_u)$ is a functional on $R, T, T, \cdots, T$, and the integral is an n-dimensional Lebesgue integral. The development necessitates the proof of certain theorems about the integral, and this first paper is devoted to that. The familar expression for the transformation of an integral under a mapping of the set on which it is defined to another (measurable) set is established. Green's theorem is established. A topological differential equivalent to a Fréchet differential when T is a Banach space is studied. The theorem of the mean is established for certain functionals, and continuity under the integral sign is considered. (Received March 14, 1942.)

181. E. J. Pinney: *Calculus of variations in abstract spaces and related topics*. II.

In this continuation of the first paper under this title, the theory there developed is applied to the calculus of variations problem mentioned there. The function $f(x, h, h_u)$ is defined, and an allowable class of functions $y(x)$ is defined. The abstract analogues of the necessary conditions of Euler, Weierstrass, Legendre, and Jacobi are established, and in addition the analogue of the necessary condition of Carathéodory is established. This condition, which reduces to that of Weierstrass in the case of the single integral, doesn't seem to be very well known as yet. Two "corner" conditions are established, and the condition of transversality on the boundaries is established. An investigation of sufficiency conditions is projected for the future. (Received March 14, 1942.)

182. Harry Pollard: *An inversion formula for the Stieltjes transform.*

This treatment of the Stieltjes transform (1) $f(x) = \int_0^\infty (x+t)^{-1}d\alpha(t)$ is intended to parallel a recent study of the Laplace integral made by Boas and Widder. The underlying idea, suggested by the work of Paley and Wiener, is to consider the iterate of the transform and to apply known results to this. In this way a new inversion operator can be derived for the transform (1). In terms of this operator necessary and sufficient conditions are obtained for the representation of a function as a Stieltjes transform with $\alpha(t)$ of general or of preassigned type. All these results depend on a knowledge of $f(x)$ on the real axis. (Received March 3, 1942.)

183. Harry Pollard: *On subseries of a convergent series.*

In a recent paper J. D. Hill (this Bulletin, vol. 48 (1942), p. 103) has defined the mean-value m of all the subseries of a given absolutely convergent series $\sum u_n$. In this case he has shown that $m = s/2$, where s is the sum of the series. By use of well known properties of the Rademacher functions this result is extended to the case of all convergent series for which $\sum |a_n|^2 < \infty$. If $\sum |a_n|^2 = \infty$, m fails to be defined, so that the extension is best possible. (Received March 4, 1942.)

184. George Pólya: *On the combinatory analysis of classifications and permutations.*

Let K_n denote the number of classifications of a set of n objects, $K_n = 1, 2, 5, 15, \cdots$ for $n = 1, 2, 3, 4, \cdots$ and let $K_0 = 1$. These K_n and connected combinatorial numbers occur in many questions, for example, K_n is the coefficient of $x^n e/n!$ in the development of $\exp(\exp(x))$. Let G be a permutation group of order g and the degree n, let $\mu_0 = 1$, and μ_k be the arithmetic mean of the kth powers of the traces of its g permutations. The μ_k are integers, $\mu_1 \geqq K_1$, $\mu_2 \geqq K_2, \cdots, \mu_n \geqq K_n$. If the equality is valid in all n inequalities, G is the symmetric group. Otherwise there is just one p, $0 \leqq p < n$, such that $\mu_p = K_p$, $\mu_{p+1} < K_{p+1}$ and G is p-fold transitive. (Received March 17, 1942.)

185. H. A. Rademacher: *On the Bloch-Landau constant.*

Landau in 1929 in a paper on the Bloch constant $\mathfrak{B}$ introduced another constant $\mathfrak{L}$ through the following definition: There exists a constant $\mathfrak{L}$ such that every function $w = f(z)$ which is regular in $|z| < 1$ with $f'(0) = 1$ covers completely, for any $\epsilon > 0$, a circle of radius $\mathfrak{L} - \epsilon$ in the w-plane, whereas to every $\epsilon > 0$ there exists a function of the same specifications whose values do not completely fill any circle of radius $\mathfrak{L} + \epsilon$. Landau gave the upper estimate $\mathfrak{L} \leqq 2^{2/3}\pi\Gamma^3(1/3)\Gamma^{-4}(1/4) = 0.55488 \cdots$. In this paper we find $\mathfrak{L} \leqq \Gamma(5/6)\Gamma(1/3)\Gamma(1/6)^{-1} = 0.54325 \cdots$. This upper bound is found by a suitable example obtained through the conformal mapping of a zero-angled circular triangle on a straight equilateral triangle. The calculations can be carried out with the help of a formula given by Ahlfors and Grunsky in 1937. (Received February 27, 1942.)

186. T. Radó: *On a problem of Geöcze.*

The Lebesgue area $A(S)$ of a continuous surface S is defined as the greatest lower bound of $\liminf E(\pi_n)$ for all sequences of polyhedra π_n converging to S in the Fréchet sense, where $E(\pi_n)$ stands for the elementary area of π_n. Denote by $A^*(S)$ the quantity obtained by requiring that the polyhedra π_n be inscribed in S. The problem of determining whether $A^*(S) = A(S)$ was first attacked by Geöcze (Mathematische und Naturwissenschaftliche Berichte aus Ungarn, vol. 26 (1910), pp. 1–88). The purpose of this paper is to extend the results of Geöcze by proving the following theorem. Let S be given by $z = f(x, y)$, where $f(x, y)$ is continuous in the unit square and is absolutely continuous in x for almost every y, or alternatively, absolutely continuous in y for almost every x. Then $A^*(S) = A(S)$. (Received February 28, 1942.)

187. P. V. Reichelderfer: *On bounded variation and absolute continuity for parametric representations of continuous surfaces.*

If S be a continuous surface in $x^1x^2x^3$-space given in parametric representation, denote by ${}^1T, {}^2T, {}^3T$ the corresponding representations for the projections of S upon the x^2x^3-, x^3x^1-, x^1x^2-planes, respectively. Definitions for essential variation and essential absolute continuity are given for ${}^1T, {}^2T, {}^3T$, based upon ideas developed by T. Radó and P. Reichelderfer (Transactions of this Society, vol. 49 (1941), pp. 258–307). An essential area $eA(S)$ is defined for S; it is a lower semi-continuous functional of S. A necessary and sufficient condition that $eA(S)$ be finite is that each of ${}^1T, {}^2T, {}^3T$ be of bounded essential variation. If $eA(S)$ is finite, then the generalized Jacobians

${}^1J$, ${}^2J$, ${}^3J$ for ${}^1T$, ${}^2T$, ${}^3T$ exist almost everywhere and are summable, and $eA(S) \geq \iint [{}^1J^2 + {}^2J^2 + {}^3J^2]^{1/2}$. A necessary and sufficient condition that the sign of equality hold here is that each of ${}^1T$, ${}^2T$, ${}^3T$ be essentially absolutely continuous. If S has a non-parametric representation, then $eA(S)$ equals the Lebesgue area of S, and this theory becomes equivalent to that developed by Tonelli for this special case. (Received March 18, 1942.)

188. W. H. Roever: *Comment on the derivation of the law of perfect gases.*

In this paper the author shows that, to derive from the laws of Boyle and Charles the law of perfect gases, one encounters the problem of showing that the two two-parameter families of curves defined by these laws lie on a one-parameter family of surfaces. Also the derivation of the expression for the entropy of a perfect gas is discussed. (Received March 6, 1942.)

189. H. M. Schwartz: *On some general series expansions.* Preliminary report.

Using results obtained in two earlier papers (*Sequences of Stieltjes integrals*, this Bulletin, vol. 47 (1941), pp. 947–955, and abstract 48-1-55), the author studies some questions of convergence and summability of expansions in series of functions which form an orthogonal set with respect to a function of bounded variation, by methods similar to those employed by H. Lebesgue in his paper *Sur les intégrales singulières* (Annales de Toulouse, (3), vol. 1 (1909)). (Received March 20, 1942.)

190. M. F. Smiley: *A comparison of algebraic, metric, and lattice betweenness.*

It is shown that metric betweenness (see L. M. Blumenthal, *Distance Geometry*) and lattice betweenness (see Everett Pitcher and M. F. Smiley, *Transitivities of betweenness*, forthcoming in the Transactions of this Society) coincide in a complete normed real vector lattice if and only if this lattice is equivalent to an (L)-space (S. Kakutani, *Concrete representation of abstract* (L)*-spaces and the mean ergodic theorem*, Annals of Mathematics, (2), vol 42 (1941), pp. 523–537). The ranges of coincidence of algebraic and metric and of algebraic and lattice betweenness are also determined. Metric betweenness in the Banach space $C[0, 1]$ is investigated. (Received March 12, 1942.)

191. M. F. Smiley: *A remark on S. Kakutani's characterization of* (L)*-spaces.*

The important condition IX of Kakutani (*Concrete representation of abstract* (L)*-spaces and the mean ergodic theorem*, Annals of Mathematics, (2), vol. 42 (1941), pp. 523–537) may be replaced by the equivalent condition (1) $\|x-y\| = \|x \vee y - x \wedge y\|$. It is shown that this permits an economy of assumptions in §§3–5 of the paper cited. The proofs are based on the fact that the condition (1) is closely related to a fundamental property of modular functionals (G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, 1940, p 40). (Received March 12, 1942.)

192. W. S. Snyder: *Non-parametric surfaces of bounded variation.*

The paper shows that if $f(x, y)$ is of bounded variation in the sense of Hardy and Krause (Hobson, *Functions of a Real Variable*, vol. 1, 3rd edition, 1927, §254), then the Lebesgue area of the surface $z=f(x, y)$ is equal to the Burkill integral of a certain rectangle function analogous to the rectangle function of de Geöcze (Saks, *Theory of the Integral*, p. 171). The method shows that the area of the surface may be calculated by a single passage to the limit from the areas of certain polyhedra inscribed in the surface. (Received March 20, 1942.)

193. Otto Szász: *On sequences of polynomials and the distribution of their zeros.*

The author considers sequences of polynomials $P_n(z)=c_{n0}+c_{n1}z+c_{n2}z^2+\cdots$ with increasing degrees $m_1<m_2<\cdots$, respectively, where the roots of P_n lie in a half-plane containing the origin on its boundary line, the direction of which may depend on n. If the first three coefficients are uniformly bounded, and the c_{n0} are bounded away from 0, then the sequence is uniformly bounded in any finite domain. If, in addition $\lim c_{n\nu}$ exists as $n\rightarrow\infty$ for each ν, then the sequence converges uniformly in any finite domain, thus representing an entire function, the convergence exponent and order of which are at most two. In particular if the sequence is a subsequence of partial sums of a formal power series, then this series represents an entire function of the described type. Some previous results of G. Pólya are thus generalized. (Received March 6, 1942.)

194. S. M. Ulam and D. H. Hyers: *Approximate isometries of the space of continuous functions.*

A transformation of a metric space E into itself will be called an ϵ-isometry if it changes distances by an amount less than ϵ (see abstract 47-9-427). Properties of such transformations are studied in the case where E is the space of continuous functions. (Received March 24, 1942.)

195. F. A. Valentine: *On the extension of a vector function so as to preserve a Lipschitz condition.*

Let $f(x)$ be a vector function mapping a set S in the euclidean plane into a set S' in the euclidean plane. Moreover $f(x)$ satisfies a Lipschitz condition of the form $|f(x_1)-f(x_2)|\leq K|x_1-x_2|$ for all pairs x_1 and x_2 in S, where $|x-y|$ is the euclidean distance between the points x and y. If T is any set containing S, then $f(x)$ can be extended to T so as to preserve the Lipschitz condition. The extension $[f(x), x\in T]$ can be defined so as to be contained in any prescribed bounded, closed, convex set containing $S'=[f(x), x\in S]$. This extension is a consequence of the following: *Consider a set of circles in the plane, such that there is a point in common to all of the circles. Move these circles to new positions in the plane, subject to the condition that the distance between any pair of centers is not increased. Then all of the circles in the new positions will still have a point in common.* After proving this result for three circles, one can prove it for an arbitrary set by using a theorem of Helly for families of bounded, closed, convex sets. (Received March 6, 1942.)

196. J. L. Walsh: *On the overconvergence, degree of convergence, and zeros of sequences of analytic functions.*

The entire theory of overconvergence as developed by Porter, Jentzsch, Ostrowski, Bourion, and others is here generalized and unified by the concept of exact harmonic majorant of a sequence of analytic functions. If the function $V(z)$ is harmonic in a region R of the z-plane, if the functions $F_n(z)$ are locally single-valued and analytic in R except for branch points, and if $|F_n(z)|$ is single-valued in R, then $V(z)$ is said to be an exact harmonic majorant of the sequence $F_n(z)$ in R provided one has $\limsup_{n\to\infty}[\max |F_n(z)|, z \text{ on } Q] = [\max e^{V(z)}, z \text{ on } Q]$ for every continuum Q (not a single point) in R. Applications of this concept involve degree of convergence and properties of the zeros of functions, and include maximal sequences of polynomials and of other rational functions, and many other sequences of analytic functions. (Received March 16, 1942.)

197. M. S. Webster: *A convergence theorem for certain Lagrange interpolation polynomials.*

A convergence theorem for a sequence of Lagrange interpolation polynomials based on the zeros of a sequence of certain Jacobi polynomials is proved. The method and result are similar to a theorem of Grunwald (this Bulletin, vol. 47, (1941), pp. 271–275). (Received March 19, 1942.)

198. Hermann Weyl: *Solution of the simplest boundary-layer problems in hydrodynamics.*

For some simple configurations the hydrodynamic boundary-layer problem can be reduced to a non-linear ordinary differential equation of third order involving a parameter λ. For $\lambda = 0$ and $1/2$, solution may be obtained by a rapidly converging process of alternating successive approximations. The general case is attacked by a suitable adaptation of the method of fixed points of transformations in functional spaces. (Received February 28, 1942.)

199. František Wolf: *On the limits of harmonic and analytic functions along radii which form a set of positive measure.*

If $u(r, \theta) = \log|f(re^{i\theta})|$ and $f(z)$ is analytic in the unit circle $r < 1$, $u(r, \theta) \leq M/(1-r)^n$ for any M and n, and $\limsup_{r\to 1} u(r, \theta) \leq 0$ for $\theta \subset E$, $|E| > 0$, then $\limsup u(r, \theta) \leq 0$ in any sector at almost all points of E. Hence if $u(r, \theta)$ is harmonic and satisfies the conditions of the theorem, then $u(r, \theta)$ and its conjugate $v(r, \theta)$ have finite limits in any sector at almost all points of E. This follows from above by the well known results of Privaloff (Recueil Mathématique de Moscou, vol. 91 (1923), p. 232) and Fatou. Another corollary is: If $f(z)$ is analytic in $|z| < 1$, $|f(z)| \leq \exp [M/(1-r)^n]$, and $\lim_{r\to 1} f(re^{i\theta}) = 0$ for $\theta \subset E$, $|E| > 0$, then $f(z) \equiv 0$. (Received March 20, 1942.)

Applied Mathematics

200. Stefan Bergman: *Determination of pressure in the two-dimensional flow of an incompressible perfect fluid.*

The author considers a flow of an incompressible perfect fluid around a wing profile. The pressure distribution is determined by the function $W(z)$ which maps the exterior $\mathfrak{E}$ of the wing profile onto the exterior $\mathfrak{R}$ of a circle. Of particular interest is the evaluation of $W(z)$ on the boundary in the neighborhood of the vertex O. Let boundary in the neighborhood of O be formed by two circular arcs CO and BO which make an angle α at O. Let O and D be the intersections of the circles on which the arcs CO and BO lie. Suppose that arcs CD and BD lie inside of the profile (Hypothesis

H). Then a linear transformation followed by one of the form $\zeta^* = \zeta^{\pi/(2\pi-\alpha)}$ will map $\mathfrak{G}$ into a domain $\mathfrak{T}$ which lies in the upper half-plane, and the arcs OA and OC will become segments O_1A_1 and O_1C_1 of the real axis. Using orthogonal functions (see Bergman, Publication of Brown University, 1941, p. 118), the author determines the function $w(\zeta^*)$ mapping $\mathfrak{T}+\mathfrak{T}^*$ into the unit circle. $\mathfrak{T}^*$ is the domain obtained from $\mathfrak{T}$ by reflection on the real axis. Then $(1/2)[w(\zeta^*)+(w(\zeta^*))^{-1}]^{-1}$ maps $\mathfrak{T}$ into $\mathfrak{R}$. If H is not fulfilled, follow the linear transformation by one which maps the triangle $O_1A_1C_1$ of the ζ-plane into the upper half-plane. (Received February 21, 1942.)

201. Stefan Bergman: *Three-dimensional flow of a perfect incompressible fluid and its singularities.*

The author considers vectors $\mathfrak{S}(\mathfrak{X})$, $\mathfrak{X}=(x_1, x_2, x_3)$. The components of $\mathfrak{S}$ are harmonic functions with an algebraic singularity described in the author's papers (Mathematische Zeitschrift, vol. 24 (1925–1926), p. 655 and Mathematische Annalen, vol. 99 (1928), p. 645). At every point outside of the singularity, curl $\mathfrak{S}=0$, div $\mathfrak{S}=0$ holds. Let $\theta(\mathfrak{X})$ be a potential function such that grad $\theta(\mathfrak{X})=\mathfrak{S}(\mathfrak{X})$. The author defines a curve $l^1(\zeta)$ in the $x_1x_2x_3$-space for every $\mathfrak{S}(\mathfrak{X})$ and every complex number ζ. Let q^1 be a closed curve in the ζ-plane. $L^2(q^1)$ is the logical sum of $l^1(\zeta)$, $\zeta \epsilon q^1$, so that to every point $\mathfrak{X}$ of $L^2(q^1)$ corresponds a $\zeta=\zeta(\mathfrak{X})$. [The singularity line of $\mathfrak{S}(\mathfrak{X})$ lies on $L^2(q^1)$.] Finally the author defines for $\mathfrak{S}(\mathfrak{X})$ the residue functions $R_\nu(\zeta)$. Let $\mathfrak{i}^1$ be a closed curve in the schlicht $x_1x_2x_3$-space. $\mathfrak{i}^1$ is open in the multiply-covered space M^3 in which $\mathfrak{S}(\mathfrak{X})$ is univalent. Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be both end points of $\mathfrak{i}^1$ in M^3. Then $\int_{\mathfrak{i}^1}\mathfrak{S}(\mathfrak{X})\cdot d\mathfrak{X}+\sum[\theta_k(\mathfrak{E}_2)-\theta_k(\mathfrak{X}_k^*)]+\sum[\theta_n(\mathfrak{X}_n^*)-\theta_n(\mathfrak{E}_1)]=\sum\int_{\zeta(\mathfrak{X}_n^*)}^{\zeta(\mathfrak{X}_{n+1}^*)}R_n(\zeta)d\zeta$. Here $\theta_m(\mathfrak{X})$ are functions connected with $\theta(\mathfrak{X})$, and $\mathfrak{X}_n^*$ intersections of $\mathfrak{i}^1$ with $L^2(q^1)$. An analogous relation holds for a vector $\mathfrak{H}$, $\mathfrak{H}=\mathfrak{H}_1+\sum_{\nu=1}^{N}\mathfrak{S}_\nu$, where $\mathfrak{H}_1$ is a regular vector and $\mathfrak{S}_\nu$ the above described vectors. (Received March 6, 1942.)

202. Hilda P. Geiringer: *On the numerical solution of linear problems by group iteration.*

The so called Ph. Seidel iteration method for solving systems of linear equations converges towards their solution for all systems originated from a minimum problem; for example, in the following groups of problems: (1) statistical (least square) problems, (2) problems of mechanics, for example, statically determined or indeterminate frameworks, (3) elliptic boundary value problems. For the second group the method reduces the solution of an n-fold indeterminate system to the successive solution of simply indeterminate systems. (a) Each of these simply indeterminate systems can be solved in whatever way seems appropriate, not necessarily through the respective Maxwell equation. (b) Introduce "group-iteration" (see H. Geiringer, *Zur Praxis der Lösung linearer Gleichungen in der Statik*, Zeitschrift fur Angewandte Mathematik und Mechanik, 1928, and R. von Mises and H. Geiringer, *Praktische Verfahren der Gleichungsauflösung*, ibid., 1929); that is, instead of reducing to simply indeterminate problems, use as intermediate steps r-fold indeterminate systems whose solutions may be found by any suitable combination of equations. Thus liberty of arrangement results and the convergence is accelerated.—It seems that this procedure contains the essentials of R. V. Southwell's "relaxation method" successfully applied by him since 1933 to many mechanical problems and recently presented in a comprehensive work. The group iteration method combined with the use of computing machines to solve the small groups of equations leads to very good results. (Received February 9, 1942.)

203. A. M. Gelbart: *Bounds for pressure in a two-dimensional flow of an incompressible perfect fluid.*

It is known that the problem of the pressure distribution along a wing in the case of a two-dimensional flow of an incompressible perfect fluid can be reduced to the problem of determining the function which maps the exterior domain into the exterior of a circle. This paper deals with some properties of the function in the neighborhood of the angle of the wing. Bergman treats this problem by employing orthogonal functions and certain special transformations. (See abstract 48-5-200.) Using this approach, some inequalities previously obtained by the author for the coefficients of the mapping function, and a minimum integral, inequalities for the velocity in the neighborhood of the angle are obtained which depend only upon a suitable domain in which the boundary of the profile lies. (Received March 7, 1942.)

204. W. A. Mersman: *Heat conduction in a finite composite solid.*

The problem of one-dimensional heat conduction in a composite wall has been solved by Churchill (Duke Mathematical Journal, vol. 2 (1936), pp. 405–414, and Mathematische Annalen, vol. 115 (1938), pp. 720–739), the solution being presented in the form of a series which converges rapidly for large time values. The present paper furnishes a transformation of Churchill's solution in the form of a series which converges rapidly for small time values. This is done by expanding the Laplace transform of the solution as a geometric series and inverting term-by-term, instead of applying the Mittag-Leffler theorem and the inversion theorems of Doetsch and Churchill. (Received February 19, 1942.)

205. W. A. Mersman: *Heat conduction in an infinite composite solid with an interface resistance.*

The problem of one-dimensional heat conduction in a doubly infinite composite solid with an interface resistance is solved by the Laplace transformation method. The interface conditions are: (1) the product of conductivity and temperature gradient is continuous across the interface; (2) the temperature discontinuity across the interface is proportional to the product in (1) above, the factor of proportionality being a constant. (Received February 9, 1942.)

GEOMETRY

206. Herbert Busemann: *Spaces with convex spheres.*

In a metric space a continuous curve which is locally isometric with a euclidean straight line will be called a geodesic. The paper considers a finitely compact metric space in which there is exactly one geodesic through any two different points. With an obvious definition of a tangent of a sphere a sphere is called convex if no tangent of the sphere contains interior points of that sphere. Assume that all spheres are convex and that the space has dimension greater than or equal to 3. The space is congruent to an elliptic space, as soon as at least one geodesic is closed. If all geodesics are open, convexity of the spheres as defined above coincides with the usual idea that a segment whose end points are in a sphere lies completely in the sphere, but the space is not necessarily flat. However if the parallel axiom (properly formulated) holds, the space is flat and its metric is Minkowskian. (Received March 20, 1942.)

207. Aaron Fialkow: *Conformal differential geometry of a subspace.*

In previous papers and abstracts, the author has shown that a conformal differentiation process may be defined with respect to an arbitrary subspace V_n in any Riemann space V_m $(0<n<m;\ m>2)$. By means of this tool, the conformal fundamental forms of V_n, which completely characterize its conformal geometry, were discovered as well as the fundamental differential equations which the coefficients of these forms satisfy. The conformal differential geometry of V_n based upon these forms and upon the conformal differentiation process is now developed. As in classical differential geometry, it includes the conformal analogues of normal and geodesic curvature, conjugate and asymptotic directions, developables and spaces of constant curvature, and so on. Classical theorems such as those of Meusnier, Euler and others hold in this geometry. However, some completely new geometric theorems are derived which are based on tensors having no analogue in the classical theory. Many results of a special character (particularly if $n=2$) are obtained. Some important special conformal coordinate systems are indicated. (Received March 5, 1942.)

208. L. C. Hutchinson: *On the linear line complex in n-space.*

This paper starts from certain contradictions in the literature and from the careful distinction between two closely related, but not identical concepts, a bivector and its associated $(n-2)$-complex, and develops the general theory employing tensor methods. (Received March 7, 1942.)

209. A. D. Michal: *A theory of fluctuations in Riemannian spaces.* Preliminary report.

This paper deals with parallel displacement and differential invariants of multiple point correlation tensor fields in n-dimensional Riemannian geometry. (Received March 14, 1942.)

210. Don Mittleman: *Spin in newtonian mechanics.*

Newtonian mechanics has not considered the possibility of attributing spin properties to point particles moving in a general positional field of force. However, this may be done and the appropriate equations are obtained by considering the limiting form of the equations of motion of a rigid body of finite dimensions as the maximum diameter of the body approaches zero. The result is dependent on the limit of the ratio of the principal moments of inertia, but to fix the limit it is assumed that the ratio of the principal axes of the ellipsoid of inertia approaches one. For a particle, the trajectory is no longer a curve but a series of elements. The geometry of the trajectorial series is investigated. The program is repeated for infinitesimal rods. (Received February 2, 1942.)

211. T. W. Tomlinson: *Geometry of linear fractional polygenic functions.*

Linear fractional functions of $z=x+iy$ and $\bar{z}=x-iy$ are studied in detail. Any such correspondence T can be factored into a Moebius transformation followed by a projectivity. In general T possesses one point f_1 of direct conformality and one point f_2 of reverse conformality. The Kasner circles corresponding to the points of the perpendicular bisector of f_1f_2 pass through the origin and their centers are on a circle through the origin. (Received March 20, 1942.)

212. C. B. Tompkins: *Local imbedding of Riemannian spaces.*

The paper gives a proof of Janet's theorem on local imbedding of an n-dimensional Riemannian space in euclidean space of $n(n+1)/2$ dimensions. The proof is by induction on the number of dimensions, and the induction is made possible by strengthening the theorem slightly to state that the Riemannian space can be imbedded so that its tangent plane and the vectors obtained by differentiating the imbedding functions twice with respect to a set of $(n-1)$ of the n parameters of the Riemannian space together span the euclidean space. The proof involves an existence theorem in differential equations, a warping process somewhat analogous to the process of rolling a plane into a cylinder, and an algebraic lemma which is used to show that the warping process will furnish the independence of vectors required in the strengthened theorem of the inductive proof. (Received March 20, 1942.)

213. B. J. Topel and P. M. Pepper: *Imbedding theorems under weakened hypotheses.*

The congruent imbedding of a semi-metric space into E_n, the euclidean n-space, as implied by the imbeddability of as few as possible of its $(n+2)$-tuples is considered. For S, consisting of $n+4$ or more points, to be imbeddable into E_n, but not E_{n-1}, it is necessary and sufficient that S contain a certain nucleus S' and that all $(n+2)$-tuples containing at least n points of S' be imbeddable. Sufficient conditions are derived for the imbeddability of S when an upper bound is placed on the number of non-mapping $(n+2)$-tuples with no restriction on their distribution in S. These theorems sharpen Menger's quasi-congruence theorem. Necessary and sufficient conditions are determined for the imbeddability of a semi-metric space into E_n. Similar theorems are considered for imbeddability into an element of any congruence system. The structure of non-mapping sets with the minimum number of non-mapping $(n+2)$-tuples is studied. (Received March 23, 1942.)

214. J. E. Wilkins: *A characterization of the quadric of Wilczynski.*

The quadric of Wilczynski at a point P of a non-ruled surface S in projective three-dimensional space is characterized as the unique quadric having second-order contact with S at P (therefore intersecting S in a curve with a triple point at P), the three triple-point tangents being the tangents of Darboux, and the three triple-point osculating planes having a line in common. This line is found to be the canonical line of the first kind with $k=5/12$. (Received March 12, 1942)

Logic and Foundations

215. S. C. Kleene: *On the forms of the predicates in the theory of constructive ordinals.* Preliminary report.

In the system S_3 of notation for ordinal numbers (Journal of Symbolic Logic, vol. 3 (1938), p. 155), the class O of the natural numbers which represent ordinals, and the partial ordering relation $<_O$ between such numbers, were defined by a transfinite induction. It is now shown that the predicates $a \in O$ and $a <_O b$ are expressible explicitly in the respective forms $(x)(Ey)R(a, x, y)$ and $(x)(Ey)S(a, b, x, y)$ where R and S are primitive recursive predicates. The result can be used to exhibit the incompleteness of ordinal logics by a method presented previously. (See abstract 46-11-464. Erratum: for "for." read "for all.") The proof illustrates a technique to which recourse may be had generally in attempts to reduce inductive definitions to explicit definitions in terms of recursive predicates and quantifiers. (Received March 3, 1942.)

Statistics and Probability

216. J. H. Bushey: *The distribution function of the mean under the type α hypothesis.*

An orthogonal expansion (type α series) with the Pearson type III function as weight function is obtained in a form suitable to utilize the tables of Salvosa in representing population frequencies. The expansion differs in certain respects from that of Romanovsky. The distribution function of the mean is obtained for samples of n drawn at random from a population represented by the type α series. In special cases this distribution function reduces to that obtained for the Charlier type A series by Baker and to that obtained by Church for the type III function. (Received March 6, 1942.)

217. J. H. Bushey: *The distribution function of the sample total under the type β hypothesis.*

The orthogonal polynomials $\phi_n(x)$ are defined by the weight function $p(x)=C_{s,x}p^x(1-p)^{s-x}$, $(x=0, 1, 2, \cdots, s)$ and the orthogonal relation $\sum_{x=0}^{s}p(x)\cdot\phi_m(x)\phi_n(x)=0$, $m\neq n$, or $=1$, $m=n$. The orthogonal expansion (type β series) $f(x)=\sum_{i=0}^{k}A_i p(x)\phi_i(x)$ may be used as a statistical hypothesis. The Charlier type A series is a special case of both the type β and the type α series (the type α series is reported in another abstract). Another special case of the type β series is the Charlier type B series with the Poisson weight function $p(x)=(e^{-sp}(sp)^x)/x!$. The distribution function for the total $z=n\bar{x}$ for samples of n drawn at random from a population represented by a type β series is obtained. This result includes, as special cases, the distribution function of z for the Charlier type B series and that obtained by Baker for the Charlier type A series. (Received March 6, 1942.)

218. J. H. Bushey: *The products of certain discrete and continuous orthogonal polynomials.*

The discrete orthogonal polynomials $\phi_n(x)$, $(x=0, 1, 2, \cdots, s)$, are defined by the weight function $p(x)=C_{s,x}\cdot p^x(1-p)^{s-x}$ and the orthogonal relation $\sum_{x=0}^{s}p(x)\cdot\phi_m(x)\phi_n(x)=0$, $m\neq n$, or $=1$, $m=n$. The polynomials $\phi_n(x)$ are closely related to the continuous polynomials of Jacobi, Hermite, and Laguerre and have applications in statistics. The product $\phi_m(x)\ \phi_n(x)$ is developed in terms of the polynomials $\phi_k(x)$ for $p=q=1/2$ (symmetric polynomials). This development permits the evaluation of the sum $\sum_{x=0}^{s}p(x)\phi_m(x)\phi_n(x)\phi_r(x)$. The corresponding results of Feldheim for Hermite polynomials follow as special cases. (Received March 6, 1942.)

Topology

219. E. G. Begle (National Research Fellow): *Intersections of absolute retracts.*

Aronszajn and Borsuk have shown (Fundamenta Mathematicae, vol. 18 (1932) pp. 193–197) that if A and B are compact metric spaces such that their intersection, $A\cap B$, is an absolute retract, then their sum, $A\cup B$, is an absolute retract if and only if A and B are. They left open the following question: Is $A\cap B$ an absolute retract if A, B and $A\cup B$ are? In this paper an example is presented which answers this question in the negative. On the other hand, it is shown that if A, B and $A\cup B$ are absolute homology retracts, then so is $A\cup B$. (Received March 19, 1942.)

220. M. M. Day: *Compactness of a space with monotone closure.*

A closure c in a set X associates with each $X'\subset X$ a set $cX'\subset X$; c is monotone if $X'\subset X''$ implies $cX'\subset cX''$; c is additive if $c(X'+X'')=cX'+cX''$ for every X', X''. If c is monotone there is a unique largest additive c' such that $c'X'\subset cX'$ for every X'. A set X is called compact under c (J. W. Tukey, *Convergence and Uniformity in Topology*), if given a family of subsets of X such that finite intersections of sets of the family are non-empty, there is a point common to the closures of all sets of the family. The following theorem holds: X is compact under a monotone c if and only if it is compact under c'. The proof falls into two parts: The conditions given by Tukey for compactness of a space with additive closure still hold for monotone c; in particular, X is compact under c if and only if every ultraphalanx in X converges (in the topology of c) to a point of X. The theorem quoted can then be proved by showing that any ultraphalanx has the same limit points in the topologies of c and c'. (Received April 4, 1942.)

221. R. C. James: *Normability of topological abelian groups.*

In this paper the properties of convexity, boundedness, and normability of topological abelian groups are considered. The existence of a bounded, convex neighborhood of the identity is found to be a necessary condition for normability, and a sufficient condition if taken with connectedness or one of several generating properties. The representation of convex and normable topological abelian groups as subgroups of linear topological spaces is investigated. (Received March 14, 1942.)

222. Samuel Kaplan: *Homologies in metric separable spaces.*

It is shown that the denumerable star-finite coverings form a complete family. By developing a method originating with Čech, homology properties of arbitrary subsets can be obtained by use of "neighborhood coverings." A new type of bounding called Čech bounding is defined for Vietoris cycles as follows: To each Vietoris cycle of the space there corresponds a Čech cycle, determined up to homology; the Vietoris cycle is said to Čech bound if its corresponding Čech cycle bounds. By way of application, the following duality theorem is proved (in the case of closed sets reducing to the Alexander duality): If A is an arbitrary subset of the n-sphere S^n, and there are m k-dimensional Vietoris cycles independent in A relative to Čech homologies, then in S^n-A there are m $(n-k-1)$-dimensional Vietoris cycles independent relative to Čech homologies. Among other results are dualities established between the Čech homology properties of arbitrary subsets of S^n and the Vietoris homology properties of their complements, as well as miscellaneous results as: If a Vietoris cycle z^k fails to Čech bound in a metric separable space, there exists a homeomorphic remetrization in which z^k fails to ϵ-bound for some ϵ. (Received March 19, 1942.)

223. A. N. Milgram: *Dimension of the points with at most n rational coordinates.*

In this paper it is shown that the points in Hilbert space at most n of whose coordinates are rational is an $(n+1)$-dimensional set. This is an extension of the theorem that the points in Hilbert space all of whose coordinates are irrational form a one-dimensional set, which follows directly from the proof by Erdős for the set with all rational coordinates. (Received March 26, 1942.)

224. J. H. Roberts and Paul Civin: *Sections of continuous collections.*

The following theorem is proved: If G is a continuous collection of closed and compact sets filling a separable metric space X, and if G, regarded as a decomposition space, has dimension n, then there exists in X a closed set C such that for any g in G the set $g\cdot C$ is nonvacuous and consists of at most $n+1$ points. (Received March 20, 1942.)

225. R. L. Swain: *Proper and reductive transformations.*

A (single-valued) transformation T of a point set M is called: *proper* if for each $K\subset M$, $T(K)$ is closed and compact whenever K is; *strong proper* if, in addition, T preserves connectedness; *complete proper* (CP) if an additional condition concerning contiguous points is satisfied. A strong proper transformation of a compact continuous curve is continuous. In spaces which may contain contiguous points (and which satisfy R. L. Moore's axiom set Σ_s), for M to be a compact continuous curve it is necessary and sufficient that M should be the image of an ordinary arc under some CP transformation; also a CP transformation of an arc into itself either leaves a point fixed or interchanges two contiguous points. A transformation T of M is called *reductive* provided that if $X\in T(M)$, and $V\subset M$ is such that X is a limit point of $T(V)$, then $T^{-1}(X)$ contains a limit point of V. A reductive transformation is continuous on the nucleus of a compact closed point set. In a compact space the class of reductive transformations equals the class of proper transformations. Using reductive transformations, one may associate with each point set M an abstract measure called its topological power. (Received March 17, 1942.)

226. G. T. Whyburn: *Coherent and saturated collections.*

A theory of coherence and saturation relative to a given collection G of subsets of a connected separable metric space M is developed. Conditions are found under which the collection G will generate a unique upper semi-continuous decomposition of M and under which the associated transformation will be monotone or non-alternating. These are applied to find necessary and sufficient conditions for the mappability of M onto an arc or a circle by a monotone transformation. As a further application, an existence theorem for saturated collections of continua in a locally connected continuum L is proved which yields in a new way the known (Moore-Roberts) monotone mappability of L into a regular curve. (Received March 4, 1942.)

227. G. T. Whyburn: *On the interiority of mappings.*

A previous theorem of the author (as yet unpublished) is extended to yield the following theorem: Let $f(M)=K$ be continuous where M is locally connected separable and metric and let D be the closure of the set of all points of K of Menger-Urysohn order greater than 2. There exists a countable subset C of $K-D$ such that f is interior at every point of $M-f^{-1}(D+C)$. (Received March 4, 1942.)

228. G. T. Whyburn: *Unitary subcontinua.*

A subcontinuum N of a compact metric continuum M will be called unitary proved that if K is any subcontinuum of M intersecting N we have either $K\supset N$ or

$K \subset N$. For example, M itself (and also any single point of M) is unitary. A study of such subcontinua is made and it is shown, for example that: (1) the property of being a unitary subcontinuum containing a given subset X of M is inducible; (2) for any two given unitary subcontinua A and B there exists a unique least unitary subcontinuum $A+B$ containing both A and B; (3) no nondegenerate proper unitary subcontinuum can have a cut point; (4) the property of being a unitary subcontinuum of M is a monotone and also an interior invariant; (5) if M is irreducible between two points, any monotone interior mapping of M into the interval maps unitary subcontinua into single points of the interval. (Received March 4, 1942.)

229. J. W. T. Youngs: *On parametric representations of surfaces. II.*

This paper is a continuation of an earlier study (abstract 46-5-352). A continuous transformation $R(A)=B$ from the 2-sphere A to a set B in 3-space is called a representation. By the Eilenberg-Whyburn factor theorem, $R(A)=L(M(A))$, where $M(A)=\Sigma$ is monotone and $L(\Sigma)=B$ is light. Two representations $R_1(A)$ and $R_2(A)$ are K-equivalent (Kerekjarto, Acta Szeged, vol. 3 (1927), pp. 49–67) if there is a homeomorphism $H(\Sigma_1)=\Sigma_2$ such that $L_1(\Sigma_1)=L_2(H(\Sigma_1))$. They are F-equivalent if the Fréchet distance between them is zero. The prime end theory is used to define what is meant by the statement that R_1 and R_2 are consistent. The concept has to do with orientation. Suppose that R_1 and R_2 are themselves monotone. Then, if they are F-equivalent, they are K-equivalent and consistent. Moreover the converse is true without the restriction that R_1 and R_2 be monotone. The converse is important from the point of view of applications to the theory of area. (Received March 18, 1942.)

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Yntema, T. O. See Lange, O.

TRENDS IN RESEARCH:
THE ANALYTIC NUMBER THEORY

HANS RADEMACHER

Ladies and Gentlemen: I wish to express my gratitude for the invitation to address this meeting and to thank you for granting me this opportunity to contribute a little part to the celebration of the anniversary of our host, this renowned institution of higher learning, the University of Chicago. The research work emanating from here has in its mathematical branch strongly emphasized analysis as well as number theory. The name "analytic number theory," indicating a union of these two fields of mathematical endeavor, is about as old as this celebrating Institution. In 1894 a book was published under the title *Die Analytische Zahlentheorie*, by Paul Bachmann. The name "analytic number theory" itself was a program. It said more than "diverse applications of infinitesimal calculus to number theory" as Dirichlet, the real founder of our discipline, had modestly called one of his great memoirs (1839). The name "analytic number theory," implies, as I take it, a thorough fusion of analysis and arithmetic, in which as we shall see, analysis is not necessarily subordinate to arithmetic. Incidentally, "analysis" stands here always for function theory, especially the theory of analytic functions.

What has the theory of functions to do with the theory of integers? Let me, to gain a starting point, begin with a few examples of well established results of analytic number theory:

1. If $\pi(x)$ denotes as usual the number of prime numbers less than or equal to x, then

$$\lim_{x\to\infty}\frac{\pi(x)}{x/\log x}=1 \tag{1}$$

as first proved by Hadamard and de la Vallée Poussin in the nineties of the last century.

2. The number of representations of a positive integer n as a sum of four squares is

$$r_4(n)=8\left(\sum_{d\mid n}d-\sum_{4\mid d'\mid n}d'\right) \tag{2}$$

as Jacobi found as a corollary of his theory of elliptic functions.

An address delivered before the meeting of the Society in Chicago, Ill., on September 5, 1941, by invitation of the Program Committee; received by the editors January 6, 1942.

3. The number $h(D)$ of classes of positive primitive binary quadratic forms of determinant $D \equiv 7 \pmod 8$ is given by Dirichlet's formula

$$h(D) = \sum_a \left(\frac{a}{D}\right) \tag{3}$$

where $(a, D) = 1$, and a runs through all positive integers less than $(1/2)D$.

Now, the first of these examples makes it quite obvious that analysis has to be applied, since the result does not only contain the continuous function $x/\log x$, but moreover involves the notion of limit, so that the two fundamental concepts of infinitesimal analysis are represented in (1).

This is not so in the two other examples. All expressions appearing in the equations (2) and (3) have an arithmetical meaning and are defined in elementary number theory without any reference to analysis. Nevertheless both results appear as the goal of analytic proofs in one of which appear ϑ-functions and in the other Dirichlet series. Both theorems can therefore be claimed by analytic number theory. However, there remains a big difference in these two cases: whereas the result (2) can also be proved by purely arithmetical methods, the equation (3) has defied all attempts of an elementary arithmetical proof.

This observation raises the question, which I will state in all its logical vagueness: what is the connecting link between arithmetic and analysis, or by what means can analysis take hold of arithmetical facts? In the steady flow of real numbers, called the continuum, which is the substratum of function theory, the integers are in no way distinguished from any other real numbers. The function e^z at first glance has no "affinity" to integers. We notice, however, the relation

$$e^{2\pi iz} = e^{2\pi i(z+n)}$$

for any integer n and all complex values z. The function has *the period* n where n can be any integer. The fundamental fact is that the periods of an analytic, or for that matter, a continuous function form a discrete module or lattice. Or, speaking a little more generally: the ideas connected with period, invariance, and substitution group are basic for the application of analysis to number theory, and, as we can remark already here, also for the inverse process. Let me give another example, which is a little more involved. For the modular invariant $J(\tau)$ we know that

$$J(\tau) = J(\tau') \tag{4a}$$

implies and is implied by

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1, \tag{4b}$$

a, b, c, d being integers. The function $J(\tau)$ appears thus as an embodiment of the discrete group of modular substitutions. This fact must have implications in both directions. Indeed, the group of modular transformations defines the classes of binary quadratic forms, and so we are led to the theory of singular moduli, the theory of complex multiplication, and finally to the Kronecker-Fueter theorem that any Abelian field over an imaginary quadratic field can be generated by the values of certain modular forms which they attain for special imaginary quadratic irrationalities. On the other side, equations (4a) and (4b) lead function-theoretically to the expansion (Petersson [**33**, **37**])

$$12^3 J(\tau) = e^{-2\pi i\tau} + 744 + \sum_{n=1}^{\infty} c_n e^{2\pi i n\tau} \tag{5a}$$

with

$$c_n = \frac{2\pi}{n^{1/2}} \sum_{k=1}^{\infty} \frac{1}{k} I_1\left(\frac{4\pi n^{1/2}}{k}\right) \sum_{h \bmod k,\,(h,k)=1} e^{-(2\pi i/k)(nh+h')}, hh' \equiv -1\,(k), \tag{5b}$$

where $I_1(z) = -iJ_1(ix)$, the Bessel function of the first order.

You will always find such a group-invariance or periodicity, sometimes conspicuous, sometimes concealed, as underlying the applications of analysis to number theory. Fourier series, Fourier integrals, and Poisson sums are, therefore, natural and frequently used tools. But also the configuration of a point lattice can be taken as to express periodicity (especially in Minkowski's work in the "geometry of numbers"), and even a power series is here often better understood as a series in $e^{2\pi u}$, that is, a Fourier series with its obvious periodicity.

These analytic tools, developed in the 19th century, indicate sufficiently that our discipline must be a rather young one, so very much younger than elementary number theory, in whose history we meet the names of Euclid, Diophantos, Fermat. Indeed Lejeune Dirichlet can be regarded as the founder of our branch of research. Before him Euler, Gauss, and especially Jacobi, had found methods and results belonging to this field. After Dirichlet it was Riemann who with his epoch making memoir *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (1859) made the deepest mark in its history. In this brief survey I have also to mention the names of Hermite, Kronecker, Hadamard, de la Vallée Poussin, and Minkowski, who all contributed works of lasting value.

The *present* standing of analytic number theory and its recognition as a fruitful branch of our science is, I think, entirely due to three mathematicians of this century: Landau, Hardy, and Littlewood. To Hardy and Littlewood we owe important progress in the theory of prime numbers and in our knowledge of the distribution of the complex zeros of the ζ-function. Their greatest contribution, which they share with Ramanujan, lies, however, in the development of a fundamentally new method in additive number theory, a method which yielded their celebrated theorems concerning partitions, and about additive representations of numbers as sums of powers (Waring's problem) and as sums of primes (Goldbach's problem).

Landau has exerted a wide influence through his untiring mathematical enthusiasm and through his forceful advocacy of utter precision. The substance of his work has proved to be less permanent; his results were soon surpassed by deeper and more comprehensive ones, and his papers have very often been only second redactions of discoveries of others. How much we owe him nevertheless becomes evident if we compare a publication on analytic number theory of the the 1890's (for example, the above mentioned book of Bachmann, but also such substantial papers like those of Hadamard and de la Vallée Poussin not excepted) with a contemporaneous paper. Landau developed the technique of analytic number theory and prepared the tools; he found certain theorems about power series, Dirichlet series, Fourier integrals, which now are generally used as lemmas; and he made us familiar with the usefulness of a number of devices and lemmas discovered by others, as for example, Abel's partial summation, Mellin's formula, and Carathéodory's lemma. Landau's handbook on prime numbers will remain a classical treatise. Not a small merit of it is that it shaped our notation: It brought the symbol "O" into general use (it had been invented by Pringsheim), and Landau introduced there the symbol "o" (a constant source of quarrel between him and Pringsheim, which the German mathematicians used to watch with amusement). Both these symbols, now so well known and generally adopted, are particularly fit to express asymptotic relations which so frequently occur in analytic number theory.

Landau is no longer with us; Hardy and Littlewood about 10 years ago turned to other fields of investigation. Hence I understand it as my main task today to give a survey of developments in analytic number theory which took place after these three outstanding authors and independent of them.

This development has proceeded chiefly in two directions. One of them can roughly be characterized by words like "asymptotic formula" and "error term;" inequalities play an important part in the technique to "estimate" the error. I think here mainly of problems like the prime number theorem, lattice points in the circle, asymptotic expressions for the number of representations of a number as a sum of one or the other sort, investigations in which Hardy, Littlewood, and Landau have distinguished themselves, so that this pursuit of research can be understood as a direct continuation of their work. Investigations of this sort have attracted general attention, so much so, indeed, that the impression may have prevailed that analytic number theory deals foremost with asymptotic expressions for arithmetical functions. This view, however, overlooks another side of analytic number theory, which I may indicate by the words "identities," "group-theoretical arguments," "structural considerations." This line of research is not yet so widely known; it may very well be that methods of its type will lead to the "deeper" results, will reveal the sources of some of the results of the first direction of approach. Remarks like those which I made about applicability of analysis to arithmetic are especially pertinent here. The names Artin, Hecke, Mordell and Siegel would have to be mentioned in this connection.

The first type of analytic number theory is at present carried on in the Russian school with Vinogradoff as the leading name. Vinogradoff has become widely known for his great contributions to Waring's problem and Goldbach's problem. In this work he depends entirely on Hardy's and Littlewood's precedency; he introduced essentially two improvements of their method. One is of minor importance and merely an affair of convenience. He observed that for the additive representation of the number n you obviously do not need summands surpassing n. This means that the power series which have been used by Hardy and Littlewood (and whose use in additive number theory really dates back to Euler) can be replaced by polynomials. This has certain advantages since it discards the difficulty of the natural boundary which the unit circle represents for those power series. Vinogradoff's second and decisive advance lies in his ingenious estimate of certain sums of roots of unity, as I shall now try to explain.

I begin with Waring's problem, in whose treatment Vinogradoff reaped his first success. For the discussion of

$$f(x) = \sum_{n=0}^{\infty} x^{n^k} \tag{6}$$

in the neighborhood of a root of unity $\xi = e^{2\pi i(h/q)}$ Hardy and Littlewood need expressions for the sums

$$S_q(M) = \sum_{n=0}^{M} e^{2\pi i (h/q) n^k} \tag{6a}$$

with the emphasis upon their dependence on q and M (for fixed given $k \geqq 3$; $k=2$ is in a different class since $f(x)$ is then essentially a ϑ function and $S_q(M)$ essentially a Gaussian sum, both of which are well known from other connections). Now for an M which is large compared with q an expression with a useful error term is easily found by partial summation. The error term, however, becomes unmanageable for q comparatively large. This case, which appears on the so-called "minor arcs" proved first to be a serious obstacle to Hardy's and Littlewood's efforts. It is known that this difficulty delayed their final success for two years at a time when the required device had already been found and published by H. Weyl [58] but due to the interruption of communications during the first World War had not yet come to Hardy's and Littlewood's notice.

Now Vinogradoff recognized that a certain k-fold iteration in Weyl's method could be dispensed with and replaced by one single step. Since that iteration was the source of a factor 2^k in the estimate for $G(k)$, the smallest number of kth powers sufficing to express every sufficiently large number additively, this change meant an extraordinary diminution of the estimated order of $G(k)$, a diminution as a matter of fact which was so considerable that it enabled Dickson to establish in many cases the true value $g(k)$ of kth powers which suffice to represent additively *all* positive integers, simply by bridging the finite gap not covered by $G(k)$ through a number of elementary computational steps [6, 7].

Vinogradoff's lemma, which replaces Weyl's lemma, can be enunciated as follows (in the form which was given to it by Heilbronn [21]):

Let x run through X integers of an interval of length L, y through Y integers on an interval of length M, then

$$\left|\sum_{x,y} e^{2\pi i (xy/q)}\right|^2 \leqq \frac{XYML}{q}\left(1+\frac{q}{M}\right)\left(1+\frac{q\log q}{L}\right). \tag{7}$$

This lemma is not immediately applicable to (6a) and to (6). Vinogradoff employs the astonishing device of dropping certain Waring representations of the given number n, by selecting kth powers of a certain factorization. This diminishes the chance of a Waring representation. Nevertheless he finds that the number of representations is different from null for $s \geqq 6k \log k + (4+\log 216)k$ summands and sufficiently large n [50].

Vinogradoff's most startling achievement was his proof in 1937 that every large odd number is the sum of three primes [52, 53], a result previously obtained by Hardy and Littlewood only under a generalized Riemann hypothesis. Vinogradoff's result is so far the closest approach to Goldbach's conjecture ever reached. The difficulties are here much greater than for the Waring problem, the distinction between major and minor arcs much subtler. The major arcs require for their treatment the classical theory of the distribution of primes founded on the ζ-function and the Dirichlet L-functions. In Vinogradoff's paper, moreover, a deep theorem of Siegel about the class-number of quadratic fields [46] was used, which can be expressed in the statement

$$|L(1,\chi)| > \frac{C}{k^{\epsilon}}$$

where χ is a real character modulo k, a theorem which widens Heilbronn's result [22] that the number of imaginary quadratic fields of class-number one is finite. Walfisz thereafter showed that Siegel's theorem can be dispensed with, at the cost of some loss of precision in the error term [54].

The sum to be appraised in Goldbach's problem is

$$T_q(M) = \sum_{p \leq M} e^{2\pi i (h/q) p},$$

again in particular for the minor arcs, that is, for comparatively high denominator q. Now this sum does not fit into the scheme of Vinogradoff's lemma (7) since p alone and not a product appears in the exponent. Vinogradoff resorts here to the sieve of Eratosthenes [53] which had been so successfully used by Viggo Brun [3]. The argument runs like this:

$$T_q(M) = \sum_{\nu=1}^{M} e^{2\pi i (h/q)\nu} - \sum_{p_1 \leq M^{1/2}} \sum_{1 \leq m \leq M/p_1} e^{2\pi i (h/q) m p_1} + \sum_{p_2 < p_1 \leq M^{1/2}} \sum_{1 \leq m \leq M/p_1 p_2} e^{2\pi i (h/q) m p_1 p_2} + \cdots .$$

The first sum is a trivial geometric sum, and in the following sums we do find the desired products in the exponents. This opens the way to an application of (7). Still, the completion is not quite simple but is handled by Vinogradoff with superb technique. I have already quoted his final result regarding Goldbach's problem.

Before I drop this strand of my narration I wish to mention Tchudakoff's application of Vinogradoff's methods to the estimation of Riemann's ζ-function on $\Re(s) = 1$ [49]. Tchudakoff uses Vinogradoff's estimates [51] in the same way as Littlewood and Landau had used Weyl's method for the improvement of the error term in the prime number theorem to $O(x \exp(-a(\log x \log\log x)^{1/2}))$. Tchudakoff obtains the prime number theorem in the form

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(xe^{-a(\log x)^{\alpha}})$$

with a certain $\alpha > 1/2$. Until then the remainder term had for three decades shown the value $\alpha = 1/2$. Of course, any $\alpha < 1$ is still far off from the Riemann hypothesis which is equivalent to $\alpha = 1$, $a = (1/2) - \epsilon$.

Appraisals of sums of roots of unity have recently played an important role in several parts of analytic number theory. Mordell [31], Davenport [4], and Hasse [5, 14] discussed sums of the form

$$S_q = \sum_{n \bmod q} e^{(2\pi i/q)f(n)}$$

where $f(n)$ is a polynomial in n with integral coefficients. The aim is to give an estimate of S_q with respect to q. The problem is a generalization of the order of magnitude of Gaussian sums, that is, $f(n)$ of the second degree, in which case we have

$$S_q = O(q^{1/2}).$$

It is of interest that the exponent 1/2 which appears here in a number of cases has something to do with the abscissa $\Re(s) = 1/2$ in the Riemann hypothesis, not for the ordinary ζ-function, it is true, but for the ζ-function belonging to a field of algebraic functions over a finite field of characteristic p; these ζ-functions were introduced by E. Artin [1] in 1924 and later in a more general form defined by F. K. Schmidt [44]. The non-trivial zeros ρ of such a ζ-function turn out to be periodic and given by

$$\beta = p^{\rho}$$

where β are the roots of certain polynomials; clearly $|\beta| = p^{1/2}$ means

$$(*) \qquad \Re(\rho) = 1/2,$$

and the absolute value of β depends on the number of solutions of certain congruences modulo p, or, which is equivalent to this, on the value of certain exponential sums. Relation (*) has been proved for the Artin ζ-functions of the field of elliptic functions by Hasse [12, 13], who also outlined a program for further research. Modern algebraic concepts are fundamental in this work. Quite recently, in this year, André Weil [56, 57] published two sketches of proofs for the

Riemann hypothesis in all algebraic function fields over a finite field. We are looking forward to a complete account of his proof. Here still a wide field is open to research, a field attractive at the same time by its large generality and its richness of specific details.

A particularly interesting case of sums of roots of unity are the so-called Kloosterman sums

$$S_q(u, v) = \sum_{h \bmod q,\, (h,q)=1} e^{(2\pi i/q)(uh+vh')}$$

with $hh' \equiv 1 \pmod{q}$. Since these sums are multiplicative with respect to q, they can be broken down to those with prime-number powers q. For $q = p^\alpha$, $\alpha \geqq 2$, the order $O(q^{1/2})$ has been established by Salié [43], and moreover an explicit formula involving Legendre symbols and trigonometric functions has been found. Surprisingly, however, the case of a prime number $q = p$ offers difficulties which so far have only permitted us to prove

$$S_p(u, v) = O(p^{2/3}).$$

Weil's above-mentioned results should establish here also $O(p^{(1/2)+\epsilon})$.

The Kloosterman sums appear in the theory of partitions and in the study of the coefficients of modular functions. Recent work connected with these problems can also be understood as an extension of the Hardy-Littlewood-Ramanujan work. Whereas Hardy and Ramanujan obtained only an asymptotic expression for the number $p(n)$ of partions of n, a refinement and at the same time a simplification of their method led to the equation [35, 36]

$$p(n) = \frac{1}{\pi 2^{1/2}} \sum_{q=1}^{\infty} A_q(n) \frac{d}{dn} \left\{ \frac{\sinh \frac{K}{q}\left(n - \frac{1}{24}\right)^{1/2}}{\left(n - \frac{1}{24}\right)^{1/2}} \right\},$$

$$A_q(n) = \sum_{h \bmod q,\, (h,q)=1} \omega_{h,q} e^{-2\pi i (hn/q)}, \quad K = \pi(2/3)^{1/2},$$

where $\omega_{h,q}$ are certain roots of unity taken from the theory of transformation of the modular form $\eta(\tau)$, which have attracted the attention of quite a number of mathematicians since Hermite. The series in the equation just mentioned is absolutely convergent. It has been used successfully by D. H. Lehmer for the numerical computation of $p(n)$ for high values of n [26, 27]. Analogous results for other problems of partitions with certain congruence conditions imposed on the summands have been obtained by Zuckerman [59], Niven [32], Hua [23], Miss Haberzetle [9], and Lehner [28].

These investigations form a natural bridge between the two different ways of investigation I mentioned before. Technically they require estimates of error terms, but also some group-theoretical arguments. And they lead to identities, which in turn invite group-theoretical treatment and thus point in the second direction in our field.

Identities can be used as sources of other identities. In this way the famous Ramanujan identities, of which I quote here only

$$(^{**})\qquad \sum p(5l+4)x^l = 5\left[\prod(1-x^{5m})^5\big/\prod(1-x^m)^6\right],$$

can be proved by direct comparison of coefficients which are known in forms of series through the just mentioned methods. But the true source of the Ramanujan identities lies in the theory of transformation of modular forms, as it is seen at once if we rewrite the identity (**)

$$\sum_{\lambda=0}^{4} \frac{1}{\eta\left(\dfrac{\tau+24\lambda}{5}\right)} = 5^2\,\frac{\eta(5\tau)^5}{\eta(\tau)^6}$$

[30, 39].

The formula (5) is an identity, which can also be derived by the Hardy-Littlewood-Ramanujan method. Here the problem arises to understand the invariance

$$J\left(\frac{a\tau+b}{c\tau+d}\right) = J(\tau)$$

as an outcome of the Fourier series on the right-hand member [38].

Hardy himself has contributed a remarkable paper to similar investigations [10], showing that for example,

$$(8a)\qquad \begin{aligned} \vartheta_3^8(0,\tau) &= 1 + \frac{\pi^4}{3!}\sum_{n=1}^{\infty} n^3 S_8(n) e^{2\pi i n\tau}, \\ S_8(n) &= \sum_{q=1}^{\infty}\frac{1}{q^8}\sum_{h \bmod q,\,(h,q)=1}\left(\sum_{j=1}^{q} e^{2\pi i(hj^2/q)}\right)^8 e^{-2\pi i(hn/q)}. \end{aligned}$$

Here $S_8(n)$ is the "singular series" of the problem of expressing a number as a sum of 8 squares. The proof of (8a) has to establish the fact that the Fourier series on the right of (8a) is a modular form of dimension -4.

The "singular series," first introduced into analytic number theory by Hardy and Littlewood, form objects of great mathematical interest of their own. They appear in problems of representation of integers in an additive way, for example, as $n = x_1^k + \cdots + x_r^k$. For the number of representations the Hardy-Littlewood method yields an asymptotic result of the form

$$N_r(n) \sim f_r(n)\cdot\mathfrak{S}_r(n),$$

where $f_r(n)$ is a continuous function of n, which can be interpreted as a density with respect to the r-dimensional point lattice. The other factor is the singular series $\mathfrak{S}_r(n)$, an arithmetical function of n. A remarkable feature of all singular series is that they are factorable according to prime numbers:

$$\mathfrak{S}_r(n) = \prod_p A_p^{(r)}(n). \tag{8b}$$

This fact finds its explanation in the meaning of $A_p^{(r)}(n)$. Whereas for the problem

$$n = x_1^k + \cdots + x_r^k$$

the function $f_r(n)$ means the derivative

$$f_r(n) = \frac{d}{dn}V_r(n)$$

of the volume

$$V_r(n) = \int\cdots\int_{x_1^k+\cdots+x_r^k\leq n;\,0\leq x_j} dx_1\cdots dx_r$$

or, in other words, a measure of the area of the surface

$$n = x_1^k + \cdots + x_r^k, \qquad 0 \leqq x_j;\ j = 1, \cdots, r,$$

the $A_p^{(r)}(n)$ give the density of solutions of the congruence

$$n \equiv x_1^k + \cdots + x_r^k \pmod{p^l} \tag{9}$$

with respect to high powers of p as modulus, or explicitly:

$$A_p^{(r)}(n) = \lim_{l\to\infty}\frac{b^{(r)}(n;\,p^l)}{g^{(r)}(p^l)}$$

where $b^{(r)}(n;\,p^l)$ is the number of different solutions of (9) and $g^{(r)}(p^l)$ is the average of the number of solutions of (9), distributed over p^l different n's,

$$g^{(r)}(p^l) = \frac{p^{lr}}{p^l} = p^{l(r-1)}.$$

The multiplicativity of the $A_p^{(r)}(n)$ now becomes understandable, and the product (8*b*) means that $\mathfrak{S}_r(n)$ is the density of solutions of the congruence for, let us say, $m!$ as modulus with $m \to \infty$.

Recently Siegel has emphasized this interpretation and used it with great success in his analytic theory of quadratic forms [47]. In that connection the singular series had appeared before, in the works of Smith and Minkowski, as measure of genera of quadratic forms. Through this connection it becomes clear why the Hardy-Littlewood method leads to an exact formula for the number of representations of n as a sum of r squares for $5 \leqq r \leqq 8$, whereas for higher r it yields only an asymptotic result.

This interpretation of the singular series is possible in all those problems of additive number theory which preassign the number of summands or variables, as in Goldbach's problem, Waring's problem, or mixed problems, as for example, the problem of representation as a sum of a square and two primes (Estermann [8]), or as a sum of r squares and s cubes. It fails, however, for problems with an undetermined number of variables, as in the case of unrestricted partitions, since there the transition from equation to congruence destroys the boundedness of the number of variables for each n, so that the number of solutions modulo p becomes meaningless. Indeed, the formula for $p(n)$, given above, is of quite a different structure.

Singular series appear as coefficients of Fourier series as in (8a). Siegel's analytic theory of quadratic forms can be regarded as a brilliant example of the study of such series. He shows that his series can be combined of ϑ-functions

$$\vartheta(\tau) = \sum_{m_1,\cdots,m_r} e^{2\pi i\tau Q(m_1,\cdots,m_r)}$$

where Q is a positive quadratic form with integral coefficients; such a combination includes ϑ-functions constructed with quadratic forms Q of all classes of the same genus. It is known that such ϑ-series are modular forms of dimension $-r/2$ and of a certain "level" (Stufe) depending essentially on the discriminant of the quadratic form $Q(m_1, \cdots, m_r)$ [45].

A general study of power series with singular series as coefficients has not yet been undertaken. It would be desirable to know more specific details about these functions than the general properties which can immediately be derived from Kac's, van Kampen's, and Wintner's remark [24, 25] that the singular series give rise to almost periodic arithmetical functions of n and from Bochner's and Bohnenblust's [2] theorems about power series with almost periodic coefficients.

The modular functions have from the beginning of our discipline played an important part in *additive* problems; I need in this respect only point to Euler's formula, fundamental for the whole theory of partitions

$$1+\sum_{n=1}^{\infty} p(n)x^{n} = 1/\prod_{1}^{\infty}(1-x^{m})$$

in which the denominator is essentially the modular form

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{1}^{\infty}(1-e^{2\pi i m\tau}),$$

or to equation (2), obtained by Jacobi from an expression for the 4th power of $\vartheta_3(0, \tau)$. Recently, however, they have also appeared in *multiplicative* number theory through the discovery of a surprising and most fruitful connection between modular functions and prime numbers. This connection is elaborated by Hecke in his beautiful theory, of which he published the first account in 1935 in the Royal Danish Academy [16]. As a matter of fact, a special result of Hecke's theory was anticipated by Ramanujan [41, 11] in a conjecture which was then proved by Mordell in 1917 [29]. This is again an instance of Ramanujan's gift of divination; not only that he foresaw or unveiled hitherto unknown facts, but most of his conjectures lead back to sources of great depth and abundant yield.

The "discriminant" in the theory of elliptic functions

$$\Delta(1, \tau) = g_2^3 - 27g_3^2 = (2\pi)^{12}x((1-x)(1-x^2)\cdots)^{24}$$

with $x = e^{2\pi i\tau}$ is a modular form of dimension -12, since for a modular substitution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$\Delta\left(1, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12}\Delta(1, \tau).$$

The infinite product can be expanded into a power series

$$x\prod_{1}^{\infty}(1-x^{m})^{24} = \sum_{n=1}^{\infty}\tau(n)x^{n}.$$

If we now, following Ramanujan, associate with this power series the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s},$$

it turns out that this series can be written as an infinite product

$$F(s) = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}, \tag{10}$$

extended over all primes p. It is appropriate to call such a product an "Euler product" in view of the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}},$$

first found and utilized by Euler. I shall have to make a remark about this latter product a little later. Not only the Euler product (10), but also a functional equation for $F(s)$: $(2\pi)^{-s}\Gamma(s)F(s) = (2\pi)^{s-12}\Gamma(12-s) F(12-s)$ reminds us strongly of the ζ-function and allied functions.

The coordination of a modular function on the one side and a Dirichlet series satisfying a functional equation on the other has now been systematized by Hecke [17, 18, 19]. Let us take an entire modular form $F(\tau)$ of dimension $-k$, k a positive even integer,

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k F(\tau)$$

having the expansion

$$F(\tau) = c_0 + \sum_{n=1}^{\infty} c_n x^n, \qquad x = e^{2\pi i \tau},$$

which is everywhere convergent in the upper τ-half-plane. Then let us put

$$\phi(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \text{ (without } c_0).$$

Then the following statements are valid:

(1) The series for $\phi(s)$ is absolutely convergent in the half-plane $\Re(s) > k$.

(2) $\phi(s)$ is an entire function if $c_0 = 0$; if $c_0 \neq 0$ then $\phi(s)$ has a pole of the first order at $s = k$ with the residue $(-1)^{k/2}c_0(2\pi)^k/\Gamma(k)$ and $(s-k)\phi(s)$ is an entire function.

(3) $\phi(s)$ satisfies a functional equation:

$$(2\pi)^{-s}\Gamma(s)\phi(s) = (-1)^{k/2}(2\pi)^{s-k}\Gamma(k-s)\phi(k-s).$$

(4) $(s-k)\phi(s)$ is of finite order.

Of these statements (1) is implied by $c_n = O(n^{k-1+\epsilon})$, which in turn is a simple consequence of the dimension $-k$. For (2) and (3) we have the pattern of Riemann's classical proof for the continuation and the functional equation of the ζ-function by means of a ϑ-function. In its application here the proof runs like this:

We have from Euler's integral

$$\frac{\Gamma(s)}{(2\pi n)^s} = \int_0^\infty e^{-2\pi n y} y^{s-1} dy$$

and therefore for $\Re(s) > k$

$$(2\pi)^{-s}\Gamma(s)\phi(s) = \int_0^\infty (F(iy) - c_0) y^{s-1} dy$$

with

$$F(iy) - c_0 = \sum_{n=1}^{\infty} c_n e^{-2\pi n y}.$$

Since for the modular substitution

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we have $F(i/y) = (iy)^k F(iy)$ we can write

$$\begin{aligned}
&(2\pi)^{-s}\Gamma(s)\phi(s) \\
&\quad = \int_1^\infty (F(iy) - c_0) y^{s-1} dy + \int_0^1 ((iy)^{-k} F(i/y) - c_0) y^{s-1} dy \\
&\quad = \int_1^\infty (F(iy) - c_0) y^{s-1} dy + i^{-k} \int_1^\infty F(iw) w^{k-s-1} dw - \frac{c_0}{s} \\
&\quad = \int_1^\infty (F(iy) - c_0) y^{s-1} dy + (-1)^{k/2} \int_1^\infty (F(iw) - c_0) w^{k-s-1} dw \\
&\qquad + (-1)^{k/2} c_0 \int_1^\infty w^{k-s-1} dw - \frac{c_0}{s} \\
&\quad = \int_1^\infty (F(iy) - c_0)(y^s + (-1)^{k/2} y^{k-s}) \frac{dy}{y} \\
&\qquad - c_0 \left(\frac{1}{s} + (-1)^{k/2} \frac{1}{k-s} \right).
\end{aligned}$$

So far our reasoning required $\Re(s) > k$. Now we have arrived at an expression which remains meaningful for any complex s. We have, therefore, obtained an analytic continuation over the whole s-plane for $\phi(s)$. Moreover, the substitution $s \to k - s$ changes the right-hand member by the factor $(-1)^{k/2}$, which proves (3). At the same time we infer the validity of statement (2).

The argument runs also in reverse, from the Dirichlet series of properties (1) to (4) to a modular form of dimension $-k$. Instead of the Euler integral we have only to apply Mellin's formula.

Up to this point we have not encountered any trace of prime numbers. Hecke came to an Euler product by considering not a single modular form, but the whole set of all entire modular forms of dimension $-k$. They constitute a linear family of finite rank κ; the number κ of linearly independent modular forms in this set is a known function of k. A linear operator T will map the whole set on itself. If the system $F^{(\rho)}(\tau)$, $\rho = 1, 2, \cdots, \kappa$ forms a basis then there exists a matrix $(\lambda_{\rho\sigma})$ such that

$$T(F^{(\rho)}(\tau)) = \sum_{\sigma=1}^{\kappa} \lambda_{\rho\sigma} F^{(\sigma)}(\tau).$$

As linear operator Hecke takes, for any natural number n,

$$T_n(F) = n^{k-1} \sum_{ad=n} \sum_{b \bmod d} F\left(\frac{a\tau + b}{d}\right) d^{-k},$$

for which it can be proved, by consideration of the subgroup

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n}$$

and its factor-group in the full modular group, that it furnishes again a modular form of dimension $-k$.

Hence we have

$$T_m(F^{(\rho)}(\tau)) = \sum_{\sigma=1}^{\kappa} \lambda_{\rho\sigma}(m) F^{(\sigma)}(\tau).$$

The application of the operator T_m on the Fourier expansion

$$F^{(\rho)}(\tau) = \sum_{n=0}^{\infty} a^{(\rho)}(n) e^{2\pi i n\tau}$$

yields by comparison of coefficients,

$$\sum_{\sigma=1}^{\kappa} \lambda_{\rho\sigma}(m) a^{(\sigma)}(n) = \sum_{\delta|m,\,\delta|n} \delta^{k-1} a^{(\rho)}\left(\frac{mn}{\delta^2}\right), \qquad m \geq 1,\ n \geq 1. \tag{11}$$

It is decisive that here the right-hand member shows symmetry in m and n, which therefore must also prevail in the left-hand member:

$$\sum_{\sigma=1}^{\kappa} \lambda_{\rho\sigma}(m)a^{(\sigma)}(n) = \sum_{\sigma=1}^{\kappa} \lambda_{\rho\sigma}(n)a^{(\sigma)}(m).$$

The matrix $(\lambda_{\rho\sigma}(0))$, which appears here for the first time, can be uniquely determined as to fulfill also this relation. Now we obtain

$$\begin{aligned} T_m(F^{(\rho)}(\tau)) &= \sum_{\sigma=1}^{\kappa} \lambda_{\rho\sigma}(m) \sum_{n=0}^{\infty} a^{(\sigma)}(n)e^{2\pi i n\tau} \\ &= \sum_{\sigma=1}^{\kappa} a^{(\sigma)}(m) \sum_{n=0}^{\infty} \lambda_{\rho\sigma}(n)e^{2\pi i n\tau}. \end{aligned}$$

This leads to the introduction of the functions

$$f_{\rho\sigma}(\tau) = \sum_{n=0}^{\infty} \lambda_{\rho\sigma}(n)e^{2\pi i n\tau}$$

which span up the same linear set $\{F^{(1)}(\tau), \cdots, F^{(\kappa)}(\tau)\}$ and of which it is easily seen that they belong to the same set. With $\lambda(n) = (\lambda_{\rho\sigma}(n))$ we have in

$$B(\tau) = \sum_{n=0}^{\infty} \lambda(n)e^{2\pi i n\tau}$$

a matrix of modular forms. In connection with (11) it is found that

$$\lambda(m)\cdot\lambda(n) = \sum_{d\mid(m,n)} d^{k-1}\lambda\left(\frac{mn}{d^2}\right), \tag{12a}$$

which in particular for $(m, n) = 1$ furnishes

$$\lambda(m)\cdot\lambda(n) = \lambda(mn) \tag{12b}$$

and also shows that the different matrices $\lambda(n)$ are commutative.

Now we go over with Hecke to a set of associated Dirichlet series

$$\Phi(s) = (\phi_{\rho\sigma}(s)) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s},$$

a matrix which, of course, stands for

$$\phi_{\rho\sigma}(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\rho\sigma}(n)}{n^s}, \qquad \rho, \sigma = 1, \cdots, \kappa.$$

Since each element of this matrix is associated to the modular form

$f_\mu(\tau)$ we infer the validity of statements (1) to (4) for each element and hence the functional equation

$$(2\pi)^{-s}\Gamma(s)\Phi(s) = (-1)^{k/2}(2\pi)^{s-k}\Gamma(k-s)\Phi(k-s)$$

in matrix notation. The relation (12b) makes it evident that we can write

$$\Phi(s) = \prod_{p}\sum_{l=0}^{\infty}\frac{\lambda(p^l)}{p^{ls}}$$

in which the summation of each infinite series can be effectuated by means of (12a) with the result:

$$\Phi(s) = \prod_{p}(\lambda(1) - \lambda(p)p^{-s} + \lambda(1)p^{k-1-2s})^{-1},$$

where $\lambda(1)$, according to definition, is the unit matrix of order κ. Here we have the analogue of the Euler product.

If we would start the whole process with a new basis, obtained from the former one by a linear substitution A, we would obtain the transformed matrices

$$\lambda^*(n) = A\cdot\lambda(n)\cdot A^{-1}$$

in our results. Such a transformation can be used to get matrices $\lambda(n)$ of an especially simple type. Since the $\lambda(n)$ are commutative they can simultaneously be transformed into matrices in which all elements beneath the principal diagonal are zeros. Beyond this H. Petersson [34] has shown that the λ's can be brought simultaneously into diagonal form, which means that we can start the process with κ linearly independent "eigen-functions" of all the linear operators

$$T_n(F^{(\rho)}(\tau)) = \lambda^*_{\rho\rho}(n)F^{(\rho)}(\tau), \qquad n = 1, 2, \cdots .$$

Petersson proves this in a very elegant manner by introducing a definition of orthogonality of modular forms based on integrals extended over the fundamental region. The $\lambda^*_{\rho\rho}$ are real algebraic numbers. We have now the Dirichlet series

(13a) $$\phi_{\rho\rho}(s) = \sum_{n=1}^{\infty}\frac{\lambda^*_{\rho\rho}(n)}{n^s}$$

with an ordinary Euler product

(13b) $$\phi_{\rho\rho}(s) = \prod_{p}\frac{1}{1-\lambda^*_{\rho\rho}(p)p^{-s}+p^{k-1-2s}}, \qquad \rho = 1, \cdots, \kappa,$$

which is exactly of the form as it appears in Ramanujan's work on $\Delta(1, \tau)$. Summarizing, we can roughly say that the Riemann functional equation of a Dirichlet series already implies the Euler factorization with respect to prime numbers, certainly a result in which analytic and arithmetic elements are inseparably welded together.

Unfortunately, Riemann's original coordination of a modular form and a Dirichlet series, namely,

$$\vartheta(\tau) = \frac{1}{2} + \sum_{n=1}^{\infty} e^{\pi i n^2 \tau}, \qquad \zeta(2s) = \sum_{n=1}^{\infty} \frac{1}{n^{2s}}$$

with the Euler product

$$\prod_{1}^{\infty} (1 - p^{-2s})^{-1}$$

does not fit into the scheme, as already the comparison of this Euler product with (13b) shows; indeed, $\vartheta(\tau)$ has the dimension $-1/2$, and for fractional dimensions the complicated transformation formulae introduce multipliers which interfere with the definition of the operators T_n.

Let us go back for a moment to the functions $\phi_{\mu\nu}(s)$ in (13a) and (13b). We are here at once confronted with all the problems familiar from the theory of the Riemann ζ-function: What is the distribution of the zeros of $\phi_{\mu\nu}(s)$? The critical strip is $0 \leqq \Re(s) \leqq k$; do all nontrivial zeros of $\phi_{\mu\nu}(s)$ lie on the middle line $\Re(s) = k/2$? In case $\phi_{\mu\nu}(s)$ belongs to a modular form with $c_0 = 0$ ("cusp form") we know a few facts. The function $\phi_{\mu\nu}(s)$ has no pole, of course, its series is absolutely convergent for $\Re(s) > (k+1)/2$, $\phi_{\mu\nu}(s)$ has no zeros on $\Re(s) = (k+1)/2$ but infinitely many ones on $\Re(s) = k/2$, results all due to R. A. Rankin [42]. Ramanujan has for $\phi_{\mu\nu}(s) = \Delta(1, \tau)$ made the conjecture $|\tau(p)| \leqq 2p^{11/2}$ and Petersson [34] has generalized it to

$$|\lambda_{\nu\nu}(p)| \leqq 2p^{(k-1)/2}$$

or, which is equivalent, to the conjecture that the quadratic equation $z^2 - \lambda_{\nu\nu}(p)z + p^{k-1} = 0$ (derived from the denominator in (13b) with $p^s = z$) does not have two different real roots, which, since $\lambda_{\nu\nu}(p)$ is real would mean that the roots have to be of absolute value $p^{(k-1)/2}$. This conjecture has a close similarity to Hasse's theorem which confirms the Riemann hypothesis for elliptic function fields over finite fields.

Hecke has extended his theory also to modular forms of level (Stufe) Q. The operators T_n with $(n, Q) = 1$ have the same behavior as the operators for $Q=1$; the other ones, however, introduce complications on which I cannot dwell. Hecke applies this extension of his theory to a study of the ϑ-functions generated by quadratic forms of several variables [20].

So far I have spoken only of the theory of rational integers on one side and of functions of one variable on the other. I would overstep the limits which I had to set myself in this discussion if I were to enter upon the problems presented by the fields of algebraic numbers. In many respects a full analogy with the rational field can act as guide. However, some new features appear here, as, for example, the group of units. An algebraic field of degree n can be illustrated by an n-dimensional point lattice. Some problems, in particular additive ones, lead to analytic functions of n complex variables; some other problems, in particular the multiplicative ones, are handled by means of a set of ζ-functions with one complex variable s and $(n-1)$ integer parameters (Hecke's $\zeta(s, \lambda)$-functions) [15]. The multiplicative theory of algebraic fields is, through Hecke's fundamental investigations, about as far advanced as that for the rational field; the additive number theory in algebraic fields, however, is still in its beginnings.

The theory of quadratic forms offers another way to come to functions of several variables, as Siegel has recently shown in his introduction of modular functions of several variables [48].

This report could, of course, mention only selected topics of analytic number theory. I do not wish to imply that no recent progress has been made in subjects which I did not discuss. There are, for example, the important contributions to the estimates of lattice points in the circle and in other areas due to van der Corput and his successors, and the investigations of Siegel and Gelfond concerning the transcendentality of certain types of numbers. I have purposely chosen such portions of analytic number theory which seemed to me to exhibit most clearly the present tendency, the trend, in contemporary research work. Our discipline has grown beyond the stage of mere application of analysis to number theory; we can characterize the present tendency as a deep mutual penetration of analysis and arithmetic. It is still so that the theory of analytic functions helps to solve arithmetical problems which otherwise appear as inaccessible; but number theoretical problems are the sources of new functions, and number theoretical arguments lead to a deeper understanding of the structure of known functions. I think that analytic number theory is a branch of mathematics that fascinates through its harmony and beauty and appeals to the working mathematician through its variety and fruitfulness.

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UNIVERSITY OF PENNSYLVANIA

NORMAL VARIETIES AND BIRATIONAL CORRESPONDENCES

OSCAR ZARISKI

1. **Introduction.** As one advances into the general theory of algebraic varieties, one reluctantly but inevitably reaches the conclusion that there does not exist a general theory of birational correspondences. This may sound too reckless a statement or too harsh a criticism, especially if one thinks of the fundamental role which birational transformations are supposed to have in algebraic geometry. Nevertheless our conclusion is in exact agreement with the facts and it is made with constructive rather than with critical intentions. It is true that the geometers have a fairly good intuitive idea of what happens or what may happen to an algebraic variety when it undergoes a birational transformation; but the only thing they know with any certainty is what happens in a thousand and one special cases. All these special cases—and they include all Cremona transformations—are essentially reducible to one special but very important case, namely, the case in which the varieties under consideration are nonsingular(that is, free from singular points). One can give many reasons for regarding as inadequate any theory which has been developed exclusively for nonsingular varieties. One rather obvious reason is that we have as yet no proof that every variety of dimension greater than 3 can be transformed birationally into a nonsingular variety.[1] But there are other, less transient, reasons. Were such a proof available, it would still be advisable to develop the theory of algebraic varieties, *as far as possible*, without restricting oneself to nonsingular projective models. This certainly would be the correct program of work from an arithmetic standpoint. I have a distinct impression that my friends the algebraists have not much use anyway for the resolution of the singularities. All they want is a general uniformization theorem, and now that they have it, they are content.

The following consideration will perhaps carry greater weight with the geometers. It turns out, as I have found out at some cost to myself, that we have to know a lot more about birational correspondences than we know at present before we can even attempt to carry

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An address delivered before the meeting of the Society in Bethlehem, Pa., on December 31, 1941, by invitation of the Program Committee; received by the editors January 22, 1942.

[1] The resolution of the singularities of three-dimensional varieties will be carried out in a forthcoming paper of mine.

out the resolution of the singularities of higher varieties. A general theory of birational correspondences is a necessary prerequisite for such an attempt. I shall have occasion later on to indicate some difficult questions concerning birational correspondences which arise in connection with the resolution of singularities.

2. **Birational correspondences and valuations.** From a formal point of view, there is nothing mysterious about a birational transformation. If $V=V_r^n$ is an irreducible r-dimensional algebraic variety in an n-dimensional projective space, the coordinates $\xi_1, \xi_2, \cdots, \xi_n$ of its general point are algebraic functions of r independent variables, and the field $\Sigma=K(\xi_1, \xi_2, \cdots, \xi_n)$ generated by these functions is *the field of rational functions on* V. Here K denotes the field of constants (in the classical case K is the field of complex numbers). If $V'=V_r'^m$ is another irreducible algebraic variety, with general point $(\xi_1', \xi_2', \cdots, \xi_m')$ and associated field $\Sigma'=K(\xi_1', \xi_2', \cdots, \xi_m')$, then the two varieties V and V' are *birationally equivalent* if the two fields Σ and Σ' are simply isomorphic: $\Sigma/K\cong\Sigma'/K$. A birational transformation is merely the process of passing from one variety to another, birationally equivalent, variety.

The difficulties begin when we wish to associate with this purely formal process a *geometric transformation*, that is, a correspondence between the points of the two varieties. From the equations of the transformation, in which the ξ''s are given as rational functions of the ξ's and vice versa, it is not difficult to conclude that the transformation sets up a (1, 1) correspondence between the non-special points of V and the non-special points of V'. The points of either variety for which the equations of the transformation fail to define corresponding points on the other variety are referred to as special points in the sense that they lie on certain algebraic subvarieties, of dimension less than r. In the classical case, considerations of continuity allow us to complete the definition of the correspondence also for these special points. In the abstract case we use valuation theory instead, as follows:

A *valuation* of the field Σ is an homomorphic mapping v of the multiplicative group $\Sigma-0$ (that is, the element zero excluded) upon an ordered additive abelian group Γ, which satisfies the well known valuation axioms: (1) $v(\omega_1\,\omega_2)=v(\omega_1)+v(\omega_2)$; (2) $v(\omega_1\pm\omega_2)\geqq\min\{v(\omega_1), v(\omega_2)\}$; (3) $v(\omega)\neq 0$, for some ω in Σ; (4) $v(c)=0$, for all constants $c\neq 0$. We put $v(0)=+\infty$.

In the case of algebraic functions of one variable, every valuation arises from a branch of our curve V. The value $v(\omega)$ is then *the order* *of* ω at the branch and is an integer. Positive and negative $v(\omega)$ signify, respectively, that the center of the branch is a zero or a pole of the function ω, while if $v(\omega)=0$ then the function-theoretic value of ω at the center of the branch is a finite constant, different from zero. In this special case, it is clear that from an algebraic standpoint the function-theoretic values of the elements of the field are the cosets of the *valuation ring* $\mathfrak{B}$ $\{\omega\in\mathfrak{B}\leftrightarrow v(\omega)\geq 0\}$, with respect to its subset consisting of the elements ω such that $v(\omega)>0$. This subset is a prime divisorless ideal $\mathfrak{p}$, and so the cosets form indeed a field; the field of complex numbers, in the classical case. This consideration is independent of the dimension of the field and can therefore be applied directly to the general case. It is therefore always possible to associate with any valuation v of Σ a mapping f of the elements of Σ upon the elements of another field (and the symbol ∞), *the field of residual classes* of $\mathfrak{B}$ mod $\mathfrak{p}$, and we may speak of $f(\omega)$, $\omega\in\Sigma$, as being the *function-theoretic value* of ω (if $v(\omega)<0$, then $f(\omega)=\infty$). This field is the so-called *residue field* of the valuation. However, if $r>1$ then the residue field may be a transcendental extension of the ground field K. Its degree of transcendence s over K, or briefly, *its dimension*, is at most $r-1$, and is referred to as the dimension of the valuation. A zero-dimensional valuation is called *a place* of the field Σ. The function-theoretic values of the elements of Σ at a given place are *constants*, that is, either elements of K or algebraic quantities over K.

Given a valuation v and a projective model V of Σ, with general point $(\xi_1, \xi_2, \cdots, \xi_n)$, it is permissible to assume that the function-theoretic values of the ξ's are different from ∞, since we may subject the coordinates ξ_i to an arbitrary projective transformation. Then the polynomials in the ξ's which have function-theoretic value zero form a prime ideal in the ring of all polynomials in the ξ's. This prime ideal defines an irreducible algebraic subvariety W of V. This subvariety W we call *the center of the valuation* v on the variety V. The dimension of W cannot exceed the dimension of the valuation. In particular, *the center of a place is always a point of* V.

The following geometric picture of a valuation is suggestive, although not entirely adequate. A zero-dimensional valuation, that is, a place, with center at a point P, corresponds to a way of approaching P along some one-dimensional branch, which may be algebraic, analytic, or transcendental. Similarly an s-dimensional valuation with an s-dimensional center W corresponds to a way of approaching W along an $(s+1)$-dimensional branch through W.

After these preliminaries, we define the birational correspondence between two birationally equivalent varieties V and V' as follows:

DEFINITION 1. *Two subvarieties W and W' of V and V', respectively, correspond to each other if there exists a valuation of the field Σ whose center on V is W and whose center on V' is W'.*

Note that our definition does not treat points in any privileged fashion. Any subvariety of V is treated as an element, rather than as a set of points. This procedure is much more convenient than the usual one in which corresponding loci are defined as loci of corresponding points.

In the study of the birational correspondence between V and V', it is found convenient to introduce a third variety $\bar{V}$ which is birationally related to both V and V', the so-called *join of V and V'* (or *the variety of pairs of corresponding points* of V and V'). $\bar{V}$ is defined as follows. We adjoin to Σ a new transcendental η_0 and we regard the $n+1$ quantities η_0, $\eta_1=\eta_0\xi_1, \cdots, \eta_n=\eta_0\xi_n$ as the *homogeneous coordinates* of the general point of V. Similarly, the $m+1$ quantities $\eta_0'=\eta_0$, $\eta_1'=\eta_0\xi_1', \cdots, \eta_m'=\eta_0\xi_m'$ will be the homogeneous coordinates of the general point of V'. The $(n+1)(m+1)$ quantities $\omega_{ij}=\eta_i\eta_j'$ can be regarded as the homogeneous coordinates of the general point of a variety birationally equivalent to V and to V'. This variety is our $\bar{V}$, the join of V and V'. The birational correspondence between $\bar{V}$ and V has the property that to any subvariety of $\bar{V}$ there corresponds a unique subvariety of V; in particular, to every point of $\bar{V}$ there corresponds a unique point of V. Similarly, for $\bar{V}$ and V'. Thus, both V and V' are single-valued transforms of $\bar{V}$. The properties of the birational correspondence between V and V' can be readily derived from the properties of the birational correspondences between $\bar{V}$ and V and between $\bar{V}$ and V'. We therefore replace one of the two varieties V, V', say V', by the join $\bar{V}$, that is, from now on we shall always assume that *V is a single-valued transform of V'.*

3. **Fundamental loci; geometric preliminaries.** On the basis of Definition 1, it is easy to prove that the points of V to which there correspond more than one point on V' constitute an algebraic subvariety F of V, and that every subvariety W of V to which there correspond more than one subvariety W' on V' must lie on F. This variety F is called the *fundamental locus* of the birational correspondence, and every W which lies on F is a *fundamental* variety. This is not our final definition, but it will do for the moment. Note that the fundamental locus on V' is an empty set, in view of our assumption that V is a single-valued transform of V'.

What corresponds on V' to a fundamental variety W of V? To this question we have a complete answer in the case of a nonsingular

V. We have, namely, in this case the following two fundamental theorems (see van der Waerden, *Algebraische Korrespondenzen und rationale Abbildungen*, Mathematische Annalen, vol. 110 (1934)):

A. *If W is an irreducible s-dimensional fundamental variety of V, then the transform of W is an algebraic subvariety of V' whose irreducible components are all of dimension greater than s.*

B. *The transform of the fundamental locus F is a pure $(r-1)$-dimensional subvariety of V'.*

The following examples show that both theorems fail to hold for singular models.

(1) If P is a point of V at which V is *locally reducible*, that is, if in the neighborhood of P the variety V consists of ν $(\nu>1)$ analytical r-dimensional branches, then in *special cases* it turns out that V is the projection of another variety V' on which these ν branches become separated.[3] Then the point P will be the projection of ν distinct points of V', and this contradicts Theorem A.

(2) Let Q be a ruled quadric surface in S_3 and let V be the three-dimensional cone which projects Q from a point O not in S_3. We take another copy of Q, say Q', which we now imagine as being immersed in an S_5. Let l be a line in S_5 which does not meet the S_3 containing the quadric. We set up a (1, 1) projective correspondence between the points P' of l and the lines p of one ruling of the quadric. Let V' be the irreducible three-dimensional variety generated by the planes (P', p), where P' and p are corresponding elements in the above projectivity. It is easy to set up a birational correspondence between V and V' in which to the planes (O, p) there correspond the planes (P', p). There will be no fundamental points on V', while O will be the only fundamental point on V. To the point O there corresponds on V' the line l, in contradiction with Theorem B.

One has the feeling that the second example does more damage than the first, because the first counterexample could be explained away on the basis that the point P, as origin of ν analytical branches, should not be regarded at all as a "point" of the variety, but rather as a point of the ambient projective space at which ν "points" of the variety accidentally happened to come together. This explanation, if stripped of all metaphysics, can have only one mathematical meaning, namely, it means, by implication, that in the general theory of birational correspondence, we should restrict ourselves to varieties which are locally irreducible at each point. What interests us in this argument is the formal admission that some kind of restriction as to the type of varieties to be studied is necessary in the theory of birational correspondences. Whether or not the restriction to locally irreducible varieties is the right one, is a debatable matter. For one thing, it is not certain that in making such a restriction we are not being too stingy, unless we can prove that the branches of a variety can always be separated by the method of projection. This is probably true and should not be too difficult to prove. Actually we regard this restriction as being too generous. For, besides the requirement that the varieties V of our hypothetical restricted class satisfy Theorem A, we find it essential that these varieties also satisfy the following additional condition:

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[3] Whether this is true generally, is not at all obvious and, in fact, has never been proved.

C. *If to a point P of V there corresponds a unique point P' of V', then the birational correspondence, regarded as an analytical transformation, is regular at P.*

The geometric meaning of this condition can be roughly indicated as follows. If this condition and condition A are satisfied for a given variety V, then the analytical structure of the neighborhood of any point P of V cannot be affected by a birational transformation, unless this transformation blows up P into a curve, or a surface and so on (always provided we replace the transform V' of V by the join $\bar{V}$; compare with §2). In particular, it is not possible to simplify any further the type of singularity which V possesses at P without doing a thing as radical as that of spreading out that singular point into a variety of dimension greater than 0. From this point of view, condition C can be looked upon as a sort of maximality condition.

In the case of algebraic curves it follows readily from this geometric interpretation that the only curves which satisfy conditions A and C are the nonsingular curves. But already in the case of surfaces we get a much wider variety of types. For instance, it can be proved that the surfaces in S_3 which satisfy our conditions are the surfaces which have only isolated singularities.

We now proceed to define arithmetically the varieties of our restricted class. We call these varieties *locally normal.* Included among these varieties are those which I have called *normal* varieties. The algebraic operation which plays a fundamental role in our arithmetic approach to the geometric questions just outlined is that of the *integral closure of a ring in its quotient field.* Theorem C follows in a relatively simple fashion from our arithmetic definitions and from some well known theorems in valuation theory. Theorem A lies much deeper and its proof is more difficult. As to Theorem B, it definitely must be sacrificed when we are dealing with singular varieties.

4. **Locally normal and normal varieties.** We use the homogeneous coordinates $\eta_0, \eta_1, \cdots, \eta_n$ of the general point of V (§2) and we define the *quotient ring* $Q(P)$ of any point P of V as the ring of all quotients $f(\eta)/g(\eta)$, where f and g are forms of like degree in $\eta_0, \eta_1, \cdots, \eta_n$ and where $g \neq 0$ at P. In other words, the quotient ring $Q(P)$ consists of all functions in our field Σ which have a definite and finite value at P. In a similar fashion, we define the quotient ring $Q(W)$ of any irreducible algebraic subvariety W of V by the condition that $g(\eta) \neq 0$ on W (that is, that g should not vanish at every point of W).

One is led to the consideration of quotient rings when one examines the equations of a birational correspondence between V and another variety V'. For it is seen immediately that if the nonhomogeneous coordinates $\xi_1', \cdots, \xi_m'$ of the general point of V' belong to the quotient ring $Q(W)$ of a given W on V, then to W there corresponds a unique subvariety W' on V' and moreover $Q(W')$ will be a subring of $Q(W)$. If $Q(W') = Q(W)$, then also W will be the only subvariety of V which corresponds to W'. In this case we say that the birational correspondence is *regular at* W, or *along* W. In particular, if the birational correspondence is regular at a point P of V, then as an analytical transformation it is regular in the neighborhood of P. Therefore, the quotient ring of a point determines uniquely the analytical structure of the neighborhood of the point. In the sequel we shall say that a birational correspondence is *regular* on V if it is regular at each point of V.

DEFINITION 2. *A variety V is locally normal along a subvariety W, if the quotient ring $Q(W)$ is integrally closed in its quotient field* (*that is, in* Σ).

DEFINITION 3. *V is locally normal, if it is locally normal at each point.*

The last definition refers only to points. The reason for this is the following: If V is locally normal at one point P of a subvariety W of V, then it is also locally normal along W. Hence if V is locally normal, in the sense of Definition 3, it is also locally normal along any W.

It is not difficult to show that V is locally normal if and only if the following condition is satisfied: *If $\mathfrak{C}$ is the conductor of the ring $K[\eta_0, \eta_1, \cdots, \eta_n]$ with respect to the integral closure of this ring in its quotient field, then the subvariety of V determined by the* (*homogeneous*) *ideal $\mathfrak{C}$ is empty.* This implies that either $\mathfrak{C}$ has no zeros at all, or its only zero is the trivial one: $(0, 0, \cdots, 0)$. Therefore, $\mathfrak{C}$ is either the unit ideal or is a primary ideal belonging to the irrelevant prime ideal $(\eta_0, \eta_1, \cdots, \eta_n)$.

If $\mathfrak{C}$ is the unit ideal, we say that V is *normal*, that is, we give the following definition:

DEFINITION 4. *A variety V is normal if the ring $K[\eta_0, \eta_1, \cdots, \eta_n]$ is integrally closed in its quotient field.*

It can be proved that the singular manifold of a locally normal r-dimensional variety is of dimension less than or equal to $r-2$ (in particular, *a locally normal curve is nonsingular*). The converse is not generally true, except in the case $r=1$, *since a nonsingular curve is always locally normal*. However, *the converse is true for hypersurfaces*, that is, for V_r's in an S_{r+1}. Thus any surface in S_3 is locally normal if and only if it has a finite number of singularities.

It is important to point out that *nonsingular varieties are always locally normal.*

Our definitions clearly indicate that normal varieties can differ from locally normal varieties only by some property at large, since locally they cannot be distinguished from each other. This difference at large is put into evidence by the following characterization of normal varieties, due to Muhly: *A variety V is normal if and only if the hypersurfaces of its ambient space, of any given order m, cut out on V a complete linear system.* Now it can be proved that locally normal varieties are characterized by the completeness of the above linear systems *for sufficiently high values of m.* This last result, in conjunction with the fact that every nonsingular V is locally normal, contains as a special case of the well known lemma of Castelnuovo concerning nonsingular curves. This lemma plays an important role in Severi's proof of Riemann-Roch's theorem for surfaces.

We have already pointed out that for locally normal varieties Theorems A and C hold true. Moreover it can be shown that these are the only varieties for which these theorems are true. It may be added that there is really no great loss of generality in confining the theory of birational correspondences to locally normal varieties. For it can be shown that any variety V determines uniquely, *to within regular birational transformations*, a locally normal variety V' which is birationally equivalent to V and which is such that: (a) to each point P' of V' there corresponds a unique point P of V, and we have always: $Q(P') \supseteq Q(P)$; (b) to any point P of V there corresponds a finite number of points on V'; (c) if V is locally normal at P, then the birational correspondence between V and V' is regular at P. It is not difficult to show that these properties of the birational correspondence between

V and V' imply that, to within a regular birational transformation, *V is the projection of V' from a center S_k which does not meet V'.*

It would be of interest to characterize the locally normal varieties for which Theorem B holds. It can be proved that the following condition is sufficient for the validity of Theorem B: If P is any point of V and if W is any $(r-1)$-dimensional subvariety through P, then a sufficiently high multiple of W should be *locally* (that is, at P) complete intersection of V with an hypersurface of the ambient space. In terms of ideal theory, this means that in the quotient ring of P a sufficiently high power of any minimal prime ideal should be *quasi-gleich* (in the sense of van der Waerden) to a principal ideal. This condition gives us a good insight into the "real" reason of the validity of Theorem B for nonsingular varieties; for we know that if P is a simple point, then every minimal prime ideal in $Q(P)$ is itself a principal ideal.

5. **Monoidal transformations.** I should now like to discuss briefly a special class of birational correspondences which seem to be very useful in the theory of singularities, whether we deal with the resolution of singularities or with the analysis of the composition of a singularity from the standpoint of infinitely near points. These special transformations are the hyperspace analogue of plane quadratic transformations, and they are therefore of importance also for the general theory of birational correspondences.

When we are dealing with locally normal varieties V, we find it most convenient to define fundamental varieties of a birational correspondence between V and another variety V', as follows:

DEFINITION 5. *An irreducible subvariety W of V is fundamental if a corresponding subvariety W' of V' exists such that $Q(W) \not\supseteq Q(W')$.*

We know from §4 that if $Q(W) \supseteq Q(W')$, then W' is the only subvariety which corresponds to W. Hence if W is fundamental, then the relation $Q(W) \not\supseteq Q(W')$ is true for any W' which corresponds to W. If the birational correspondence has no fundamental points on V and on V', then $Q(W) = Q(W')$, for *any* two corresponding subvarieties W and W', and the transformation is regular on V (and on V'). We do not regard two birational transformations as being essentially distinct if they differ only by a regular birational transformation.

We consider a birational correspondence between two locally normal varieties V and V', and we again restrict ourselves to the case in which the given correspondence has no fundamental points on V'. Then V' is in regular birational correspondence with the join $\mathbf{V}$ of V'

and V, and we may replace V' by V. Consequently, we assume that the equations of the birational correspondence between V and V' are of the form:

$$\rho\eta_{ij} = \eta_i\phi_j, \qquad i = 0, 1, \cdots, n; j = 0, 1, \cdots, m,$$

where $\eta_0, \eta_1, \cdots, \eta_n$ are the homogeneous coordinates of the general point of V and where the ϕ_j are forms of like degree in the η's. Here ρ is a factor of proportionality and the η_{ij} are the homogeneous coordinates of the general point of V'. It can be shown that the fundamental locus on V is given by the base manifold of the *linear system* $\lambda_0\phi_0 + \cdots + \lambda_m\phi_m = 0$, provided we first drop all fixed $(r-1)$-dimensional components of the system. In terms of ideal theory, it means that we first write each principal ideal (ϕ_j) as a power product of minimal primes, say $(\phi_j) = \mathfrak{A}\mathfrak{B}_j$, where $\mathfrak{A}$ is the highest common divisor of $(\phi_0), \cdots, (\phi_m)$. Then the fundamental locus is given by the ideal $(\mathfrak{B}_0, \mathfrak{B}_1, \cdots, \mathfrak{B}_m)$. This ideal is of dimension less than or equal to $r-2$.

The special transformations which we wish to discuss are those for which $(\phi_0, \phi_1, \cdots, \phi_m)$ is itself a prime ideal, of dimension $s \leqq r-2$ or differs from a prime ideal by an irrelevant primary component. For the lack of a better name, we call them *monoidal transformations*.[3] The irreducible subvariety W of V defined by the ideal $(\phi_0, \phi_1, \cdots, \phi_m)$ is called the *center* of the transformation. It is not difficult to see that a change of the base of the ideal $(\phi_0, \cdots, \phi_m)$ does not essentially affect the transformation. A *quadratic transformation* is a special case of a monoidal transformation, the center is in that case a point.

The effect of a monoidal transformation consists in that the center W is spread out into an $(r-1)$-dimensional irreducible subvariety W' of V'. Moreover, *points of W which are simple both for V and W, correspond to simple points of W'.* This is the main reason why a monoidal transformation is a useful tool in the resolution of singularities, since while it may conceivably simplify some singular points which lie on its center, it does not introduce new singularities.[4]

There are two outstanding problems concerning monoidal transformations which play a role in the problem of resolution of singularities, but which at the same time are decisively of interest in themselves. We proceed to outline these questions.

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[3] With some non-essential modifications, and without their projective trimmings, the space Cremona transformations, known as monoidal transformations, are monoidal transformations in our sense.

[4] There is one exception: to a simple point of V which is *singular* for W there may correspond singular points of V'! For this reason it is usually advisable to "smooth out" W, that is, to resolve the singularities of W, before one applies the monoidal transformation.

PROBLEM 1. *Given any birational correspondence between two nonsingular models V and V', and assuming that there are no fundamental points on V', show that the birational transformation can be decomposed into monoidal transformations.*

In other words, the question is to show that for nonsingular models the monoidal transformations form a set of generators of the birational group. It is very likely that this decomposition exists also when only V' is nonsingular.

PROBLEM 2. *Given any birational correspondence between two arbitrary (not necessarily nonsingular) models V and V', and assuming as before that there are no fundamental points on V', show that the fundamental varieties on V can be eliminated by monoidal transformations.*

By this I mean that it is asked to transform V by a sequence of monoidal transformations into another variety V^* such that the birational correspondence between V^* and V' has no fundamental points on V^*.

As to Problem 1, we have a proof in the case of surfaces. In this case, the result can be regarded as a generalization of the well known theorem of Noether on the decomposition of plane Cremona transformations into quadratic transformations, although Noether's theorem is not a special case of this general result. It may be well to clarify the connection between the two results. In the first place, a quadratic Cremona transformation is not at all a quadratic transformation in our sense. Our quadratic transformation has only one ordinary fundamental point, and its inverse has no fundamental points at all, while a plane quadratic transformation and its inverse both have three fundamental points, which in special cases may be infinitely near points. For this reason a plane Cremona transformation can never be a quadratic transformation in our sense. The transform of a plane π under a quadratic transformation in our sense is not a plane, but a certain rational surface M in S_5, or any other surface in regular birational correspondence with M. Of course, an ordinary quadratic transformation between two planes π and π' can be expressed as a product of quadratic transformations in our sense, or more precisely as the product of 3 successive quadratic transformations and of 3 inverses of quadratic transformations. Since our proof takes care of surfaces over abstract fields K, it yields immediately a corresponding result for the fundamental varieties W of dimension $r-2$ in the general case, for the adjunction of certain $r-2$ transcendentals to the ground field will make a surface out of the variety V and a point out of W. In particular, the decomposition into monoidal transformations is thus established for birational correspondences between nonsingular three-dimensional varieties, provided the correpondence has only fundamental curves, but no isolated fundamental points.

In applications of Problem 2, the main interest lies in the elimination of the *simple* fundamental varieties of V. In this case we have a complete proof, provided the resolution theorem is granted for varieties of *dimension two less* than the dimension of V. Thus, in the case of three-dimensional varieties we have to use only a thing as little as that of the resolution of singularities of an algebraic curve. It is clear that Problem 2 is to be viewed as a step in an inductive proof of the general theorem of the resolution of singularities rather than as a problem for the solution of which we first need that general theorem. The really important problem is Problem 1. Its solution seems to be essential for the resolution of singularities of higher varieties. Thus, it is possible to carry out the resolution of singularities of three-dimensional varieties, because we have the theorem of local uniformization for the varieties of this dimension, plus the solution of Problem 1 for surfaces over abstract fields of constants.

THE JOHNS HOPKINS UNIVERSITY

ON DIFFERENTIATION OF INTEGRALS AND APPROXIMATE CONTINUITY

ARTHUR ROSENTHAL

The following discussion is closely connected with Lebesgue's theorem that the derivative of an integral is equal to the integrand almost everywhere. It is well known that in generalizing this theorem to higher dimensions, great care must be exercised in the choice of the systems of intervals or sets used for n-dimensional differentiation.

Lebesgue[1] had already observed that arbitrary intervals (parallel to the axes) cannot be used for the generalization of that theorem, but only such intervals whose edges have a bounded ratio, or, more generally, such sets which are *regular* relative to the cubes. This also corresponds to the behavior of the most essential tool used in the proof, namely, Vitali's covering theorem. Saks[2] and, independently, Busemann and Feller[3] found later that there is a remarkable difference between the integrals of bounded[4] and unbounded functions: in the first (but not generally in the last) case, differentiation relative to arbitrary intervals (parallel to the axes) furnishes the integrand almost everywhere. But, according to Zygmund and Nikodym[5] and to Busemann and Feller,[3] even in the case of bounded integrands, differentiation relative to the system of all rectangular parallelopipeds (arbitrarily oriented) does not always furnish the integrand almost everywhere.

As to integrals in abstract spaces—in the case of bounded integrands, de Possel[6] gave necessary and sufficient conditions for the systems of sets used in differentiation to permit a generalization of Lebesgue's theorem; while in case of arbitrary integrands, de Possel[6]

Presented to the Society, September 5, 1941; received by the editors July 14, 1941.

[1] H. Lebesgue, Annales de l'École Normale, (3), vol. 27 (1910), pp. 363, 387.

[2] S. Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 232; *Theory of the Integral*, Warsaw, Lwow, 1937, p. 132.

[3] H. Busemann and W. Feller, Fundamenta Mathematicae, vol. 22 (1934), pp. 226–256.

[4] Further generalizations: A. Zygmund, Fundamenta Mathematicae, vol. 23 (1934), pp. 143–149; B. Jessen, J. Marcinkiewicz, A. Zygmund, Fundamenta Mathematicae, vol. 25 (1935), pp. 217–234.

[5] O. Nikodym, Fundamenta Mathematicae, vol. 10 (1927), pp. 167–168 (note cf. A. Zygmund).

[6] R. de Possel, Comptes Rendus de l'Académie des Sciences, Paris, vol. 201 (1935), pp. 579–581; Journal de Mathématiques, (9), vol. 15 (1936), pp. 391–409.

and Hahn (in a still unpublished manuscript[7]) gave sufficient conditions suggested by Vitali's covering theorem.

On the other hand, in the case of bounded integrands, the original result of Lebesgue has been supplemented in an essential way by a theorem of Denjoy,[8] namely, that differentiation of an integral furnishes the integrand certainly wherever the integrand is *approximately continuous*. Denjoy proved this theorem for the case of one dimension, Saks[3] generalized it for more dimensions, namely—as in Lebesgue's theorem for bounded functions—using arbitrary intervals. Of course, the definition of approximate continuity depends on the notion of the density of a set M, and as the density of M results from differentiation of the characteristic function of M, the notion of approximate continuity also depends on the systems of sets used for this differentiation. Therefore, it is natural to use the same systems for definition of density and consequently of approximate continuity as for differentiation of the integral, especially if we are speaking (as in the theorem of Denjoy) about approximate continuity and differentiation simultaneously.

In preparing the second volume of Hahn's *Reelle Funktionen*,[7] I observed that there is an essential difference between the generalization of Lebesgue's theorem and that of Denjoy: While in the case of *Lebesgue's* theorem (also for bounded integrands) only rather *special* systems of sets can be used for differentiation, it is possible for the generalization of *Denjoy's* theorem, in any metric space (as will be shown), to ***differentiate relative to quite arbitrary***, only ***indefinitely fine, systems of sets*** (see Theorem 1).[9] In a certain sense (namely, for ***regular*** derivatives), the approximate continuity at a point is even ***necessary and sufficient*** for the derivative of the integral to be equal to the integrand, assumed as bounded (see Theorem 4).

Another theorem of Denjoy[10] states that every measurable function is approximately continuous almost everywhere. The systems of sets used in the generalization of this theorem can be the same[11] as, but cannot be more general than, those used in the generalization of Lebesgue's theorem for bounded integrands, for the two theorems of Denjoy taken together immediately imply Lebesgue's theorem for bounded integrands.

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[7] This manuscript forms the basis of the second volume of Hahn's *Reelle Funktionen* which is now being elaborated and prepared for publication by the author of the present paper. The more general systems used by de Possel (see Footnote 6) in the case of bounded integrands, or the quite arbitrary "indefinitely fine" systems to be used here, had not been considered by Hahn. The notations of this second volume are used here.

[8] A. Denjoy, *Sur les fonctions dérivées sommables*, Bulletin de la Société Mathématique de France, vol. 43 (1915), p. 172.

[9] For instance, in the n-space relative to the system of all the rectangular parallelopipeds (arbitrarily oriented).

[10] A. Denjoy, loc. cit., p. 170; see also W. Sierpinski, Fundamenta Mathematicae, vol. 3 (1922), p 320.

Thus we have the following situation: For the generalization of Denjoy's theorem that the derivative of an integral furnishes the integrand (assumed as bounded) certainly wherever the integrand is approximately continuous, the systems of sets used for differentiation can be chosen quite arbitrarily; yet in order to get Lebesgue's result that the integrand is obtained almost everywhere (or Denjoy's second result that the integrand is approximately continuous almost everywhere), the systems of sets employed must be rather special.

Let R be a metric space, $\mathfrak{M}$ a σ-field consisting of subsets of R, with $R \in \mathfrak{M}$; let ϕ and ψ be totally additive, finite set functions in $\mathfrak{M}$; let $\psi(M) \geqq 0$ for $M \in \mathfrak{M}$, (so that $\psi(M)$ is monotone increasing in $\mathfrak{M}$), and let $\mathfrak{M}$ be complete for ψ.

A system $\mathfrak{Q} \subseteq \mathfrak{M}$ of sets may be called an ***indefinitely fine system of sets***, if to every point $a \in R$ there corresponds a certain subsystem $\mathfrak{Q}_a \subseteq \mathfrak{Q}$ consisting of non-empty sets such that for every $\rho > 0$ there is a set $Q \in \mathfrak{Q}_a$ contained in the sphere $S_{a\rho}$ (with center a and radius ρ).

If there is a sequence of sets $Q_\nu \in \mathfrak{Q}_a$ converging to a so that the sequence[12] $\phi(Q_\nu)/\psi(Q_\nu)$ converges to a limit d,[13] then d will be called a ***derivate*** of ϕ with respect to ψ at the point a relative to $\mathfrak{Q}$. The largest and smallest of these numbers d[13] (if a is fixed) are called the ***upper*** and ***lower derivate*** of ϕ with respect to ψ at a relative to $\mathfrak{Q}$:

$$\overline{D}(a, \phi, \psi, \mathfrak{Q}), \qquad \underline{D}(a, \phi, \psi, \mathfrak{Q}).$$

If

$$\overline{D}(a, \phi, \psi, \mathfrak{Q}) = \underline{D}(a, \phi, \psi, \mathfrak{Q}),$$

we call this value the ***derivative*** $D(a, \phi, \psi, \mathfrak{Q})$ of ϕ with respect to ψ at a relative to $\mathfrak{Q}$.

Following Lebesgue[1] we say that a sequence of sets $M_\nu \in \mathfrak{M}$ ***converges regularly*** to a (for ψ relative to $\mathfrak{Q}$), if there exists a number $\zeta(a) > 0$ and a sequence of sets $Q_\nu \in \mathfrak{Q}_a$ converging to a, such that for almost all ν we have:

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[11] We shall not prove it here.

[12] If $\psi(Q_\nu) = 0$, this quotient is to be 0, $+\infty$, $-\infty$, according as $\phi(Q_\nu) = 0$, >0, <0.

[13] Which may be finite or infinite.

$$(1) \qquad M_\nu \subseteq Q_\nu, \qquad \psi(M_\nu) \geqq \zeta\cdot\psi(Q_\nu).$$

If we have $\phi(M_\nu)/\psi(M_\nu) \to D(a, \phi, \psi, \mathfrak{O})$ for every sequence M_ν converging regularly to a (for ψ relative to $\mathfrak{O}$), we say that the *derivative* $D(a, \phi, \psi, \mathfrak{O})$ is *regular* at a.

If $A \in \mathfrak{M}$, let us set $\phi(M) = \psi(AM)$; then $\overline{D}(a, \phi, \psi, \mathfrak{O})$, $D(a, \phi, \psi, \mathfrak{O})$, and $D(a, \phi, \psi, \mathfrak{O})$ become the *upper density* $\bar{d}(a, A, \psi, \mathfrak{O})$, the *lower density* $\underline{d}(a, A, \psi, \mathfrak{O})$, and the *density* $d(a, A, \psi, \mathfrak{O})$ of A at a for ψ relative to $\mathfrak{O}$; all these values are greater than or equal to 0 and less then or equal to 1.

Let R^* be the set of all those $a \in R$ for which there is a sequence of sets $Q_\nu \in \mathfrak{O}_a$ converging to a, such that $\psi(Q_\nu) = 0$. Then we have:

$$(2) \qquad \underline{d}(a, A, \psi, \mathfrak{O}) = 0, \qquad \text{for all } a \in R^*.$$

LEMMA 1. *For every* $a \in R - R^*$ *we have*:

$$\underline{d}(a, A, \psi, \mathfrak{O}) + \bar{d}(a, R - A, \psi, \mathfrak{O}) = 1;$$

$$\bar{d}(a, A, \psi, \mathfrak{O}) + \underline{d}(a, R - A, \psi, \mathfrak{O}) = 1.$$

PROOF. If $Q \in \mathfrak{O}_a$, then $\psi(AQ) + \psi((R-A)Q) = \psi(Q)$; thus, if $\psi(Q) \neq 0$: $\psi(AQ)/\psi(Q) + \psi((R-A)Q)/\psi(Q) = 1$.

LEMMA 2. *If* $d(a, A_1, \psi, \mathfrak{O}) = 0$ *and* $d(a, A_2, \psi, \mathfrak{O}) = 0$, *then also* $d(a, A_1 + A_2, \psi, \mathfrak{O}) = 0$.

PROOF. Let $Q_\nu \in \mathfrak{O}_a$ be a sequence converging to a and choose $\epsilon > 0$; then $\psi(A_1 Q_\nu) \leqq \epsilon\psi(Q_\nu)$, $\psi(A_2 Q_\nu) \leqq \epsilon\psi(Q_\nu)$ for almost all ν; thus also $\psi((A_1 + A_2)Q_\nu) \leqq 2\epsilon\psi(Q_\nu)$, that is, $\psi((A_1 + A_2)Q_\nu)/\psi(Q_\nu) \to 0$.

LEMMA 3. *If* $d(a, A_1, \psi, \mathfrak{O}) = 1$ *and* $d(a, A_2, \psi, \mathfrak{O}) = 1$, *then also* $d(a, A_1 A_2, \psi, \mathfrak{O}) = 1$.

PROOF. According to (2): $a \in R - R^*$; therefore because of Lemma 1, $d(a, R - A_1, \psi, \mathfrak{O}) = 0$, $d(a, R - A_2, \psi, \mathfrak{O}) = 0$, which implies by Lemma 2, $d(a, R - A_1 A_2, \psi, \mathfrak{O}) = 0$; therefore again because of Lemma 1, $d(a, A_1 A_2, \psi, \mathfrak{O}) = 1$.

If α, β are two real numbers ($\pm\infty$ permitted), we write:[14] $\|\alpha - \beta\| = |S(\alpha) - S(\beta)|$, where $S(\xi)$ means the "bounding transformation" $\xi/(1 + |\xi|)$. Furthermore, $[f(x) < y]$ denotes the set of all $x \in R$, for which $f(x) < y$ holds; similar notations are used in analogous cases.

In the following, let f always be a *ψ-measurable* function (that is, measurable with respect to ψ as measure).

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[14] H. Hahn, *Reelle Funktionen*, I, Leipzig, 1932, p. 178.

The function f is called ***approximately continuous***[15] at the point a (for ψ relative to $\mathfrak{Q}$), if for every $\epsilon>0$ the set $[\|f(x)-f(a)\|<\epsilon]$ has density 1 at a (for ψ relative to $\mathfrak{Q}$). If $f(a)$ is finite, in this definition $\|f(x)-f(a)\|<\epsilon$ may be replaced by $|f(x)-f(a)|<\epsilon$.

Equation (2) immediately implies the following lemma.

LEMMA 4. *If f is approximately continuous at a (for ψ relative to $\mathfrak{Q}$), then $a\in R-R^*$.*

LEMMA 5.[16] *In order that f be approximately continuous at a (for ψ relative to $\mathfrak{Q}$), it is necessary and sufficient that for every $y>f(a)$ and for every $y'<f(a)$ the sets $[f(x)<y]$ and $[f(x)>y']$ have density 1 at a (for ψ relative to $\mathfrak{Q}$).*

NECESSITY. Let $\epsilon>0$ be sufficiently small; then $y>f(a)$, $y'<f(a)$ imply $[\|f(x)-f(a)\|<\epsilon]\subseteq[f(x)<y]$ and $\subseteq[f(x)>y']$.

SUFFICIENCY. If for every $y>f(a)$ and every $y'<f(a)$ the sets $[f(x)<y]$ and $[f(x)>y']$ have density 1 at a, then, because of Lemma 3, the same is true for the set $[f(x)<y]\cdot[f(x)>y']$ and thus also for the set $[\|f(x)-f(a)\|<\epsilon]$.

LEMMA 6. *In order that f be not approximately continuous at the point $a\in R-R^*$ (for ψ relative to $\mathfrak{Q}$), it is necessary and sufficient that there exists either a $y>f(a)$ or a $y'<f(a)$, so that the upper density (for ψ relative to $\mathfrak{Q}$) of the set $[f(x)\geqq y]$ or $[f(x)\leqq y']$ is positive at a.*

NECESSITY. Because of Lemma 5, there exists either a $y>f(a)$ or a $y'<f(a)$, such that the set $[f(x)<y]=A$ or $[f(x)>y']=A'$ has a lower density less than 1 at a. But $R-A=[f(x)\geqq y]$, $R-A'=[f(x)\leqq y']$, and, because of Lemma 1, we have either $\bar{d}(a, R-A, \psi, \mathfrak{Q})>0$ or $\bar{d}(a, R-A', \psi, \mathfrak{Q})>0$.

SUFFICIENCY. Using the same notation, we have either $\bar{d}(a, R-A, \psi, \mathfrak{Q})>0$ or $\bar{d}(a, R-A', \psi, \mathfrak{Q})>0$. Thus because of Lemma 1, either

$$\underline{d}(a, A, \psi, \mathfrak{Q}) < 1$$

or

$$\underline{d}(a, A', \psi, \mathfrak{Q}) < 1.$$

Therefore, because of Lemma 5, f is not approximately continuous at a.

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[15] A. Denjoy, loc. cit., p. 165.

[16] A. Denjoy, loc. cit., p. 169.

LEMMA 7. *If $|f| \leqq k$ holds, except perhaps on a set of ψ-measure zero, and if f is approximately continuous at a, then also $|f(a)| \leqq k$.*

PROOF. We set $[f(x) > k] = A$; then $\psi(A) = 0$. If $f(a) > k$, then, according to Lemma 5, A has density 1 at a; thus there exists a set $Q \in \mathfrak{Q}_a$, such that $\psi(AQ) > 0$; therefore we have $\psi(A) > 0$, contrary to $\psi(A) = 0$. In the same way we see that $f(a) < -k$ is impossible.

Now let the function f be not only ψ-measurable in R, but also *ψ-integrable* (that is, integrable with respect to the measure ψ) for all $M \in \mathfrak{M}$ in the sense of Radon and Fréchet;[17] we set $\phi(M) = (M)\int f d\psi$, where f is integrated with respect to ψ over the set $M \in \mathfrak{M}$. If the ψ-measurable function f is *ψ-bounded* (that is, bounded, except perhaps on a set of ψ-measure zero), then f is also ψ-integrable.

THEOREM 1. *Let $\mathfrak{Q}$ be an indefinitely fine system, let f be ψ-measurable and ψ-bounded*[18] *in R, and $\phi(M) = (M)\int f d\psi$. If f is approximately continuous at a for ψ relative to $\mathfrak{Q}$, then ϕ has the derivative $D(a, \phi, \psi, \mathfrak{Q}) = f(a)$ at a.*

PROOF. Let $Q_\nu \in \mathfrak{Q}_a$ be a sequence of sets converging to a. As we have $a \in R - R^*$ according to Lemma 4, we can assume $\psi(Q_\nu) \neq 0$. As f is ψ-bounded, because of Lemma 7 $f(a)$ is finite. For a given $\epsilon > 0$, we write: $B = [|f(x) - f(a)| < \epsilon]$. Then we have

$$\phi(Q_\nu) = (Q_\nu B)\int f d\psi + (Q_\nu - B)\int f d\psi. \tag{3}$$

Because of $|f(x) - f(a)| < \epsilon$ on B, we have

$$(Q_\nu B)\int f d\psi = (f(a) + \theta_1 \epsilon)\cdot\psi(Q_\nu B), \qquad |\theta_1| \leqq 1. \tag{4}$$

As f is ψ-bounded, there is a finite c, such that $|f(x)| \leqq c$, except perhaps for a set of ψ-measure zero; thus we have

$$(Q_\nu - B)\int f d\psi = \theta_2 c \psi(Q_\nu - B), \qquad |\theta_2| \leqq 1. \tag{5}$$

As f is approximately continuous at a, we have $d(a, B, \psi, \mathfrak{Q}) = 1$, thus $\psi(Q_\nu B)/\psi(Q_\nu) \to 1$, and therefore

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[17] J. Radon, Sitzungsberichte, Akademie der Wissenschaften, Vienna, IIa, vol. 122 (1913), pp. 1324–1332; M. Fréchet, Comptes Rendus de l'Académie des Sciences, Paris, vol. 160 (1915), pp. 839–840; Bulletin de la Société Mathématique de France, vol. 43 (1915), pp. 248–265.

[18] That the condition, f be ψ-bounded, is essential is already shown by an example given by A. Denjoy, loc. cit., p. 173 (for the 1-dimensional case and ψ meaning Lebesgue's linear measure).

$$\frac{\psi(Q_\nu - B)}{\psi(Q_\nu)} \to 0. \tag{6}$$

As $\epsilon > 0$ is arbitrarily small, both these relations, together with (3), (4), (5) give $\phi(Q_\nu)/\psi(Q_\nu) \to f(a)$.

THEOREM 2. *In Theorem* 1, *the derivative* $D(a, \phi, \psi, \mathfrak{Q})$ *is regular.*

PROOF. Let $((M_\nu))$ be a sequence of sets regularly converging to a (for ψ relative to $\mathfrak{Q}$); that is, there must exist a sequence $Q_\nu \in \mathfrak{Q}_a$ converging to a, and a $\zeta > 0$, such that (1) hold. Because of Lemma 4, we can assume: $\psi(Q_\nu) \neq 0$, thus according to (1) also $\psi(M_\nu) \neq 0$. Now (1), (6) imply: $\psi(M_\nu - B)/\psi(M_\nu) \to 0$, and therefore $\psi(M_\nu B)/\psi(M_\nu) \to 1$. As (3), (4), (5) hold, if we substitute M_ν for D_ν, we get as before: $\phi(M_\nu)/\psi(M_\nu) \to f(a)$.

An inverse of the Theorems 1 and 2 is contained in the following theorem, for which, however, we do not have to assume that f is ψ-bounded, but only that it is ψ-integrable.

THEOREM 3. *Let* $\mathfrak{Q}$ *be an indefinitely fine system and* $\phi(M) = (M)\int f d\psi$. *If at the point* $a \in R - R^*$, *the derivative* $D(a, \phi, \psi, \mathfrak{Q})$ *is regular and equal to* $f(a)$, *then* f *is approximately continuous at* a *(for* ψ *relative to* $\mathfrak{Q}$).

PROOF. Assume that f is not approximately continuous at a (for ψ relative to $\mathfrak{Q}$). Then, because of Lemma 6, there exists either a $y > f(a)$ or a $y' < f(a)$, such that the upper density (for ψ relative to $\mathfrak{Q}$) of the set $[f(x) \geq y] = C$ or $[f(x) \leq y'] = C'$ is positive at a. Suppose, we have the first case (the second case is to be treated in the same way). There exists a sequence of sets $Q_\nu \in \mathfrak{Q}_a$, converging to a, so that $\psi(C\ Q_\nu)/\psi(Q_\nu) \to \bar{d}(a, C, \psi, \mathfrak{Q}) > 0$; thus there exists a $\zeta(a) > 0$, so that $\psi(CQ_\nu) > \zeta(a) \cdot \psi(Q_\nu) > 0$ for almost all ν, that is, the sequence $((C\ Q_\nu))$ converges regularly to a (for ψ relative to $\mathfrak{Q}$). Because $f(x) \geq y$ on C and $y > f(a)$, we therefore have $\phi(CQ_\nu) = (CQ_\nu)\int f d\psi \geq y\psi(CQ_\nu) > f(a)\psi(CQ_\nu)$ for almost all ν; thus the sequence $\phi(CQ_\nu)/\psi(CQ_\nu)$ cannot converge to $f(a)$.

The Theorems 1, 2, 3, together give the following result:

THEOREM 4. *Let* $\mathfrak{Q}$ *be an indefinitely fine system, let* f *be* ψ*-measurable and* ψ*-bounded in* R, *and* $\phi(M) = (M)\int f d\psi$. *In order that at the point* $a \in R - R^*$, ϕ *have the regular derivative* $D(a, \phi, \psi, \mathfrak{Q}) = f(a)$, *it is necessary and sufficient, that* f *be approximately continuous at* a *for* ψ *relative to* $\mathfrak{Q}$.

UNIVERSITY OF NEW MEXICO

A NOTE ON HILBERT'S OPERATOR

H. KOBER

The transformation

$$(1)\quad \mathfrak{H}f = \frac{1}{\pi} P\,V \int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{dt}{t} \{f(x+t) - f(x-t)\}$$

is well known to have the following properties:

LEMMA 1.[1] *When $1 < p < \infty$, then $\mathfrak{H}f$ is a continuous (bounded) linear transformation with both domain and range $L_p(-\infty, \infty)$, and $\mathfrak{H}^2 f = -f$.*

LEMMA 2.[2] *When $f(t) \in L_1(-\infty, \infty)$, then $\mathfrak{H}f$ exists for almost all x in $(-\infty, \infty)$, but does not necessarily belong to $L_1(a, b)$, where a, b are arbitrary numbers $(-\infty \leqq a < b \leqq \infty)$; however $(1+x^2)^{-1}|\mathfrak{H}f|^q \in L_1(-\infty, \infty)$ when $0 < q < 1$. When f and $\mathfrak{H}f$ belong to $L_1(-\infty, \infty)$, then $\mathfrak{H}^2 f = -f$.*

The case $p = 1$ appears to present the greatest difficulties. In the present note I shall deal with the set of elements $f(t) \in L_1(-\infty, \infty)$ for which $\mathfrak{H}f \in L_1(-\infty, \infty)$. In consequence of the lemmas, in this set or in $L_p(-\infty, \infty)$ $(1 < p < \infty)$, $\mathfrak{H}f$ has no characteristic values other than $\pm i$. We shall start from the sets of characteristic functions and, incidentally, from the class $\mathfrak{H}_p$, the theory of which has been developed by E. Hille and J. D. Tamarkin; $\mathfrak{H}_p$ is the set of functions $F(z)$ $(z = x + iy)$ which, for $y > 0$, are regular and satisfy the inequality

$$(2)\qquad \int_{-\infty}^{\infty} |F(x+iy)|^p dx \leqq M^p \quad \text{or} \quad |F(z)| \leqq M$$

for $0 < p < \infty$ or $p = \infty$, respectively, where M depends on F and p only.[3] By $\mathfrak{K}_p$ we denote the corresponding class defined for $y < 0$, and by $F(t)$, $G(t)$ the limit-functions[3] $(y \to 0;\ x = t)$ of elements $F(z) \in \mathfrak{H}_p$, $G(z) \in \mathfrak{K}_p$. By $\mathfrak{H}_p'$ and $\mathfrak{K}_p'$, respectively, we denote the two sets of those limit-functions, and by $\mathfrak{H}_p' + \mathfrak{K}_p'$ the smallest linear manifold

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Received by the editors August 5, 1941.

[1] M. Riesz, Mathematische Zeitschrift, vol. 27 (1928), pp. 218–244.

[2] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford, 1937, §5.14. E. Hille and J. D. Tamarkin, Fundamenta Mathematicae, vol. 25 (1935), pp. 329–352. Comparing our notation with that of Hille-Tamarkin, we have $\mathfrak{H}f = -\tilde{f}$.

[3] Loc. cit., $1 \leqq p < \infty$. T. Kawata, Japanese Journal of Mathematics, vol. 13 (1936), pp. 421–430, $0 < p < \infty$. The limit-functions exist for almost all t in $(-\infty, \infty)$ and belong to $L_p(-\infty, \infty)$.

containing both $\mathfrak{H}_p'$ and $\mathfrak{K}_p'$. Obviously an element $f(t)$ belongs to $\mathfrak{H}_p'+\mathfrak{K}_p'$ if and only if it can be represented in the form

$$f(t) = F(t) + G(t) = F(t) + \overline{F}_1(t), \qquad F \in \mathfrak{H}_p', F_1 \in \mathfrak{H}_p', G \in \mathfrak{K}_p', \tag{3}$$

and this representation is unique, except for a constant when $p=\infty$.

Theorem 1 (b), as yet unpublished, is due to H. R. Pitt, Aberdeen, to whom I am greatly indebted.

We obtain the following results:

LEMMA 3. *Let* $1 \leqq p \leqq \infty$; *let the norm of an element* $\phi(z)$ *belonging to* $\mathfrak{H}_p$ *or* $\mathfrak{K}_p$ *be defined by*

$$| \phi(t) |_p = \left\{ \int_{-\infty}^{\infty} | \phi(t) |^p dt \right\}^{1/p} \quad \text{or} \quad | \phi(t) |_p = \operatorname*{ess.u.b.}_{-\infty<t<\infty} | \phi(t) | \tag{4}$$

for $1 \leqq p < \infty$ *or* $p=\infty$, *respectively. Then* $\mathfrak{H}_p$ *and* $\mathfrak{K}_p$ *are complete normed linear spaces, that is to say,* (*B*) *spaces in the terminology of Banach.*

THEOREM 1. *Let* $f(t) \in L_1(-\infty, \infty)$. (a) *A necessary condition that* $\mathfrak{H}f \in L_1(-\infty, \infty)$ *is*

$$\int_{-\infty}^{\infty} f(t) dt = 0. \tag{5}$$

(b) (*Pitt's theorem.*) *The condition is not sufficient.*

THEOREM 2. (a) *A necessary and sufficient condition that both f and* $\mathfrak{H}f$ *belong to* $L_1(-\infty, \infty)$ *is that f belongs to* $\mathfrak{H}_1'+\mathfrak{K}_1'$. (b) *With domain* $\mathfrak{H}_1'+\mathfrak{K}_1'$, $\mathfrak{H}f$ *is a linear closed*[4] *unbounded transformation in* $L_1(-\infty, \infty)$.

THEOREM 3. *The set* $\mathfrak{H}_1'+\mathfrak{K}_1'$ *is a non-closed subspace of* L_1 *and is nowhere dense in* L_1. *Its closure is the subset of* L_1 *satisfying* (5).

We note that, by Lemma 1 and by the argument which will be employed in the proof of Theorem 2(a), $\mathfrak{H}_p'+\mathfrak{K}_p'=L_p$ for $1<p<\infty$.

We shall now give the proofs of the above results; some examples will be given at the end of this paper.

Proof of Lemma 3. We need only show that the space $\mathfrak{H}_p$ is complete. Let $\{F_n(z)\} \in \mathfrak{H}_p$, $n=1, 2, \cdots$, be a sequence satisfying the condition of convergence $| F_n(t) - F_m(t) |_p \to 0$ $(m>n\to\infty)$. Then there exists an element $F(t) \in L_p$ such that $| F(t) - F_n(t) |_p \to 0$ as $n\to\infty$; we have to show that $F(t)$ is the limit-function of an element $\varphi(z) \in \mathfrak{H}_p$.

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[4] That is to say, $f_n \in \mathfrak{H}_1'+\mathfrak{K}_1'$, $g_n = \mathfrak{H}f_n$ $(n=1, 2, \cdots)$, $|f-f_n|_1 \to 0$ and $|g-g_n|_1 \to 0$ imply $g = \mathfrak{H}f$.

By a result due to Hille and Tamarkin,[5] $F_n(z)$ is represented by its "proper Poisson integral"

$$F_n(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yF_n(t)dt}{(t-x)^2+y^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yF_n(t+x)}{t^2+y^2}\,dt, \qquad z = x+iy;\ y \to 0. \tag{6}$$

Let

$$\varphi(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yF(t+x)}{t^2+y^2}\,dt. \tag{7}$$

For $y \geqq \epsilon > 0$, by Hölder's theorem, we have uniformly

$$\left|\varphi(z) - F_n(z)\right| \leqq \pi^{-1}\epsilon^{-1/p}\left|(t^2+1)^{-1}\right|_{p'}\left|F(t) - F_n(t)\right|_p \to 0 \tag{8}$$

as $n \to \infty$, where $1/p + 1/p' = 1$. Hence $\varphi(z)$ is a regular function for $y > 0$, and it is obviously bounded when $p = \infty$.

Now let $1 \leqq p < \infty$. By a well known convexity theorem,

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\varphi(x+iy)\right|^p dx &\leqq \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y\,dt}{t^2+y^2}\int_{-\infty}^{\infty}\left|F(t+x)\right|^p dx\\ &= \int_{-\infty}^{\infty}\left|F(x)\right|^p dx.\end{aligned} \tag{9}$$

Thus $\varphi(z) \in \mathfrak{H}_p$. By the same argument and by Fatou's theorem,

$$\begin{aligned}\left|\varphi(t) - F_n(t)\right|_p &\leqq \liminf_{y\to 0}\left\{\int_{-\infty}^{\infty}\left|\varphi(z) - F_n(z)\right|^p dx\right\}^{1/p}\\ &\leqq \left|F(t) - F_n(t)\right|_p \to 0.\end{aligned}$$

By (8), the result holds for $p = \infty$. Therefore $F(t) \equiv \varphi(t)$, which completes the proof.

To prove Theorem 2(a) we require a result which we deduce from theorems by Hille and Tamarkin:

LEMMA 4. *A necessary and sufficient condition that $f(t)$ belongs to $L_p(-\infty, \infty)$ $(1 \leqq p < \infty)$ and that $\mathfrak{H}f = if$ or $\mathfrak{H}f = -if$ is that $f(t)$ belongs to $\mathfrak{H}_p'$ or $\mathfrak{K}_p'$, respectively.*

Let $f(t) \in \mathfrak{H}_p'$ and $f(t) = \varphi(t) + i\psi(t)$. Since $f(t)$ is the limit-function $(y \to 0,\ x = t)$ of an element $F(z) \in \mathfrak{H}_p$, and since $F(z)$ is represented by its proper Poisson integral, we have[6] $\psi(x) = -\mathfrak{H}\varphi$ and, by Lemmas 1 and 2, $\varphi(x)=\mathfrak{H}\psi$; therefore $\mathfrak{H}f=\mathfrak{H}(\varphi+i\psi)=-\psi+i\varphi=if$. Conversely, let $f(t)\in L_p$ and $\mathfrak{H}f=if$. For $y>0$, the function

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[5] Loc. cit., Theorem 2.1 (ii), $1 \leqq p < \infty$. The result holds for $p = \infty$.

[6] Hille and Tamarkin, loc. cit., Theorem 3.1. For $p = 2$, the lemma is an easy consequence of Theorem 95, Titchmarsh, loc. cit.

$$F(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}\,dt$$

is regular and representable by its proper Poisson integral, and its limit function is $(1/2)\{f(x)-i\mathfrak{H}f\}=(1/2)(f(x)+f(x))=f(x)$.[6] Hence $F(z)\in\mathfrak{H}_p$, and so $f(t)\in\mathfrak{H}_p'$, which proves the lemma.

Proof of Theorem 2(a). Let $f\in L_1$ and $\mathfrak{H}f\in L_1$, then the functions

$$F=(1/2)(f-i\mathfrak{H}f),\qquad G=(1/2)(f+i\mathfrak{H}f)$$

belong to L_1; by Lemma 2, $\mathfrak{H}^2f=-f$, and so $\mathfrak{H}F=iF$, $\mathfrak{H}G=-iG$. By Lemma 4, we have $F\in\mathfrak{H}_1'$, $G\in\mathfrak{K}_1'$, and so $f=F+G\in\mathfrak{H}_1'+\mathfrak{K}_1'$.

Conversely, let $f\in\mathfrak{H}_1'+\mathfrak{K}_1'$. Then, by (3) with $p=1$, and by Lemma 4,

$$\mathfrak{H}f=\mathfrak{H}F+\mathfrak{H}G=i(F-G)\in L_1(-\infty,\infty),$$

which proves Theorem 2(a). Part (b) will be proved after Theorem 3.

To prove Theorem 1, we need a further result due to Hille and Tamarkin.[7]

LEMMA 5. *Let* $1\leqq p<\infty$, *Let* $\phi(t)$ *belong to* L_p *and possess a Fourier transform* $\psi(x)$,

$$\psi(x)=(2\pi)^{-1/2}\ \underset{N\to\infty}{\text{l.i.m.}}\ \text{index}\ p'\int_{-N}^{N}\phi(t)e^{-itx}dt,\qquad 1/p+1/p'=1.$$

Then $\phi(t)\in\mathfrak{H}_p'$ *or* $\mathfrak{K}_p'$ *if and only if* $\psi(x)$ *vanishes in* $(-\infty,0)$ *or in* $(0,\infty)$, *respectively.*

For completeness we add the following result:

LEMMA 5′. *Let* $2<p<\infty$ *and let* $\phi(t)$ *belong to* L_p *and have no Fourier transform in* $L_{p'}$. *Then* $\phi(t)\in\mathfrak{H}_p'$ *or* $\mathfrak{K}_p'$ *if and only if there is a sequence* $\{\phi_n(t)\}$ *belonging to* $\mathfrak{H}_p'$ *or* $\mathfrak{K}_p'$ *and satisfying the hypotheses of Lemma* 5 *and such that* $|\phi(t)-\phi_n(t)|_p\to 0$ *as* $n\to\infty$.[8]

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[7] Loc. cit., Lemma 4.2, and Annals of Mathematics, (2), vol. 34 (1933), pp. 606–614, Theorem 3.

[8] The proof is similar to that given for the generalization of a theorem due to Paley-Wiener; H. Kober, Quarterly Journal of Mathematics, vol. 11 (1940); pp. 66–80, Theorem 2(b). Let $\phi(t)\in\mathfrak{H}_p'$ and $|\phi(t)-f_n(t)|_p\to 0$, where $f_n(t)$ $(n=1,2,\cdots)$ has a Fourier transform in L_p'; then the functions $\phi_n(t)=(1/2)(f_n-i\mathfrak{H}f_n)$ have the desired properties. The converse is proved by Lemma 3.

By Theorem 2 and Lemma 5, $f(t)$ can be represented as the sum of two functions $F(t)$, $G(t)$ belonging to L_1 and such that their Fourier transforms $\varphi(x)$, $\psi(x)$ vanish for $x<0$ or $x>0$, respectively. By continuity, they also vanish at $x=0$; so does the Fourier transform of $f(t)$, which gives (5). For the Fourier transform of an element $F(t)\in L_1$ is continuous in $(-\infty, \infty)$.

To prove Theorem 1(b), take $f_1(t)=t^{-1}\log^{-2} t$ and $f_2=2/\log 2$ in $(0, 1/2)$, $f_1(t)=f_2(t)=0$ otherwise. Let $f(t)=f_1(t)-f_2(t)$, then obviously $f(t)$ belongs to $L_1(-\infty, \infty)$ and satisfies (5). But $\mathfrak{H}f$ does not belong to $L_1(-\infty, \infty)$, since

$$\int_{-1/2}^{0} |\mathfrak{H}f_1|\, dx = \infty,$$

$$\int_{-1/2}^{0} |\mathfrak{H}f_2|\, dx = \frac{2}{\pi\log 2}\int_{-1/2}^{0}\left|\log\left|1-\frac{1}{2x}\right|\right| dx < \infty.$$

For, in $(0, 1/2)$, we have

$$\mathfrak{H}[f_1; -x] = \frac{1}{\pi}\int_0^{1/2}\frac{\log^{-2} t\, dt}{(x+t)t} \geqq \frac{1}{\pi}\int_0^{x}\frac{\log^{-2} t\, dt}{(x+t)t}$$

$$\geqq \frac{1}{\pi}\int_0^{x}\frac{\log^{-2} t\, dt}{2xt} = \frac{1}{2\pi x|\log x|};$$

hence $\mathfrak{H}f_1$ does not belong to $L_1(-1/2, 0)$, which proves the theorem.

Proof of Theorem 3. Let E be the subset of L_1 satisfying (5). By Theorems 1 and 2, $\mathfrak{H}_1'+\mathfrak{K}_1'$ is a subset of E and different from E. It is easy to see that E is closed in L_1. We are left to show that E is the closure of $\mathfrak{H}_1'+\mathfrak{K}_1'$.

Let $f(t)$ be a step-function belonging to E. Denoting by $\mathfrak{e}(t)$ the step-function which is equal to 1 in $(0, 1)$ and to zero otherwise, we can represent $f(t)$ by a finite sum $\sum a_n\mathfrak{e}(t/b_n)$ $(b_n \lessgtr 0$, a_n complex). By (5), $\sum a_n|b_n|=0$, and so

$$\pi\mathfrak{H}f = \sum a_n\left(\log\left|1-\frac{b_n}{x}\right|\right)\operatorname{sgn} b_n = O(x^{-2}), \qquad x\to\pm\infty.$$

Hence $\mathfrak{H}f\in L_1$, $f\in\mathfrak{H}_1'+\mathfrak{K}_1'$. We can now approximate to any $f(t)\in E$ by a sequence $\{f_n(t)\}$ $(n=1, 2, \cdots)$ of step-functions belonging to $\mathfrak{H}_1'+\mathfrak{K}_1'$. Let $f(t)$ satisfy (5), and let $\{g_n(t)\}$ be a sequence of step-functions such that $|f(t)-g_n(t)|_1\to 0$ as $n\to\infty$. Take

$$f_n(t) = g_n(t) - \mathfrak{e}(t)\int_{-\infty}^{\infty} g_n(\xi)d\xi, \qquad n = 1, 2, \cdots.$$

Then $f_n(t)$ is a step-function, $f_n(t)$ satisfies (5); therefore $f_n \in \mathfrak{H}_1' + \mathfrak{K}_1'$. Finally, by (5), we have

$$|f - f_n|_1 = \left| f - g_n - e\int_{-\infty}^{\infty} \{f(\xi) - g_n(\xi)\}\, d\xi \right|_1$$

$$\leqq |f - g_n|_1 + \left| \int_{-\infty}^{\infty} \{f(\xi) - g_n(\xi)\}\, d\xi \right| \leqq 2|f - g_n|_1,$$

which tends to zero as $n \to \infty$. Thus E is the closure of $\mathfrak{H}_1' + \mathfrak{K}_1'$.

The set E is nowhere dense in L_1. For when $f \in L_1$, then, given $\epsilon > 0$, any element $g(t) = f(t) + \delta e(t)$ $(0 < |\delta| < \epsilon)$ belongs to the sphere $|g - f|_1 < \epsilon$, while g does not belong to E; when f belongs to L_1 but not to E, then no element g of the sphere $|g - f|_1 < |\int f(t)dt|$ belongs to E. Thus we have proved the theorem.

Proof of Theorem 2(b). In the domain $\mathfrak{H}_1' + \mathfrak{K}_1'$, by Lemma 2, we have $i\mathfrak{H}(i\mathfrak{H}f) = f$; hence $i\mathfrak{H}f$ is involutory. By Lemma 3, both $\mathfrak{H}_1'$ and $\mathfrak{K}_1'$ are closed spaces in L_1. Therefore $i\mathfrak{H}f$, and therefore $\mathfrak{H}f$, is closed; for a linear involutory transformation in a (B) space is closed if and only if the spaces of the characteristic functions are closed.[9] By Theorem 3, $\mathfrak{H}_1' + \mathfrak{K}_1'$ is not closed. Therefore $\mathfrak{H}f$ is not bounded in this domain; for a linear closed transformation in a (B) space is continuous if and only if its domain is closed.[10] Thus we have proved the theorem.

The following are examples for the case $f \in L_1(-\infty, \infty)$, $\mathfrak{H}f \in L_1(-\infty, \infty)$. We may start from Lemma 4,[11] but it is easier to make use of Theorem 2.

(1) Let $T_1(z)$ or $T_2(z)$ be polynomials of degree $\alpha > 0$ or $\beta > 0$ and such that they have no zeros for $y \geqq 0$ or $y \leqq 0$, respectively; let a, b be any numbers such that $-\infty < a < -1/\alpha$, $-\infty < b < -1/\beta$. Then, on a suitable Riemann surface, any branch of $\{T_1(z)\}^a$ $(y > 0)$ or $\{T_2(z)\}^b$ $(y < 0)$ belongs to $\mathfrak{H}_1$ or $\mathfrak{K}_1$, respectively. When $f(t) = \{T_1(t)\}^a + \{T_2(t)\}^b$, by Lemma 4, we have $\mathfrak{H}f = i\{T_1(x)\}^a - i\{T_2(x)\}^b \in L_1$.

(2) Let $\varphi_1(z) = (1 - \cos \alpha z)z^{-2}$, $\varphi_2(z) = \{\sin \alpha z - 2\sin(\alpha z/2)\}z^{-2}$, $\alpha > 0$, and let $f(t) = A\varphi_1(t)e^{i\alpha t} + B\varphi_2(t)e^{-i\alpha t}$; then $\mathfrak{H}f = iA\varphi_1(x)e^{i\alpha x} - iB\varphi_2(x)^{-i\alpha x}$, and $f \in L_1$, $\mathfrak{H}f \in L_1$. It can be shown that this result holds when $\varphi_j(z)$ $(j = 1, 2)$ are integral functions such that $\varphi_j(t) \in L_1$ and that, for any $\epsilon > 0$, $|\varphi_j(z)| < K_\epsilon \exp\{(\alpha + \epsilon)|z|\}$; in this way we can construct

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[9] H. Kober, Proceedings of the London Mathematical Society, (2), vol. 44 (1938), pp. 453–465, Theorem 6′(a).

[10] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 41, Theorem 7. Probably the converse is well known.

[11] Or from Theorem 3.1, Hille and Tamarkin, loc. cit.

all integral functions $f(z)$ satisfying the conditions $f(t) \in L_1$, $\mathfrak{H}f \in L_1$, $|f(z)| < K_{f,\epsilon} \exp\{(2\alpha+\epsilon)|z|\}$. The proof is based upon a result due to Plancherel and Pólya.[12]

THE UNIVERSITY,
EDGBASTON, BIRMINGHAM, ENGLAND

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[12] Commentarii Mathematici Helvetici, vol. 10 (1937–1938), pp. 110–163, §27.

THE BEHAVIOR OF CERTAIN STIELTJES CONTINUED FRACTIONS NEAR THE SINGULAR LINE

H. S. WALL

1. Introduction. We consider here continued fractions of the form[1]

$$f(z) = \frac{g_0}{1\ +}\ \frac{g_1 z}{1\ +}\ \frac{(1-g_1)g_2 z}{1}\ +\ \frac{(1-g_2)g_3 z}{1}\ + \cdots, \tag{1.1}$$

in which $g_0 \geqq 0$, $0 \leqq g_n \leqq 1$, $(n=1, 2, 3, \cdots)$, it being agreed that the continued fraction shall terminate in case some partial numerator vanishes identically. There exists a monotone non-decreasing function $\phi(u)$, $0 \leqq u \leqq 1$, such that

$$f(z) = \int_0^1 \frac{d\phi(u)}{1+zu}; \tag{1.2}$$

and, conversely, every integral of this form is representable by such a continued fraction. Put $M(f) = \text{l.u.b.}_{|z|<1} |f(z)|$. Then $M(f) \leqq 1$ if and only if the continued fraction can be written in the form

$$f(z) = \frac{h_1}{1\ +}\ \frac{(1-h_1)h_2 z}{1}\ +\ \frac{(1-h_2)h_3 z}{1}\ + \cdots, \tag{1.3}$$

in which $0 \leqq h_n \leqq 1$, $(n=1, 2, 3, \cdots)$. These functions are analytic in the interior of the z-plane cut along the real axis from $z=-1$ to $z=-\infty$.

The principal object of this paper is to prove the following theorem:

THEOREM 1.1. *If* $0 < h_n < 1$, $(n=1, 2, 3, \cdots)$, *and* $h_n \rightarrow 1/2$ *in such a way that the series* $\sum |h_n - 1/2|$ *converges, then the function* $f(z)$ *given*

Presented to the Society, October 25, 1941; received by the editors August 14, 1941.

[1] H. S. Wall, *Continued fractions and totally monotone sequences*, Transactions of this Society, vol. 48 (1940), pp. 165–184.

by (1.3) *approaches a finite limit* $\alpha(s)$ *as* $z \to -s$, $s \geqq 1$, *from the upper half-plane, and the limit* $\overline{\alpha(s)}$, *the complex conjugate of* $\alpha(s)$, *as* $z \to -s$ *from the lower half-plane. The function* $\alpha(s)$ *is continuous, and is real if and only if* $s=1$. *There is a constant C such that* $|f(z)| < C$ *over the entire plane of* z *exterior to the cut along the real axis from* $z=-1$ *to* $z=-\infty$.

Inasmuch as the function (1.1) can be written in the form $f(z)=g_0/[1+zf^*(z)]$, where $f^*(z)$ has the form (1.3), one may conclude at once that if $g_0>0$, $0<g_n<1$, $(n=1, 2, 3, \cdots)$, $\sum|g_n-1/2|$ converges, then the function $f(z)$ given by (1.1) approaches a finite limit $\beta(s)$ as $z \to -s$, $s>1$, from the upper half-plane, and the limit $\overline{\beta(s)}$ as $z \to -s$ from the lower half-plane. The function $\beta(s)$ is continuous and not real for $s>1$. The function $f(z)$ given by (1.1) may become infinite as $z \to -1$, for example, if $g_0=1$, $g_n=1/2$ $(n=1, 2, 3, \cdots)$, then $f(z)=(1+z)^{-1/2}$.

2. **Proof of Theorem** 1.1. There is a one to one correspondence between functions of the form (1.3) and functions $e(x)$ which are real when x is real, analytic for $|x|<1$, and for which $M(e) \leqq 1$, such that if $f(z) \leftrightarrow e(x)$ then[2]

$$(2.1)\quad \frac{1}{2}(1-x)\frac{1-e(x)}{1+xe(x)} = f(z), \quad z = 4x/(1-x)^2, \quad |x|<1.$$

(i) The transformation $z=4x/(1-x)^2$ maps the interior of the circle $|x|=1$ one to one upon the interior of the z-plane cut along the real axis from $z=-1$ to $z=-\infty$. Hence it follows at once from (2.1) that if $M(e)<1$, then

$$|f(z)| \leqq \frac{1+M(e)}{1-M(e)} = C,$$

over the entire domain of analyticity of $f(z)$.

(ii) In (2.1) put $x=\xi+i\eta$, $e(x)=u+iv$, $f(z)=P+iQ$, where ξ, η, u, v, P, Q are all real. We then find for Q the value

$$(2.2)\qquad Q = \frac{\eta(u^2+v^2-1)+v(\xi^2+\eta^2-1)}{2|1+xe(x)|^2}.$$

If $s \geqq 1$, $\sigma=[s-2+2i(s-1)^{1/2}]/s$, so that $|\sigma|=1$, then as $x \to \sigma$ from the interior of the circle $|x|=1$, z must approach $-s$ from the upper half-plane. If $M(e)<1$, and $e(x)$ approaches a limit $e(\sigma)$ as $x \to \sigma$, $|x|<1$, then it follows from (2.1) that $f(z)$ approaches a finite limit

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[2] H. S. Wall, *Some recent developments in the theory of continued fractions*, this Bulletin, vol. 47 (1941), pp. 405–423; Theorem 5.1, p. 415.

$\alpha(s)$ as $z \to -s$ from the upper half-plane; and from (2.2) it follows that Q has the limit

$$\frac{(s-1)^{1/2}}{s} \cdot \frac{|e(\sigma)|^2 - 1}{|1 + \sigma e(\sigma)|^2}, \tag{2.3}$$

where is zero if and only if $s = 1$. Hence $\alpha(s)$ is real if and only if $s = 1$. Inasmuch as $f(\bar{z}) = \overline{f(z)}$, it follows that $f(z)$ has the limit $\overline{\alpha(s)}$ as $z \to -s$ from the lower half-plane. Clearly $\alpha(s)$ is continuous if $e(x)$ is continuous for $|x| \leqq 1$.

(iii) To complete the proof of Theorem 1.1 it remains to be proved that when $\sum |h_n - 1/2|$ converges then $M(e) < 1$ and $e(x)$ is continuous for $|x| \leqq 1$. Put $e_0(x) = e(x)$,

$$e_{n+1}(x) = \frac{1}{x} \frac{t_n - e_n(x)}{1 - t_n e_n(x)}, \qquad t_n = e_n(0);\ n = 0, 1, 2, \cdots. \tag{2.4}$$

Then $t_{n-1} = 1 - 2h_n$ $(n = 1, 2, 3, \cdots)$. Now, Schur[3] proved that if $|t_{n-1}| < 1$, $(n = 1, 2, 3, \cdots)$, and $\sum |t_n|$ is convergent, then $M(e) < 1$, and $e(x)$ is continuous for $|x| \leqq 1$. Since $0 < h_n < 1$ by hypothesis, it follows that $-1 < t_{n-1} < 1$; and since the series $\sum |h_n - 1/2|$ converges by hypothesis, it follows that $\sum |t_n|$ converges.

This completes the proof of Theorem 1.1.

It will be seen from (2.3) that if $f(z)$ has a real limit as $z \to -s$, $s > 1$, then $M(e) = 1$. This is true also if $f(z)$ becomes infinite as $z \to -s$, $s \geqq 1$, and in this case $e(x) \to -1/\sigma$ as $x \to \sigma$. Inasmuch as $\lim_{z \to -s} (z + s) f(z) = 0$, $\lim_{z \to \infty} f(z) = 0$, if $M(e) < 1$, it follows that the corresponding mass function $\phi(u)$ (cf. (1.2)), is continuous for $0 \leqq u \leqq 1$ in this case.[4]

3. **An example.** If we apply the transformation (2.4) to a function $f(z)$ of the form (1.3) we obtain a sequence of functions $f_0(z) = f(z)$, $f_1(z)$, $f_2(z)$, $\cdots$ all having continued fraction expansions of the same character as that of $f(z)$. Suppose that in (1.3), $0 < h_n < 1$, $(n = 1, 2, 3, \cdots)$, and that the series $\sum |h_n - 1/2|$ converges. On applying Theorem 1.1 we find at once that as $z \to -s$, $s \geqq 1$, $I(z) > 0$:

$$\lim f_1(z) = -\frac{1}{s} \frac{g_1 - \alpha(s)}{1 - g_1 \alpha(s)} = \alpha_1(s);$$

and that $\alpha_1(s)$ is real if and only if $s = 1$; $\alpha_1(1) = 1$; $\alpha_1(s)$ is continuous for $s \geqq 1$. By mathematical induction, $f_2(z)$, $f_3(z)$, $\cdots$ also have this property. Let $h_1'/1+(1-h_1')h_2'z/1+(1-h_2')h_3'z/1+\cdots$ be the continued fraction for $f_1(z)$. *Then we shall prove that* $\sum|h_n'-1/2|$ *may diverge although* $\sum|h_n-1/2|$ *converges, and that the convergence of the series* $\sum|h_n-1/2|$ *is not necessary in order that the conclusion in Theorem* 1.1 *shall hold.* For this purpose, let $h_n=1/2$, $(n=1, 2, 3, \cdots)$. Then $f(z)=1/[1+(1+z)^{1/2}]$, $(f(0)=1/2)$, and $f_1(z)=1/[1+(1+z)^{1/2}][1+2(1+z)^{1/2}]$, $(f_1(0)=1/6)$. The function $f_1(z)$ has the properties stated in Theorem 1.1 for the function $f(z)$ of that theorem, excepting that, as we shall see, the series $\sum|h_n'-1/2|$ diverges. In fact, $h_{2n}'=(4n+3)/2(4n+1)$, $h_{2n-1}'=(4n-3)/2(4n-1)$, $(n=1, 2, 3, \cdots)$, in consequence of the following theorem:

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[3] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind*, Journal fur die reine und angewandte Mathematik, vol. 147 (1916), pp. 205–232, and vol. 148 (1917), pp. 122–145.

[4] I. J. Schoenberg, *Über die asymptotische Verteilung reeller Zahlen* mod 1, Mathematische Zeitschrift, vol. 28 (1928), pp. 171–199; p. 179.

THEOREM 3.1. *Let k be a parameter subject only to the conditions*

$$k \neq (3-4n)/2, \qquad (1-4n)/2, \qquad n=1, 2, 3, \cdots, \tag{3.1}$$

and put

$$\begin{aligned} h_{2n-1}^{(k)} &= (4n-3)/2(4n-3+2k), \\ h_{2n}^{(k)} &= (4n-1+4k)/2(4n-1+2k), \qquad n=1, 2, 3, \cdots. \end{aligned}$$

Then the continued fraction $f_k(z)=h_1^{(k)}/1+(1-h_1^{(k)})h_2^{(k)}z/1+(1-h_2^{(k)})h_3^{(k)}z/1+\cdots$ *converges uniformly in a sufficiently small neighborhood of* $z=0$, *and the analytic function* $f_k(z)$ *satisfies the relation*

$$f_{k+1}(z) = \frac{1}{z}\,\frac{h_1^{(k)}-f_k(z)}{1-h_1^{(k)}f_k(z)}\,. \tag{3.2}$$

PROOF. The uniform convergence follows from the fact that all the partial numerators after the first are numerically less than or equal to 1/4 for z in a sufficiently small neighborhood of the origin. To prove (3.2), write the right-hand member in the form:

$$\frac{1}{z}\left\{h_1^{(k)}-\frac{1-(h_1^{(k)})^2}{-h_1^{(k)}+1/f_k(z)}\right\} = \frac{1}{z}\left\{h_1^{(k)}-\frac{h_1^{(k)}}{1}\,+\,\frac{h_2^{(k)}z/(1+h_1^{(k)})}{1}\right.$$
$$\left.+\,\frac{(1-h_2^{(k)})h_3^{(k)}z}{1}\,+\,\frac{(1-h_3^{(k)})h_4^{(k)}z}{1}\,+\cdots\right\}.$$

We are to show that this is equal to $f_{k+1}(z)$. This can be done by showing that the odd part of the last continued fraction is identical with the even part of the continued fraction for $f_{k+1}(z)$. We omit here the details of the calculation.[6]

Let $e_n(x) \leftrightarrow f_n(z)$, $(n=0, 1, 2, \cdots)$. Then we find for the e_n's the following recursion formulas:

$$e_{n+1}(x) = \frac{1}{x} \frac{k_n + (2 - k_n)x + (3x - 1)e_n(x)}{(3 - x) + (2 - k_n + k_n x)e_n(x)}, \quad k_n = e_n(0), \tag{3.3}$$

$(n=0, 1, 2, \cdots)$. For the special example under consideration, $e(x) = e_0(x) \equiv 0$ and $e_1(x) = 2/(3-x)$. Hence, although $M(e) < 1$ in this case, nevertheless $M(e_1) = 1$. From the way in which (3.3) was obtained it follows that if $e_0(x)$ is an arbitrary function which is real when x is real, analytic for $|x| < 1$, and such that $M(e_0) \leqq 1$, then the functions $e_1(x)$, $e_2(x)$, $\cdots$ are all of this same character.

Northwestern University

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[6] O. Perron, *Die Lehre von den Kettenbruchen*, 2d edition, Leipzig and Berlin, 1929 p. 201.

ON LOCAL CONVEXITY IN HILBERT SPACE

I. J. SCHOENBERG

1. **Introduction.** Let M be a set of points in a normed vector space V. M is said to be convex if p and q being any two points of M, the whole rectilinear segment $\overline{pq}$ belongs to M. A weakened form of convexity, due to H. Tietze,[1] is as follows:

Definition of local convexity. *Let $p\in M$. The set M is said to be locally convex at p if there exists a positive $\rho=\rho(p)$ such that the intersection $M\cdot S(p;\rho)$, of M with the open sphere $S(p;\rho)$ of center p and radius ρ, is convex in the ordinary sense. The set M is called locally convex if M is locally convex at all its points.*

Every convex set is locally convex. The converse is not true since every open set is obviously locally convex. Tietze's chief result concerning local convexity is as follows:

Theorem 1 (of Tietze). *Let E_k denote the k-dimensional euclidean space. A closed and connected set M in E_k which is locally convex is also convex in the ordinary sense.*

By means of his concept of local euclidean dimension of a set M at a point $p\in M$, Tietze reduces the proof of Theorem 1 to the case of locally convex continua with interior points and which coincide with the closure of their set of interior points. Tietze then proves the theorem for continua in E_2 and finally extends the proof to cover any E_k. Tietze's method does not seem to be applicable for sets in Hilbert space.

The following lines contain a simpler method of dealing with this problem which allows the establishment of Tietze's theorem in any real or complex normed vector space whether separable or not. For the sake of definiteness we state and prove our theorem for real Hilbert space.[2]

Presented to the Society, September 8, 1939 under the title *On local convexity in euclidean spaces*; received by the editors August 13, 1941. The present note is an improved version (see Footnote 6).

[1] H. Tietze, *Über Konvexheit im kleinen und im grossen und uber gewisse den Punkten einer Menge zugeordneten Dimensionszahlen*, Mathematische Zeitschrift, vol. 28 (1928), pp. 697–707.

[2] Recent results have demonstrated the breakdown of some classical properties of convex sets when we pass from euclidean spaces to Hilbert spaces. See, for example, David Moskowitz and L. L. Dines, *Convexity in linear spaces with an inner product*, Duke Mathematical Journal, vol. 5 (1939), pp. 520–534, in particular the example on

THEOREM 2. *A closed and connected set M in the real Hilbert space H which is locally convex is also convex in the ordinary sense.*

2. On δ-convexity and three lemmas.

All our sets are in Hilbert space unless otherwise stated. A strengthened form of local convexity is as follows:

DEFINITION OF δ-CONVEXITY.[3] *Let δ be a positive number. A set is said to be δ-convex if p and q being any two points of M, of distance $pq<\delta$, the whole segment $\overline{pq}$ belongs to M.*

It is clear that δ-convexity implies local convexity. The converse is not true. Thus the following two sequences of points on the x-axis (space E_1)

$$M_1: \quad \left\{\frac{1}{n}\right\}, \qquad n = 1, 2, \cdots,$$

$$M_2: \quad \left\{1+\frac{1}{2}+\cdots+\frac{1}{n}\right\}, \qquad n = 1, 2, \cdots,$$

are clearly locally convex but not δ-convex, no matter how small δ may be. Notice that M_1 is bounded but not closed and M_2 is closed but not bounded, hence none of these two sets is self-compact. For self-compact sets both types of convexity are equivalent, a fact which we now state.

LEMMA 1. *A self-compact set M which is locally convex is also δ-convex for some appropriate value of δ.*

Indeed, the contrary assumption to the effect that M is never δ-convex, no matter how small δ is, implies that for every integer n our set M contains a pair of points a_n, b_n with $a_nb_n<1/n$ and such that the segment $\overline{a_nb_n}$ does not wholly belong to M. Since M is self-compact we may assume $a_n\rightarrow p$, $b_n\rightarrow p$ and $p\in M$. But this clearly contradicts the local convexity of M at the limiting point p.

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pp. 531–532. This makes it the more remarkable that Tietze's theorem does hold in Hilbert space.

[3] The δ-convexity is readily seen to be equivalent to *uniform* local convexity. By this we mean local convexity of a set in the sense of our first definition with a radius $\rho=\rho(p)>0$ which is independent of p.

A δ-convex set is a very special instance of a *metric space with elementary arcs* in the sense of Marston Morse, *The Calculus of Variations in the Large*, American Mathematical Society Colloquium Publications, vol. 18, New York, 1934, p. 298. See also J. H. C. Whitehead, *Convex regions in the geometry of paths*, Quarterly Journal of Mathematics, vol. 3 (1932), pp. 33–42.

Our principal tool will be the following elementary geometrical lemma in euclidean space:

LEMMA 2.[4] *Let* $p_0, p_1, \cdots, p_n$ $(p_0 \neq p_n)$ *be points in the euclidean space* E_k. *Let* Π *denote the open polygonal line joining successively the given points* $p_0, p_1, \cdots, p_n$. *Regarding* E_k *as a vector space we derive a new sequence of points by setting*

$$p'_{-1} = p_0, \quad p'_0 = \tfrac{1}{2}(p_0 + p_1), \quad p'_1 = \tfrac{1}{2}(p_1 + p_2), \cdots,$$
$$p'_{n-1} = \tfrac{1}{2}(p_{n-1} + p_n), \quad p'_n = p_n.$$

Repeating the same operation on the newly derived broken line $\Pi' = p'_{-1}p'_0 \cdots p'_n$, *let*

$$p''_{-2} = p'_{-1} = p_0, \quad p''_{-1} = \tfrac{1}{2}(p'_{-1} + p'_0), \quad p''_0 = \tfrac{1}{2}(p'_0 + p'_1), \cdots,$$
$$p''_{n-1} = \tfrac{1}{2}(p'_{n-1} + p'_n), \quad p''_n = p'_n = p_n.$$

Repeating this operation indefinitely we obtain an infinite sequence of broken lines

$$\Pi^{(m)} = p^{(m)}_{-m}p^{(m)}_{-m+1} \cdots p^{(m)}_n, \qquad m = 1, 2, 3, \cdots,$$

all joining the points p_0 *and* p_n. *Let* N *be an open set containing the closed segment* $\overline{p_0p_n}$. *Then* N *will contain the whole broken line* $\Pi^{(m)}$ *for all sufficiently large values of* m.

Since our proof will be independent of the dimension k we may without loss of generality assume that $k=2$. Let $p_i^{(0)} = p_i$ and $\Pi^{(0)} = \Pi$. For convenience we now extend the definition of the points $p_i^{(m)}$ $(-m \leqq i \leqq n)$ for all integral values of i $(=0, \pm 1, \pm 2, \cdots)$ by setting

$$(1) \qquad \begin{aligned} p_i^{(m)} &= p_{-m}^{(m)} = p_0 \quad \text{if} \quad i \leqq -m, \\ p_i^{(m)} &= p_n^{(m)} = p_n \quad \text{if} \quad i \geqq n. \end{aligned}$$

We then have without exception the relation

$$p_i^{(m)} = \tfrac{1}{2}(p_i^{(m-1)} + p_{i+1}^{(m-1)}), \qquad m = 1, 2, \cdots; i = 0, \pm 1, \pm 2, \cdots,$$

and therefore

$$(2) \qquad p_i^{(m)} = \frac{1}{2^m}\left\{p_i + C_{m,1}p_{i+1} + C_{m,2}p_{i+2} + \cdots + p_{i+m}\right\}.$$

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[4] Lemma 2 was suggested by the related problem (Number 3547) proposed by Martin Rosenman, American Mathematical Monthly, vol. 39 (1932), p. 239, and by its elegant solution due to R. E. Huston, ibid., vol. 40 (1933), pp. 184–185.

We now choose in our plane E_2 a coordinate system (x, y) such that $p_0=(0, 0)$, $p_n=(1, 0)$. Furthermore, let the open set N contain the open rectangle

$$(3) \qquad R:\quad -\epsilon < x < 1+\epsilon, \qquad -\epsilon < y < \epsilon.$$

Finally let $p_i=(x_i, y_i)$. In view of (2) and (3) it suffices to establish the two inequalities

$$(4) \qquad -\epsilon < \frac{1}{2^m}\{x_i + C_{m,1}x_{i+1} + \cdots + x_{i+m}\} < 1+\epsilon,$$

$$(5) \qquad -\epsilon < \frac{1}{2^m}\{y_i + C_{m,1}y_{i+1} + \cdots + y_{i+m}\} < \epsilon,$$

for sufficiently large values of m. From (1), for $m=0$, we get

$$\cdots = x_{-2} = x_{-1} = x_0 = 0, \qquad x_n = x_{n+1} = x_{n+2} = \cdots = 1,$$
$$\cdots = y_{-2} = y_{-1} = y_0 = 0, \qquad y_n = y_{n+1} = y_{n+2} = \cdots = 0.$$

But then our inequalities (4) and (5) obviously hold for sufficiently large values of m, for the sum of any set of at most $n-1$ consecutive ones among the binomial coefficients $1, C_{m,1}, C_{m,2}, \cdots, 1$, divided by their total sum $2^m=1+C_{m,1}+\cdots+1$, tends uniformly to zero as $m\rightarrow\infty$. Hence our lemma is proved.[5]

Our geometrical Lemma 2 will be made effective for our purpose by the following third and last lemma which states the effect of the assumption of local convexity on connected sets.

LEMMA 3. *Let M be a set in Hilbert space which is closed, connected and locally convex. Then M is also connected in the following stronger sense: Any two points a and b of M may be joined by a finite polygonal line lying wholly in M.*[6]

Indeed, denote by N the set of points of M which may be joined to the fixed point a by a finite polygonal line lying in M.

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[5] A proof of Theorem 2 in the complex Hilbert space requires a proof of Lemma 2 in the k-dimensional complex euclidean (unitary) space U_k. The proof just given is readily adaptable to the space U_2 of two complex coordinates $x=x'+ix''$, $y=y'+iy''$. The only change is that the rectangular neighborhood (3) of the segment $\overline{p_0p_n}$ is to be replaced by the open set defined by $-\epsilon<x'<1+\epsilon$, $-\epsilon<x''<\epsilon$, $-\epsilon<y'<\epsilon$, $-\epsilon<y''<\epsilon$. This is essentially again the real case for the space E_4.

[6] Open connected sets are known to enjoy this stronger connectivity property. In fact Lemma 3 implies this classical property since open sets are locally convex as already mentioned.

Lemma 3 was pointed out to the author by S. Kakutani and J. W. Tukey. The earlier version of this paper was thereby simplified and the method became applicable to Hilbert space.

N is closed in M. Indeed, let $q_i \to q$, $q_i \in N$, $q \in M$. Since $M \cdot S(q; r)$ is convex for a certain $r > 0$, as soon as $q_i \in S(q; r)$ we have $\overline{q_i q} \subset M$, hence $q \in N$.

N is open in M. For if $q \in N$ clearly all points of $M \cdot S(q; r)$ will be in N, provided r is sufficiently small.

The connectedness of M now implies that $N = M$.

3. **Proof of Theorem** 2. Let M be a closed and connected set in H which is also locally convex. Let a and b be any two points of M. We want to show that $\overline{ab} \subset M$. By Lemma 3 there is a sequence of points $p_1, \cdots, p_{n-1}$ such that the polygonal line $\Pi = a p_1 p_2 \cdots p_{n-1} b$ lies in M. Let E_n be a euclidean subspace of H which contains this polygonal line Π. Let furthermore $\bar{S}(a; r)$ be a closed sphere of H of sufficiently large radius r so as to contain our polygonal line Π. The intersection

$$M_1 = M \cdot E_n \cdot \bar{S}(a; r)$$

is obviously closed and bounded, hence also self-compact as being a subset of E_n. But M_1 is also locally convex, being the intersection of three locally convex sets. By Lemma 1 we know that M_1 is δ-convex for some appropriate value of δ. On the other hand our assumptions insure that $\Pi \subset M_1$. By subdividing, if necessary, the sides of Π we may assume that the sides $p_i p_{i+1}$ of the line Π are all of length less than δ. We now perform the construction of Lemma 2. Since $\Pi \subset M_1$, the δ-convexity of M_1 and the triangle inequality imply that also $\Pi' \subset M_1$. Since all sides of all derived polygonal lines $\Pi^{(m)}$ are of length less than δ, induction with respect to m will show that $\Pi^{(m)} \subset M_1$ for all values of m. By Lemma 2 $\Pi^{(m)}$ converges to the segment $\overline{ab}$ as $m \to \infty$. Since M_1 is closed we indeed have $ab \subset M_1$, hence $\overline{ab} \subset M$. Thus M is convex and our theorem is proved.

UNIVERSITY OF PENNSYLVANIA

ON 3-DIMENSIONAL MANIFOLDS

C. E. CLARK

Let P be a 3-dimensional manifold.[1] Let Q be a 2-dimensional manifold imbedded in P. Moreover, let P and Q admit of a *permissible simplicial division* K, that is, a simplicial division of P such that some subcomplex of K, say L, is a simplicial division of Q. Let K_i and L_i denote the ith normal subdivisions of K and L, respectively. We define the *neighborhood* N_i of L_i to be the simplicial complex consisting of the simplexes of K_i that have at least one vertex in L_i together with the sides of all such simplexes. By the *boundary* B_i of N_i we mean the simplicial complex consisting of the simplexes of N_i that have no vertex in L_i. Our purpose is to prove the following theorem.

THEOREM. *The boundary B_1 is a two-fold but not necessarily connected covering of Q, and change of permissible division K replaces B_1 by a homeomorph of itself.*

PROOF. The neighborhood N_1 is the sum of a set of 3-dimensional simplexes. Some of these 3-simplexes, say $a_1, a_2, \cdots$, have exactly one vertex in L_1, others, say $b_1, b_2, \cdots$, have exactly two vertices in L_1, while the remaining, say $c_1, c_2, \cdots$, have three vertices in L_1. Since K_1 is a normal subdivision of K, the intersection of L_1 and b_i or c_i is a 1-simplex or 2-simplex, respectively. Let α_i, β_i, and γ_i be the intersections of B_1 and a_i, b_i, and c_i, respectively. We shall regard α_i and γ_i as triangles with vertices on the 1-simplexes of a_i and c_i. Also we shall regard β_i as a square with vertices on the 1-simplexes of b_i.

Any 2-simplex of L_1, say ABC, is incident to exactly two of the c_i. Let $c_1 = ABCM$. There is a unique 3-simplex of N_1, say σ, that is incident to ABM and different from c_1. This σ is either a c_i, say c_2, or a b_i, say b_2. If σ is c_2, then the triangles γ_1 and γ_2 have a common side. Suppose that σ is $b_2 = ABMN$. The 2-simplex ABN is incident to a unique 3-simplex of N_1, say τ, with $\tau \neq ABMN$. This τ is either c_3 or b_3. If $\tau = b_3$, there is a c_4, or b_4. Finally we must find a $c_p = ABDS$, D in L_1, S in B_1. We now consider $\beta_2, \beta_3, \cdots$, and β_{p-1}. The sum of these squares is topologically equivalent to a square. One side of the square is coincident with a side of γ_1 and the opposite side coincident with a side of γ_p.

Received by the editors July 21, 1941.

[1] Our terminology is that of Seifert-Threlfall, *Lehrbuch der Topologie*. Manifolds are finite, while simplexes and cells are closed point sets.

Since K_1 is a manifold, we can repeat the construction and associate with ABC and ABD a second pair of triangles in B_2 that are either incident along a common side or incident to opposite sides of a square. But there is not a third such configuration associated with ABC and ABD. We repeat the construction for all pairs of adjacent 2-simplexes of L_1. Then to each 2-simplex of L_1 there correspond two triangles in B_2. Moreover, if two 2-simplexes of L_1 are incident along a side, the four corresponding triangles can be paired so that the two triangles of each pair either have a common side or are incident to opposite sides of a square.

Since P and Q are 3- and 2-manifolds, respectively, we can say that Q is two-sided in P in the neighborhood of any point of Q. Moreover, the two γ's of B_2 that correspond to a 2-simplex of L_1 lie on opposite sides of Q (in the neighborhood of this 2-simplex).

Consider a vertex X of L_1 and the 2-simplexes Δ_i of L_1 that have X as a vertex. On one side of Q (in the neighborhood of X) there corresponds to each Δ_i a unique γ_i, and the γ's have the same incidences as the corresponding Δ's (we say that two γ's are incident if they are incident to opposite sides of a square). Let us denote by R the points of these γ's and the squares incident to pairs of these γ's. Let A denote the points of all α_i's that are in a_i's incident to X and on the side of Q that we are considering.

We shall show that $R+A$ is a 2-cell. To do this we shall show that $R+A$ is a manifold relative to its boundary, that its boundary consists of one or more circles, and that any 1-cycle of $R+A$ bounds in $R+A$. First we observe that B_2 is a manifold; this fact follows from the structure of B_2 and the fact that K_1 is a manifold; the argument is elementary and we omit it. Since $R+A$ is the sum of 2-cells α, β, and γ, the set $R+A$ is a manifold relative to its boundary.

To show that this boundary of $R+A$ consists of one or more circles we shall study the incidences among the cells of $R+A$. First, let a_i have X as a vertex. If a 2-dimensional side of a_i is not in B_1, this side must be a side of an a_j or b_j. Furthermore, this a_j or b_j has X as a vertex. Hence, any side of an α_i is also a side of an α_j or β_j of $R+A$. Next, let c_i have vertices $XABM$, M in B_1. The sides of γ_i that are in XAM and XBM are sides of γ_j's or β_j's of $R+A$. But the side of γ_i in ABM is not incident to any other 2-cell of $R+A$. This side is part of the boundary of $R+A$. Finally, let b_i have vertices $XAMN$, A in L_1. The sides of β_i in XAM and XAN are incident to sides of β_j's or γ_j's of $R+A$; the side of β_i in XMN is incident to an α_j or β_j of $R+A$; but the side of β_i in AMN is not incident to any other 2-cell of $R+A$. This side is part of the boundary of $R+A$. Examination of the segments of the boundary of $R+A$ shows that they fit together to form one or more circles.

We next show that if C is a 1-dimensional cycle of $R+A$, then C bounds in $R+A$. We shall find it convenient to replace A by a new set that will never be empty. We define A' to be A together with all vertices of γ's of R that are not in the boundary of $R+A$ and all sides of squares of R that are not sides of γ's of R and not in the boundary of $R+A$. If A is not empty, the set A' is the same as A. But in any case A' is not empty, and $R+A'$ is the same set as $R+A$. The set $(R+A')-\bar{A}'$ is homeomorphic to a 2-cell with an inner point removed because $(R+A')-\bar{A}'$ can be obtained from the configuration of the 2-simplexes of L_1 that have X as a vertex by removing X and replacing some of the 1-simplexes by squares (open along one side). Hence, the cycle C is homologous in $R+A'$ to a cycle on A', and we assume that C is on A'. The set A' is part of b, the boundary of the combinatorial neighborhood of X in K_2. Since K_2 is a manifold, the set b is a 2-sphere. Assume that C does not bound in A'. Then C must surround a 2-simplex of b that is not in A'. We easily find a 2-simplex of $R+A'$ that is not incident along one of its sides to another 2-simplex of the manifold B_2. This contradiction proves that C bounds, and the proof that $R+A$ is a 2-cell is complete.

Now we draw some lines on $R+A$. If two γ's have a common side, we draw a line coincident with this common side. If two γ's are incident to a square, we draw a line across the square half way between the γ's. All these lines are continued so that they meet at a point of A. These lines give a subdivision of $R+A$ that is combinatorially equivalent to the combinatorial neighborhood of X in L_1. The lines can be drawn for all $R+A$ of B_2 and we get a subdivision of B_2 that is combinatorially equivalent to a two-fold but not necessarily connected covering of L_1.

A triangle of the covering is associated with a 2-simplex of L_1 and a side of Q (in the neighborhood of this simplex). Hence, a homeomorphism is determined between this covering and any covering obtained by changing the permissible division K.

The theorem is not true with B_1 rather than B_2. For example, let Q be the boundary of a 3-simplex of K. Then B_1 is a sphere and a point.

We can prove the following theorem in the same way but with much less effort.

THEOREM. *The above theorem is true if P and Q are replaced by* 2- *and* 1-*dimensional manifolds.*

PURDUE UNIVERSITY

A LINEAR TRANSFORMATION WHOSE VARIABLES AND COEFFICIENTS ARE SETS OF POINTS

S. T. SANDERS, JR.

Introduction. While the theory of the linear transformation has been developed in great detail, attention has seldom[1] been called to the transformation T in which variables and coefficients are sets of points. Doubtless the nonexistence of a unique inverse transformation has occasioned this neglect. In this paper the writer studies the iteration of T.

Consider first the transformation

$$T: \quad \begin{aligned} x_1 &= a_{11}x_1' + a_{12}x_2' \\ x_2 &= a_{21}x_1' + a_{22}x_2', \end{aligned}$$

whose *set matrix* is

$$M = \left\| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right\|,$$

where the a's and x's are sets of points, and the indicated sums and products refer to set operations. Applying T to the primed variables, we have the product transformation

$$T^2: \quad \begin{aligned} x_1 &= \overset{(2)}{a_{11}}\, x_1'' + \overset{(2)}{a_{12}}\, x_2'', \\ x_2 &= \overset{(2)}{a_{21}}\, x_1'' + \overset{(2)}{a_{22}}\, x_2'', \end{aligned}$$

of set matrix

$$M^2 = \left\| \begin{matrix} \overset{(2)}{a_{11}} & \overset{(2)}{a_{12}} \\ \overset{(2)}{a_{21}} & \overset{(2)}{a_{22}} \end{matrix} \right\|,$$

where

$$\begin{aligned} \overset{(2)}{a_{11}} &= a_{11} + a_{12}a_{21}, & \overset{(2)}{a_{12}} &= a_{11}a_{12} + a_{12}a_{22}, \\ \overset{(2)}{a_{21}} &= a_{21}a_{11} + a_{22}a_{21}, & \overset{(2)}{a_{22}} &= a_{21}a_{12} + a_{22}. \end{aligned} \tag{1}$$

Transforming in turn each new set of variables, we obtain product transformations T^3, T^4, $\cdots$, whose set matrices are M^3, M^4, $\cdots$.

Presented to the Society, December 30, 1941 under the title *On powers of a matrix whose elements are sets of points;* received by the editors August 25, 1941.

[1] Lowenheim, *Über Transformationen im Gebietskalkul*, Mathematische Annalen, vol. 73 (1913), pp. 245–272; *Gebietsdetermination*, Mathematische Annalen, vol. 79 (1919), pp. 223–236.

Set matrices $M(a_{ij})$ and $M'(a'_{ij})$ are defined to be *equal* if $a_{ij}=a'_{ij}$, $i, j=1; 2, \cdots, n$, while if $a_{ij}\subset a'_{ij}$,[2] then we shall write $M\subset M'$. A sequence M_k, $k=1, 2, \cdots$, of set matrices is *increasing* or *decreasing* according as $M_k\subset M_{k+1}$, or $M_{k+1}\subset M_k$.

From (1) follow $a_{11}\subset a_{11}^{(2)}$ and $a_{22}\subset a_{22}^{(2)}$. The assumptions $a_{12}\subset a_{12}^{(2)}$ and $a_{21}\subset a_{21}^{(2)}$ are equivalent to

$$a_{12} + a_{21} \subset a_{11} + a_{22}, \tag{2}$$

which is a necessary and sufficient condition that $M\subset M^2$. But from (1) we have $a_{12}^{(2)}+a_{21}^{(2)}\subset a_{11}^{(2)}+a_{22}^{(2)}$, whence by (2) the inclusion $M^2\subset M^4$, and the following theorem is established.

THEOREM 1. *The sequences M^{2k} and M^{2k+1}, $k=1, 2, \cdots$, of second order set matrices are increasing.*

From (1) follow also $a_{12}^{(2)}\subset a_{12}$ and $a_{21}^{(2)}\subset a_{21}$, while the assumptions $a_{11}^{(2)}\subset a_{11}$ and $a_{22}^{(2)}\subset a_{22}$ are equivalent to

$$a_{12}a_{21} \subset a_{11}a_{22}, \tag{3}$$

a necessary and sufficient condition that $M^2\subset M$. From (1) comes $a_{12}^{(2)}a_{21}^{(2)}\subset a_{11}^{(2)}a_{22}^{(2)}$, whence by (3) the inclusion $M^4\subset M^2$, and the following theorem.

THEOREM 2. *The sequences M^{2k} and M^{2k+1}, $k=1, 2, \cdots$, of second order set matrices are decreasing.*

This theorem follows at once.

THEOREM 3. *Even powers of a second order transformation are identical; likewise odd powers beyond the first.*

The general case. Consider the transformation

$$T:\quad x_i = \sum_{j=1}^{n} a_{ij}x'_j, \qquad i = 1, 2, \cdots, n,$$

of set matrix

$$M = \left\| \begin{matrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad \cdot \cdot \cdot \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{matrix} \right\|,$$

where the variables and coefficients are sets of points in a space whose generality is limited only by the following assumption.

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[2] The symbol $\subset$ denotes inclusion.

(A) *The elements a_{ij} of M are independent point sets.* That is, if products $P_1, P_2, \cdots, P_k, \cdots$ of elements of M are such that

$$P_1 \subset \sum_{k \neq 1} P_k,$$

there is a subscript $k = k'$ such that[3] $P_1 \subset P_{k'}$.

The following theorem will presently be established.

THEOREM 4. *The iteration of the transformation T of order n leads to at most $(n-1)^2+N$ distinct transformations, where N is the least common multiple of $1, 2, \cdots, n$, and at most N of these transformations recur periodically. If the coefficients of T are independent, there are precisely $(n-1)^2+N$ distinct transformations of which N recur periodically.*

We shall denote by $a_{ij}^{(\nu)}$ the element in the ith row and jth column of M^ν. Clearly $a_{ij}^{(\nu)}$ is a sum of products of the form

$$P = a_{ik_1}a_{k_1k_2}a_{k_2k_3} \cdots a_{k_\nu j},$$

for all distinct ways of selecting the subscripts $k_1, k_2, \cdots, k_\nu$, from the integers $1, 2, \cdots, n$. The characteristic interlocking form k_rk_s, k_sk_t of the subscripts of consecutive factors of P will be expressed in the word *proper*. Thus $a_{12}a_{23}a_{32}a_{23}a_{32}$ is a term of $a_{12}^{(5)}$ and a proper product, as opposed to the identical point set $a_{12}a_{23}a_{32}$.

The *order* of a product P is the number of factors occurring in the product. The orders of products $P, C, \cdots$, will be denoted by $p, c, \cdots$. A *cycle* of P is a proper product $a_{k_1k_2}a_{k_2k_3} \cdots a_{k_ck_1}$ of factors of P in which the subscripts $k_1, k_2, \cdots, k_c$ are distinct. It is convenient to denote such a cycle by C_{k_1}. Thus $a_{12}a_{23}a_{33}a_{32}$ involves the cycles $C_2 = a_{23}a_{32}$, $C_3 = a_{33}$, and $C_3' = a_{32}a_{23}$. A *closed cycle* of P is one whose factors occur consecutively in P. A product P is *c-cyclic* if every product of c consecutive factors is a closed cycle. It follows that these closed cycles are cyclic permutations of a single cycle C, known as the defining cycle of the product P. Thus $a_{23}a_{31}a_{12}a_{23}a_{31}$ is 3-cyclic, with the defining cycle $C_2 = a_{23}a_{31}a_{12}$.

Proper products are *coterminal* if corresponding terminal subscripts are equal. Thus $P = a_{12}a_{23}a_{34}$ and $P_1 = a_{12}a_{24}$ are coterminal. Such products clearly are terms of elements similarly situated in set matrices M^p and $M^{p'}$. It is convenient to denote a proper product $a_{ik_1}a_{k_1k_2} \cdots a_{k_pj}$ by P_{ij}. If coterminal proper products P_{ij}, P_{ij}' are such that $p'_{ij}<p_{ij}$ while $P_{ij}\subset P'_{ij}$, we say that P'_{ij} is a *contraction* of P_{ij}, while P_{ij} is an *expansion* of P'_{ij}. Thus $a_{13}a_{33}$ is a contraction of $a_{13}a_{34}a_{43}a_{33}$, while the latter product is an expansion of the former.

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[3] Thus, under (A), such inclusions as $a_{12}a_{23} \subset P = a_{14} + a_{13}a_{32}$ cannot hold, since P does not involve one of the terms a_{12}, a_{23}, $a_{12}a_{23}$.

The following lemmas are immediate consequences of the preceding definitions.

LEMMA 1. *A proper product of order exceeding $n-1$ involves a closed cycle.*

LEMMA 2. *The deletion of a closed cycle from a proper product of greater order yields a contraction of the product.*

LEMMA 3. *The insertion of a cycle C_{k_s} into a proper product immediately following a factor $a_{k_rk_s}$, or immediately preceding a factor $a_{k_sk_t}$, yields an expansion of the product. Thus the appropriate insertion of $C_1=a_{14}a_{41}$ into $P_{13}=a_{13}a_{31}a_{13}$ yields $P'_{13}=a_{13}a_{31}a_{14}a_{41}a_{13}$, or $a_{14}a_{41}a_{13}a_{31}a_{13}$.*

Consider the sequence

$$P_{ij},\ P'_{ij},\ P''_{ij},\ \cdots,\ P^{(k)}_{ij}, \tag{4}$$

in which each element after the first is obtained from its predecessor by the deletion of a closed cycle. The last element can involve no cycle, or is itself a cycle, and is called a *stem* of P_{ij}. The definition is not unique, since $P^{(k)}_{ij}$ clearly varies with the sequence of cycles of P_{ij} whose deletion leads to (4). Thus $a_{13}a_{33}a_{33}a_{34}a_{43}$ has the stems $a_{13}a_{33}$ and $a_{13}a_{34}a_{43}$.

From Lemmas 1 and 2 follows this lemma.

LEMMA 4. *A stem of a proper product P is a contraction of P which has no contraction. The order of a stem cannot exceed n.*

Increasing sequences. We first prove this lemma.

LEMMA 5. *For every integer c not exceeding p nor n, there occurs in M^p, $p\geqq 2$, an element involving a term which is a c-cyclic product.*

If $p=mc$ each diagonal element $a_{ii}^{(p)}$ involves certain c-cyclic terms in which a cycle C is repeated m times. If $p=mc+r$, $1\leqq r<c$, each element $a_{ij}^{(p)}$, $i\neq j$, involves certain c-cyclic terms in which the rth factor, and hence the pth, is a_{kj}. Thus, for $n\geqq 4$, $a_{33}^{(4)}$ involves the 2-cyclic term $a_{31}a_{13}a_{31}a_{13}$, while $a_{34}^{(5)}$ involves the 3-cyclic term $a_{31}a_{14}a_{43}a_{31}a_{14}$.

THEOREM 5. *Let M be a set matrix of order n whose elements are independent. The sequence $M^{p_1}, M^{p_2}, \cdots, M^{p_r}, M^{p_{r+1}}, \cdots, p_r<p_{r+1}$, is increasing if and only if $p_1>n-1$, while $p_{r+1}-p_r$ is a multiple of $1, 2, \cdots, n$.*

By Lemma 1 a term P of $a_{ij}^{(p_r)}$ involves a closed cycle C. Since $p_{r+1}-p_r$ is a multiple of c we can insert appropriately into P sufficient repetitions of C to yield by Lemma 3 an expansion $\overline{P}=P$ of order p_{r+1}. Thus $\overline{P}$ is the required term of $a_{ij}^{(p_{r+1})}$.

Conversely, if $p_1 \leqq n-1$ there is a term P' of $a_{ij}^{(p_1)}$ involving no cycle, and which by Lemma 1 and (A) is not contained in $a_{ij}^{(p)}$ for $p=p_2 \geqq n$. And if $p_1 > n-1$ while $p_{r+1}-p_r$ is not a multiple of $c \leqq n$, there is by Lemma 5 an element $a_{ij}^{(p_r)}$ of M^{p_r} involving a term P_r which is a c-cyclic product. Now a product P_{r+1} which contains P_r can by (A) involve only factors of P_r, and is hence c-cyclic. But since $p_{r+1}-p_r$ is not a multiple of c, it follows that P_{r+1} is not coterminal with P_r, and so is not a term of $a_{ij}^{(p_{r+1})}$. From (A) we conclude that P_r is not contained in the set $a_{ij}^{(p_{r+1})}$ and the theorem is established.

Decreasing sequences. Before proceeding we prove two lemmas.

LEMMA 6. *Let P be a proper product, S a sequence of cycles of P determining a stem P', $h=h(S)$ the highest common factor of the orders of the cycles of S, and c the greatest of these orders. There is a contraction $\overline{P}$ of P involving every subscript of P, and such that $\bar{p} \leqq c(n-c+2)-1$, $p \equiv \bar{p}$ (mod h).*

Consider the following sequence

$$S': \quad P', C_1, C_2, \cdots, C_b,$$

in which: (i) C_1 is a cycle of S involving a subscript not found in P', and has the maximum order of all such cycles. (ii) Each C following C_1 is a cycle of S involving a subscript which has previously appeared in S', and one which has not done so. Further, all such cycles of S are in the sequence S'.

It is easily shown that every subscript involved in P occurs in some cycle of S'. For let k_s be the first subscript of P not found in S'. Since k_s cannot occur in the stem P', it must first appear in P in a factor of the form $a_{k_r k_s}$. But k_s occurs in some cycle C' of S, while k_r occurs in a cycle of S'. We infer from (ii) that C' is a cycle of S', and a contradiction is reached.

Now if the cycles $C_1, C_2, \cdots, C_b$ exist, at least one involves a subscript of P'; for the contrary assumption leads to a contradiction, as in the above argument, on consideration of the first appearance in P of a subscript of $C_1, C_2, \cdots, C_b$. It follows that the cycles of S' can be combined with P' into a contraction $\overline{P}$ of P which involves every subscript of P.

Consider now the order of $\overline{P}$. We have

$$\bar{p} = p' + \sum_{i=1}^{k} c_i, \qquad c_i \leqq c.$$

If $p' \leqq c-2$, P' and C_1 together involve at least c distinct subscripts, whence $k \leqq n-c+1$, and $\bar{p} \leqq c-2+c(n-c+1) < c(n-c+2)-1$. While if $p' > c-2$, P' and C_1 together involve at least $p'+1$ distinct subscripts, whence $k \leqq n-p'$, and $\bar{p} \leqq p'+c(n-p') = cn-p'(c-1) \leqq cn-(c-1)^2 = c(n-c+2)-1$.

The congruence $p \equiv \bar{p} \pmod{h}$ follows from the definition of P' and $\bar{P}$. The lemma is established.

LEMMA 7. *Let* $c_1 > c_2 > \cdots > c_m$ *be a set* S_1 *of positive integers whose highest common factor is* $h = h(S_1)$. *The equation* $c_1x_1 + c_2x_2 + \cdots + c_mx_m = k$ *has a non-negative integral solution* $(x_1', x_2', \cdots, x_m')$ *for every integer* k *which is a multiple of* h *exceeding* $(c_1c_2/h) - c_1 - c_2$.[4]

THEOREM 6. *Let* M *be a set matrix of order* n *whose elements are independent. The sequence* $M^{p_1}, M^{p_2}, \cdots, M^{p_r}, M^{p_{r+1}}, \cdots, p_r < p_{r+1}$, *is decreasing if and only if* $p_1 > (n-1)^2$, *while* $p_{r+1} - p_r$ *is a multiple of* $1, 2, \cdots, n$.

Sufficiency. For any term P of $a_{ij}^{(p_{r+1})}$; the theorem asserts the existence of a term P_1 of $a_{ij}^{(p_r)}$ such that $P \subset P_1$. Our procedure is to insert into a contraction of P appropriate cycles of P of the precise total order required to yield the desired product P_1.

Let P' be a stem of P, determined by a sequence S of cycles of P. Let $\bar{P}$ be the contraction of P presented in Lemma 6; and let $S_1: C_1, C_2, \cdots, C_m$, be cycles of S among whose orders, $c_1 > c_2 > \cdots > c_m$ occur all orders of cycles of S. If $m = 1$, the required term P_1 is clearly encountered in the sequence (4) of products defining P'.

Case 1. $c_1 < n$.

By Lemma 6 we have

$$\begin{aligned} p_r - \bar{p} &> (n-1)^2 - c_1(n - c_1 + 2) + 1 \\ &= c_1(c_1 - 1) - c_1 - (c_1 - 1) + (n-1)(n - c_1 - 1), \end{aligned}$$

whence

$$p_r - \bar{p} > c_1(c_1 - 1) - c_1 - (c_1 - 1) \geqq \frac{c_1c_2}{h} - c_1 - c_2. \tag{5}$$

Now by the same lemma, $p_{r+1} - \bar{p} \equiv 0 \pmod{h}$, hence from $p_{r+1} - p_r \equiv 0 \pmod{h}$ follows

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[4] This lemma is readily established by mathematical induction. However, a better bound on k, namely, $(c_1 c_m/h) - c_1 - c_m$ has been communicated to the author by Dr. Alfred Brauer.

$$p_r - \bar{p} \equiv 0 \pmod{k}. \tag{6}$$

By (5), (6), and Lemma 7 there is established the existence of non-negative integers, $x_1', x_2', \cdots, x_m'$ such that

$$\bar{p} + \sum_{i=1}^{m} c_i x_i' = p_r.$$

By Lemmas 6 and 3 we can insert the cycles $C_1, C_2, \cdots, C_m$ into $\bar{P}$ and obtain the required product P_1.

Case 2. $c_1 = n$, $c_2 < n - 1$.

Again by Lemma 6

$$p_r - \bar{p} > (n-1)^2 - c_1(n - c_1 + 2) + 1$$
$$= c_1(c_1 - 2) - c_1 - (c_1 - 2) + (n-2)(n - c_1),$$

whence

$$p_r - \bar{p} > c_1(c_1 - 2) - c_1 - (c_1 - 2) \geq \frac{c_1 c_2}{k} - c_1 - c_2,$$

and the proof proceeds as in Case 1.

Case 3. $c_1 = n$, $c_2 = n - 1$.

Here the contraction P', instead of $\bar{P}$, is employed. We have by Lemma 4

$$p_r - p' > (n-1)^2 - n = c_1 c_2 - c_1 - c_2.$$

Since $k = 1$ it follows by Lemma 7 that non-negative integers, x_1', x_2', exist such that $p' + c_1 x_1' + c_2 x_2' = p_r$. Thus since P' must involve a subscript of C_1 and C_2, it is possible to insert the cycles C_1, C_2, into P' and obtain by Lemma 3 the required product P_1.

Necessity. As in Theorem 5 it can be shown that $p_{r+1} - p_r$ must be a multiple of $1, 2, \cdots, n$; while for the condition $p_1 > (n-1)^2$, it will suffice to produce a term of an element of M^{p_2} which is not contained in the corresponding element of M^{p_1}, although $p_2 - p_1$ is a multiple of $1, 2, \cdots, n$.

Consider the n-cyclic product P_1' of order $p_1 + n - 1$ whose defining cycle is $C_1 = a_{12} a_{23} \cdots a_{n1}$, and the $(n-1)$-cyclic product P_2' of order $p_2 - p_1 - n + 1$ whose defining cycle is $C_2 = a_{23} a_{34} \cdots a_{n2}$. By inserting P_2' into P_1' following any factor a_{12}, a proper product P of order p_2 is obtained which is a term of an element of the first row of M^{p_2}. Now from (A) and the structure of P it follows that any proper coterminal product containing P can be had from P by deletion of the cycles C_1, C_2. We are thus led to the equation

$$n x_1 + (n-1) x_2 = p_2 - p_1, \tag{7}$$

with the restrictions

$$(8)\qquad nx_1 \leqq p_1 + n - 1,$$

$$(9)\qquad (n-1)x_2 \leqq p_2 - p_1 - n + 1.$$

By (7) and (9), x_1 is a positive multiple of $n-1$, while from $p_1 \leqq (n-1)^2$ we have by (8), $nx_1 \leqq n(n-1)$. Thus $x_1 = n-1$, but it is clear that the deletion of $n-1$ cycles C_1 from P will yield a product[5] whose first factor is a_{nn}. From (A) it follows that P is not contained in the corresponding element of M^{p_1}, and the theorem is established.

Equality of matrices. Theorems 5 and 6 provide conditions for increase and decrease, respectively, in a sequence of ascending powers of M. In summation we have the following theorem:

THEOREM 7. *Let M be a set matrix of order n whose elements are independent. The equality $M^{p_1} = M^{p_2}$, $p_2 > p_1$, holds if and only if $p_1 > (n-1)^2$, while $p_2 - p_1$ is a multiple of* 1, 2, $\cdots$, n.

Theorem 4 is an immediate consequence.

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[5] Thus for $n=3$, $p_1=4$, $p_2=10$, we have $P = a_{12}a_{23}a_{33}a_{32}a_{22}a_{23}a_{31}a_{12}a_{23}a_{31}$ which has no contraction of order 4.

A THEOREM ON LIE GROUPS

DEANE MONTGOMERY AND LEO ZIPPIN

It is the purpose of this note to prove the following theorem:

THEOREM 1. *Let G be a Lie group*[1] *and G^* a compact subgroup of G. Then there exists in G an open set O containing G^* with the property that for each subgroup H of G lying in O there is an element g of G such that $g^{-1}Hg$ is in G^*.*

Roughly, the theorem says that each subgroup near enough to G^* can be transformed into G^* by an appropriate element of G. This result can be regarded as a generalization of the known fact that Lie groups cannot have arbitrarily small subgroups (other than the identity), although it was not from this point of view that our interest arose. To make our meaning clear, assume that G^* is an invariant subgroup so that the factor group G/G^* is also a Lie group. If there were in G a subgroup H near G^* it would go, by the homomorphism taking G into G/G^*, into a subgroup near the identity of G/G^*. The only subgroup of G/G^* near the identity is the identity itself which means that if H is to be near G^* it must actually be a subgroup of G^*. We see that when G^* is an invariant compact subgroup of G, the conclusion of the theorem is true in a trivial sense.

Our proof of Theorem 1 in the more general situation is based on certain facts about the way in which G operates on the coset space G/G^* which will be denoted by M. This is the space whose points are the cosets gG^* of G^* in G. The group G acts transitively on M which can be regarded as a Riemannian space and Cartan[2] has shown that there exists in M a Riemannian metric for which G is a group of isometries. This fact will be of great importance in what follows.

We begin, as we may, by supposing that M is endowed with a Riemannian metric invariant under G and, furthermore, we assume that M has been made into a metric space (Fréchet) in the usual way

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Presented to the Society, September 5, 1941; received by the editors August 29, 1941.

[1] For all definitions and theorems on topological groups used in this paper see Pontrjagin, *Topological Groups*. The term compact, as used here, implies that the set is closed.

[2] *La Théorie des Groupes Finis et Continus et l'Analysis Situs*, Mémorial des Sciences Mathématiques, vol. 42, p. 43. For an excellent summary of properties of isometries and geodesics which will be useful here, see the paper by Myers and Steenrod, *The group of isometries of a Riemannian manifold*, Annals of Mathematics, (2), vol. 40 (1939), pp. 400–416.

by letting the distance $d(x, y)$, between two points x and y be the greatest lower bound of the lengths of geodesic paths joining a and b. This distance is invariant under G. Moreover, for small enough neighborhoods this distance is equal to the length of the geodesic joining x and y since in such neighborhoods geodesics furnish a proper absolute minimum for the lengths of paths.

We shall use the symbol $S(x, r)$, x a point of M and r a positive real number, to designate the set of points of M whose distance from x is at most r, and we shall call that set the sphere about x of radius r. The boundary of this sphere will be the set of points for which the distance is precisely r, the *inside* (or *interior*) will be the set for which the distance is less than r. We shall say that a sphere is *convex* if each pair of its points are the end points of a geodesic every inner point of which is *inside* the sphere. It will be worth keeping in mind that, on our definition, when a geodesic belongs to a convex sphere only its end points can belong to the boundary of the sphere.

It will be convenient to use a number of simple lemmas and we now turn to them.

LEMMA 1. *If x is any point of M there exists a neighborhood U of x such that if a, b, and c are three distinct points of U which lie on one geodesic in that order, then*

$$d(b, x) < \max\,[d(a, x), d(c, x)].$$

Let r be a positive number such that every sphere of center x and radius $r' \leq r$ is convex.[3] Let U be the interior of $S(x, r)$. If a, b, and c are in U, consider the sphere $S(x, r_1)$ where $r_1 = d(x, b)$. The boundary of this sphere meets the geodesic abc at b and this means that either a or c is outside $S(x, r_1)$ as otherwise the fact of convexity of $S(x, r_1)$ would be contradicted. But if a point is outside $S(x, r_1)$ its distance from x is greater than r, and this concludes the proof.

LEMMA 2. *If x is any point of M, there exists a neighborhood U of x such that if a, b, c, and e are four distinct points in U and a, b, and c are on a geodesic in that order then*

$$d(b, e) < \max\,[d(a, e), d(c, e)].$$

Here we shall let U designate the interior of $S(x, r/2)$ where r has the same meaning as in the preceding lemma. Since G is transitive over M there will be an element g in G such that $g(e) = x$. The points

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[3] For the existence of such a number r in a Riemann space see the article by Whitehead, *Convex regions in the geometry of paths*, Quarterly Journal of Mathematics, vol. 3 (1932), pp. 33–42.

$g(a)$, $g(b)$, and $g(c)$ are inside $S(x, r)$, and they are on a geodesic since the isometry g preserves geodesics. Making use of Lemma 1, and using again the fact the g is an isometry, we complete the proof of this lemma.

LEMMA 3. *If x is a point of M there is a neighborhood U of x such that if a, b, c are distinct points lying in that order on a geodesic in U and if d, e, f are distinct and lie in that order on a geodesic in U then*

$$d(b, e) < \max\,[d(a, d), d(a, f), d(c, d), d(c, f)].$$

First we conclude from Lemma 2 that $d(b, e)$ is either less than $d(a, e)$ or less than $d(c, e)$. Say the notation is such that

$$d(b, e) < d(a, e).$$

Now using Lemma 2 again we have

$$d(a, e) < \max\,[d(a, d), d(a, b)],$$

and Lemma 3 is proved.

LEMMA 4. *If x is any point in M, there exists a neighborhood U of x such that if H is a closed subgroup of G and $H(x)$[4] is in U then H has a fixed point in U.*

Let U denote some convex sphere about x which satisfies the conclusion of Lemma 3. We shall show that this U fulfills the conclusion of the present lemma. Suppose, therefore, that H is a subgroup of G such that $H(x)$ is contained in $\bar{U}$.

Let q be a point in $\bar{U}$ such that $H(q)$ is of minimum diameter with respect to all orbits of H which lie entirely in $\bar{U}$. If $H(q) = q$ our proof is completed so we may take it that $H(q) \neq q$.

In this case $H(q)$ contains at least two points. Let b denote the midpoint of some geodesic determined by a pair of points of $H(q)$; now, because H is a group of isometries, $H(b)$ is entirely included in the set of such midpoints, and $H(b)$ lies in U. It will now be shown that $H(b)$ has a smaller diameter than $H(q)$ and this will be a contradiction to our choice of q. This contradiction will complete the proof of our lemma.

The diameter of $H(b)$ coincides with the distance between two points of $H(b)$. This implies that there is an h in H such that the diameter of $H(b)$ is

$$d[b, h(b)].$$

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[4] This is the orbit of the point x under the group H.

The point b is the midpoint of a geodesic abc where a and c are in $H(q)$. The point $h(b)$ is on the geodesic $h(a)$, $h(b)$, $h(c)$. Calling these points d, e, and f we may apply Lemma 3 and conclude that

$$d[b, h(b)] < \max\{d[a, h(a)], d[a, h(c)], d[c, h(a)], d[c, h(c)]\}.$$

This shows that the diameter of $H(b)$ is smaller than that of $H(q)$ and completes the proof of this lemma.

It is now a simple matter to conclude the proof of our theorem. We have to recall that the "points" of the space M are the cosets gG^*, g in G. Let us now interpret Lemma 4 in this light, with the group G^* taken as the point x. The open set U of the lemma corresponds to a system of cosets gG^* and the set of elements of G which belong to one of these cosets constitutes an open set O of G. The set O is a neighborhood of G^*.

Now let H be a subgroup of G which lies in O and let h be an element of H. Then, by the construction of O, every element of the coset hG^* belongs to O and the point hG^* of M belongs to U in M. Therefore the premises of Lemma 4 are satisfied and we may suppose that H has a fixed point in M.

This means that there is a "point" gG^* of M such that

$$H(gG^*) = gG^*.$$

This, however, is an equation in sets in G. We may write it

$$HgG^* = gG^*.$$

From this we conclude that

$$g^{-1}HgG^* = G^*.$$

But this means that for every choice of an element k in G^* and element h in H: $g^{-1}hgk$ is contained in G^*.

This means that for every choice of h in H: $g^{-1}hg$ is contained in G^* and then finally, that the group $g^{-1}Hg$ is a subgroup of G^*. This concludes the proof of our theorem.

COROLLARY. *The element g of Theorem 1 may be chosen within any prescribed neighborhood of the identity if the open set O is chosen thereafter sufficiently small.*

This corollary is a consequence of our method of proof. If the neighborhood U of the last lemma is small H will have a fixed point y near x. Because of the transitivity of G over M there is a g in G such that $g=g(x)$ and this g may be chosen near the identity in G if y is near enough to x.

These results may be extended slightly in certain directions as follows:

THEOREM 2. *Let G be a compact connected finite-dimensional group and let G^* be a closed connected subgroup. There exists an open set O containing G^* such that if H is a closed connected group in O then H is the transform of a subgroup of G^*.*

Let Z be an invariant zero-dimensional subgroup of G such that $f(G)=G/Z$ is a Lie group. By Theorem 1 there exists an open set O_1 containing $f(G^*)$ such that any subgroup of $f(G)$ which is in O_1 can be transformed into a subgroup of $f(G^*)$. Let $O=f^{-1}(O_1)$, and let H be a closed connected subgroup of G which is contained in O. There is a g_1 in $f(G)$ such that $g_1^{-1}f(H)g_1$ is in $f(G^*)$. Let g be an element in $f^{-1}(g_1)$. Then $g^{-1}Hg$ is a connected group such that $f(g^{-1}Hg)$ is in $f(G^*)$. This means that every element of $g^{-1}Hg$ belongs to the collection of elements $G^*Z=f^{-1}f(G^*)$.

The set of elements G^*Z forms a group which must have the same dimension as G^* for both are taken by f into $f(G^*)$ and the homomorphism f, since it has a zero-dimensional kernel, preserves dimension.[5] Hence the component of the identity of G^*Z must be G^*. For in any case this component certainly includes G^* and it cannot be larger than G^*, because if a finite-dimensional connected compact group contains another connected closed group as a *proper* subgroup it must be of higher dimension than this subgroup. This fact can be seen generally by observing that a similar fact is true for Lie groups.

If the connected group $g^{-1}Hg$ is in G^*Z it must be in the component of the identity of G^*Z. Therefore $g^{-1}Hg$ is in G^* as we wished to prove.

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[5] See van Kampen, *Note on a theorem of Pontrjagin*, American Journal of Mathematics, vol. 58 (1936), pp. 177–180, especially p. 178.

A NOTE ON COMPLEMENTATION IN LATTICES

L. R. WILCOX

The results of this note concern the notions of complementation and modularity in lattices. In the theories of modular lattices, for example, the theory of continuous geometries of von Neumann, the assumption of complementation has proved extremely powerful; the author has found that some analogue of this assumption is usually necessary in the study of non-modular lattices.[1] If L is a modular lattice with 0, 1, and if $a \in L$, then x is a complement of a in case $a+x=1$, $ax=0$. If L has the property

$$\text{(1)} \qquad \text{each } a \in L \text{ has a complement,}$$

then it may be proved that

$$\text{(2)} \qquad a, b \in L \text{ implies the existence of } b_1 \leqq b \text{ such that}$$
$$a + b_1 = a + b, \qquad ab_1 = 0.$$

Property (2) is the really useful one. Now if L is any lattice with 0, 1, then (1) does not imply (2); moreover, even (2) seems too weak for most purposes. However, if in (2) we assume in addition that (a, b_1) (or (b_1, a)) is a modular pair, then the usefulness of this assumption in analyzing the structure of L is considerably increased. At the same time, of course, limitations are placed on L, and it is these which we study here.

Let L be a lattice with 0 and 1. If $b, c \in L$, we say that (b, c) is a *modular pair* and write $(b, c)M$ if $(a+b)c=a+bc$ for $a \leqq c$. We write $(b, c)\perp$ if $bc=0$ and $(b, c)M$. The negation of M is M'. If $bc=0$ and $(b, c)M'$ we write $b\|c$. If $\perp$ is a symmetric relation, L is said to be $\perp$-*symmetric*; if M is symmetric, L is said to be M-*symmetric*.[2]

DEFINITION 1. *Let $a, b, b_1 \in L$. Then b_1 is a right complement within b of a in $a+b$ if*

$$b_1 \leqq b, \qquad a + b_1 = a + b, \qquad (a, b_1)\perp.$$

We say that b_1 is a left complement within b of a in $a+b$ if

$$b_1 \leqq b, \qquad a + b_1 = a + b, \qquad (b_1, a)\perp.$$

If $a \leqq b$, the phrase "within b" is omitted; if $a+b=1$, the phrase

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Presented to the Society, September 5, 1941; received by the editors September 24, 1941.

[1] This Bulletin, abstract 47-5-208

[2] See L. R. Wilcox, *Modularity in the theory of lattices*, Annals of Mathematics, (2), vol. 40 (1939), pp. 490–505, for the elementary properties of the relations M and $\perp$.

"in $a+b$" is omitted. L is right (left) complemented if $a, b \in L$ implies the existence of b_1 such that b_1 is a right (left) complement within b of a in $a+b$. If L is $\perp$-symmetric, right and left complements coincide, and the words "right" and "left" will be omitted.

THEOREM 1. *Let $(a, b)M$. Then b_1 is a right complement within b of a in $a+b$ if and only if b_1 is a right complement of ab in b. The forward implication holds also for left complements.*

PROOF. Suppose $b_1 \leqq b$, $b_1+a=b+a$, $(a, b_1) \perp$ (or $(b_1, a) \perp$). Then

$$b_1 + ab = (b_1 + a)b = (b + a)b = b;$$

$(ab, b_1) \perp$ (or $(b_1, ab) \perp$) is immediate. Suppose that $ab+b_1=b$, $(ab, b_1) \perp$. Then $b_1 \leqq b$; moreover, $a+b_1=a+ab+b_1=a+b$. Clearly $ab_1=abb_1=0$. Let $d \leqq b_1$, whence

$$\begin{aligned}(d + a)b_1 &= (d + a)bb_1 = (d + ab)b_1 \\ &= d + abb_1 = d + ab_1,\end{aligned}$$

and $(a, b_1)M$. Hence $(a, b_1) \perp$, and b_1 is a right complement within b of a in $a+b$.

We turn now to consequences of the assumption that L is left complemented. The fundamental result follows.

THEOREM 2. *If L is left complemented, then L is right complemented, the notions of right and left complement coincide, and L is M-symmetric.*

PROOF. Let $(b, c)M$. There exists a left complement c_1 within c of b in $b+c$. Now let $b_1 \leqq b$. Then by the second part of Theorem 1 c_1 is a left complement of bc in c, and

$$(b_1 + c)b = (b_1 + bc + c_1)b = b_1 + bc,$$

whence $(c, b)M$. The remainder of the theorem is immediate.

COROLLARY. *If L is $\perp$-symmetric and complemented, then L is M-symmetric.*

REMARK. It is false that if L is right complemented then L is left complemented. For, in the right complemented lattice consisting of 0, 1, three points p, q, r, and a line s with $s>p, q$ but $s \not> r$, the element s has no left complement (in 1).

We shall henceforth assume that L is left complemented. It is now possible to prove a converse of Theorem 1.

THEOREM 3. *Let $a, b \in L$. If every complement b_1 within b of a in $a+b$ is a complement of ab in b, then $(a, b)M$.*

PROOF. Let $d \leqq b$, and let d_1 be a complement within d of a in $a+d$. Then

$$d_1 + a = d + a, \qquad (d_1, a) \perp.$$

Now let b' be a complement within b of $d+a$ in $b+a$. Then

$$b' + d_1 + a = b' + d + a = b + a;$$

moreover, $(b', d+a) \perp$, whence $(b', d_1, a) \perp$, and $(b'+d_1, a) \perp$. Define $b_1 = b'+d_1$. Then $b_1 \leqq b$, $b_1+a = b+a$, $(b_1, a) \perp$, that is, b_1 is a complement within b of a in $a+b$. By the hypothesis, b_1 is a complement of ab in b, whence $b_1+ab=b$. Now

$$\begin{aligned}(d + a)b &= (d + a)(b_1 + ab)\\ &= ((ab + d_1) + b')(d + a)\\ &= ab + d_1 \leqq d + ab\end{aligned}$$

since $ab+d_1 \leqq d+a$, $(b', d+a) \perp$. But $d+ab \leqq (d+a)b$, whence $(d+a)b = d+ab$, and $(a, b)M$.

COROLLARY 1. *$(a, b)M$ if and only if every complement within b of a in $a+b$ is a complement of ab in b.*

COROLLARY 2. *$(a, b)M'$ if and only if there exists a complement b_1 within b of a in $a+b$ such that $b_1+ab<b$.*

COROLLARY 3. *$a \| b$ if and only if $ab=0$ and there exists $b_1<b$ such that $b_1+a=b+a$ and $(b_1, a) \perp$.*

REMARK. It should be observed that Corollary 3 justifies our use of the symbol $\|$. Let, for example, L be a euclidean (or affine) geometry. Parallel spaces in L are then characterized by the condition $ab=0$ and the existence of a space $b_1<b$ such that $b_1+a=b+a$; if b_1 is chosen as a minimal space with these properties, and it can be so chosen, then $(b_1, a) \perp$.

It is easily seen[8] that an affine geometry is a left complemented lattice. Indeed, it may be shown that the property

$$bc \neq 0 \quad \text{implies} \quad (b, c)M \tag{3}$$

holds in an affine geometry. Property (3) may be stated also in the following way. Define the set M_L' as the set $[bc; (b, c)M']$. Then

$$M_L' \subset [0]. \tag{3'}$$

It is of interest to investigate the nature of the set M_L' for our general left complemented lattice L. It might appear that M_L' is an ideal; that this is false will be shown by an example. However, the next theorem shows that M_L' has one of the properties of an ideal.

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[8] L. R. Wilcox, loc. cit., pp. 496–499.

LEMMA 1. *Let* $(b, c)M'$ *and* $a < bc$. *Then there exists* $b' < b$ *with* $a = b'c$ *and* $(b', c)M'$. *Moreover, there exists* $x \leqq b$ *with* $x \| c$.

PROOF. Let x be a complement of bc in b, whence $x + bc = b$, $(x, bc) \perp$. Define $b' = a + x \leqq b$. Then

$$b'c = (a + x)c = (a + x)bc = a.$$

Suppose that $(b', c)M$, that is, $(a + x, c)M$. Then $c_1 \leqq c$ implies

$$\begin{aligned}(c_1 + b)c &= (c_1 + x + bc)c \\ &= (c_1 + x + a + bc)c \\ &= c_1 + bc + (x + a)c \\ &= c_1 + bc,\end{aligned}$$

contrary to $(b, c)M'$. Hence $(b', c)M'$. The second part follows by putting $a = 0$.

THEOREM 4. *If* $x \in M_L'$ *and* $y \leqq x$, *then* $y \in M_L'$.

PROOF. Immediate from Lemma 1.

We turn now to a general example of a left complemented lattice. Let Λ be a complemented modular lattice with operations $\oplus$, $\otimes$, and let ∞ be any element of Λ; let L be a subset of Λ with the following properties:

(4) $\mathfrak{a} = [x; x \leqq \infty] \subset L$; $1 \in L$;

(5) $x \in L$, $x \notin \mathfrak{a}$, $y \geqq x$ implies $y \in L$;

(6) $x \in L$, $y \leqq x$ implies the existence of $z \in L$ such that

$$z \oplus y = x, \qquad z \otimes y = 0.$$

A special case is the following. Λ is a projective geometry, and ∞, $h \in \Lambda$, $0 < \infty < h < 1$, $L' = [x; x \leqq h, x \text{ not } \leqq \infty]$, $L = \Lambda - L'$. Then L obviously satisfies (4) and (5), and (6) holds by virtue of the irreducibility of Λ. If h is a hyperplane, that is, h is a maximal element less than 1, then L is an affine geometry (for example, Λ with h as ideal hyperplane) with the ideal elements of $\mathfrak{a}$ restored.

THEOREM 5. *L is a lattice in which the operations* $+$, $\cdot$ *are given by*

$$(7) \qquad a + b = a \oplus b, \qquad ab = \begin{cases} a \otimes b & \text{if } a \otimes b \in L \\ a \otimes b \otimes \infty & \text{otherwise.} \end{cases}$$

PROOF. To show that $a \oplus b$ is a least upper bound of a, b in L it suffices to prove that $a, b \in L$ implies $a \oplus b \in L$. If both $a, b \in \mathfrak{a}$, then $a \oplus b \in L$ by (4). Hence suppose $a \notin \mathfrak{a}$; by (5) $a \oplus b \in L$. We shall prove directly that ab as defined in (7) is effective as greatest lower bound of a, b in L. Clearly $ab \leqq a, b$; by (4) $ab \in L$. Let $c \in L$, $c \leqq a, b$. Then $c \leqq a \otimes b$. If $c \notin \mathfrak{a}$, then $a \otimes b \in L$, and $c \leqq ab$ by (7). If $c \in \mathfrak{a}$, then $c \leqq a \otimes b \otimes \infty \leqq ab$.

COROLLARY. *If* $a \in \mathfrak{a}$, $b \in L$, *then* $ab = a \otimes b$.

LEMMA 2. *If* $b, c \in L$, *then* $(b, c)M$ *if and only if* $b \otimes c \in L$.

PROOF. First let $(b, c)M$, and suppose $b \otimes c \notin L$. Let $a \in L$ be a complement of $b \otimes c$ in c, existent by (6). Then

$$(a \oplus b) \otimes c = a \oplus (b \otimes c) = c,$$

whence $(a \oplus b) \otimes c \in L$; thus by (7)

$$a \oplus (b \otimes c) = (a \oplus b) \otimes c = (a + b)c = a + bc = a \oplus (b \otimes c \otimes \infty),$$

and it follows from the theory of modular lattices that $(b \otimes c) \otimes \infty = b \otimes c$. By (4) this contradicts the hypothesis, whence $b \otimes c \in L$. Conversely, let $b \otimes c \in L$, and let $a \leqq c$, $a \in L$. If $a \notin \mathfrak{a}$, then $(a \oplus b) \otimes c \in L$, and by (7) and the modular law in Λ we have

$$(a + b)c = (a \oplus b) \otimes c = a \oplus (b \otimes c) = a + bc. \tag{8}$$

Suppose $a \in \mathfrak{a}$. If $(a \oplus b) \otimes c \in L$, equalities (8) hold. If $(a \oplus b) \otimes c \notin L$, then

$$\begin{aligned}(a + b)c = (a \oplus b) \otimes c \otimes \infty &= (a \oplus (b \otimes c)) \otimes \infty \\ &= a \oplus (b \otimes c \otimes \infty) \leqq a + bc \leqq (a + b)c.\end{aligned} \tag{9}$$

By (8) and (9) we have $(b, c)M$.

COROLLARY. *L is M-symmetric.*

THEOREM 6. *The lattice L is left complemented.*

PROOF. Let $a, b \in L$, and let $b_1 \in L$ be a complement of $a \otimes b$ in b, existent by (6). Then

$$a + b_1 = a \oplus (a \otimes b) \oplus b_1 = a \oplus b = a + b;$$

$$a \otimes b_1 = a \otimes b \otimes b_1 = 0 \in \mathfrak{a} \subset L,$$

whence $(a, b_1)M$ by Lemma 2. But $ab_1 = 0$, and it follows from the corollary to Lemma 2 that $(b_1, a) \perp$. Thus b_1 is a (left) complement within b of a in $a + b$.

THEOREM 7. *The set* $\mathfrak{a}$ *is a modular ideal in* L *containing the set* M_L'.

PROOF. It is obvious that $\mathfrak{a}$ is an ideal; that it is modular follows from the corollary to Theorem 5, equations (7) and Lemma 2. Now if $(b, c)M'$, then by Lemma 2 $b \otimes c \notin L$, whence $bc = (b \otimes c) \otimes \infty$, and $bc \in \mathfrak{a}$. Hence $M_L' \subset \mathfrak{a}$.

It should be observed that both possibilities $M_L' = \mathfrak{a}$ and $M_L' \neq \mathfrak{a}$ can occur. In our special example where Λ is a projective geometry and $\Lambda - L$ consists of all subelements x of a hyperplane h with x not $\leqq \infty$, $M_L' = \mathfrak{a}$ if there exists $k \notin L$ with $\infty < k < h$. On the other hand, if no element exists (in Λ) between ∞ and h, then $M_L' \neq \mathfrak{a}$, for then $\infty \notin M_L'$. This example shows also that M_L' need not be an ideal.

ILLINOIS INSTITUTE OF TECHNOLOGY

SUFFICIENT CONDITIONS THAT POLYNOMIALS IN SEVERAL VARIABLES BE POSITIVE

IRWIN E. PERLIN

1. **Introduction.** Let $T(x)$ denote a polynomial in a single real variable x with real coefficients. T. Popoviciu[1] established sufficient conditions that $T(x)$ be positive for all real x. In this paper we shall consider polynomials in several real variables with real coefficients, and establish sufficient conditions that the polynomials be positive for real values of the variables.

2. **Polynomials in two real variables.** In this section we shall develop sufficient conditions that a polynomial in two real variables with real coefficients be positive for all real values of the variables. Let us consider then,

$$T(x, y) = \sum_{i=0}^{2m} \sum_{j=0}^{2n} b_{ij} c_{ij} x^i y^j,$$

where the b_{ij} are positive constants, and the c_{ij} are real numbers. Introducing parameters, we write $T(x, y)$ in the following form

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Presented to the Society, April 13, 1940; received by the editor June 26, 1941.

[1] T. Popoviciu, *Sur une condition suffisante pour qu'un polynôme soit positif*, Mathematica, vol. 11 (1935), pp. 247–256.

$$
\begin{aligned}
T(x, y) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Bigg\{ & \alpha_{ij} c_{2i\,2j} x^{2i} y^{2j} \left[\left(1 + \beta_{ij} \frac{c_{2i+1\,2j}}{c_{2i\,2j}}\, x\right)^2 \right. \\
& \left. + \left(1 + \gamma_{ij} \frac{c_{2i\,2j+1}}{c_{2i\,2j}}\, y\right)^2 \right] \\
& + \zeta_{ij} \frac{(1/\lambda^2) c_{2i\,2j} c_{2i+2\,2j} - c_{2i+1\,2j}^2}{c_{2i\,2j}}\, x^{2i+2} y^{2j} \\
& + \eta_{ij} \frac{(1/\lambda^2) c_{2i\,2j} c_{2i\,2j+2} - c_{2i\,2j+1}^2}{c_{2i\,2j}}\, x^{2i} y^{2j+2} \Bigg\} \\
& + \sum_{j=0}^{n-1} \Bigg\{ 2\alpha_{mj} c_{2m\,2j} x^{2m} y^{2j} \left(1 + \gamma_{mj} \frac{c_{2m\,2j+1}}{2c_{2m\,2j}}\, y\right)^2 \\
& + \eta_{mj} \frac{(1/\lambda^2) c_{2m\,2j} c_{2m\,2j+2} - (1/2) c_{2m\,2j+1}^2}{c_{2m\,2j}}\, x^{2m} y^{2j+2} \Bigg\} \\
& + \sum_{i=0}^{m-1} \Bigg\{ 2\alpha_{in} c_{2i\,2n} x^{2i} y^{2n} \left(1 + \beta_{in} \frac{c_{2i+1\,2n}}{2c_{2i\,2n}}\, x\right)^2 \\
& + \zeta_{in} \frac{(1/\lambda^2) c_{2i\,2n} c_{2i+2\,2n} - (1/2) c_{2i+1\,2n}^2}{c_{2i\,2n}}\, x^{2i+2} y^{2n} \Bigg\} \\
& + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} b_{2i+1\,2j+1} c_{2i+1\,2j+1} x^{2i+1} y^{2j+1} + 2\alpha_{mn} c_{2m\,2n} x^{2m} y^{2n}.
\end{aligned}
$$

By comparing coefficients, we find

$$
\begin{aligned}
& 2\alpha_{00} = b_{00}, \\
& 2\alpha_{ij}\beta_{ij} = b_{2i+1\,2j}, \qquad \zeta_{ij} = \beta_{ij}^2 \alpha_{ij}, \\
& \qquad\qquad i = 0, 1, \cdots, m-1;\ j = 0, 1, \cdots, n, \\
& 2\alpha_{ij}\gamma_{ij} = b_{2i\,2j+1}, \qquad \eta_{ij} = \gamma_{ij}^2 \alpha_{ij}, \\
& \qquad\qquad i = 0, 1, \cdots, m;\ j = 0, 1, \cdots, n-1, \\
& \frac{1}{\lambda^2} \zeta_{i-1\,0} + 2\alpha_{i0} = b_{2i\,0}, \qquad i = 1, 2, \cdots, m, \\
& \frac{1}{\lambda^2} \eta_{0\,j-1} + 2\alpha_{0j} = b_{0\,2j}, \qquad j = 1, 2, \cdots, n, \\
& \frac{1}{\lambda^2} \zeta_{i-1\,j} + \frac{1}{\lambda^2} \eta_{i\,j-1} + 2\alpha_{ij} = b_{2i\,2j}, \\
& \qquad\qquad i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n.
\end{aligned}
$$

We define $\zeta_{-1j}=\eta_{i-1}=0$ ($i=0, 1, \cdots, m; j=0, 1, \cdots, n$). The last three equations can be written

$$\frac{1}{\lambda^2}\zeta_{i-1\ j}+\frac{1}{\lambda^2}\eta_{i\ j-1}+2\alpha_{ij}=b_{2i\ 2j},$$

$$i=0, 1, \cdots, m; j=0, 1, \cdots, n.$$

The parameters α_{ij} are determined by the recursion relations

$$\alpha_{ij}=\frac{4\lambda^2 b_{2i\ 2j}\alpha_{i-1\ j}\alpha_{i\ j-1}-b_{2i-1\ 2j}^2\alpha_{i\ j-1}-b_{2i\ 2j-1}^2\alpha_{i-1\ j}}{8\lambda^2\alpha_{i-1\ j}\alpha_{i\ j-1}},$$

where we have defined $b_{-1j}=b_{i-1}=0$ and $\alpha_{i-1}=\alpha_{-1j}=1$ ($i=0, 1, \cdots, m$; $j=0, 1, \cdots, n$). Let us write

$$\alpha_{ij}=\frac{P_{ij}(\lambda)}{2\lambda Q_{ij}(\lambda)},$$

where $P_{ij}(\lambda)$ and $Q_{ij}(\lambda)$ are polynomials in λ. The polynomials $P_{ij}(\lambda)$ and $Q_{ij}(\lambda)$ satisfy the recursion relations

$$\begin{aligned} P_{00}(\lambda) &= b_{00}\lambda, \qquad\qquad Q_{00}(\lambda) = 1, \\ P_{ij}(\lambda) &= \lambda b_{2i\ 2j}Q_{ij}(\lambda)-\tfrac{1}{4}b_{2i-1\ 2j}^2P_{i\ j-1}(\lambda)Q_{i-1\ j}(\lambda) \\ &\quad -\tfrac{1}{4}b_{2i\ 2j-1}^2P_{i-1\ j}(\lambda)Q_{i\ j-1}(\lambda), \\ Q_{ij}(\lambda) &= P_{i-1\ j}(\lambda)P_{i\ j-1}(\lambda), \end{aligned}$$

where we define $P_{-1j}(\lambda)=P_{i-1}(\lambda)=Q_{-1j}(\lambda)=Q_{i-1}(\lambda)=1$ ($i=0, 1, \cdots, m$; $j=0, 1, \cdots, n$). Let λ_{ij} be the greatest positive zero of $P_{ij}(\lambda)$. Then we establish the following lemma.

LEMMA 1. *$\bar{\lambda}_{ij}$ is greater than or equal to the greater of $\bar{\lambda}_{i-1\ j}$ and $\bar{\lambda}_{i\ j-1}$.*

We proceed to prove this lemma by induction. Obviously $\bar{\lambda}_{10}\geqq\bar{\lambda}_{00}$ and $\bar{\lambda}_{01}\geqq\bar{\lambda}_{00}$. Since the degree of $P_{i-1\ j}(\lambda)$ is greater than that of $P_{i-1\ j-1}(\lambda)P_{i-2\ j}(\lambda)$ and the degree of $P_{i\ j-1}(\lambda)$ is greater than that of $P_{i-1\ j-1}(\lambda)P_{i\ j-2}(\lambda)$, it follows that the degree of $P_{ij}(\lambda)$ is the same as that of $\lambda b_{2i\ 2j}P_{i-1\ j}(\lambda)P_{i\ j-1}(\lambda)$. We see that the coefficient of the highest power of λ in $P_{ij}(\lambda)$ is positive. Let τ equal the greater of $\bar{\lambda}_{i-1\ j}$ and $\bar{\lambda}_{i\ j-1}$. Then $P_{ij}(\tau)\leqq 0$. Hence, $\bar{\lambda}_{ij}$ is greater than or equal to the greater of $\bar{\lambda}_{i-1\ j}$ and $\bar{\lambda}_{i\ j-1}$.

THEOREM 1. *Let r be a real constant. If $r>\bar{\lambda}_{mn}$, and*

$$c_{00}>0,$$

$$\frac{1}{r^2}c_{2i\,2j}c_{2i+2\,2j}-c_{2i+1\,2j}^2>0,$$
$$i=0,1,\cdots,m-1;\ j=0,1,\cdots,n-1,$$
$$\frac{1}{r^2}c_{2i\,2n}c_{2i+2\,2n}-\tfrac{1}{2}c_{2i+1\,2n}^2>0,\qquad i=0,1,\cdots,m-1,$$
$$\frac{1}{r^2}c_{2i\,2j}c_{2i\,2j+2}-c_{2i\,2j+1}^2>0$$
$$i=0,1,\cdots,m-1;\ j=0,1,\cdots,n-1,$$
$$\frac{1}{r^2}c_{2m\,2j}c_{2m\,2j+2}-\tfrac{1}{2}c_{2m\,2j+1}^2>0;\qquad j=0,1,\cdots n,-1,$$
$$c_{2i+1\,2j+1}=0,\ i=0,1,\cdots,\qquad m-1;\ j=0,1\cdots,n-1.$$

then $T(x, y)>0$ *for all real values of* x *and* y.

By the preceding lemma $P_{ij}(r)>0$ ($i=0, 1, \cdots, m; j=0, 1, \cdots, n$). Hence $\alpha_{ij}>0$ ($i=0, 1, \cdots, m; j=0, 1, \cdots, n$), $\zeta_{ij}>0$ ($i=0, 1, \cdots, m-1; j=0, 1, \cdots, n$), and $\eta_{ij}>0$ ($i=0, 1, \cdots, m; j=0, 1, \cdots, n-1$). It follows that $T(x, y)>0$ for all real x and y.

3. **Applications.** In this section we shall study the case where $b_{ij}=1$ ($i=0, 1, \cdots, 2m; j=0, 1, \cdots, 2n$), and deduce a sufficient condition that the polynomial

$$L(x, y) = \sum_{i=0}^{2m}\sum_{j=0}^{2n} c_{ij}x^iy^j$$

be positive for all real values of x and y. For this case, the polynomials $P_{ij}(\lambda)$ satisfy the recursion relations

$$P_{00}(\lambda) = \lambda,$$
$$P_{i0}(\lambda) = \lambda P_{i-1\,0}(\lambda) - \tfrac{1}{2}P_{i-2\,0}(\lambda), \qquad i = 1, 2, \cdots, m,$$
$$P_{0j}(\lambda) = \lambda P_{0\,j-1}(\lambda) - \tfrac{1}{2}P_{0\,j-2}(\lambda), \qquad j = 1, 2, \cdots, n,$$
$$P_{ij}(\lambda) = \lambda P_{i-1\,j}(\lambda)P_{i\,j-1}(\lambda) - \tfrac{1}{2}P_{i\,j-1}(\lambda)P_{i-1\,j-1}(\lambda)P_{i-2\,j}(\lambda)$$
$$- \tfrac{1}{2}P_{i-1\,j}(\lambda)P_{i-1\,j-1}(\lambda)P_{i\,j-2}(\lambda),$$
$$i = 1, 2, \cdots, m; j = 1, 2, \cdots, n,$$

where we define $P_{-1j}(\lambda) = P_{i\,-1}(\lambda) = 1$ ($i=0, 1, \cdots, m; j=0, 1, \cdots, n$).

LEMMA 2. $\bar{\lambda}_{ij}<2$, $i = 0, 1, \cdots, m; j=0, 1, \cdots, n$.

We shall prove this lemma by showing that

$$P_{ij}(\lambda) \geqq P_{i-1\,j}(\lambda)\cdot P_{i\,j-1}(\lambda) > 0, \qquad \lambda \geqq 2.$$

We establish this result by induction. It is seen that the above is true for $i+j=0$ and 1. Now

$$P_{i0}(\lambda) = \lambda P_{i-1\,0}(\lambda) - \tfrac{1}{2}P_{i-2\,0}(\lambda),$$

$$P_{i0}(\lambda) \geqq (\lambda - \tfrac{1}{2})P_{i-1\,0}(\lambda) > 0, \qquad \lambda \geqq 2,$$

$$P_{i0}(\lambda) \geqq P_{i-1\,0}(\lambda) > 0, \qquad \lambda \geqq 2.$$

And since $P_{0j}(\lambda) = P_{j0}(\lambda)$ it follows that

$$P_{0j}(\lambda) \geqq P_{0\,j-1}(\lambda) > 0, \qquad \lambda \geqq 2.$$

Now

$$\begin{aligned} P_{ij}(\lambda) = {} & \lambda P_{i-1\,j}(\lambda)P_{i\,j-1}(\lambda) - \tfrac{1}{2}P_{i\,j-1}(\lambda)P_{i-1\,j-1}(\lambda)P_{i-2\,j}(\lambda) \\ & - \tfrac{1}{2}P_{i-1\,j}(\lambda)P_{i-1\,j-1}(\lambda)P_{i\,j-2}(\lambda), \end{aligned}$$

$$P_{ij}(\lambda) \geqq (\lambda - 1)P_{i-1\,j}(\lambda)P_{i\,j-1}(\lambda) > 0, \qquad \lambda \geqq 2,$$

$$P_{ij}(\lambda) \geqq P_{i-1\,j}(\lambda)P_{i\,j-1}(\lambda) > 0, \qquad \lambda \geqq 2.$$

Hence $\bar{\lambda}_{ij} < 2$ $(i = 0, 1, \cdots, m; j = 0, 1, \cdots, n)$. From Theorem 1 and Lemma 2 we have immediately the following result.

THEOREM 2. *If* $c_{00} > 0$,

$$c_{2i\,2j}c_{2i+2\,2j} - 4c_{2i+1\,2j}^{2} > 0,$$

$$i = 0, 1, \cdots, m-1;\ j = 0, 1, \cdots, n-1,$$

$$c_{2i\,2n}c_{2i+2\,2n} - 2c_{2i+1\,2n}^{2} > 0, \qquad i = 0, 1, \cdots, m-1,$$

$$c_{2i\,2j}c_{2i\,2j+2} - 4c_{2i\,2j+1}^{2} > 0,$$

$$i = 0, 1, \cdots, m-1;\ j = 0, 1, \cdots, n-1,$$

$$c_{2m\,2j}c_{2m\,2j+2} - 2c_{2m\,2j+1}^{2} > 0, \qquad j = 0, 1, \cdots, n-1,$$

$$c_{2i+1\,2j+1} = 0, \qquad i = 0, 1, \cdots, m-1;\ j = 0, 1, \cdots, n-1,$$

then $L(x, y) > 0$ *for all real values of* x *and* y.

This theorem is a generalization of the theorem of E. B. Van Vleck[2] for the case of a polynomial in a single real variable x.

4. **Polynomials in several real variables.** Let

$$(1) \qquad T(x_1, x_2, \cdots, x_n) = \sum_{p=0}^{2u}\sum_{q=0}^{2v}\cdots\sum_{r=0}^{2w} b_{pq\cdots r}c_{pq\cdots r}x_1^{p}x_2^{q}\cdots x_n^{r},$$

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[2] E. B. Van Vleck, *A sufficient condition for the maximum number of imaginary roots of an equation of the nth degree*, Annals of Mathematics (2), vol. 4 (1902–1903), p. 191.

where the $b_{pq\cdots r}$ are positive constants, and the $c_{pq\cdots r}$ are real numbers. Let A denote the class of all the distinct permutations $ij\cdots k$ where i ranges from 0 to u, j ranges from 0 to v, $\cdots$, and k ranges from 0 to w. We define the following $n+1$ functions over the class A

$$I_{ij\cdots k} = \begin{matrix} 1, & i \neq u \\ 0, & i = u \end{matrix}, \quad J_{ij\cdots k} = \begin{matrix} 1, & j \neq v \\ 0, & j = v \end{matrix}, \quad \cdots, \quad K_{ij\cdots k} = \begin{matrix} 1, & k \neq w \\ 0, & k = w \end{matrix},$$

$$S_{ij\cdots k} = \frac{n}{n - d_{ij\cdots k}}, \qquad ij\cdots k \neq uv\cdots w,$$

$$S_{uv\cdots w} = n,$$

where $d_{ij\cdots k}$ represents the number of the indices $i, j, \cdots, k$ which are equal to $u, v, \cdots, w$, respectively. Introducing parameters we rewrite the polynomial (1) as

$$\begin{aligned} T(x_1, x_2, \cdots, x_n) = \sum_A \Bigg[& S_{ij\cdots k}\alpha_{ij\cdots k}c_{2i\,2j\cdots 2k}x_1^{2i}x_2^{2j}\cdots x_n^{2k} \\ & \Bigg\{ I_{ij\cdots k}\left(1 + \beta_{ij\cdots k}\frac{c_{2i+1\,2j\cdots 2k}}{S_{ij\cdots k}c_{2i\,2j\cdots 2k}}\,x_1\right)^2 \\ & + J_{ij\cdots k}\left(1 + \gamma_{ij\cdots k}\frac{c_{2i\,2j+1\cdots 2k}}{S_{ij\cdots k}c_{2i\,2j\cdots 2k}}\,x_2\right)^2 + \cdots \\ & + K_{ij\cdots k}\left(1 + \delta_{ij\cdots k}\frac{c_{2i\,2j\cdots 2k+1}}{S_{ij\cdots k}c_{2i\,2j\cdots 2k}}\,x_n\right)^2\Bigg\} + I_{ij\cdots k}\zeta_{ij\cdots k} \\ & \cdot\frac{(1/\lambda^2)c_{2i\,2j\cdots 2k}c_{2i+2\,2j\cdots 2k} - c^2_{2i+1\,2j\cdots 2k}/S_{ij\cdots k}}{c_{2i\,2j\cdots 2k}}\,x_1^{2i+2}x_2^{2j}\cdots x_n^{2k} \\ & + J_{ij\cdots k}\eta_{ij\cdots k}\frac{(1/\lambda^2)c_{2i\,2j\cdots 2k}c_{2i\,2j+2\cdots 2k} - c^2_{2i\,2j+1\cdots 2k}/S_{ij\cdots k}}{c_{2i\,2j\cdots 2k}} \\ & \cdots x_1^{2i}\cdots x_2^{2j+2}\cdots x_n^{2k} + \cdots + K_{ij\cdots k}\mu_{ij\cdots k} \\ & \cdot\frac{(1/\lambda^2)c_{2i}\,c_{2j\cdots 2k}c_{2i\,2j\cdots 2k+2} - c^2_{2i\,2j\cdots 2k+1}/S_{ij\cdots k}}{c_{2i\,2j\cdots 2k}}\,x_1^{2i}x_2^{2j}\cdots x_n^{2k+2}\Bigg] \\ & + \sum_{p=0}^{u-1}\sum_{q=0}^{v-1}\cdots\sum_{r=0}^{w-1} b_{2p+1\,2q+1\cdots 2r+1}c_{2p+1\,2q+1\cdots 2r+1}x_1^{2p+1}x_2^{2q+1}\cdots x_n^{2r+1}, \end{aligned}$$

where $\sum_A$ means that the summation shall extend over all permutations $ij \cdots k \in A$. By comparing coefficients we obtain

$$\begin{aligned}
n\alpha_{00\cdots 0} &= b_{00\cdots 0},\\
2\alpha_{ij\cdots k}\beta_{ij\cdots k} &= b_{2i+1\,2j\cdots 2k},\\
2\alpha_{ij\cdots k}\gamma_{ij\cdots k} &= b_{2i\,2j+1\cdots 2k},\\
&\cdots\cdots\cdots\cdots\cdots,\\
2\alpha_{ij\cdots k}\delta_{ij\cdots k} &= b_{2i\,2j\cdots 2k+1},\\
\zeta_{ij\cdots k} &= \alpha_{ij\cdots k}\beta^2_{ij\cdots k},\\
\eta_{ij\cdots k} &= \alpha_{ij\cdots k}\gamma^2_{ij\cdots k},\\
&\cdots\cdots\cdots\cdots\cdots,\\
\mu_{ij\cdots k} &= \alpha_{ij\cdots k}\delta^2_{ij\cdots k},
\end{aligned}$$

$$n\alpha_{ij\cdots k} + \frac{1}{\lambda^2}\left(\zeta_{i-1\,j\cdots k} + \eta_{i\,j-1\cdots k} + \cdots + \mu_{ij\cdots k-1}\right) = b_{2i\,2j\cdots 2k},$$

where we define $\zeta_{ij\cdots k} = \eta_{ij\cdots k} = \cdots = \mu_{ij\cdots k} = 0$, $\alpha_{ij\cdots k} = 1$, $ij \cdots k \notin A$; $b_{pq\cdots r} = 0$ if $p, q, \cdots, r$ are not in the ranges 0 to $2u$, 0 to $2v$, $\cdots$, 0 to $2w$, respectively. The parameters $\alpha_{ij\cdots k}$ are determined by the recursion relations

$$n\alpha_{00\cdots 0} = b_{00\cdots 0},$$

$$4n\lambda^2\alpha_{ij\cdots k} = 4\lambda^2 b_{2i\,2j\cdots 2k} - \left(\frac{b^2_{2i-1\,2j\cdots 2k}}{\alpha_{i-1\,j\cdots k}} + \frac{b^2_{2i\,2j-1\cdots 2k}}{\alpha_{i\,j-1\cdots k}} + \cdots \right.$$
$$\left. + \frac{b^2_{2i\,2j\cdots 2k-1}}{\alpha_{ij\cdots k-1}}\right).$$

We write $\alpha_{ij\cdots k} = P_{ij\cdots k}(\lambda)/n\lambda Q_{ij\cdots k}(\lambda)$, where $P_{ij\cdots k}(\lambda)$ and $Q_{ij\cdots k}(\lambda)$ are polynomials in λ. The polynomials $P_{ij\cdots k}(\lambda)$ and $Q_{ij\cdots k}(\lambda)$ satisfy the recursion relations

$$P_{00\cdots 0}(\lambda) = b_{00\cdots 0}\lambda, \qquad Q_{00\cdots 0}(\lambda) = 1,$$

$$\begin{aligned}
P_{ij\cdots k}(\lambda) = Q_{ij\cdots k}(\lambda)\Bigg\{\lambda b_{2i\,2j\cdots 2k} - \frac{n}{4}\Bigg[&\frac{b^2_{2i-1\,2j\cdots 2k}Q_{i-1\,j\cdots k}(\lambda)}{P_{i-1\,j\cdots k}(\lambda)}\\
&+ \frac{b^2_{2i\,2j-1\cdots 2k}Q_{i\,j-1\cdots k}(\lambda)}{P_{i\,j-1\cdots k}(\lambda)} + \cdots\\
&+ \frac{b^2_{2i\,2j\cdots 2k-1}Q_{ij\cdots k-1}(\lambda)}{P_{ij\cdots k-1}(\lambda)}\Bigg]\Bigg\},
\end{aligned}$$

$$Q_{ij\cdots k}(\lambda) = P_{i-1\,j\cdots k}(\lambda)P_{i\,j-1\cdots k}(\lambda) \cdots P_{ij\cdots k-1}(\lambda),$$

where we define $P_{ij\cdots k}(\lambda)=Q_{ij\cdots k}(\lambda)=1$ for $ij\cdots k\notin A$. Let $\bar{\lambda}_{ij\cdots k}$ be the greatest positive zero of $P_{ij\cdots k}(\lambda)$. Then we can establish the following lemma.

LEMMA 3. *$\bar{\lambda}_{ij\cdots k}$ is greater than or equal to the greater of $\bar{\lambda}_{i-1\,j\cdots k}$, $\bar{\lambda}_{i\,j-1\cdots k}, \cdots, \bar{\lambda}_{ij\cdots k-1}$.*

The proof of this lemma is similar to that given for Lemma 1.

THEOREM 3. *Let r be a real constant. If $r\geqq\bar{\lambda}_{uv\cdots w}$, and*

$$c_{00\cdots 0}>0,$$

$$\frac{1}{r^2}c_{2i\,2j\cdots 2k}c_{2i+2\,2j\cdots 2k}-\frac{c^2_{2i+1\,2j\cdots 2k}}{S_{ij\cdots k}}>0,$$

$$\frac{1}{r^2}c_{2i\,2j\cdots 2k}c_{2i\,2j+2\cdots 2k}-\frac{c^2_{2i\,2j+1\cdots 2k}}{S_{ij\cdots k}}>0,$$

$$\frac{1}{r^2}c_{2i\,2j\cdots 2k}c_{2i\,2j\cdots 2k+2}-\frac{c^2_{2i\,2j\cdots 2k+1}}{S_{ij\cdots k}}>0,$$

$$c_{2i+1\,2j+1\cdots 2k+1}=0,$$

then

$$T(x_1,x_2,\cdots,x_n)>0$$

for all real values of $x_1, x_2, \cdots, x_n$.

This theorem follows from the preceding lemma in the same manner that Theorem 1 follows from Lemma 1.

If we take the coefficients

$$b_{pq\cdots r}=1,$$

we can deduce a result generalizing Theorem 2. We consider the polynomial

$$L(x_1,x_2,\cdots,x_n)=\sum_{p=0}^{2u}\sum_{q=0}^{2v}\cdots\sum_{r=0}^{2w}c_{pq\cdots r}x_1^p x_2^q\cdots x_n^r.$$

We define over the class A the following n functions

$$f_{ij\cdots k}=\begin{matrix}0, & i=0\\ 1, & i\neq 0\end{matrix},\quad g_{ij\cdots k}=\begin{matrix}0, & j=0\\ 1, & j\neq 0\end{matrix},\quad\cdots,\quad h_{ij\cdots k}=\begin{matrix}0, & k=0\\ 1, & k\neq 0\end{matrix}.$$

The recursion relations that the polynomials $P_{ij\cdots k}(\lambda)$ satisfy in this case are

$$P_{00\cdots 0}(\lambda) = \lambda,$$

$$\begin{aligned} P_{ij\cdots k}(\lambda) = Q_{ij\cdots k}(\lambda)\biggl(\lambda - \frac{n}{4}\biggl[& f_{ij\cdots k}\frac{Q_{i-1\,j\cdots k}(\lambda)}{P_{i-1\,j\cdots k}(\lambda)} \\ & + g_{ij\cdots k}\frac{Q_{i\,j-1\cdots k}(\lambda)}{P_{i\,j-1\cdots k}(\lambda)} + \cdots + h_{ij\cdots k}\frac{Q_{ij\cdots k-1}(\lambda)}{P_{ij\cdots k-1}(\lambda)}\biggr]\biggr), \end{aligned}$$

$$Q_{ij\cdots k}(\lambda) = P_{i-1j\cdots k}(\lambda)P_{ij-1\cdots k}(\lambda)\cdots P_{ij\cdots k-1}(\lambda),$$

where we have defined $P_{ij\cdots k}(\lambda) = Q_{ij\cdots k}(\lambda) = 1$ for $ij\cdots k \notin A$.

LEMMA 4. $\bar{\lambda}_{ij\cdots k} < n^2/4 + 1$.

The proof of this lemma is similar to that given for Lemma 2.

THEOREM 4. *If* $c_{00\cdots 0} > 0$, *and*

$$\begin{aligned} \frac{16}{(n^2+4)^2}c_{2i\,2j\cdots 2k}c_{2i+2\,2j\cdots 2k} - \frac{c^2_{2i+1\,2j\cdots 2k}}{S_{ij\cdots k}} &> 0, \\ \frac{16}{(n^2+4)^2}c_{2i\,2j\cdots 2k}c_{2i\,2j+2\cdots 2k} - \frac{c^2_{2i\,2j+1\cdots 2k}}{S_{ij\cdots k}} &> 0, \\ \frac{16}{(n^2+4)^2}c_{2i\,2j\cdots 2k}c_{2i\,2j\cdots 2k+2} - \frac{c^2_{2i\,2j\cdots 2k+1}}{S_{ij\cdots k}} &> 0, \\ c_{2i+1\,2j+1\cdots 2k+1} &= 0, \end{aligned}$$

then $L(x_1, x_2, \cdots, x_n) > 0$ *for all real values of* $x_1, x_2, \cdots, x_n$.

ILLINOIS INSTITUTE OF TECHNOLOGY

ON THE MAPPING OF n QUADRATIC FORMS

LLOYD L. DINES

If $Q_1(z), Q_2(z), \cdots, Q_n(z)$ are quadratic forms in the real variables $z^1, z^2, \cdots, z^m$ with real coefficients, the question arises as to the conditions under which there exists a linear combination

$$\lambda_1 Q_1(z) + \lambda_2 Q_2(z) + \cdots + \lambda_n Q_n(z)$$

which is positive definite. A number of recent papers have considered this question.

For the case $n=2$ a satisfactory answer was obtained by Paul Finsler,[1] and independently by a group of interested persons at the University of Chicago.[2]

For any finite n, Finsler obtained sufficient conditions under the restriction that the number of independent variables does not exceed four. The conditions are quite involved, and are in terms of a certain type of algebraic manifold which Finsler designates "Freigebilde" and of which he had made an elaborate study in an earlier paper.

For any finite n and any finite number of independent variables Hestenes and McShane, in a joint paper,[3] obtained sufficient conditions. They are obviously not necessary, and though they seem exactly suited to the application which the authors desired to make, their lack of symmetry perhaps leaves something to be desired.

In a recent paper[4] the present author called attention to the suitability of the theory of convexity as a means for studying this type of question, and treated the case of two quadratic forms in m variables from this point of view. The purpose of the present paper is to make a similar study of the general case of n quadratic forms in m variables.

1. **The map $\mathfrak{M}$.** The n quadratic forms $Q_1(z), Q_2(z), \cdots, Q_n(z)$ may be thought of as mapping the m-dimensional space $\mathfrak{Z}_m$ of points $z=(z^1, z^2, \cdots, z^m)$ onto an n-dimensional space $\mathfrak{X}_n$ of points

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Presented to the Society, December 31, 1941; received by the editors September 5, 1941.

[1] *Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen*, Commentarii Mathematici Helvetici, vol. 9 (1937), pp. 188–192.

[2] From this group two different proofs of the essential theorem appeared: one by A. A. Albert, this Bulletin, vol. 44 (1938), pp. 250–253; and one by W. T. Reid, ibid., pp. 437–440.

[3] *A theorem on quadratic forms and its application in the calculus of variations*, Transactions of this Society, vol. 47 (1940), pp. 501–512.

[4] *On the mapping of quadratic forms*, this Bulletin, vol. 47 (1941), pp. 494–498.

$x=(x^1, x^2, \cdots, x^n)$. The map $\mathfrak{M}$ consists of the set of points x defined by

$$x = (Q_1(z), Q_2(z), \cdots, Q_n(z)), \qquad z \text{ on } \mathfrak{Z}_m,$$

a formula which may be condensed to

$$\mathfrak{M}: \quad x = Q(z), \qquad z \text{ on } \mathfrak{Z}_m.$$

In view of the homogeneity property

$$Q(tz) = t^2 Q(z) \tag{1}$$

the map consists of half-lines emanating from the origin $x=(0)$. In the particular case $n=2$ it is convex, as was shown in the earlier paper. For $n>2$ it is not necessarily convex. For example, the map of three quadratic forms in two variables lies on a conical surface. It is not necessarily closed, but it may be shown to be closed if the forms $Q_i(z)$ have no common zero except $z=(0)$. However, more useful for our present purpose than the map $\mathfrak{M}$ itself is a certain subset of it which we now define.

2. **The map $\mathfrak{M}_1$.** By $\mathfrak{M}_1$ we shall denote the subset consisting of those points of $\mathfrak{M}$ which correspond to points z on the unit $(m-1)$-sphere in $\mathfrak{Z}_m$. The formula for the points of $\mathfrak{M}_1$ is then

$$\mathfrak{M}_1: \quad x = Q(z), \qquad \|z\| = 1.$$

The particular usefulness of the map $\mathfrak{M}_1$ is due to the facts that it is *bounded* and *closed*, and that it contains the origin $x=(0)$ only in exceptional cases. That it is bounded follows immediately from the continuity of $Q(z)$ on the $(m-1)$-sphere. That it is closed may be seen as follows.

Suppose $\{x_j\}$ is a sequence of points in $\mathfrak{M}_1$ defined by

$$x_j = Q(z_j), \qquad \|z_j\| = 1, \tag{2}$$

and suppose there is in the space $\mathfrak{X}_n$ a point $\bar{x}$ such that

$$\bar{x} = \lim_{j\to\infty} x_j. \tag{3}$$

We will prove that $\bar{x}$ belongs to $\mathfrak{M}_1$.

The sequence of points $\{z_j\}$ on the $(m-1)$-sphere $\|z\|=1$ has an accumulation point $\bar{z}$, which must itself be on the $(m-1)$-sphere (that is, $\|\bar{z}\|=1$), since otherwise it would be a definite distance from it. And from the sequence $\{z_j\}$ there can be chosen a subsequence $\{z_{j_k}\}$ such that

$$\lim_{k\to\infty} z_{j_k} = \bar{z}. \tag{4}$$

Then from (2), (3), (4), and the continuity of $Q(z)$ it follows that $\bar{x} = Q(\bar{z})$, and hence $\bar{x}$ belongs to $\mathfrak{M}_1$.

3. **The convex extension of $\mathfrak{M}_1$.** The convex extension of a set of points may be defined as the smallest convex set which contains it, or (equivalently) as consisting of those points which can be the centroids of positive mass distribution (of total mass unity) at suitably chosen points of the given set. According to the latter definition, $C(\mathfrak{M}_1)$, the convex extension of $\mathfrak{M}_1$, consists of the points represented by

$$C(\mathfrak{M}_1): \quad x = \sum_{j=1}^{r} m_j Q(z_j)$$

where

$$\|z_j\| = 1, \; m_j > 0, \quad \textit{and} \quad \sum_{j=1}^{r} m_j = 1. \tag{5}$$

It may also be remarked that whenever a representation (5) exists, there exists an equivalent one in which the number r of terms in the summation does not exceed $n+1$.

4. **On the existence of a definite $\sum \lambda_i Q_i(z)$.** It is well known that through each point exterior to the convex extension of a bounded and closed set of points there passes a bounding plane[5] (Schranke) for the set, while through no point of the convex extension does there pass such a plane. Hence if the origin $x = (0)$ does not belong to $C(\mathfrak{M}_1)$, there is a plane $\sum_{i=1}^{n} \lambda_i x^i = 0$ such that

$$\sum_{i=1}^{n} \lambda_i Q_i(z) > 0 \quad \textit{whenever} \quad \|z\| = 1, \tag{6}$$

while if $x = (0)$ does belong to $C(\mathfrak{M}_1)$ there is no such plane. And in view of the homogeneity relation (1), the validity of (6) is obviously equivalent to the validity of the same relation with z restricted only by $z \neq (0)$.

The condition that $x = (0)$ shall belong to $C(\mathfrak{M}_1)$ is that there shall exist valid relations

$$\sum_{j=1}^{r} m_j Q_i(z_j) = 0, \qquad i = 1, 2, \cdots, n, \tag{7}$$

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[5] For simplicity we use the term "plane" to designate the locus of a linear equation in the x-coordinates, regardless of the dimensions of the space $\mathfrak{X}_n$.

where the z_j and m_j are restricted as in (5). And again in view of the homogeneity property (1) and the homogeneity of (7) in the m_j, the only essential restrictions are $z_j \neq (0)$ and $m_j > 0$. We have then the following theorem:

THEOREM 1. *A necessary and sufficient condition that there exist a linear combination* $\sum \lambda_i Q_i(z)$ *such that*

$$\sum_{i=1}^{n} \lambda_i Q_i(z) > 0 \quad \textit{when} \quad z \neq (0)$$

is that there exist in $\mathfrak{Z}_m$ *no set of points* $z_j \neq (0)$, $(j=1, 2, \cdots, r)$ *such that*

$$\sum_{j=1}^{r} m_j Q_i(z_j) = 0, \qquad i = 1, 2, \cdots, n, \tag{8}$$

the coefficients m_j *being positive.*

5. **On the existence of a semi-definite** $\sum \lambda_i Q_i(z)$. We have seen that when $x=(0)$ is exterior to $C(\mathfrak{M}_1)$ there exist definite linear combinations $\sum \lambda_i Q_i(z)$. It is easily seen that under these circumstances there also exist *semi*-definite linear combinations. We seek here however the more critical conditions under which there exists a linear combination which is semi-definite but none which is definite. This is the case when $x=(0)$ is a *boundary* point of $C(\mathfrak{M}_1)$. For in that case there passes through $x=(0)$ a *supporting* plane (Stützebene) $\sum \lambda_i x^i = 0$ for $\mathfrak{M}_1$, such that

$$\sum_{i=1}^{n} \lambda_i Q_i(z) \geqq 0, \qquad z \textit{ on } \mathfrak{Z}_m, \tag{9}$$

the restriction $\|z\| = 1$ being removable in view of the homogeneity property (1). The "=" sign in (9) is required for at least one point $z \neq (0)$, and hence the linear combination is truly semi-definite unless the $Q_i(z)$ are linearly dependent. We may now state this theorem:

THEOREM 2. *If the quadratic forms* $Q_i(z)$ *are linearly independent, necessary and sufficient conditions that they admit a linear combination which is semi-definite but none which is definite are*: (a) *that there exist in* $\mathfrak{Z}_m$ *a set of points* $z_j \neq (0)$, $(j=1, 2, \cdots, r)$ *and associated positive constants* m_j *such that*

$$\sum_{j=1}^{r} m_j Q(z_j) = 0,$$

and (b) *that there exist a set of real constants* $a=(a^1, a^2, \cdots, a^n)$ *not representable in the form*

$$a = \sum_{j=1}^{r} m_j' Q(z_j'), \tag{10}$$

each m_j' *being positive and each* $z_j' \neq (0)$.

We have already seen that condition (a) is the condition that $x=(0)$ belong to $C(\mathfrak{M}_1)$. Condition (b) is necessary and sufficient to assure that it be a *boundary* point.

It is sufficient, for if some point a of $\mathfrak{X}_n$ is not representable in form (10), then no point on the half-line $x=at$, $(t>0)$, can belong to $C(\mathfrak{M}_1)$, and the origin $x=(0)$ can be approached by a sequence of exterior points on this half-line.

It is necessary, for in its absence every point a of the space $\mathfrak{X}_n$ is representable in form (10), and as a result some point on every half-line $x=at$, $(t>0)$, belongs to $C(\mathfrak{M}_1)$. And since the origin $x=(0)$ itself belongs to $C(\mathfrak{M}_1)$, it follows that there is a hypercube with this origin at its center and all of its vertices belonging to $C(\mathfrak{M}_1)$. The origin would then be an inner point of $C(\mathfrak{M}_1)$.

6. **Generalization.** Of the various properties of quadratic forms, only their continuity and their quadratic homogeneity were used in the above discussion. Indeed it remains valid if: (1) the m-dimensional space $\mathfrak{Z}_m$ is replaced by any normed linear space $\mathfrak{Z}$ in which the points on the unit hypersphere $\|z\|=1$ form a compact subspace; and (2) the n functions $Q_i(z)$ are real-valued functions on $\mathfrak{Z}$, continuous on the hypersphere $\|z\|=1$, and having the property that for real t, $Q_i(tz)=p(t)Q_i(z)$, where $p(t)$ is a non-negative real-valued function which vanishes when and only when $t=0$ and which has a real-valued inverse $t(p)$, not necessarily unique, but defined for all positive p.

CARNEGIE INSTITUTE OF TECHNOLOGY

MEASURE AND OTHER PROPERTIES OF A HAMEL BASIS

F. B. JONES

A Hamel basis[1] is a set $a, b, c, \cdots$ of real numbers such that if x is any real number whatsoever then x may be expressed uniquely in the form $\alpha a+\beta b+\gamma c+\cdots$ where $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero. Since each of these sums is formed from a finite number of nonzero terms and the coefficients $\alpha, \beta, \gamma, \cdots$ are rational and therefore form a countable set, it seems intuitively plausible that not only should the basis set be of the same power as the continuum but in some way be of the same "thickness" as the continuum. However, this intuitive feeling is seemingly contradicted by the only known results along this line, namely: the inner measure of a Hamel basis is zero and its outer measure may also be zero.[2] Nevertheless, this intuition is justified to some extent by Theorems 2, 4, and 5. A natural question arises: In order for a set of real numbers to contain a Hamel basis, what is both necessary and sufficient? For a certain family of sets (including the Borel and analytical sets) this question is answered in two ways. Certain other properties of a Hamel basis are investigated, the most interesting being an example of a Hamel basis which contains a nonvacuous perfect set. Finally, some rather curious discontinuous solutions of the equation $f(x)+f(y)=f(x+y)$ are given.

Measure. No Hamel basis of positive exterior measure is measurable.[3] The next few theorems show this to be true also of certain transforms of every Hamel basis.

DEFINITION. *If M is a set of real numbers, by $T(M)$ is meant the set of all numbers x' such that $x'=x+(y'-y)$, where x, y, and y' belong to M.*

With M considered as a linear set, $T(M)$ is the sum of all translations of M which intersect M. For convenience, $T[T(M)]$ is abbreviated $T^2(M)$, $T[T^2(M)]$ is abbreviated $T^3(M)$, and so on and $T^0(M)=M$.

Presented to the Society, September 2, 1941; received by the editors September 8, 1941.

[1] G. Hamel, *Eine Basis aller Zahlen und die unstetigen Losungen der Funktionalgleichung*: $f(x+y)=f(x)+f(y)$, Mathematische Annalen, vol. 60 (1905), pp. 459–462.

[2] W. Sierpinski, *Sur la question de la mesurabilité de la base de M. Hamel*, Fundamenta Mathematicae, vol. 1 (1920), pp. 105–111.

[3] Ibid.

THEOREM 1. *If H is a Hamel basis, then for each positive integer n, $m_i T^n(H)=0$.*[4]

PROOF. Suppose, on the contrary, that for some positive integer n, $m_i T^{n-1}(H)>0$. Let a and b denote two numbers of H. Since the set of distances of the set $T^{n-1}(H)$ contains an entire interval with its left end at zero,[5] $T^n(H)$ contains an interval V whose midpoint is b. Since a/b is irrational and a is not zero, there exists a rational number r which is not an integer such that ra belongs to V and hence to $T^n(H)$. But every number of $T^n(H)$ can be expressed in the form $n_1x_1+n_2x_2+n_3x_3+\cdots$ where $x_1, x_2, x_3, \cdots$ belong to H and $n_1, n_2, n_3, \cdots$ are *integers* of which only a finite number are different from zero. This is contrary to the properties of H.

THEOREM 2. *If H is a Hamel basis, then for some positive integer n, $m_e T^n(H)>0$.*

PROOF. Suppose, on the contrary, that for each positive integer n, $mT^n(H)=0$. Hence $m\sum_0^\infty T^n(H)=0$. Let M_0 denote $\sum T^n(H)$, and let a denote a fixed number of M_0. Evidently every number of the form $a+n_1(b-a)+n_2(c-a)+n_3(d-a)+\cdots$ (where $b, c, d, \cdots$ belong to H and $n_1, n_2, n_3, \cdots$ are integers of which only a finite number are different from zero) belongs to M_0. Hence the set of all numbers of this form are of measure zero. So the set of all numbers of the form $a+n_1(b-a)+n_2(c-a)+n_3(d-a)+\cdots+n_0a$ (where n_0 is an integer and all other symbols have the same meaning as before) is of measure zero. Now any real number x can be expressed in the form $\alpha a+\beta b+\gamma c+\cdots$ (where $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero) and hence there exists an integer q (which depends on x) such that $q\alpha, q\beta, q\gamma, \cdots$ are integers of which only a finite number are different from zero. Consequently, qx is of the form $a+n_1(b-a)+n_2(c-a)+n_3(d-a)+\cdots+n_0a$. It follows that the set of all real numbers x is of measure zero, which is false.

DEFINITION. *If M is a set of real numbers, then $D(M)$ denotes the set of all numbers $x-y$ where x and y belong to M and $x\geqq y$. The set $D(M)$ is called the set of distances of the set M.*[6]

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[4] Throughout this paper, if Q is a set, m_iQ and m_eQ denote the interior measure and exterior measure, respectively, of Q and mQ denotes the measure of Q if measurable (in the sense of Lebesgue).

[5] H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fundamenta Mathematicae, vol. 1 (1920), pp. 93–104.

[6] Steinhaus, loc. cit.

THEOREM 3. *If H is a Hamel basis, then for each positive integer n, $m_iD^n(H)=0$.*

Theorem 3 may be established by substituting D for T in the proof of Theorem 1.

THEOREM 4. *If H is a Hamel basis, then for some positive integer n, $m_iD^n(H)>0$.*

PROOF. Suppose the contrary. Let $N=\sum_1^\infty D^n(H)$. Then $mN=0$. Let Q denote the set of all points of H which are limit points of H from both sides, and let M denote a countable subset of H such that (1) M contains $H-Q$ and (2) every point of $H-M$ is a limit point of M. It follows that each point of $H-M$ is a limit point of M from both sides. Let x denote a real number. Then x is of the form $\alpha a+\beta b+\gamma c+\cdots$ (where $a, b, c, \cdots$ belong to H and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero). Hence there exists an integer q such that qx is of the form $n_1a+n_2b+n_3c+\cdots$ (where $n_1, n_2, n_3, \cdots$ are integers of which only a finite number are different from zero). So qx is of the form $\sum_1^je_ix_i$ (where for each i, $i=1, 2, \cdots, j$, x_i belongs to H and e_i is ± 1). For each integer i, $i=1, 2, \cdots, j$ (j is finite and depends on x), there exists a number y_i of M such that $\sum_1^je_i(x_i-y_i)$ belongs to N.[7] Hence $qx=z+\sum_1^je_iy_i$ where z belongs to N. Since there are only countably many numbers of the form $\sum_1^je_iy_i$, it follows that the set of all real numbers x is of measure zero, which is a contradiction.

LEMMA 1. *If M is a set of numbers and every number of some number interval V can be expressed in the form $\alpha a+\beta b+\gamma c+\cdots$ (where $a, b, c, \cdots$ belong to M and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero), then M contains a Hamel basis.*

Lemma 1 may be established by well-ordering M and applying Hamel's argument to this well-ordering.

THEOREM 5. *If M is a set of real numbers and for some positive integer n, either $m_iT^n(M)>0$ or $m_iD^n(M)>0$, then M contains a Hamel basis.*

Theorem 5 follows from Lemma 1 and one of Steinhaus' theorems on the set of distances of a set.[8] The condition in Theorem 5 under which the set M contains a Hamel basis is not necessary. This can be seen from Theorems 1 and 3.

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[7] This can be done by first choosing y_1 so that $e_1(x_1-y_1)\geqq 0$. This makes $e_1(x_1-y_1)$ belong to $D(H)$. If it is zero, choose y_2 so that $e_2(x_2-y_2)\geqq 0$; but if it is positive choose y_2 so that $e_2(x_2-y_2)\leqq 0$ but $e_1(x_1-y_1)+e_2(x_2-y_2)\geqq 0$. This makes $e_1(x_1-y_1)+e_2(x_2-y_2)$ belong to $D^2(H)$. Continue this process.

[8] Loc. cit., Theorem 8, p. 99.

EXAMPLE 0. *The Cantor discontinuum*[9] *contains a Hamel basis, because its set of distances contains the interval from zero to one.*

DEFINITIONS. *Let F_T denote the family of all sets M of real numbers such that for infinitely many different positive integral values of n, $T^n(M)$ is measurable; and let F_D denote the family of all sets M of real numbers such that for infinitely many different positive integral values of n, $D^n(M)$ is measurable.*

THEOREM 6. *If M is a set of the family F_T, then in order that M contain a Hamel basis it is necessary and sufficient that for some positive integer n, $mT^n(M) > 0$.*

THEOREM 7. *If M is a set of the family F_D, then in order that M contain a Hamel basis it is necessary and sufficient that for some positive integer n, $mD^n(M) > 0$.*

LEMMA 2. *If M is an analytical set of real numbers, then both $T(M)$ and $D(M)$ are analytical sets.*

PROOF. Being an analytical set of real numbers, M is the set of values of a function $f_1(x)$ of a real variable, defined and continuous in the set of all irrational numbers.[10] Furthermore, $D(M)$ is analytical[11] and the set N consisting of all real numbers x such that either x or $-x$ belongs to $D(M)$ is an analytical set. Hence N is the set of values of a function $f_2(x)$ of a real variable, defined and continuous in the set of all irrational numbers. Let $f(x, y) = f_1(x) + f_2(y)$ where x and y are irrational numbers. Then $T(M)$ is the set of values of $f(x, y)$ and $f(x, y)$ is defined and continuous in the set of all points of the number plane whose coordinates are both irrational numbers. It follows that $T(M)$ is analytical.[12]

Perfect sets, analytical sets. Since every analytical set is measurable,[13] the following three theorems may be easily established using Lemma 2 and Theorems 6 and 7.

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[9] The subset of the interval from 0 to 1 (of real numbers) remaining after deleting in succession every middle-1/3 segment (open interval).

[10] W. Sierpinski, *Introduction to General Topology*, The University of Toronto Press, Toronto, 1930, translated by C. C. Krieger, Theorem 73, p. 145.

[11] W. Sierpinski, *Sur l'ensemble de distances entre les points d'un ensemble*, Fundamenta Mathematicae, vol. 7 (1925), pp. 144–148, p. 146 in particular.

[12] W. Sierpinski, *General Topology*, loc. cit., Theorem 89, p. 185

[13] N. Lusin, *Leçons sur les ensembles analytiques*, Gauthier-Villars, Paris, 1930, p. 152.

THEOREM 8. *In order that an analytical set A of real numbers shall not contain a Hamel basis it is necessary and sufficient that for each positive integer n, both $mT^n(A)$ and $mD^n(A)$ be zero.*

THEOREM 9. *No Hamel basis is an analytical set.*

THEOREM 10. *If an analytical set M is a subset of a Hamel basis, then for each positive integer n, both $mT^n(M)$ and $mD^n(M)$ are zero.*

THEOREM 11. *Suppose that $f(x)$ is a discontinuous real function which is defined for all real values of x and which satisfies the equation $f(x)+f(y)=f(x+y)$. If $f(x)$ is bounded over the analytical set K of real numbers, then for each positive integer n, $mT^n(K)=0$.*

PROOF. Since $f(x)$ is bounded over K, there exists a positive number B such that $|f(x)|<B$ if x belongs to K. It follows from the elementary properties of $f(x)$ (which are imposed upon it by the functional equation) that for any positive integer n, $|f(x)|<3^nB$ if x belongs to $T^n(K)$. Suppose that for some integer n, $mT^n(K)\neq 0$. Since $T^n(K)$ is analytical, it is measurable, and hence, is of *positive* measure. Then $T^{n+1}(K)$ contains an interval. Since $f(x)$ is bounded over this interval, by one of Darboux's theorems it must be continuous.[14] This is a contradiction.

COROLLARY 1. *If the discontinuous solution $f(x)$ of the functional equation $f(x)+f(y)=f(x+y)$ is continuous in an analytical set K of real numbers, then for each positive integer n, $mT^n(K)=0$.*

COROLLARY 2. *Suppose that A is an analytical subset of the plane image of a discontinuous solution of the equation $f(x)+f(y)=f(x+y)$ and that K is the projection of A onto the x-axis. Then for each positive integer n, $mT^n(K)=0$.*

THEOREM 12. *No discontinuous solution of the equation $f(x)+f(y)$ $=f(x+y)$ is continuous in an analytical set which contains a Hamel basis.*

Theorem 12 follows immediately from Theorem 8 and Corollary 1.

Remarks and examples. Burstin showed the existence of a Hamel basis H which intersects every perfect set of real numbers.[15] It follows that H does not contain a perfect set. But every uncountable analytical set contains a perfect set.[16] Consequently, this particular Hamel basis H does not contain an uncountable analytical set. This example and Theorems 9 and 10 might lead one to conjecture that no Hamel basis whatsoever contains an uncountable analytical set. That this conjecture would be false is shown by the following example.

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[14] Darboux, *Sur la composition des forces in statique*, Bulletin des Sciences Mathématiques, vol. 9 (1875), p. 281.

[15] C. Burstin, *Die Spaltung des Kontinuum in c in L Sinne nichtmessbare Mengen*, Sitzungsberichte der Akademie der Wissenschaften, Vienna, Mathematisch-naturwissenschaftliche Klasse, Abt. IIa, vol. 125 (1916).

EXAMPLE 1. *There exists a Hamel basis which contains a perfect set.*[17]

CONSTRUCTION. Let $r_1, r_2, r_3, \cdots$, denote a simple well-ordering of the rational numbers such that $r_1=0$. Let I_{11} denote a closed interval of real numbers not containing zero. The interval I_{11} contains two intervals I_{21} and I_{22} such that (1) I_{21} precedes I_{22} and (2) no number of the form $n_1x_1+n_2x_2$ except the forms x_1 or x_2 belongs to $I_{21}+I_{22}$, where $x_1, x_2 \in I_{21}+I_{22}$ and where $n_i=r_j$, $i, j=1$ or 2. Likewise I_{21} contains two intervals I_{31} and I_{32}, and I_{22} contains two intervals I_{33} and I_{34} such that (1) I_{31} and I_{33} precede I_{32} and I_{34}, respectively, and (2) no number of the form $n_1x_1+n_2x_2+n_3x_3$ except the forms x_1, x_2, or x_3 belongs to $\sum I_{3n}$ where $x_1, x_2, x_3 \in \sum I_{3n}$ and where $n_i=r_j$, $i, j=1$, 2, or 3. This process may be continued. For each positive integer k, let G_k denote the collection of mutually exclusive intervals $I_{k1}, I_{k2}, \cdots, I_{kq}$, where $q=2^{k-1}$. For each k, each element of G_k contains two elements of G_{k+1} and if $x_1, x_2, \cdots, x_k$ denote numbers of G_k^*, then no number of the form $\sum_1^k n_i x_i$ except the form x_i belongs to G_k^*, where $n_i=r_j$, $i, j=1, 2, \cdots, k$.[18] Now let M denote $\prod_1^\infty G_i^*$. Evidently M is a perfect set such that if $\bar{x}$ is a number of M, then $\bar{x} \neq \alpha A+\beta B+\gamma C+\cdots$ where $A, B, C, \cdots$ are numbers of $M-\bar{x}$ and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero. It follows from this that if $\bar{x}$ is a real number which can be expressed in the form $\alpha A+\beta B+\gamma C+\cdots$ where $A, B, C, \cdots$ belong to M and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero, then it can be thus expressed in only one way, that is, the form is unique.

Let Γ denote a well-ordering of the real numbers not belonging to M and let a denote the first number of Γ which is not of the form $\alpha A+\beta B+\gamma C+\cdots$ where $A, B, C, \cdots$ belong to M and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero. Let b denote the first number of Γ which is not of the form $\alpha A+\beta B+\gamma C+\cdots$ where $A, B, C, \cdots$ belong to $M+a$. This process may be continued. It follows from Hamel's argument that $M+a+b+c+\cdots$ is a Hamel basis.[19]

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[16] Lusin, loc. cit., p. 151.

[17] R. L. Swain, in a conversation with me, demonstrated the existence of a perfect set whose set of distances contains no rational number. His method of construction forms the kernel of the one that I use here.

[18] G_k^* denotes the sum of the elements of G_k.

Since, from Theorem 11, no discontinuous (real) solution of the functional equation $f(x)+f(y)=f(x+y)$ can be continuous at any point or continuous in any set of positive measure, could such a discontinuous solution be continuous in some *perfect* set? And could its image in the number plane (the graph of $y=f(x)$) at the same time be *connected*? Example 2 shows that the answer is *yes*.

EXAMPLE 2. *There exist a discontinuous (real) solution of the functional equation* $f(x)+f(y)=f(x+y)$ *and a perfect set* M *of real numbers such that* (1) $f(x)=0$ *if* x *belongs to the perfect set* M *and* (2) *the plane image of* $f(x)$ *is connected.*

CONSTRUCTION. Let M denote a perfect subset of a Hamel basis H such that $H-M$ contains a perfect set. For each number x of M, let $f(x)$ be defined to be zero. Define the function $f(x)$ for each of the numbers of $H-M$ in such a way so that if Q is a continuum in the number plane not lying wholly in a vertical line, then for some number x of $H-M$, the point $(rx, rf(x))$ belongs to Q.[20] Now if x is any number not belonging to H, $x=\alpha a+\beta b+\gamma c+\cdots$ where $a, b, c, \cdots$ belong to H and $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero. Let $f(x)=\alpha f(a)+\beta f(b)+\gamma f(c)+\cdots$. It follows from Hamel's argument that $f(x)$ satisfies the functional equation.[21] Furthermore, $f(x)$ has the following properties: (0) $f(x)$ is totally discontinuous but (1) $f(x)$ is zero for all numbers in the perfect set M and (2) the plane image of $f(x)$ is connected.[22]

In order for the plane image I of *any* discontinuous solution of $f(x)+f(y)=f(x+y)$ to be connected, I must intersect every[23] continuum in the plane which does not lie in some vertical line. However, I need not intersect every perfect subset of the plane not lying in the sum of a countable number of vertical lines, for the image of $f(x)$ in Example 2 does not contain any point of the line $y=1$ whose abscissa belongs to M. So by virtue of the fact that I may be connected and still *contain* a perfect set not lying in the sum of countably many vertical lines, it follows that I need not *intersect* every such perfect set. Since, by Theorem 11, Corollary 2, I can *not* contain a perfect set whose projection on the x-axis is of positive measure, one might suppose that I must intersect every such perfect set in order to be connected. The following example shows this supposition to be false.

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[19] G. Hamel, loc cit. $M+a+b+c+\cdots$ denotes the sum of the *sets*, $M, a, b, c, \cdots$.

[20] F B. Jones, *Connected and disconnected plane sets and the functional equation* $f(x)+f(y)=f(x+y)$, this Bulletin, vol. 48 (1942), pp. 115–120, Theorems 2 and 4; r is rational.

[21] G. Hamel, loc. cit

[22] F. B. Jones, loc cit.

[23] Ibid., Theorem 2.

EXAMPLE 3. *There exists a discontinuous solution of the equation* $f(x)+f(y)=f(x+y)$ *such that* (1) *the image* I *of* $f(x)$ *in the number plane* E *is connected but* (2) *the intersection of* I *with the* x-*axis is exactly the set of rational numbers.*

Two lemmas are necessary.

LEMMA 3. *Suppose that* M *is a bounded, closed subset of the number plane. If uncountably many horizontal lines intersect* M *in an uncountable set, then each of* c *horizontal lines intersects* M *in an uncountable set.*[24]

PROOF. Let E denote the number plane. Let W denote the set of all points w of the y-axis for which there exist sets M_w such that (1) M_w is a subset both of a horizontal line in E through w and of M and (2) every point of M_w is a point of condensation of M_w from both sides. There exists a pair of vertical lines L_{11} and L_{12} (L_{11} being to the left of L_{12}) and an uncountable subset W_1 of W such that for each element w of W_1, M_w contains points between L_{11} and L_{12}, to the left of L_{11} and to the right of L_{12}. Let P_{11} denote a point of M lying between L_{11} and L_{12} such that for uncountably many different elements w of W_1, M_w is a subset of a horizontal line lying above P_{11} in E and for uncountably many elements w of W_1, M_w is a subset of a horizontal line lying below P_{11} in E. Let W_1^+ and W_1^- denote the set of all of those elements w of W_1 for which M_w is on a line above P_{11} and below P_{11}, respectively. Select one of the sets W_1^+ and W_1^- and denote the selection by $W_1^\pm$. Then there exist vertical lines L_{21}, L_{22}, L_{23}, and L_{24} having that order from left to right (with P_{11} between L_{22} and L_{23}) and an uncountable subset W_2 of $W_1^\pm$ such that for each w of W_2, M_w has points to the left of L_{21}, between L_{21} and L_{22}, between L_{22} and the vertical line through P_{11}, between this line and L_{23}, between L_{23} and L_{24}, and to the right of L_{24}. Let P_{21} and P_{22} denote two points of M which lie on the same horizontal line such that uncountably many elements of W_2 lie above and uncountably many elements of W_2 lie below this line, denoting them by W_2^+ and W_2^-. Select *one* of the sets W_2^+ and W_2^- and denote the selection by $W_2^\pm$. This process may be continued and an infinite sequence of points P_{11}, P_{21}, P_{22}, P_{31}, P_{32}, P_{33}, P_{34}, $\cdots$ obtained whose limiting set Q is uncountable and lies both in M and in a horizontal line. Since for each n, $n=2, 3, 4, \cdots$ there are *two* selections possible in the selection of $W_n^\pm$, there are $2^{\aleph_0}$ such limiting sets of which no three are identical. Hence there are c different such sets Q.

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[24] c is the cardinal number of the continuum. It would follow from this lemma that any closed subset of the plane which cannot be covered by a countable collection of horizontal and vertical lines can be covered only by a collection of horizontal and vertical lines which contains at least c lines.

LEMMA 4. *No compact continuum M lying in the number plane which is not wholly in a vertical line and whose common part with every horizontal line is totally disconnected is a subset of the sum of less than c vertical and horizontal lines.*

Lemma 4 may be established by an indirect argument with the help of Lemma 3 and the fact that an uncountable closed plane set contains c points.

CONSTRUCTION OF EXAMPLE 3. Let Γ denote a well-ordering of type Ω_0 (Ω_0 is the smallest ordinal having c ordinals less than it) of the set of all nondegenerate compact subcontinua of E not lying in a vertical line and let $x_1, x_2, x_3, \cdots, x_\omega, x_{\omega+1}, \cdots, x_z, \cdots, z<\Omega_0$, denote a well-ordering of a Hamel basis H such that $x_1=1$ and H contains a number of every perfect set of real numbers. Let $y_1=f(x_1)=f(1)=0$. If r is a rational number, let $f(r)=rf(1)=0$. Suppose that, for each ordinal $z<z_0<\Omega_0, f(x_z)$ is defined to be a number y_z. Then if $x=\sum r_z x_z$, $z<z_0$, $f(x)=\sum r_z y_z$, $z<z_0$, where r_z is a rational number and *not* zero for only a finite number of different values of z. Let I_{z_0} denote the image of $f(z)$ as defined so far. Let M_{z_0} denote the first element of Γ which contains no point of I_{z_0}. Let z_1 denote the smallest ordinal such that (1) $f(x_{z_1})$ is not defined so far and (2) there exists a number y_{z_1} such that (x_{z_1}, y_{z_1}) belongs to M_{z_0} and y_{z_1} is not the ordinate of any point of I_{z_0}. Since $z<\Omega_0$, the existence of z_1 may be established with the help of Lemma 4 and the theorem that every uncountable inner-limiting set of real numbers contains c mutually exclusive perfect sets. Then let $f(x_{z_1})=y_{z_1}$. If $\bar{z}$ is an ordinal less than z_1 such that for each ordinal $z<\bar{z}, f(x_z)$ is defined to be y_z, then let $f(x_{\bar{z}})$ be a real number $y_{\bar{z}}$ such that $y_{\bar{z}}$ is not $r_{z_1}y_{z_1}+\sum r_z y_z$, $z<\bar{z}$, where r_z is a rational number which is different from zero for only a finite number of different values of z and r_{z_1} is a rational number.

This completes the induction in the definition of $f(x)$ if x belongs to H. If x is any real number, then $x=\sum r_z x_z$, where r_z is a rational number which is different from zero for only a finite number of different values of s. This expression for x being unique, $f(x)$ is defined to be $\sum r_s y_s$. By Hamel's argument[25] $f(x)$ satisfies the equation $f(x)+f(y)=f(x+y)$. Obviously the image I of $f(x)$ intersects every compact subcontinuum of E not in a vertical line and hence intersects every such subcontinuum of E whether compact or not. Hence I is connected.[26] Suppose that for two different numbers a_1 and a_2, $f(a_1)=f(a_2)$. But $a_1=\sum r_{1s}x_s$ and $a_2=\sum r_{2s}x_s$, where r_{1s} and r_{2s} are rational numbers which are different from zero for only a finite number of different values of s. Hence $\sum r_{1s}y_s=\sum r_{2s}y_s$. Let $\bar{s}$ be the largest value of s such that $r_{1s}-r_{2s}\neq 0$. Then $y_{\bar{s}}=\sum y_s(r_{2s}-r_{1s})/(r_{1\bar{s}}-r_{2\bar{s}})$, $s<\bar{s}$. This however is impossible if $\bar{s}>1$. Hence $r_{1s}-r_{2s}=0$ if $s>1$ and $a_1-a_2=r_{11}-r_{21}$. It follows from this that if L is a horizontal line intersecting I at the point (x, y) then $I\cdot L$ is the set of all points $(x+r, y)$ where r is rational.

THE UNIVERSITY OF TEXAS

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[25] Loc. cit.

[26] See Footnote 23.

ON THE COMPOSITION OF FIELDS

CLAUDE CHEVALLEY

Let K/k, K'/k be two extensions of a basic field k. By a *composite extension* of these two extensions, we understand the complex notion formed of an extension $\mathfrak{K}/k$ of k, of an isomorphism τ of K/k into $\mathfrak{K}/k$ and of an isomorphism τ' of K'/k into $\mathfrak{K}/k$, provided the following conditions are verified:

(1) $\mathfrak{K}$ is generated by the two fields K^{τ}, $K'^{\tau'}$.

(2) If A, A' are subsets of K, K' respectively which are algebraically independent over k, the set $A^{\tau}\cup(A')^{\tau'}$ is algebraically independent over k. In other words, the algebraic relations which hold in $\mathfrak{K}$ between elements of K^{τ}, $K'^{\tau'}$ are consequences of the algebraic relations which hold between elements of K^{τ} alone or of $K'^{\tau'}$ alone.[1]

THEOREM 1. *Any two given extensions K/k, K'/k have at least one composite extension.*

Let B' be a transcendence basis for K'/k. We can find a purely transcendental extension Ω/K which has a transcendence basis $B'^{\tau'}$ with the same cardinal number as B' (τ' stands for a one-to-one mapping of B' onto $B'^{\tau'}$). The algebraic closure $\bar{\Omega}$ of Ω contains the algebraic closure $\bar{\mathrm{P}}$ of the field $\mathrm{P}=k(B'^{\tau'})$. The mapping τ' may be extended to an isomorphism of K'/k with an extension $K'^{\tau'}/k$ contained in $\bar{\mathrm{P}}/k$, and a fortiori in $\bar{\Omega}$. We set $\mathfrak{K}=KK'^{\tau'}$, and denote by τ the identity mapping of K/k into $\mathfrak{K}/k$. We claim that the system $(\mathfrak{K}/k, \tau, \tau')$ is a composite extension of K/k, K'/k.

It is sufficient to check the condition (2), and we may assume without loss of generality that A, A' are finite. There exists a finite subset B_1' of B' such that $k(A', B_1')$ is algebraic over $k(B_1')$. Let d, d', e be the number of elements in A, A_1', B_1'. The elements of B_1', being algebraically independent over K, are *a fortiori* algebraically independent over $k(A)$. Therefore, the degree of transcendency of $k(A, A'^{\tau'}, B_1'^{\tau'})$ over k is $d+e$. The degree of transcendency of $k(A_1'^{\tau'}, B_1'^{\tau'})$ over $k(A_1'^{\tau'})$ is $e-d'$. The degree of transcendency f of $k(A, A'^{\tau'}, B_1'^{\tau'})$ over $k(A)$ is therefore less than or equal to $e-d$. It follows that the degree of transcendency of $k(A, A'^{\tau'})$ over k, which

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Received by the editors October 1, 1941.

[1] The problem of composite extensions has been considered by Zariski (*Algebraic varieties over ground fields of characteristic zero*, American Journal of Mathematics, vol. 62 (1940), pp. 187–221) in the case when one of the extensions K/k, K'/k is algebraic and normal.

is $d+e-f$, is at least equal to $d+d'$, which proves that $A \cup A'^{\tau'}$ is algebraically independent over k.

Let $(\mathfrak{K}/k, \tau, \tau')$ and $(\mathfrak{K}_1/k, \tau_1, \tau_1')$ be two composite extensions of K/k, K'/k. We shall say that these extensions are isomorphic if there exists an isomorphism σ of $\mathfrak{K}/k$ with $\mathfrak{K}'/k$ such that $\tau_1 = \sigma\tau$, $\tau_1' = \sigma\tau'$.

The consideration of the case where K/k, K'/k are algebraic over k (but not normal) shows immediately that there are in general several non-isomorphic types of composite extensions. The extreme opposite case occurs when k is algebraically closed in K and K' (k is said to be algebraically closed in K if every element of K which is algebraic over k lies already in k). In that case, the composite extension $\mathfrak{K}/k$ turns out to be unique; but, unfortunately, k may fail to be algebraically closed in $\mathfrak{K}$. For instance, let us take $K = K' = k(x, (a+bx^p)^{1/p})$, where $p \neq 0$ is the characteristic of k, and where a, b are elements of k such that $k((a)^{1/p}, (b)^{1/p})$ is of degree p^2 over k. It is easy to verify that k is algebraically closed in K; on the other hand, the composite extension is $k(x, y, (a+bx^p)^{1/p}, (a+by^p)^{1/p})/k = k(x, y, (a)^{1/p}, (b)^{1/p})/k$.

We shall get around this difficulty by introducing the following notion:

DEFINITION 1. *A field k is said to be quasi-algebraically closed (q.a.c.) in K if every element a of K which is algebraic over k is purely inseparable over k (that is, is the unique root of some equation with coefficients in k).*

We shall prove the following theorem:

THEOREM 2. *Let $(\mathfrak{K}/k, \tau, \tau')$ and $(\mathfrak{K}_1/k, \tau_1, \tau_1')$ be two composite extensions of the extensions K/k, K'/k. Let L, L' be fields such that: (1) $k \subset L \subset K$, $k \subset L' \subset K'$; (2) L, L' are algebraic over k; (3) L is q.a.c. in K and L' is q.a.c. in K'. If an isomorphism σ_0 of $L^\tau L'^{\tau'}/k$ with $L^{\tau_1} L'^{\tau_1'}/k$ is such that $\sigma_0\tau$ coincides on L with τ_1 and $\sigma_0\tau'$ coincides on L' with τ_1', then σ_0 may be extended to an isomorphism of $(\mathfrak{K}/k, \tau, \tau')$ with $(\mathfrak{K}_1/k, \tau_1, \tau_1')$.*

In other words, the type of the composite extension $(\mathfrak{K}/k, \tau, \tau')$ is determined by the type of the composite extension $(L^\tau L'^{\tau'}/k, \tau, \tau')$ of L/k, L'/k.

We shall first prove three lemmas. The fields which are considered in the first two of these lemmas are assumed to be all subfields of some all-inclusive field.

LEMMA 1. *Let k be a q.a.c. sub-field of a field K. Let Z be a field such that $k \subset Z \subset K$, and y be an element of K which is a root of an irreducible equation $F = 0$ with coefficients in Z. Let Ω/k be an algebraic extension* *of k, α be an element of Ω and $\Phi=0$, the irreducible equation in k which is satisfied by α. Then the polynomial F is a power of an irreducible polynomial in the field ΩZ, and Φ is a power of an irreducible polynomial in K. Moreover, Ω is q.a.c. in the field $K\Omega$.*

We may assume without loss of generality that $\Omega=k(\alpha)$. Let L be the algebraic closure of k in K (that is, the field consisting of the elements of K which are algebraic over k). Let $\Phi_1=0$ be the irreducible equation satisfied by α in L. Therefore Φ_1 divides Φ. On the other hand, if p is the characteristic of k, the polynomial $\Phi_1^{p^u}$ has its coefficients in k if u is large enough, and therefore Φ divides $\Phi_1^{p^u}$. It follows that Φ is a power of Φ_1.

If $\Phi_1'=0$ is the irreducible equation satisfied by α in K then Φ_1' divides Φ_1. Therefore, every root of the equation $\Phi_1'=0$ is also a root of $\Phi_1=0$, which shows that the coefficients of Φ_1' are algebraic over k. It follows that $\Phi_1'=\Phi_1$ which proves that Φ is a power of an irreducible polynomial in K.

Let $F_1=0$ be the irreducible equation satisfied by y in LZ, and let m be the degree of F_1. If n is the degree of ϕ_1, we have

$$[LZ(y,\alpha):LZ] = [LZ(y,\alpha):LZ(y)][LZ(y):LZ] = mn$$

because Φ_1' is irreducible in K, and *a fortiori*, in $LZ(y)$. It follows that

$$[LZ(y,\alpha):LZ(\alpha)][LZ(\alpha):LZ] = mn.$$

But $[LZ(\alpha):LZ]=n$, since Φ_1 is irreducible in LZ; therefore we have $[LZ(y,\alpha):LZ(\alpha)]=n$ which shows that F_1 is irreducible in $LZ(\alpha)$. On the other hand, F_1 divides F; since LZ is purely inseparable over Z, the same argument which was used above for Φ_1 shows that F is a power of F_1. Since $Z\subset\Omega Z=Z(\alpha)\subset LZ(\alpha)$, F is also a power of an irreducible polynomial in ΩZ.

There remains to prove that $k(\alpha)$ is q.a.c. in $K(\alpha)$. Let β be an element of $K(\alpha)$ which is algebraic over $k(\alpha)$, and therefore also on k. There exists a power $\alpha^{p^u}=\alpha_1$ of α which is separable over k; we set $\beta_1=\beta^{p^u}$, whence $\beta_1\in K(\alpha_1)$ and

$$\beta_1 = \xi_0+\xi_1\alpha_1+\cdots+\xi_{h-1}\alpha_1^{h-1}, \qquad \xi_i\in K,$$

where $h=[K(\alpha_1):K]$. If we write the corresponding formulas for the conjugates of β_1 with respect to K, and observe that α_1 is different from its conjugates, we see that the ξ_i's may be expressed rationally by means of the conjugates of α_1, β_1. It follows that $\xi_0, \xi_1, \cdots, \xi_{h-1}$ are algebraic over k, and therefore belong to L. Since L is purely inseparable over k, we may conclude that some power $\beta_1^{p^s}$ of β_1 lies in $k(\alpha_1)$, where p is the characteristic of k $(\beta_1 \in k(\alpha_1))$ (if $p=0$). We have $\beta^{p^{s+t}} \in \Omega$ which completes the proof of the lemma.

Before stating Lemma 2, we have to introduce another notion. Two extensions K/k, Ω/k of the field k which are contained in some larger field are said to be *algebraically dissociated* if the following condition is realized: If A, B are subsets of K, Ω, respectively, which are algebraically independent over k, the set $A \cup B$ is also algebraically independent.

LEMMA 2. *The Lemma* 1 (*abstraction made of what concerns* α *and* Φ) *remains valid when* Ω/k *is any extension of* k, *provided the extensions* Ω/k, K/k *are algebraically dissociated.*

Let C be a transcendence basis of Ω/k and Ω_0 the field $k(C)$. Under our assumption the extension $K\Omega_0/K$ is purely transcendental. We claim that Ω_0 is q.a.c. in $K\Omega_0$. It will of course be sufficient to prove it in the case where C consists[3] in a single element t. Let $P(t)/Q(t)=\omega$ be an element of $K\Omega_0=K(t)$ (where $P(t)$, $Q(t)$ are polynomials with coefficients in K). We shall prove that if ω is algebraic over Ω_0, it can be expressed as a rational function in t with coefficients in L (the algebraic closure of k in K). From this result it will follow that $L\Omega_0$ is algebraically closed in $K\Omega_0$, and therefore that Ω_0 is q.a.c. in $K\Omega_0$.

The proof will proceed by induction on the number $l=d^\circ P+d^\circ Q$ where $d^\circ P$, $d^\circ Q$ denote the degrees of P, Q with respect to t. It is obvious for $l=0$; assume that the result holds for $l-1$. If either one of the elements $P(0)$, $Q(0)$ is null, we can reduce ourselves to the case $l-1$ by considering instead of ω one of the elements ω/t, $t\omega$ (these elements are also algebraic over Ω_0). So, let us assume that $P(0)Q(0)\neq 0$. We have by assumption a relation of the form

$$A_0(t)P^n(t)+A_1(t)P^{n-1}(t)Q(t)+\cdots+A_n(t)Q^n(t)=0,$$

where $A_0(t), \cdots, A_n(t)$ are polynomials in t with coefficients in k, not all divisible by t. Putting $t=0$, we conclude that $P(0)/Q(0)$ is algebraic over k, and therefore belongs to L. The element $\omega'=P(t)/Q(t)-P(0)/Q(0)$ is again algebraic over Ω_0 and may be written in the form $tP'(t)/Q'(t)$ with $d^\circ P'+d^\circ Q'=l-1$. Therefore $\omega'/t\in L(t)$ and $\omega\in L(t)$, which proves our assertion.

The extension $Z\Omega_0/Z$ being purely transcendental (because $K\Omega_0/K$ is), the polynomial F, which is irreducible in Z, remains irreducible in $Z\Omega_0$. The extension Ω/Ω_0 being algebraic, it follows from Lemma 1 (applied with Ω_0 instead of k) that F becomes a power of an irreducible polynomial in $Z\Omega$ and that Ω is q.a.c. in $K\Omega$; Lemma 2 has been proved.

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[3] This property was proved in the paper, *Pencils on an algebraic variety and a new proof of a theorem of Bertini*, by Zariski, Transactions of this Society, vol. 50 (1941), pp. 48–70.

We pass now to the third lemma. The notations used in this lemma are the same as those introduced in the statement of Theorem 2.

LEMMA 3. *Let Z be a field such that $L\subset Z\subset K$, and let y be an element of K.* (1) *If y is transcendental over Z, y^{τ} is transcendental over $Z^{\tau}K'^{\tau'}$.* (2) *If y is a root of the irreducible equation $F=0$ in Z, the polynomial F^{τ} is a power of an irreducible polynomial in $Z^{\tau}K'^{\tau'}$.*

(1) Let B' be a transcendence basis of K'/k and C be a transcendence basis of Z/k; hence $C^{\tau}\cup B'^{\tau'}$ is a transcendence basis of $Z^{\tau}K'^{\tau'}/k$. If y is transcendental over Z, the set $C\cup\{y\}$ is algebraically independent; therefore $C^{\tau}\cup B'^{\tau'}\cup\{y^{\tau}\}$ is algebraically independent over k, which proves that y^{τ} is transcendental over $Z^{\tau}K'^{\tau'}$.

(2) Since the extensions Z^{τ}/k, $K'^{\tau'}/k$ are clearly algebraically dissociated over k, assertion (2) results from Lemma 2.

We pass now to the proof of Theorem 2. We consider the set Σ of the systems $(Z, Z', \sigma(Z, Z'))$ composed (a) of fields Z, Z' such that $L\subset Z\subset K$, $L'\subset Z'\subset K'$; (b) of an isomorphism $\sigma(Z, Z')$ of $Z^{\tau}Z'^{\tau'}$ with $Z^{\tau_1}Z'^{\tau_1'}$ such that $\sigma(Z, Z')\tau$ coincides with τ_1 on Z, that $\sigma(Z, Z')\tau'$ coincides with τ_1' on Z', and that $\sigma(Z, Z')$ coincides with σ_0 on $L^{\tau}L'^{\tau'}$. We order the set Σ by the convention that

$$(Z, Z', \sigma(Z, Z')) \leq (U, U', \sigma(U, U'))$$

if $Z\subset U$, $Z'\subset U'$ and $\sigma(U, U')$ coincides with $\sigma(Z, Z')$ on $Z^{\tau}Z'^{\tau'}$. It is trivial to verify that in this ordered set every completely ordered subset has an upper bound. Hence, by Zorn's theorem, Σ has a maximal element, which we denote from now on by (Z, Z', σ). Theorem 2 will be proved if we can show that $Z=K$, $Z'=K'$.

Let y be an element of K, and assume for a moment that y is transcendental over Z. Then y^{τ} is transcendental over Z^{τ} and y^{τ_1} is transcendental over Z^{τ_1}. By Lemma 3, y is also transcendental over $Z^{\tau}Z'^{\tau'}$, and *a fortiori*, over $Z^{\tau}Z'^{\tau'}$. Similarly, y^{τ_1} is transcendental over $Z^{\tau_1}Z'^{\tau_1'}$. We set $U=Z(y)$; then it is possible to extend σ to an isomorphism σ^* of $U^{\tau}Z'^{\tau'}$ with $U^{\tau_1}Z'^{\tau_1'}$ in such a way that $\sigma^*(y^{\tau})=y^{\tau_1}$. It follows that $\sigma^*\tau$ coincides with τ_1 on U and that $\sigma^*\tau'$ coincides with τ_1' on Z'. But this is contrary to the maximality of (Z, Z', σ).

It follows that K is algebraic over Z, and similarly that K' is algebraic over Z'.

Let us again consider the element $y\in K$; it is a root of an irreducible equation $F=0$ in Z. By Lemma 3, F^τ becomes in $Z^\tau Z'^{\tau'}$ a power of an irreducible polynomial F_1, and we have $F_1(y^\tau)=0$. The polynomial F_1^σ is irreducible in $Z^{\tau_1}Z'^{\tau_1'}$, and $F^{\sigma\tau}$ is a power of F_1^σ. On the other hand, we have $F^{\sigma\tau}=F^{\tau_1}$ and $F^{\tau_1}(y^{\tau_1})=0$, whence $F_1^\sigma(y^{\tau_1})=0$. Therefore, we may extend σ to an isomorphism σ^* of $Z^\tau Z'^{\tau'}(y^\tau)$ with $Z^{\tau_1}Z'^{\tau_1'}(y^{\tau_1})$ such that $\sigma^*(y^\tau)=y^{\tau_1}$. The isomorphism $\sigma^*\tau$ of $Z(y)$ into $K^{\tau_1}K'^{\tau_1'}$ coincides with the automorphism induced by τ_1 and $\sigma^*\tau'$ coincides on Z' with τ_1'. By the maximality property of (Z, Z', σ), we have $Z(y)=Z$, whence $K=Z$, and we see in the same way that $K'=Z'$, which completes the proof of Theorem 2.

Corollary. *Let K/k, K'/k be two extensions of k, and assume that k is q.a.c. in at least one of them. Then there exists only one type of composite extension of our two extensions.*

In fact, if k is q.a.c. in K', we may apply Theorem 2 with $L'=k'$. If we set $\sigma_0=\tau_1\tau^{-1}$, σ_0 is an isomorphism of L^τ with L^{τ_1} and $\sigma_0\tau$ coincides with τ_1 on L; it follows that σ_0 may be extended to an isomorphism of $(\mathfrak{K}, \tau, \tau')$ with $(\mathfrak{K}_1, \tau_1, \tau_1')$.

Theorem 3. *Let $(\mathfrak{K}/k, \tau, \tau')$ be a composite extension of K/k, K'/k and let L, L' be fields which satisfy the conditions* (1), (2), (3) *of Theorem* 2. *Then $L^\tau L'^{\tau'}$ is q.a.c. in $\mathfrak{K}$.*

By Lemma 1, $L^\tau L'^{\tau'}$ is q.a.c. in $L^\tau K'^{\tau'}$; by Lemma 2, we see that $L^\tau K'^{\tau'}$ is q.a.c. in $K^\tau K'^{\tau'}$, because the extensions K^τ/k, $K'^{\tau'}/k$ are algebraically dissociated. Let α be an element of $\mathfrak{K}$ which is algebraic over $L^\tau L'^{\tau'}$; then, if p is the characteristic of k, the second result shows that, for s large enough, $\alpha^{p^s}\in L^\tau K'^{\tau'}$. Since α^{p^s} is also algebraic over $L^\tau L'^{\tau'}$, the first result shows that $(\alpha^{p^s})^{p^{s'}}=\alpha^{p^{s+s'}}\in L^\tau L'^{\tau'}$, for s' large enough, which proves Theorem 3.

Corollary. *If k is q.a.c. in both K and K', it is also q.a.c. in $K^\tau K'^{\tau'}$.*

Princeton University

NOTE ON CLOSURE FOR ORTHOGONAL POLYNOMIALS

J. SHOHAT

In this short note we give a simple theorem concerning closure for orthogonal polynomials (OP). The importance of closure in the theory of OP, particularly in the study of expansions of functions in series of OP, can hardly be overestimated.

The notations employed below are those of my monograph, *Théorie générale des polynômes orthogonaux de Tchebicheff* (Mémorial des Sciences Mathématiques, vol. 66, 1934), referred to as M.

We write generally

$$f(x) \sim \sum_{n=0}^{\infty} f_n \phi_n(x)$$

to signify that

$$f_n = \int_a^b f(x)\phi_n(x)d\psi, \qquad n = 0, 1, 2, \cdots,$$

$$\phi_n(x) \equiv \phi_n(x; a, b; d\psi),$$

$$\int_a^b \phi_m(x)\phi_n(x)d\psi = \delta_{mn}, \quad m, n = 0, 1, 2, \cdots.$$

The hereafter assumed existence of $\int_a^b f^2(x)d\psi$ implies, as is known, the convergence of the series $\sum_{n=0}^{\infty} f_n^2$. By closure we mean the following equality, Parseval's formula:

$$\int_a^b f^2(x)d\psi = \sum_{n=0}^{\infty} f_n^2.$$

THEOREM. *If closure holds for the "symmetric" sequence*

$$(1) \qquad \{\phi_n(x; -h, h; d\psi)\}, \qquad \psi(\infty) - \psi(x) \equiv \psi(-x)$$

[or

$$\{\phi_n(x; -h, h; p(x))\}, \qquad p(x) \equiv p(-x)]$$

then each of the following two sequences of OP is closed

$$(2) \qquad \begin{gathered} \{\phi_n(x; 0, h^2; d\psi_1)\}, \qquad \{\phi_n(x; 0, h^2; d\psi_2)\} \\ d\psi_1(x) = 2d\psi(x^{1/2}), \qquad d\psi_2(x) = x\,d\psi_1(x) \end{gathered}$$

[or

Received by the editors October 2, 1941.

$$\{\phi_n(x;\,0,\,h^2;\,p_1(x))\},\qquad \{\phi_n(x;\,0,\,h^2;\,p_2(x))\}$$
$$p_1(x) = x^{-1/2}p(x^{1/2}),\qquad p_2(x) = xp_1(x)],$$

and conversely.

It suffices to give the proof for $d\psi(x)$ proper, for the case where $d\psi(x)=p(x)dx$ is quite similar.

We have, for any $n=0, 1, 2, \cdots$ [M, pp. 19–20],

$$\begin{aligned} \phi_{2n}(x;\,-h,\,h;\,d\psi) &\equiv \phi_n(x^2;\,0,\,h^2;\,d\psi_1), \\ \phi_{2n+1}(x;\,-h,\,h;\,d\psi) &\equiv x\phi_n(x_n^2;\,0,\,h^2;\,d\psi_2). \end{aligned} \tag{3}$$

This yields at once, for any $n=0, 1, 2, \cdots$,

$$\begin{aligned} f_{2n} &= \int_{-h}^{h} f(x)\phi_{2n}(x;\,-h,\,h;\,d\psi)d\psi \\ &= \int_{0}^{h^2} F_1(x)\phi_n(x;\,0,\,h^2;\,d\psi_1)d\psi_1 \equiv f_{n,1}, \\ f_{2n+1} &= \int_{-h}^{h} f(x)\phi_{2n+1}(x;\,-h,\,h;\,d\psi)d\psi \\ &= \int_{0}^{h^2} F_2(x)\phi_n(x;\,0,\,h^2;\,d\psi_2)d\psi_2 \equiv f_{n,2}, \end{aligned} \tag{4}$$

where

$$F_1(x^2) = \frac{f(x)+f(-x)}{2},\qquad xF_2(x^2) = \frac{f(x)-f(-x)}{2}. \tag{5}$$

We have further

$$f(x) = F_1(x^2) + xF_2(x^2) \sim \sum_{n=0}^{\infty} f_n\phi_n(x;\,-h,\,h;\,d\psi).$$

Making use of (3), (4) and (5), we get readily

$$\begin{aligned} F_1(x^2) &\sim \sum_{n=0}^{\infty} f_{2n}\phi_{2n}(x;\,-h,\,h;\,d\psi), \\ xF_2(x^2) &\sim \sum_{n=0}^{\infty} f_{2n+1}\phi_{2n+1}(x;\,-h,\,h;\,d\psi), \\ F_1(x) &\sim \sum_{n=0}^{\infty} f_{n,1}\phi_n(x;\,0,\,h^2;\,d\psi_1), \\ F_2(x) &\sim \sum_{n=0}^{\infty} f_{n,2}\phi_n(x;\,0,\,h^2;\,d\psi_2), \end{aligned} \tag{6}$$

$$\int_{-h}^{h} xF_1(x^2)F_2(x^2)\,d\psi = 0,$$

$$\int_{-h}^{h} f^2(x)\,d\psi - \sum_{n=0}^{\infty} f_n^2 = \left[\int_0^{h^2} F_1^2(x)\,d\psi_1 - \sum_{n=0}^{\infty} f_{n,1}^2\right] + \left[\int_0^{h^2} F_2^2(x)\,d\psi_2 - \sum_{n=0}^{\infty} f_{n,2}^2\right]. \tag{7}$$

By Bessel's inequality, *both brackets on the right side of* (7) *are non-negative*, and this directly proves the theorem stated.

REMARK. In the above discussion h may be assumed to be infinite, for closure always holds for a finite interval.

The most interesting application of the foregoing is to Hermite and Laguerre polynomials. In fact, it follows from the above theorem that the closure of both sequences

$$\left\{\phi_n(x;\,0,\,\infty;\,e^{-x}x^{\alpha-1})\right\}, \qquad \left\{\phi_n(x;\,0,\,\infty;\,e^{-x}x^{\alpha})\right\}, \qquad \alpha \geqq 1/2,$$

implies that of $\left\{\phi_n(x;\,-\infty,\,\infty;\,e^{-x^2}|x|^{2\alpha-1})\right\}$, and conversely. In particular, taking $\alpha=1/2$, we conclude that the closure of Hermite polynomials $\left\{\phi_n(x;\,-\infty,\,\infty;\,e^{-x^2})\right\}$ implies that either of the two sequences of Laguerre polynomials $\left\{\phi_n(x;0,\,\infty;\,e^{-x}x^{-1/2})\right\}$, $\left\{\phi_n(x;0,\,\infty;\,e^{-x}x^{1/2})\right\}$ is closed, and conversely.

THE UNIVERSITY OF PENNSYLVANIA

THE APRIL MEETING IN NEW YORK

The three hundred eighty-sixth meeting of the American Mathematical Society was held at Columbia University on Friday and Saturday, April 3–4, 1942. The attendance was approximately two hundred seventy-five including the following two hundred four members of the Society:

Milton Abramowitz, C. R. Adams, Felix Adler, Leonidas Alaoglu, C. B. Allendoerfer, Lawrence Annenberg, R. G. Archibald, B. H. Arnold, L. A. Aroian, Y. K. Bal, J. D. Bankier, J. J. Barron, Alfred Basch, P. T. Bateman, M. F. Becker, P. O. Bell, Stefan Bergman, Garrett Birkhoff, Gertrude Blanch, M. L. Boas, R. P. Boas, H. F. Bohnenblust, Samuel Borofsky, C. C. Bramble, Richard Brauer, Jewell H. Bushey, J. H. Bushey, W. E. Byrne, W. B. Campbell, Claude Chevalley, D. E. Christie, Alonzo Church, M.D. Clement, Harvey Cohn, J. B. Coleman, L. M. Comer, T. F. Cope, Richard Courant, W. H. H. Cowles, H. B. Curry, M. D. Darkow, Norman Davids, A. H. Diamond, R. P. Dilworth, J. L. Doob, Jesse Douglas, C. H. Dowker, Jacques Dutka, John Dyer-Bennet, J. E. Eaton, W. F. Eberlein, M. L. Elveback, Benjamin Epstein, Paul Erdös, J. M. Feld, W. K. Feller, Aaron Fialkow, F. A. Ficken, W. B. Fite, R. M. Foster, G. A. Foyle, J. S. Frame, K. O. Friedrichs, Orrin Frink, Guido Fubini, R. E. Fullerton, H. P. Geiringer, J. H. Giese, B. P. Gill, A. M. Gleason, M. C. Gray, H. J. Greenberg, Lewis Greenwald, C. C. Grove, Laura Guggenbuhl, N. G. Gunderson, Jacques Hadamard, Theodore Hailperin, F. C. Hall, P. R. Halmos, K. E. Hazard, M. H. Heins, Edward Helly, Olaf Helmer, E. H. C. Hildebrandt, Banesh Hoffmann, T. R. Hollcroft, Harold Hotelling, E. M. Hull, Witold Hurewicz, W. A. Hurwitz, L. C. Hutchinson, R. P. Isaacs, Nathan Jacobson, R. A. Johnson, G. K. Kalisch, Edward Kasner, S. A. Kiss, S. C. Kleene, R. B. Kleinschmidt, J. R. Kline, E. R. Kolchin, B. O. Koopman, Arthur Korn, M. E. Ladue, J. S. Le Caine, Solomon Lefschetz, B. A. Lengyel, Howard Levi, Norman Levinson, E. R. Lorch, A. N. Lowan, N. H. McCoy, J. C. C. McKinsey, Brockway McMillan, L. A. MacColl, J. K. L. MacDonald, C. C. MacDuffee, G. W. Mackey, G. R. MacLane, H. M. MacNeille, M. S. Macphail, P. T. Maker, Imanuel Marx, W. P. Mason, A. E. Meder, H. L. Mintzer, R. E. von Mises, E. C. Molina, Deane Montgomery, Richard Morris, Marston Morse, G. W. Mullins, F. J. Murray, C. A. Nelson, N. D. Nelson, Philip Newman, K. L. Nielsen, C. O. Oakley, Rufus Oldenburger, Oystein Ore, J. C. Oxtoby, N. G. Parke, L. G. Peck, A. S. Peters, George Polya, E. L. Post, Willy Prager, M. H. Protter, R. G. Putnam, J. F. Randolph, C. J. Rees, M. S. Rees, Eric Reissner, R. W. Rempfer, H. J. Riblet, Moses Richardson, R. G. D. Richardson, J. F. Ritt, H. E. Robbins, J. H. Roberts, M. S. Robertson, S. L. Robinson, Benjamin Rosenbaum, J. E. Rosenthal, Raphael Salem, Hans Samelson, Arthur Sard, L. J. Savage, Rubin Schatten, M. G. Scherberg, I. J. Schoenberg, Alberta Schuettler, I. E. Segal, I. M. Sheffer, Seymour Sherman, C. A. Shook, P. A. Smith, D. C. Spencer, Wolfgang Sternberg, J. J. Stoker, R. E. Street, R. G. Sturm, Alvin Sugar, J. L. Synge, Otto Szász, J. D. Tamarkin, M. E. Taylor, J. W. Tukey, J. L. Vanderslice, G. L. Walker, Henry Wallman, R. M. Walter, W. R. Wasow, Louis Weisner, Hermann Weyl, A. L. Whiteman, D. V. Widder, V. A. Widder, Norbert Wiener, John Williamson, Audrey Wishard, Oscar Zariski, Leo Zippin.

The meeting opened Friday morning with a section for papers in analysis and applied mathematics, Professor Norman Levinson presiding, and another for algebra, geometry and logic, Professor Alonzo Church presiding.

Friday afternoon there was a symposium on topics in the theory of functions, at which Vice President C. C. MacDuffee and President Marston Morse presided. These addresses were given: *On entire functions of exponential type*, by Dr. R. P. Boas of Duke University; and, *On the zeros of the derivatives of a function and its analytic character*, by Professor George Polya of Smith College. The discussion was led by Professor Norman Levinson of the Massachusetts Institute of Technology and Professor I. J. Schoenberg of the University of Pennsylvania.

Saturday morning was devoted to a symposium on aeronautics at which Vice President J. D. Tamarkin presided. There was an address on *Theory of wing flutter*, by Dr. Theodore Theodorsen of Langley Field, Virginia, and on *New approach to the three-dimensional wing theory*, by Dr. R. E. von Mises of Harvard University. The discussion was led by Professor Willy Prager of Brown University, Dr. Eric Reissner of the Massachusetts Institute of Technology, Professor K. O. Friedrichs of New York University, and Professor J. L. Synge of the University of Toronto.

Titles and cross references to the abstracts of papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers numbered 1–8 were read before the section for analysis and applied mathematics; those numbered 9–17 before the section for algebra, geometry and logic; those numbered 18–39 were read by title. Paper 7 was read by Professor Roberts, and paper 12 by Mr. Tuan. Dr. Bers was introduced by Dr. A. M. Gelbart.

1. Lipman Bers: *A property of harmonic functions.* (Abstract 48-5-170.)

2. Otto Szász: *On sequences of polynomials and the distribution of their zeros.* (Abstract 48-5-193.)

3. J. H. Bushey: *The distribution function of the sample total under the type β hypothesis.* (Abstract 48-5-217.)

4. Hermann Weyl: *Solution of the simplest boundary-layer problems in hydrodynamics.* (Abstract 48-5-198.)

5. Hilda P. Geiringer: *On the numerical solution of linear problems by group iteration.* (Abstract 48-5-202.)

6. Stefan Bergman: *Three-dimensional flow of a perfect incompressible fluid and its singularities.* (Abstract 48-5-201.)

7. J. H. Roberts and Paul Civin: *Sections of continuous collections.* (Abstract 48-5-224.)

8. Witold Hurewicz: *Ergodic theorem without invariant measure.* (Abstract 48-7-235.)

9. B. H. Arnold: *Rings of transformations of certain vector spaces.* Preliminary. report. (Abstract 48-5-142.)

10. F. A. Ficken: *On Rosser's generalization of Euclid's algorithm.* Preliminary report. (Abstract 48-5-151.)

11. Rufus Oldenburger: *The index of a quadratic form for an arbitrary field.* (Abstract 48-5-162.)

12. Richard Brauer and Hsio-Fu Tuan: *Some remarks on simple groups of finite order.* Preliminary report. (Abstract 48-5-147.)

13. J. S. Frame: *Double coset matrices and group characters.* (Abstract 48-5-152.)

14. S. C. Kleene: *On the forms of the predicates in the theory of constructive ordinals.* Preliminary report. (Abstract 48-5-215.)

15. L. C. Hutchinson: *On the linear line complex in n-space.* (Abstract 48-5-208.)

16. Aaron Fialkow: *Conformal differential geometry of a subspace.* (Abstract 48-5-207.)

17. R. P. Dilworth: *The partition lattice.* (Abstract 48-5-150.)

18. Stefan Bergman: *Determination of pressure in the two-dimensional flow of an incompressible perfect fluid.* (Abstract 48-5-200-*t*.)

19. J. H. Bushey: *The distribution function of the mean under the type α hypothesis.* (Abstract 48-5-216-*t*.)

20. J. H. Bushey: *The products of certain discrete and continuous orthogonal polynomials.* (Abstract 48-5-218-*t*.)

21. Leonard Carlitz: *On the reciprocal of certain types of Hurwitz series.* (Abstract 48-5-148-*t*.)

22. F. G. Dressel: *The fundamental solution of the parabolic equation.* II. (Abstract 48-5-171-*t*.)

23. A. M. Gelbart: *Bounds for pressure in a two-dimensional flow of an incompressible perfect fluid.* (Abstract 48-5-203-*t*.)

24. P. R. Halmos and Hans Samelson: *On monothetic groups.* (Abstract 48-7-250-*t*.)

25. Nathan Jacobson: *Classes of restricted Lie algebras of characteristic p.* II. (Abstract 48-7-231-*t*.)

26. Meyer Karlin: *Characterization of the extremals of a variation problem of higher order in the plane.* Preliminary report. (Abstract 48-5-175-*t*.)

27. Joseph Lehner: *On certain irrational modular equations.* (Abstract 48-5-156-*t*.)

28. F. A. Lewis: *Generators of permutation groups simply isomorphic with $LF(2, p^n)$.* (Abstract 48-5-157-*t*.)

29. A. N. Lowan, H. E. Salzer, and Abraham Hillman: *Coefficients of differences in the expansion of derivatives in terms of advancing differences.* (Abstract 48-5-176-*t*.)

30. Rufus Oldenburger: *Repeated linear factors of forms.* (Abstract 48-5-161-*t*.)

31. Harry Pollard: *An inversion formula for the Stieltjes transform.* (Abstract 48-5-182-*t*.)

32. Harry Pollard: *On subseries of a convergent series.* (Abstract 48-5-183-*t*.)

33. H. A. Rademacher: *On the Bloch-Landau constant.* (Abstract 48-5-185-*t*.)

34. M. F. Smiley: *A remark on S. Kakutani's characterisation of* (L)*-spaces.* (Abstract 48-5-191-*t*.)

35. M. F. Smiley and W. R. Transue: *Applications of transitivities of betweenness in lattice theory.* (Abstract 48-5-165-*t*.)

36. J. L. Walsh: *On the overconvergence, degree of convergence, and zeros of sequences of analytic functions.* (Abstract 48-5-196-*t*.)

37. G. T. Whyburn: *Coherent and saturated collections.* (Abstract 48-5-226-*t*.)

38. G. T. Whyburn: *On the interiority of mappings.* (Abstract 48-5-227-*t*.)

39. G. T. Whyburn: *Unitary subcontinua.* (Abstract 48-5-228-*t*.)

T. R. HOLLCROFT,
Associate Secretary

THE APRIL MEETING IN BERKELEY

The three hundred eighty-seventh meeting of the American Mathematical Society was held at the University of California, Berkeley, on Saturday, April 11, 1942. The attendance was about seventy, including the following thirty-six members of the Society:

S. P. Avann, H. M. Bacon, E. W. Barankin, K. E. Benson, B. A. Bernstein, H. F. Blichfeldt, T. C. Doyle, G. C. Evans, A. L. Foster, D. L. Fuller, S. M. Hallam, M. A. Heaslet, Einar Hille, Alfred Horn, R. C. James, D. H. Lehmer, A. O. Leuschner, S. H. Levy, Hans Lewy, W. R. Mann, W. A. Manning, A. D. Michal, E. M. Morenus, C. B. Morrey, Jr., A. P. Morse, A. R. Noble, R. M. Robinson, E. B. Roessler, P. C. Rosenbloom, A. C. Schaeffer, W. H. Simons, Gabor Szegö, S. P. Timoshenko, F. A. Valentine, W. M. Whyburn, A. R. Williams.

The meeting opened in the morning with a general session for contributed papers at which Professor W. M. Whyburn presided. By invitation of the Program Committee, Professor W. A. Manning of Stanford University delivered an hour address on *The simply transitive permutation groups*. Professor B. A. Bernstein presided at this lecture.

The afternoon session was devoted to a symposium on applied mathematics, Professor Gabor Szegö presiding. Professor Theodore von Kármán spoke on *Some problems in dynamics of compressible fluids*; Professor E. E. Weibel on *The photo-elastic method of stress determination*; and Professor C. F. Richter on *Mathematical questions in seismology*.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1–5 were read at the morning session; those numbered 6–14 were read by title.

1. G. E. Forsythe and A. C. Schaeffer: *A remark on Toeplitz matrices*. (Abstract 48-3-108.)

2. F. A. Valentine: *On the extension of a vector function so as to preserve a Lipschitz condition*. (Abstract 48-5-195.)

3. R. C. James: *Normability of topological abelian groups*. (Abstract 48-5-221.)

4. S. P. Avann: *On complete classes of projective quotients in a modular lattice*. (Abstract 48-5-145.)

5. Einar Hille: *Notes on linear transformations*. IV. *Representation of semi-groups*. (Abstract 48-7-234.)

6. W. A. Mersman: *Heat conduction in an infinite composite solid with an interface resistance*. (Abstract 48-5-205-*t*.)

7. Walter Leighton and W. J. Thron: *On value regions of continued fractions.* (Abstract 48-3-113-*t*.)

8. R. A. Beaumont: *Projections of the prime-power abelian group of order* p^m *and type* $(m-1, 1)$. (Abstract 48-3-103-*t*.)

9. W. A. Mersman: *Heat conduction in a finite composite solid.* (Abstract 48-5-204-*t*.)

10. Knox Millsaps: *Abstract polynomials in non-abelian groups.* (Abstract 48-5-158-*t*.)

11. S. P. Avann: *A numerical condition for modularity of a finite lattice.* (Abstract 48-5-144-*t*.)

12. E. J. Pinney: *Calculus of variations in abstract spaces and related topics.* I. (Abstract 48-5-180-*t*.)

13. E. J. Pinney: *Calculus of variations in abstract spaces and related topics.* II. (Abstract 48-5-181-*t*.)

14. A. D. Michal: *A theory of fluctuations in Riemannian spaces.* Preliminary report. (Abstract 48-5-209-*t*.)

A. D. MICHAL,
Acting Associate Secretary

THE APRIL MEETING IN CHICAGO

The three hundred eighty-eighth meeting of the American Mathematical Society was held at the University of Chicago, Chicago, Illinois, on Friday and Saturday, April 17–18, 1942.

The attendance was one hundred eighty-three including the following one hundred sixty-one members of the Society:

A. A. Albert, J. V. Atanasoff, W. L. Ayres, Reinhold Baer, R. M. Ballard, R. H. Bardell, Walter Bartky, E. F. Beckenbach, J. C. Bell, D. L. Bernstein, S. F. Bibb, E. E. Blanche, G. A. Bliss, Henry Blumberg, L. M. Blumenthal, R. W. Brink, E. L. Buell, H. E. Burns, I. W. Burr, L. E. Bush, Herbert Busemann, Albert Cahn, W. C. Carter, E. W. Chittenden, L. W. Cohen, B. H. Colvin, J. J. Corliss, L. L. Cronvich, A. R. Craw, E. L. Crow, D. R. Curtiss, W. M. Davis, M. M. Day, L. E. Dickson, J. M. Dobbie, Roy Dubisch, W. D. Duthie, Phillip Eddy, Samuel Eilenberg, R. S. Embree, W. S. Erickson, E. B. Escott, H. P. Evans, H. S. Everett, G. M. Ewing, J. V. Finch, L. R. Ford, R. H. Fox, Bernard Friedman, H. L. Garabedian, B. H. Gere, H. A. Giddings, B. E. Gillam, J. W. Givens, E. L. Godfrey, R. A. Good, G. D. Gore, L. M. Graves, L. W. Griffiths, V. G. Grove, G. J. Haltiner, R. W. Hamming, O. G. Harrold, W. L. Hart, M. L. Hartung, G. E. Hay, C. T. Hazard, A. E. Heins, E. D. Hellinger, M. R. Hestenes, J. F. Heyda, T. H. Hildebrandt, A. S. Householder, H. K. Hughes, D. H. Hyers, M. H. Ingraham, Dunham Jackson, R. D. James, R. N. Johanson, Fritz John, R. E. Johnson, G. K. Kalisch, Samuel Kaplan, Wilfred Kaplan, William Karush, D. E. Kibbey, Lois Kiefer, J. R. Kline, W. C. Krathwohl, H. N. Laden, J. C. Lapsley, R. A. Leibler, A. L. Lewis, D. R. Lintvedt, N. M. Livesay, M. I. Logsdon, A. T. Lonseth, Morris Marden, C. W. Mendel, Karl Menger, A. N. Milgram, W. L. Mitchell, M. G. Moore, C. W. Moran, Marston Morse, E. J. Moulton, S. B. Myers, Ivan Niven, E. N. Oberg, J. W. Odle, Rufus Oldenburger, C. D. Olds, Isaac Opatowski, G. H. Peebles, P. M. Pepper, I. E. Perlin, Samuel Perlis, G. B. Price, Tibor Radó, P. V. Reichelderfer, W. T. Reid, Haim Reingold, C. V. Robinson, M. D. P. Rochford, W. H. Roever, A. E. Ross, R. G. Sanger, R. D. Schafer, Peter Scherk, O. F. G. Schilling, K. C. Schraut, G. E. Schweigert, H. A. Simmons, A. H. Smith, W. S. Snyder, N. E. Steenrod, R. C. Stephens, B. M. Stewart, E. B. Stouffer, H. P. Thielman, C. J. Thorne, C. B. Tompkins, P. L. Trump, A. W. Tucker, A. E. Turner, S. M. Ulam, V. J. Varineau, Bernard Vinograde, R. W. Wagner, H. S. Wall, K. W. Wegner, J. V. Wehausen, G. W. Whitehead, L. R. Wilcox, J. E. Wilkins, František Wolf, R. S. Wolfe, Ruth Wolkow, Y. K. Wong, J. W. T. Youngs, Antoni Zygmund.

The meeting opened Friday morning with two sections, analysis with Professor L. M. Graves presiding, and geometry and the theory of numbers with Professor V. G. Grove presiding. Saturday morning there were three sections, analysis with Professor J. V. Atanasoff presiding, algebra, Professor Reinhold Baer presiding, and geometry and topology, Professor N. E. Steenrod presiding. All sectional sessions were held in Eckhart Hall.

On Friday afternoon the Symposium Lecture was given by Professor L. M. Blumenthal with the title *Some imbedding theorems and characterization problems of distance geometry*. Professor Karl Menger presided at this lecture which was given in Rosenwald Hall.

On Friday afternoon following the Symposium Lecture, a tea was given by the ladies of the Department of Mathematics of the University of Chicago.

The attendance at the dinner on Friday evening at the Quadrangle Club was one hundred two. Professor Dunham Jackson served as toastmaster and introduced President Marston Morse who spoke on the present status of deferments for teachers of mathematics and stressed the danger of a serious shortage of qualified teachers at the college level. Following this talk Professor L. W. Cohen and Dean Walter Bartky gave short talks.

At the various sectional meetings on Saturday morning there was read a telegram from a group of Russian mathematicians addressed to all American mathematicians, expressing feelings of scientific unity and a belief in the triumph of our common cause.

There were two meetings of the Council, one being a joint session with the Board of Governors of the Mathematical Association of America. The joint session was held at 12:15 P.M. in the Quadrangle Club and the regular meeting of the Council at 8:45 P.M. in the Common Room of Eckhart Hall at the University of Chicago on April 17, 1942.

At the joint meeting of the Council and Board of Governors President Marston Morse reported on the steps which had been taken concerning the deferment of mathematicians and the procedures recommended to secure proper classification of competent teachers and graduate students by Selective Service Boards. The question of the use of all mathematicians who are inducted into the armed forces was also discussed and President Morse was authorized to make further investigations concerning this problem.

The Secretary announced the election of the following seventy-six persons to membership in the Society:

Mr. Blair Miller Bennett, College of the City of New York;
Dr. Ferdinand Charles Biesele, University of Utah;
Professor Conrad B. Bilgery, Regis College, Denver, Colo.;
Professor Warren Hill Brothers, Jr., Talladega College, Talladega, Ala.;
Mr. Hugh Daniel Brunk, The Rice Institute;
Dr. Arthur Walter Burks, University of Pennsylvania;
Sister Mary Roberdette Burns, Clarke College, Dubuque, Iowa;
Mr. Albert Cahn, Jr., University of Chicago;
Dr. Walter Francis Cassidy, St. John's University, Brooklyn, N. Y.;
Mr. A. John Coleman, University of Toronto;
Miss Clara Marie Cooper, St. Vincent's College, Shreveport, La.;
Miss Jane E. Crawford, American Mathematical Society, New York, N. Y.;
Mr. Anthony B. Crawley, West Virginia State College, Institute, W. Va.;
Professor Ray Harbaugh Dotterer, Pennsylvania State College;
Dr. Ralph Waldo Erickson, Hibbing Junior College, Hibbing, Minn.;
Mr. Edward Allen Fay, University of Rochester;
Mr. Edwin Howard Feather, Valley Forge Military Academy, Wayne, Pa.;
Dr. Hans Fried, Sproul Observatory, Swarthmore College;
Professor Israel Everette Glover, Fayetteville State Teachers College, Fayetteville N. C.;
Professor John M. Gonzâlez, Spring Hill College, Spring Hill, Ala.;
Sister Agnes Ann Green, Immaculate Heart College, Los Angeles, Calif.;
Dr. Frederick William Green, St. Louis Southwestern Railway Lines, St. Louis, Mo.;
Professor William V. Houston, California Institute of Technology;
Mr. Oscar Branche Jackson, Shell Oil Company, Tulsa, Okla.;
Dr. Haym Jaffe, Penn Treaty Junior High School, Philadelphia, Pa.;
Mr. William Katz, New York, N. Y.;
Dr. David Milton Krabill, Potomac State School of West Virginia University, Keyser, W. Va.;
Sister Mary Pachomia Lackay, St. Teresa's College, Kansas City, Mo.;
Miss Jane Marie Lawler, Kansas City, Mo.;
Mr. Herbert R. Leifer, United States Department of Labor, Philadelphia, Pa.;
Mr. Harold Wilbert Linscheid, Bluffton College, Bluffton, Ohio;
Mr. Robert Lorenzen, New York, N. Y.;
Professor John Joseph Lynch, Fordham University, New York, N. Y.;
Dr. Henry Berthold Mann, Columbia University;
Mr. Ernest Percy Miles, Jr., Naval Station, New Orleans, La.;
Professor Carman Edgar Miller, University of New Brunswick;

Mr. Arthur George Montgomery, University of Minnesota;
Dr. Elliott Waters Montroll, Cornell University;
Mr. Joseph Baxter Nelson, University of Southern California;
Sister Mary Cyprian Nightwine, Xavier University, New Orleans, La.;
Mr. Henry Mortimer Nodelman, College of Pharmacy, University of Connecticut;
Dr. Nilan Norris, Hunter College;
Mr. Kenneth Alva Norton, Federal Communications Commission, Washington, D. C.;
Sister Anna Concilio O'Neill, College of Saint Elizabeth, Convent, N. J.;
Mr. Milton Oliver Peach, Carnegie Institute of Technology;
Dean Henry William Pietenpol, Central College, Pella, Iowa;
Mr. George Piranian, The Rice Institute;
Mrs. Marie Helen Polanyi, Bennington College, Bennington, Vt.;
Professor Joseph William Porter, Bluefield State Teachers College, Bluefield, W. Va.;
Miss Ruth Ellen Porter, Albany Senior High School, Albany, Ore.;
Professor Francis McConnell Pulliam, Lambuth College, Jackson, Tenn.;
Mr. Lawrence Lee Rauch, Princeton University;
Dr. Helene Reschovsky, Baldwin School, Bryn Mawr, Pa.;
Professor Charles Kendall Robbins, Purdue University;
Mr. Ira Rosenbaum, Camp Croft, S. C.;
Mr. Joshua Heshel Rosenbloom, Federal Government Service, Philadelphia, Pa.;
Professor Samuel Thomas Sanders, Jr., Southwestern Louisiana Institute, Lafayette, La.;
Mr. William Leete Sandidge, Lynchburg, Va.;
Mr. Milner Baily Schaefer, University of Washington;
Mr. Charles W. Seekins, North Carolina State College;
Mr. Samuel Shulits, United States Engineer Office, Charleston, S. C.;
Mr. Daniel Harold Simmons, University of California at Los Angeles;
Dr. Rolland George Sturm, Aluminum Research Laboratories, New Kensington, Pa.;
Mr. Leonard W. Swanson, University of Minnesota;
Professor Alejandro Terracini, Universidad di Tucuman, Tucuman, Argentina;
Professor Louis F. Tolle, St. John's University, Brooklyn, N. Y.;
Dr. Lewis Fletcher Walton, San Diego State College;
Mr. Alan Wayne, Rhodes School, New York, N. Y.;
Professor Ernst Weber, Polytechnic Institute of Brooklyn;
Mr. Donald Ward Western, Brown University;
Mr. Robert L. White, University of California at Los Angeles;
Mr. Donald Ransom Whitney, Mary Washington College, Fredericksburg, Va.;
Mr. Lloyd Bayard Williams, Georgia School of Technology;
Miss Ruth Wolkow, Barret Junior High School, Louisville, Ky.;
Professor Cecil Benjamin Wright, East Texas State Teachers College, Commerce, Tex.;
Dr. Alexander Wundheiler, College of the City of New York.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

University of California: Mr. Edward William Barankin, Miss Sarah Margaret Hallam, Messrs. Alfred Horn and Robert Mann.
California Institute of Technology: Messrs. Sherwin Parker Avann, Gerald Harrison, Robert Clarke James, Chia Chiao Lin, Mark Muir Mills, Knox Taylor Millsaps, Wolfgang Kurt Herman Panofsky, Edmund Pinney, William Shand, Jr., Stanley

Cooper Snowdon, Clifford Ambrose Truesdale, and Donald Bingham Wheeler, Jr.
Harvard University: Mr. Robert Wayne Byerly.
New York University: Messrs. Imanuel Marx and Wolfgang Richard Wasow.
Pennsylvania State College: Mr. Ralph P. Bentz.
Wayne University: Mr. Gerald S. Heller.
University of Wisconsin: Messrs. Robert Phillip Eddy, Raymond Scott Embree, John Vernor Finch, Donald R. Lintvedt, and Theodore Edgar Rodd, Miss Marion Dell Wetzel, Mr. Norman Arthur Wiegmann.

Dr. Andrew Paul Guinand of the Royal Canadian Air Force, Chatham, New Brunswick, Canada, was admitted to the Society in accordance with the reciprocity agreement with the London Mathematical Society and Professor Charles Blanc of the University of Lausanne was admitted to the Society in accordance with the reciprocity agreement with the Swiss Mathematical Society.

The following appointments by President Marston Morse were reported: as representative of the Society at the Celebration of the One Hundredth Anniversary of the Founding of Hollins College (Virginia) on May 17–19, 1942, Dr. Nancy Cole; as auditors of the Society's books for 1942, Professors Samuel Borofsky and A. E. Meder, Jr.; as Committee on Arrangements for the Summer Meeting of 1942 at Vassar College, Professors Mary E. Wells (chairman), W. D. Cairns, T. R. Hollcroft, Gertrude Smith, H. S. White; as Committee on Arrangements for the Annual Meeting of 1942 at New York City, Professors G. W. Mullins (chairman), W. D. Cairns, B. P. Gill, T. R. Hollcroft, E. R. Lorch, Mina Rees, J. F. Ritt.

The Secretary reported that the following had been chosen as the voting representatives on the Council of the editorial committees for the year 1942: Bulletin—Dean Tomlinson Fort; Transactions—Professor Einar Hille; Colloquium—Professor M. H. Stone; American Journal of Mathematics—Professor R. L. Wilder; Mathematical Reviews—Professor Oswald Veblen.

An invitation from Oregon State College to hold a meeting of the Society in Corvallis in June, 1943, in connection with the meetings of the Pacific Division of the American Association for the Advancement of Science, was accepted.

The Secretary reported the receipt of a gift of $100 from Professor R. C. Archibald to be used as deemed most desirable by the Librarian.

It was reported by the Bulletin Editorial Committee that, in the future the Index Number of the Bulletin will include indices of abstracts, meetings and papers, but will omit the Annual List of Published Papers and the index to Notes.

The Secretary reported that Professor E. J. McShane had replaced

Professor J. L. Walsh as chairman of the editorial committee for the series, Mathematical Surveys.

Associate Secretary Ingraham reported that the state of the finances of the Society was good, in spite of the present emergency. He also reported that approximately ten persons had taken advantage of the privilege of the reduction in dues offered to enlisted men in the armed forces of the United States and Canada.

Titles and cross references to the abstracts of papers follow below. The papers were read as follows: Papers 1–8 in the section for analysis on Friday morning; papers 9–16 in the section for geometry and the theory of numbers on Friday morning; papers 17–22 in the section for analysis on Saturday morning; papers 23–29 in the section for algebra on Saturday morning; papers 30–36 in the section for geometry and topology on Saturday morning; and papers 37–47, whose abstract numbers are followed by the letter *t*, were read by title. Professor Jonah was introduced by Professor H. K. Hughes, Mr. Tomlinson by Dr. J. J. DeCicco, Mr. Rommel by Professor L. W. Cohen, Dr. Begle by Professor R. L. Wilder, Dr. Swain by Professor S. M. Ulam, and Professor Sadowsky by Professor L. R. Ford. Paper 1 was read by Dr. Bernstein, paper 5 by Professor Morse, paper 11 by Dr. Scherk, paper 13 by Professor Pepper, paper 18 by Dr. Hellinger, paper 20 by Professor Miller, paper 21 by Dr. Hyers, and paper 22 by Professor Cohen.

1. Dorothy L. Bernstein and S. M. Ulam: *On the problem of completely additive measure in classes of sets with a general equivalence relation.* (Abstract 48-5-169.)

2. W. S. Snyder: *Non-parametric surfaces of bounded variation.* (Abstract 48-5-192.)

3. P. V. Reichelderfer: *On bounded variation and absolute continuity for parametric representations of continuous surfaces.* (Abstract 48-5-187.)

4. Tibor Radó: *On a problem of Geöcze.* (Abstract 48-5-186.)

5. Marston Morse and G. M. Ewing: *The non-regular case in the variational theory in large.* (Abstract 48-5-178.)

6. Fritz John: *The character of solutions of linear partial differential equations.* (Abstract 48-5-173.)

7. František Wolf: *On the limits of harmonic and analytic functions along radii which form a set of positive measure.* (Abstract 48-5-199.)

8. William Karush: *A sufficiency theorem for an isoperimetric problem in parametric form with general end conditions.* (Abstract 48-7-236.)

9. H. F. S. Jonah: *Congruences connected with the solution of a certain Diophantine equation.* (Abstract 48-5-155.)

10. R. D. James: *On Euler's conjecture.* (Abstract 48-5-153.)

11. Emil Artin and Peter Scherk: *On the sum of two sets of integers.* (Abstract 48-5-143.)

12. Herbert Busemann: *Spaces with convex spheres.* (Abstract 48-5-206.)

13. B. J. Topel and P. M. Pepper: *Imbedding theorems under weakened hypotheses.* (Abstract 48-5-213.)

14. J. E. Wilkins: *A characterisation of the quadric of Wilcsynski.* (Abstract 48-5-214.)

15. T. W. Tomlinson: *Geometry of linear fractional polygenic functions.* (Abstract 48-5-211.)

16. A. E. Ross: *On a theorem of Kloosterman.* (Abstract 48-7-232.)

17. W. H, Roever: *Comment on the derivation of the law of perfect gases.* (Abstract 48-5-188.)

18. E. D. Hellinger and H. S. Wall: *Contributions to the analytic theory of continued fractions.* (Abstract 48-5-172.)

19. Isaac Opatowski: *Confluent hypergeometric functions and Markoff chains.* (Abstract 48-5-179.)

20. G. T. Miller and H. K. Hughes: *Analytic continuation of functions defined by factorial series.* (Abstract 48-5-177.)

21. S. M. Ulam and D. H. Hyers: *Approximate isometries of the space of continuous functions.* (Abstract 48-5-194.)

22. J. D. Rommel: *On conservative transformations of functions of two variables.* (Abstract 48-7-244.)

23. Bernard Vinograde: *Split rings and their representation theory.* (Abstract 48-5-167.)

24. Rufus Oldenburger: *Expansions of quadratic forms.* (Abstract 48-5-160.)

25. A. A. Albert: *The radical of a non-associative algebra.* (Abstract 48-5-141.)

26. Ivan Niven: *On matric polynomials.* (Abstract 48-5-159.)

27. R. D. Schafer: *Alternative algebras over an arbitrary field.* (Abstract 48-5-164.)

28. R. E. Johnson: *On structures of infinite modules.* (Abstract 48-5-154.)

29. R. W. Wagner: *Expressibility relations for bilinear operations.* Preliminary report. (Abstract 48-5-168.)

30. C. B. Tompkins: *Local imbedding of Riemannian spaces.* (Abstract 48-5-212.)

31. M. M. Day: *Compactness of a space with monotone closure.* (Abstract 48-5-220.)

32. Samuel Kaplan: *Homologies in metric separable spaces.* (Abstract 48-5-222.)

33. J. W. T. Youngs: *On parametric representations of surfaces.* II. (Abstract 48-5-229.)

34. E. G. Begle (National Research Fellow): *Intersections of absolute retracts.* (Abstract 48-5-219.)

35. R. L. Swain: *Proper and reductive transformations.* (Abstract 48-5-225.)

36. A. N. Milgram: *Dimension of the points with at most n rational coordinates.* (Abstract 48-5-223.)

37. M. S. Webster: *A convergence theorem for certain Lagrange interpolation polynomials.* (Abstract 48-5-197-*t*.)

38. M. A. Sadowsky: *Approximation by rational sequences.* (Abstract 48-5-163-*t*.)

39. H. M. Schwartz: *On some general series expansions.* Preliminary report. (Abstract 48-5-189-*t*.)

40. Walter Leighton and W. J. Thron: *On the convergence of continued fractions.* (Abstract 48-3-112-*t*.)

41. A. R. Schweitzer: *On a class of ordered* $(n+1)$*-ads relevant to the algebra of logic.* I. (Abstract 48-3-134-*t*.)

42. A. R. Schweitzer: *On a class of ordered* $(n+1)$*-ads relevant to the algebra of logic.* II. (Abstract 48-3-135-*t*.)

43. Reinhold Baer: *Radical ideals.* (Abstract 48-5-146-*t*.)

44. George Pólya: *On the combinatory analysis of classifications and permutations.* (Abstract 48-5-184-*t*.)

45. M. L. Kales: *Tauberian theorems.* (Abstract 48-5-174-*t*.)

46. R. F. Clippinger: *Matrix products of matrix powers.* (Abstract 48-5-149-*t*.)

47. R. M. Thrall: *On the decomposition of modular tensors.* (Abstract 48-5-166-*t*.)

W. L. Ayres,
Associate Secretary

ROBERT HENDERSON—IN MEMORIAM

In the death of Robert Henderson on February 16, 1942, American mathematics lost one of its ablest leaders. Those who came into close association with him were deeply impressed not only by his ability but even more by the integrity of his intellect and character; they could not but feel that it was a privilege to know a man of such extraordinary parts.

Mr. Henderson was born in Russell, Ontario, on May 24, 1871. He graduated from the University of Toronto in the Honours School of Mathematics at the head of his class in 1891 and received an appointment as Fellow in Mathematics for the following year. In 1892 the young man entered the Government Insurance Department at Ottawa and by 1896 he had passed the examinations of the Institute of Actuaries of Great Britain and had become a Fellow. Soon thereafter he entered the employ of the Equitable Life Assurance Society of the United States and transferred his residence to this country. Here he progressed rapidly to the top of the profession and became Actuary in 1911. In 1929 he was made Vice President of the company, which position he held until 1936 when he retired.

While nearly all of the two score papers which he published were concerned with actuarial problems, he never lost his interest in pure mathematics. As early as the meeting of October 1895 he presented a paper to the American Mathematical Society which was published in November of the same year. It is entitled *Moral values* and deals with some fundamental relations between probability and insurance. He was an omnivorous reader and found time to keep up with many of the modern developments of mathematics. In the Annals of Mathematics, volume 24, he has a brief paper entitled *Geodesic lines in Riemann space*, which was stimulated by his interest in the theory of relativity.

Mr. Henderson was prominent in the counsels of the Actuarial Society and served a term as Secretary, four years as Vice President, and two years as President. His contributions to the advancement of theory in the actuarial field were varied in character, but his work on interpolation and graduation and on frequency curves and moments is perhaps the best known.

In the early days of the Society, Professor Emory McClintock, who was its president for the period 1891–1894, made a profound impression on the development of actuarial science in America; Mr. Henderson proved to be a worthy successor. He was the recipient of many honors; for his signal leadership in his field his Alma Mater honored him with the degree of Doctor of Science in 1930. For the Annual Meeting of 1924 in Washington, the Society invited him to give the Josiah Willard Gibbs lecture; his title was *Life insurance as a social service and as a mathematical problem* (see this Bulletin, vol. 31 (1925), pp. 227–252, for the text). In 1927 he was elected to membership in the American Philosophical Society.

When the Society was in sore financial straits in 1924 and Professor J. L. Coolidge was requested to head a movement to raise an endowment fund, Mr. Henderson rendered valiant service; he was in that connection appointed along with G. E. Roosevelt, the banker, and Professor W. B. Fite, the treasurer, on a Committee on Investments. During the succeeding years and until 1940 he served the Society on the financial side giving sound advice regarding investments and other features of the financial problems. During much of that period he served on the Board of Trustees, where his counsel was invaluable. His interest in the Society was further evinced by his gift of one thousand dollars on retirement from the Board of Trustees in 1940. His donation was set up as a separate Fund bearing his name. He was generous also with his time in directions outside his immediate circle; as examples it may be cited that he was consultant to the Church Pension Fund of the Episcopal Church when it was established in 1915, was several times an adviser to the Director of the U. S. Census, and was for years a member of the Trustees of the Teachers Insurance and Annuity Association of America.

R. D. MURPHY
R. G. D. RICHARDSON

BOOK REVIEWS

The Philosophy of Alfred North Whitehead. Edited by Paul Arthur Schilpp. (The Library of Living Philosophers, vol. 3.) Evanston and Chicago, Northwestern University, 1941. 20+745 pp. $4.00.

This book is the third volume of a series entitled *The Library of Living Philosophers* published at Northwestern University under the direction of Professor Paul Schilpp. The basic aim of the series is to effect a clearer understanding of contemporary thinkers by permitting critics to ask them crucial questions which they are to answer. Each volume purports to be the edited record of this journalistic version of the Socratic method. Unfortunately, the present volume does not realize this aim. Professor Whitehead's health did not allow him to answer his critics. Nonetheless, there are interesting essays in the volume, and the publishing of some of Whitehead's latest papers adds to its worth. Especially valuable is the eleven page *Autobiographical notes*, excellent as a revelation of Whitehead, the man. It is difficult to agree with the editor, however, that the essays entitled *Mathematics and the good* and *Immortality* are summaries of Whitehead's philosophy. It is even more difficult to consider them answers to his critics. They should be thought of as late chapters in the Whiteheadian corpus.

But what does this volume contain by way of interest to the mathematician? There is little that will attract the pure mathematician. The only possible exception would be W. V. Quine's essay, *Whitehead and the rise of modern logic*, but even this embraces too many of the evils of secondary sources. There are some interesting points regarding Whitehead's contributions to Boolean algebra in his *Treatise on Universal Algebra*, but these can be appreciated adequately only by a study of the original. By far the largest portion of Quine's essay is devoted to the monotonous task of sketching the high points of *Principia Mathematica.* If one is familiar with mathematical logic this offers little that is new. If one is not familiar with mathematical logic, it is doubtful whether the essay will serve as an adequate introduction to the subject.

Mathematicians with interest in the philosophical aspects of relativity theory may find something to their taste in the essays by F. S. C. Northrop and E. B. McGilvary. These are entitled respectively, *Whitehead's philosophy of science*, and *Space-time, simple location, and prehension.* Those with a general interest in philosophy will profit most from the volume. There are many essays covering a variety of philosophical themes such as, *Whitehead's theory of value* and *Whitehead's idea of God.*

The book as a whole is well designed, and the print is good. There is an occasional typographical error such as formula (12) on page 150 which should read,

$$[\hat{\alpha} \sim (\alpha \in \alpha) \in \hat{\alpha} \sim (\alpha \in \alpha)] \equiv \sim [\hat{\alpha} \sim (\alpha \in \alpha) \in \hat{\alpha} \sim (\alpha \in \alpha)].$$

A. R. TURQUETTE

Introduction to Logic and to the Methodology of Deductive Sciences. By Alfred Tarski. Enlarged and Revised Edition. New York, Oxford University Press, 1941. 18+239 pp. $2.75.

This is an amplified and revised version of a book which first appeared in Polish in 1936, and was translated into German in 1937. The intention of the original book was to give an elementary but clear account of the concepts of modern mathematical logic for the benefit of readers interested in mathematics but with no technical knowledge of it beyond that possessed by a well trained college freshman. In the English version various additions have been made to make the work more suitable as a textbook for college courses.

This book and its preceding editions have been already reviewed in several places; in particular the German version was reviewed in this Bulletin (vol. 44, p. 317) by Quine. For a considerable list of other reviews see the indexes to volumes 4 (1939) and 6 (1941) of the Journal of Symbolic Logic—on pp. 193 and 187 respectively—; to the lists there given should be added the review by Frink in Mathematical Reviews, vol. 2 (1941), p. 209. In view of this fact it is superfluous for the present reviewer to do more than summarize the general purport of these reviews and to add to the criticisms certain amplifications of his own.

All reviewers, including the present one, are agreed that this is a work of exceptional merit. For the purpose for which it was originally designed it is a masterpiece of exposition. Whether the patching which the book has received to convert it into a textbook will succeed in that endeavor is doubtful—for some there will not be enough technique and for others not enough application to extra-mathematical domains—; but the value of the book for the independent reader is enhanced thereby. The exercises at the ends of the chapters are an excellent feature. For the seasoned mathematician the book is, perhaps, too easy, and it certainly does not give an adequate idea of the difficulty of some logical problems; nevertheless it contains material of interest and value. Within the limitations imposed by its objectives it is the most accurate and perspicuous survey of its subject which now exists.

A defect which has been noticed by several reviewers is an unwarranted departure from accepted terminology. This is especially patent in the case of the algebra of propositions. Here the book avoids the word 'proposition' altogether; in the place of it and its derivatives we find such words as 'sentence,' 'sentential calculus,' and so on. There are two circumstances which probably have a bearing on this choice of terminology. In the first place in the German language there is no distinction analogous to that between the English words 'proposition' and 'sentence'; there are, to be sure, two words, 'Satz' and 'Aussage,' but the distinction between them is not parallel. Now those who prefer the 'sentence' terminology have generally come under the influence of the Vienna Circle, which originated, of course, in a German-speaking community; and it is likely that some of these persons do not sense the violence they are doing to the linguistic instincts of those for whom the English language is native. (It is said that Carnap, since he has lived in America, has become more sympathetic to the use of 'proposition.') In the second place the word 'proposition' seems vague and has for some a metaphysical connotation; these propositions must be mysterious beings indeed, since we have no exact criteria for identity amongst them; moreover they are unnecessary.

In defense of the more usual terminology, reply may be made as follows. In the first place it is purely a mistake in translation to foist upon the English language an incongruous Germanism. Moreover, the notion of sentence is just as vague as that of proposition. Indeed consider the following lines of type (suggested by Church's review—see the list above cited):

(1) $2+3=5$
(2) The sum of 2 and 3 is 5.
(3) The sum of 2 and 3 is 5.
(4) **The sum of 2 and 3 is 5.**
(5) *The sum of 2 and 3 is 5.*
(6) Die Summe von 2 und 3 ist 5.
(7) Die Summe von 2 und 3 iſt 5.
(8) $3+2=5$
(9) $2+3\neq 5$

How many sentences are there? For Lesniewski presumably there would be $9n$, where n is the number of copies of this magazine; for a typographer, 8; for Quine, who says he regards a sentence as a shape, 7; for a grammarian 5; for a German logician, understanding 'sentence' as translation of 'Satz,' there might be 3. (Carnap's usage in the *Logical Syntax* is distinct from any of the first four; for when Countess Zeppelin translated his book into English she was compelled to translate inside quotation marks in order to preserve the meaning which Carnap wished to convey, which translation would have been improper if any of the first four senses had been intended.) Moreover the use of 'proposition' does not connote a belief in mysterious entities running around the universe; one can simply understand a proposition as a sentence in the above logician's sense. Finally the vagueness inherent in both the words 'proposition' and 'sentence' is irrevelevant since in logic we have no concern with equality of propositions but only their equipollence. In short the replacement of 'proposition' by 'sentence' serves no useful purpose and introduces, rather than eliminates, confusion. This is the opinion of every mathematician, so far as the reviewer knows, who has commented on the subject; Quine, who would doubtless disagree, is a philosopher. It should be noted that there is no departure from accepted usage in the German version; there 'Satz' and 'Aussagenkalkül' are used in their usual senses.

There is a related but distinct criticism in regard to the interpretation of the propositional algebra. Tarski uses the syntactical interpretation, whereby the variables 'p,' 'q' are to be replaced by sentences and the connectives ('$\vee$,' '$\rightarrow$,' '$\sim$') are conjunctions. Another interpretation is possible, namely: where these variables are replaceable by nouns, that is, noun clauses such as 'that 2 precedes 3.' On the first interpretation '$p\rightarrow q$' gives rise to the sentence

if 2 *precedes* 3, *then* 3 *follows* 2;

on the second to the complex clause

that (*that* 2 *precedes* 3) *implies* (*that* 3 *follows* 2),

which, when preceded by the assertion sign (or some similar indication) becomes the sentence

that 2 *precedes* 3 *implies that* 3 *follows* 2.

Either interpretation can be used, the second having the advantage of making the algebra have characteristics more similar to those of mathematical theories in general. As Quine points out, Tarski vacillates between these two.

A final criticism is that the first chapter, on variables, could profitably be completed by considering the types of bound variables that occur in the infinitesimal calculus—for example, in

$$\int_1^2 x^2 dx, \frac{d}{dx} x^2.$$

Such an addition might lead up to an explanation of the Church λ-quantifier, in terms of which, in connection with constants, all other quantifiers can be defined.

HASKELL B. CURRY

Two-Dimensional Potential Problems Connected with Rectilinear Boundaries. By B. R. Seth. Allahabad, India, The Allahabad Law Journal Press, 1939. 118 pp.

This attractive little volume grew out of a course of lectures delivered by Dr. Seth at the University of Lucknow in 1939. It deals primarily with the application of the Schwarz-Christoffel transformation in the solution of several problems and potential theory and related fields of mathematical physics. Among the problems discussed are special cases of the problem of torsion of a long prism as well as the Saint-Venant flexure problem, and problems of ideal fluid flow around prisms.

The author restricts himself to rectilinear boundaries for which the Schwarz-Christoffel transformation allows mapping on a half-plane. By confining himself to rectangular regions, special triangular regions such as the equilateral triangle, the 90°, 60°, 30° triangles and other similar special regions, he is able to express the mapping and the solutions of the problems for them in terms of the classical elliptic functions. Among other regions considered is the "angle-iron," the region on the outside of a rectangle, and the *L*-section.

The book will be welcomed by workers in this field of mathematical physics, as well as by mathematicians who are interested in application of elliptic functions.

H. PORITSKY

Operational Methods in Applied Mathematics. By H. S. Carslaw and J. C. Jaeger. Oxford, Clarendon Press, 1941. 16+264 pp. $5.00.

In May 1815 Simeon Denis Poisson read before the Paris Academy a memoir on the distribution of heat in solid bodies. Extracts from this memoir were at once published in the Journal de Physique and in the Bulletin de la Société Philomathique. The memoir was subsequently enlarged and became, perhaps, one of Poisson's favorites because in May 1821 the work was printed and distributed privately two years before its final publication in Journal de l'École Polytechnique, vol. 12, no. 19, pp. 1–162. Poisson here made, I think, the first use of the method of the inverse Laplace transformation. In an attempt to find the distribution of temperature in a uniform rod radiating at its ends he was led to two linear functional differential equations for the unknown function f in his general solution of the equation of the conduction of heat. To solve these equations Poisson formed two Laplacian integrals involving the function f, the argument of the exponential function being xs in one case and $-xs$ in the other. By an integration by parts, equations were found from the functional differential equations which led to the values of the integrals. Poisson then deduced the value of a Fourier integral involving the function f and finally derived f by means of the Fourier inversion formula. He gave indeed a formal derivation of the inversion formula for the Laplace integral, a formula which is usually written as an integral along a vertical line in the complex plane.

Poisson thus initiated a method which has been found particularly useful during the last forty years. When V. Pareto in 1892 used the multiplication theorem for Laplacian integrals to solve a type of integral equation and M. Lerch in 1892 proved the uniqueness theorem for the representation of a function by a Laplacian integral involving a continuous function f, the method became a powerful mathematical tool. Further progress was made in 1896 when H. Mellin gave a rigorous discussion of the inversion formula used by Poisson. In 1902 H. M. Macdonald used the inversion formula for the evaluation of some definite integrals and in 1903 Lerch made a similar application of his theorem when his work was republished in Acta Mathematica, volume 27. This volume happened to contain also an account of Fredholm's pioneer work on linear integral equations and so the men interested in integral equations became acquainted also with Lerch's work. Sketches of the method under discussion and its application to differential equations and the evaluation of definite integrals appeared in a Smith's Prize essay of 1905, in a report on integral equations and in a paper of 1910 on radioactive transformations. The subsequent history of the subject and its connection with the operational calculus of Heaviside is given in the historical introduction of the book under review. This book is particularly welcome now that the subject is widely taught because the exposition is good and there are numerous illustrative examples taken from the subjects of dynamics, electric circuits, the conduction of heat, hydrodynamics, electric waves and diffusion. There is also a large collection of examples to enable a student to develop his skill.

The appendices contain a proof of Lerch's theorem, a note on Bessel functions, a discussion of impulsive functions and a treatment of two-point boundary value problems for ordinary linear differential equations. There is also a table of Laplace transforms.

H. BATEMAN

NOTES

A pamphlet entitled *Some military applications of elementary mathematics* has been prepared by the Department of Mathematics at the United States Military Academy at West Point. The material involves mathematics of high school and freshman grade. This pamphlet is being used at the Institute of Military Studies, University of Chicago, and may be obtained by qualified institutions for $0.15 a copy. Address Director, Institute of Military Studies, University of Chicago, Chicago, Illinois.

Professor E. T. Copson of University College, Dundee, and Professor W. H. McCrea of Queen's University, Belfast, have been awarded the Keith Prize of the Royal Society of Edinburgh.

The Chauvenet Prize has been awarded to Associate Professor Saunders MacLane of Harvard University for his two papers *Modular fields* and *Some recent advances in algebra* published in the American Mathematical Monthly.

Professor W. D. Cairns, Secretary-Treasurer of the Mathematical Association of America, has announced that the Department of Mathematics of the University of Toronto has won the first prize of $400 in the fifth annual William Lowell Putnam Mathematical Competition, held March 7, 1942. The members of the team were K. S. Hoyle, H. V. Lyons, M. A. Preston. The second prize of $300 is awarded to the Department of Mathematics of Yale University, the members of whose team were F. H. Brownell, A. M. Gleason, A. E. Roberts. The third prize of $200 is awarded to the Department of Mathematics of the Massachusetts Institute of Technology, the members of whose team were E. D. Calabi, W. S. Loud, G. P. Wachtell. The fourth prize of $100 is awarded to the Department of Mathematics of the College of the City of New York, the members of whose team were Herman Chernoff, Harvey Cohn, Edward Gordon. In addition to these prizes to the departments of mathematics with winning teams, a prize of $50 each is awarded to the following five persons whose scores ranked highest in the six-hour examination (names listed alphabetically): Harvey Cohn, A. M. Gleason, W. S. Loud, H. V. Lyons, M. A. Preston. One of these five will be chosen to receive a $1000 year scholarship at Harvard University, this award to be announced later. The members of the four winning teams will receive individual cash awards according to the ranks of their teams, and all individuals receiving awards will also receive medals. Honorable mention has been awarded this year to four teams and to five individuals. The teams are from the Department of Mathematics, Cooper Union Institute of Technology, members of the team being Harold Grad, M. S. Klamkin, Kenneth Robinson; the Department of Mathematics, Harvard University, members of the team being R. M. Bloch, L. S. Shapley, J. A. Zilber; the Department of Mathematics, New York University, members of the team being Melvin Lax, Harold Lewis, Henry Shenker; and the Department of Mathematics, Swarthmore College, the team members being N. B. Hannay, W. H. Mills, M. S. Raff. The five individuals receiving honorable mention are E. D. Calabi, Massachusetts Institute of Technology; C. P. Gadsden, Tulane University; K. S. Hoyle, University of Toronto; Melvin Lax, New York University; W. H. Mills, Swarthmore College.

Mr. I. M. Vinogradov has been elected to membership in the American Philosophical Society.

Professor A. N. Whitehead has been elected president of the British Mathematical Society.

Mr. Philip Hall and Mr. A. H. Wilson of Cambridge University and Mr. G. H. Henderson of Dalhousie University have been elected fellows of the Royal Society

Dr. Vannevar Bush, president of the Carnegie Institute of Technology, received an honorary doctorate from Rutgers University.

Professor F. A. Foraker has received an honorary doctorate from the University of Pittsburgh.

Professor T. R. Hollcroft of Wells College has received an honorary doctorate from Hanover College.

Professor F. D. Murnaghan of Johns Hopkins University has been elected to membership in the National Academy of Sciences and the American Philosophical Society.

Dr. J. A. Sharpe, a member of this Society, has been elected by the Society of Exploration Geophysicists to a three year term as editor of Geophysics, a quarterly journal of general and applied geophysics.

Dean H. M. Westergaard of Harvard University has been elected to membership in the American Philosophical Society.

Dr. Howard Levi of Columbia University and Dr. L. I. Wade of Duke University have received fellowship appointments for 1942–1943 from the National Research Council. Dr. Levi and Dr. Wade will work at the Institute for Advanced Study.

Guggenheim Memorial Fellowships for work in mathematics have been awarded to Dr. J. C. C. McKinsey and Dr. Alfred Tarski.

Professor Willy Prager of Brown University has requested the correction of information about him which was received by the American Mathematical Society and published on page 25 of volume 48 of this Bulletin. He was acting director of the Institute of Applied Mechanics at Göttingen and professor of mechanics at Istanbul before coming to the United States.

Assistant Professor Harriet W. Allen of Hollins College, Hollins College, Virginia, is temporarily serving as physicist, in the Bureau of Ordnance, Navy Department, Washington, D. C.

Dr. H. C. Ayres of the United States Naval Academy has been promoted to an assistant professorship.

Mr. V. P. Barta has accepted an appointment as research physicist at the Massachusetts Institute of Technology.

Dr. Walter Bartky of the University of Chicago has been promoted to a professorship of applied mathematics.

Assistant Professor Julia W. Bower of Connecticut College has been promoted to an associate professorship.

Associate Professor R. V. Churchill of the University of Michigan has been promoted to a professorship.

Mr. E. C. Coker of Clemson College, Clemson, South Carolina, has been promoted to an assistant professorship.

Assistant Professor Marguerite D. Darkow of Hunter College has been promoted to an associate professorship.

Assistant Professor Paul Eberhart of Washburn Municipal University, Topeka, Kansas, has been promoted to an associate professorship.

Mr. B. G. Farley of Massachusetts Institute of Technology will be a research associate in electrical engineering at the Radiation Laboratory.

Professor W. B. Fite of Columbia University has retired. Professor J. F. Ritt will succeed him as chairman of the department of mathematics.

Assistant Professor J. S. Frame of Brown University has been appointed to an associate professorship at Allegheny College.

Assistant Professor E. E. Haskins of Northeastern University has been promoted to an associate professorship.

Associate Professor C. T. Hazard of Purdue University has been promoted to a professorship.

Assistant Professor E. A. Hazlewood of Texas Technological College has been promoted to an associate professorship.

Professor W. V. Houston of California Institute of Technology is working as a physicist in the Division of National Defense of Columbia University.

Mr. J. F. Kenney has been appointed to an assistant professorship at the University of Wisconsin.

Associate Professor Marie Litzinger of Mt. Holyoke College has been promoted to a professorship.

Assistant Professor Sidney McCuskey of Case School of Applied Science has been promoted to an associate professorship.

Assistant Professor H. L. Miller of the University of Cincinnati has been promoted to an associate professorship.

Associate Professor Deane Montgomery of Smith College has been promoted to a professorship.

Assistant Professor C. W. Munshower of Colgate University has been promoted to an associate professorship.

Dr. D. C. Murdoch of the University of Saskatchewan has been promoted to an assistant professorship.

Associate Professor C. O. Oakley of Haverford College has been promoted to a professorship.

Dr. George Polya has been appointed to an associate professorship at Stanford University.

Assistant Professor E. J. Purcell of the University of Arizona has been promoted to an associate professorship.

Dr. Eric Reissner of Massachusetts Institute of Technology has been promoted to an assistant professorship.

Professor F. G. Reynolds of the College of the City of New York has retired.

Assistant Professor H. J. Riblet of Hofstra College will be research associate at the Radiation Laboratory, Massachusetts Institute of Technology.

Professor P. R. Rider of Washington University is to be an exchange professor at the National University of Mexico under the auspices of the Division of Cultural Relations, United States Department of State.

Assistant Professor Robin Robinson of Dartmouth College has been promoted to a professorship.

Assistant Professor E. A. Saibel of Carnegie Institute of Technology has been promoted to an associate professorship.

Professor S. T. Sanders of Louisiana State University has retired. Professor W. V. Parker will succeed him as head of the department of mathematics.

Dr. L. W. Tordella of Loyola University, Chicago, Illinois, has been promoted to an assistant professorship.

Dr. Abraham Wald of Columbia University has been promoted to an assistant professorship.

Dr. J. F. Wardwell of Colgate University has been promoted to an assistant professorship.

Dr. Antoni Zygmund of Mt. Holyoke College has been promoted to an associate professorship.

Dr. P. R. Halmos has been appointed associate at the University of Illinois.

Mr. Thorstein Larsen has been appointed associate in electrical engineering at Johns Hopkins University.

Dr. Abe Gelbart of North Carolina State College, Raleigh, will be on leave for the academic year 1942–1943 at Brown University.

The following appointments to instructorships are announced: Air Corps Technical School, Scott Field, Illinois: Dr. Louis Garfin; Amherst College: Mr. N. D. Nelson; University of California (Berkeley): Dr. František Wolf; University of Colorado: Mr. George Ulrich; Columbia University: Miss Margaret P. Martin; Compton Junior College: Dr. A. E. Marston; Connecticut College: Miss Alice E. Turner; Cornell University: Dr. C. E. Rhodes; Harvard University: Dr. Harry Pollard; University of Illinois: Dr. R. W. Hamming; Johns Hopkins University: Dr. G. W. Whaples; Louisiana State

University: Dr. K. L. Nielsen; Loyola University, New Orleans, Louisiana: Mr. E. H. Larguier; Montana State University: Mr. Roy Dubisch; Naval Pre-Flight Training School, Chapel Hill, North Carolina: Dr. R. P. Boas; Princeton University: Mr. C. A. Truesdell; R.C.A. Institutes, Inc., New York City: Mr. H. M. Nodelman; Rutgers University: Mr. R. B. Kleinschmidt; Stanford University: Dr. J. G. Herriot; United States Merchant Marine Academy: Mr. B. M. Ingersoll; United States Naval Academy: Dr. H. T. Muhly; University of Wisconsin: Dr. R. H. Bruck, Dr. Bernard Vinograde.

Professor Emeritus Susan R. Benedict of Smith College died on April 8, 1942, at the age of sixty-eight years. She had been a member of the Society since 1912.

Professor Emeritus L. S. Hulburt of Johns Hopkins University died on March 29, 1942, at the age of eighty-four years. He had been a member of the Society for thirty-five years.

Professor Byron Ingold of Culver-Stockton College, Canton, Missouri, died March 23, 1942.

Assistant Professor Elizabeth E. Knight of Milwaukee State Teachers College died on March 26, 1942. She had been a member of the Society since 1927.

Professor W. W. Landis of Dickinson College died April 9, 1942, at the age of seventy-three years. He had been a member of the Society since 1897.

Associate Professor Roy MacKay of New Mexico State College died May 12, 1942.

Professor Maria M. Roberts of Iowa State College died on April 12, 1942, at the age of seventy-four years. She had been a member of the Society for twenty-five years.

Professor J. H. Weaver of Ohio State University died April 7, 1942. He had been a member of the Society since 1916.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

230. Reinhold Baer: *Rings with duals.*

The ring R' is said to be a right-dual of the ring R, if there exists a duality between the partially ordered set of all the right-ideals in R and the partially ordered set of all the right-ideals in R'. If the ring R possesses a right-identity element, and if there exists to every quotient-ring of R a right-dual possessing a left-identity element, then both the maximum and the minimum condition are satisfied by the right-ideals in R, and R meets a generalized uni-seriality requirement. A complete theory of the primary rings with right-duals may be developed, determining their structure and their dualities. (Received May 13, 1942.)

231. Nathan Jacobson: *Classes of restricted Lie algebras of characteristic p.* II.

Let $\mathfrak{A}=\Phi(x_1, \cdots, x_m)$ be an associative commutative algebra having the basis $x_1^{k_1}x_2^{k_2} \cdots x_m^{k_m}$, $0 \leqq k_i \leqq p-1$ such that $x_i^p=\xi_i$ is in the field Φ of characteristic p. The restricted Lie algebras considered in this paper are the derivation algebras θ of $\mathfrak{A}$'s of the above type. The case where $\mathfrak{A}$ is a field was defined in a previous paper of the author and the case where $m=1$ and $\xi_1=1$ has been considered by Witt, by Zassenhaus and by Ho-Jui Chang. The author proves that θ is normal simple except when $p=2$ and $m=1$. The derivations of θ are all inner and if $p \geqq 5$ the automorphisms of θ have the form $D \rightarrow S^{-1}DS$ where S is an automorphism of $\mathfrak{A}$. This implies that if $p \geqq 5$, two algebras $\theta(\mathfrak{A}_1)$ and $\theta(\mathfrak{A}_2)$ are isomorphic if and only if the associative algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic. (Received April 3, 1942.)

232. A. E. Ross: *On a theorem of Kloosterman.*

Kloosterman (Acta Mathematica, vol. 49 (1926)) determined conditions under which the form (1) $f=ax^2+by^2+cz^2+du^2$ would represent all large integers and pointed out that there was a finite (and actually small) number of such forms f for which his method failed to yield conclusive results. Upon close examination of these exceptional cases one notices that should one replace the original form f by a suitably chosen equivalent form $f_1=x'Ax$ (this Bulletin, abstract 45-9-369) then one can reduce the problem to the study of a related form of type (1) to which Kloosterman's sufficient conditions apply. Although the need for choosing the quotients of the upper left-hand corner principal minors of A by the order invariants as primes introduces an element of tentativeness, the actual computation is carried out without difficulty. (Received April 17, 1942.)

233. H. Schwerdtfeger: *A complete parametrization of the symplectic group.*

While all known parametrizations of the symplectic group omit certain "exceptional" elements, the parametrization derived in the present paper covers the whole group. Let ϵ be the n-rowed unit matrix and F the matrix with first row 0,0 and second row ϵ,0. A $2n$-rowed real regular matrix T is proved to be symplectic if and only if $T'FT-F=H$ is symmetric (contact condition), and satisfies the condition: $(F'-F)\cdot(H+F)$ is idempotent. To prove the sufficiency of the latter condition one has to show that for any such matrix H a symplectic T can be found with which it is associated by the said relation. By establishing a set of normal forms for H under the transformation $S'HS$ where S is a $2n$-rowed matrix for which $S'FS=F$, that is, S is the matrix with first row σ,0 and second row $0,\sigma'^{-1}$, a set of normal matrices T can be found such that RTS is the most general symplectic matrix where R is the matrix with first row ρ,0; second row $0,\rho'^{-1}$ and ρ,σ are any regular n-rowed matrices. The method has been carried through in detail for $n=2$. (Received May 1, 1942.)

ANALYSIS

234. Einar Hille: *Notes on linear transformations.* IV. *Representation of semi-groups.*

Let $\{T_s\}$ be a semi-group of linear bounded transformations on a separable Banach space to itself, defined for $s>0$. Let T_s be weakly measurable and $\|T_s\|\leqq 1$ for $s>0$. Let $T_s\,(E)$ be dense in E. Put $A_k=(1/k)[T_k-I]$. For $k\to 0$, $T_kx\to x$ everywhere in E and $A_kx\to Ax$ in a dense set $D(A)$. Here A is linear, closed and ordinarily unbounded. The resolvent $R(\lambda)$ of A is the negative of the Laplace transform of T_s and is bounded for $\mathfrak{R}(\lambda)>0$. Conversely, T_s is expressible in terms of $R(\lambda)$ by the inversion formula for Laplace integrals which gives an interpretation of T_s as $\exp\,(sA)$. A further interpretation is given by $T_sx=\lim_{k\to 0}\exp[sA_k]x$, valid in $D(A)$. The method is essentially that of Stone. (Received April 6, 1942.)

235. Witold Hurewicz: *An ergodic theorem without invariant measure.*

Let E be an abstract space carrying a completely additive measure μ defined on a completely additive field Ω of subsets of E (it is assumed that a set $X\in\Omega$ with $\mu(X)=\infty$ can always be split into a countable number of sets with finite measures). Let T be a one-to-one point transformation of E on itself satisfying the conditions: (1) $X\in\Omega$ implies $T(X)\in\Omega$; (2) $T(X)\subset X\in\Omega$ implies $\mu(X-T(X))=0$ (incompressibility condition). Finally let $F(X)$ $(X\in\Omega)$ be an additive finite-valued set function, absolutely continuous with respect to the measure μ. For $X\in\Omega$, let $F_n(X)=\sum_{i=0}^{n}F(T^i(X))$, $\mu_n(X)=\sum_{i=0}^{n}\mu(T^i(X))$ $(n=0, 1, 2, \cdots)$. Since F_n is absolutely continuous with respect to the measure μ_n, $F_n(X)=\int_X f_n d\mu_n$, where f_n is a measurable point function defined on E. It can be shown that the sequence $\{f_n\}$ converges almost everywhere to a function ϕ, invariant with respect to T. By "almost everywhere" is meant that the points of divergence form a set M such that $\mu(T^n(M))=0$ for $n=1, 2, \cdots$. If the measure μ is finite and invariant with respect to T, this theorem coincides with Birkhoff's ergodic theorem. (Received April 24, 1942.)

236. William Karush: *A sufficiency theorem for an isoperimetric problem in parametric form with general end conditions.*

The problem studied is that of minimizing an integral $I(C)=g(a)+\int_{t_1}^{t_2}f(a, y, y')dt$

in a class of admissible arcs $C: a_k, y_i(t)$ $(k=1, 2, \cdots, r; i=1, 2, \cdots, n; t_1 \leqq t \leqq t_2)$ in ay-space satisfying conditions of the form $y_i(t_s)=y_{is}(a)$ and $I_\sigma(C)=g_\sigma+\int_{t_1}^{t_2} f_\sigma dt=0$ $(s=1, 2; \sigma=1, 2, \cdots, m)$. The parameters a_k are independent of t. It is shown that a set of sufficient conditions for a proper strong relative minimum on an arc E which does not intersect itself is that E satisfy the multiplier rule including the transversality conditions, nonsingularity, the strengthened Weierstrass condition, and that the second variation of I along E be positive. The method of proof is a generalization of methods used by M. R. Hestenes for non-parametric problems. In particular, use is made of a generalized field theory and of properties of broken extremals. Among other things it is shown that in the sufficiency proof one can assume, without loss of generality, that E is an extremal of no integral of the form $c_\sigma I_\sigma$ where the constants c_σ are not all zero, that is, that E is normal relative to the fixed end point case. (Received April 17, 1942.)

237. Walter Leighton and W. J. Thron: *On the convergence of continued fractions to meromorphic functions.*

In the continued fraction $1+K(a_n x/1)$ let the numbers a_n be complex and denote by L (the point ∞ may be in this set) the set of the limit points of the sequence $\{a_{2n+1}\}$. If $\lim a_{2n}=0$, the continued fraction converges to a meromorphic function of the complex variable x in every region contained in the set D, where D is defined as follows: $x \in D$ if $|x+1/k|>\epsilon$ for all $k \in L$, $k \neq 0$, and if $|x|<M$. The constants ϵ and M are arbitrary. In this theorem the role of the sequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ can be interchanged. (Received May 28, 1942.)

238. A. N. Lowan and H. E. Salzer: *Coefficients in formulae for numerical integration.*

The values of $B_n^{(n)}(1)/n!$ and $B_n^{(n)}/n!$, where $B_n^{(n)}(1)$ denotes the nth Bernoulli polynomial of the nth order for $x=1$ and $B_n^{(n)}$ denotes the nth Bernoulli number of the nth order, were computed for $n=1, 2, \cdots, 20$. The quantities $B_n^{(n)}(1)/n!$ are required in the Laplace formula of numerical integration employing forward differences, as well as in the Gregory formula and the central difference formulae. The quantities $B_n^{(n)}/n!$ are used in the Laplace formula employing backward differences. (See Milne-Thomson, *Calculus of Finite Differences*, pp. 181–187, 191–193.) The above results were obtained in the course of work carried out by Mathematical Tables Project (Work Projects Administration, New York City), on computational tools for numerical integration. (Received May 21, 1942.)

239. Szolem Mandelbrojt and F. E. Ulrich: *On a generalization of the problem of quasi-analyticity.*

If $\{M_n\}$ is a sequence of positive numbers, let $C(M_n)$ be the class of functions infinitely differentiable on $0 \leqq x < \infty$ such that to each function $f(x)$ of the class there corresponds a positive constant k with the property that $|f^{(n)}(x)| < k^n M_n$ $(n \geqq 1)$ for all $0 \leqq x < \infty$. Let (1) $T(r)=\text{l.u.b.}(r^n/M_n)$ for $n \geqq 1$. It is known that if a function $f(x)$ belongs to a class $C(M_n)$ such that (2) $\int_1^\infty \log T(r)dr/r^2=\infty$, and if $f^{(n)}(0)=0$ $(n \geqq 0)$, then $f(x)\equiv 0$. In the present paper conditions are obtained under which a function belonging to $C(M_n)$ will be identically zero when it is supposed that only a partial subsequence of its derivatives are zero at $x=0$. For instance, suppose that $f(x)$ belongs to $C(M_n)$, (3) $\int_0^\infty |f(x)|dx < \infty$ and (4) $f^{(\nu_n)}(0)=0$ $(n \geqq 1)$ with (5) $G=\limsup_{n\to\infty}(\nu_n/n)<2$. If (6) $\limsup_{n\to\infty}|f^{(n)}(0)|^{1/n}<\infty$ and if (2) holds, then $f(x)\equiv 0$. Condition (6) can be removed and the conclusion will still be valid if in (2) $T(r)$ is replaced by $T_\omega(r) = \text{l.u.b.}\,(r^{\omega\lambda_n}/M_{\lambda_n+1})$ for $n \geqq 1$, where $\{\lambda_n\}$ is the sequence of positive integers complementary to the sequence $\{\nu_n\}$ with respect to the non-negative integers, and ω is a positive constant less than $(2/G)-1$. If $f(x)$ is even and periodic, condition (3) can be removed. (Received May 11, 1942.)

240. Marston Morse: *Unstable minimal surfaces bounded by a rectifiable contour.*

Recent theorems of Morse and Tompkins and of Shiffman establish the existence of an unstable minimal surface of disc type under appropriate hypothesis on the minimizing sets of the Douglas Dirichlet integral, and on the bounding curve g. A similar theorem is now proved by the author supposing g is no more than simple and rectifiable, and that at least two relative disjoint minimizing sets belong to the area functional A. The surfaces admitted are harmonic and of disc type. The method of the author does not require the analysis of the harmonic surfaces bounded by polygons as does the method of Courant and Shiffman (announced orally recently but not yet published). The theorem is a corollary of theorems proved by Morse and Tompkins in their recent paper in the Annals of Mathematics on ***Minimal surfaces of non-minimum type by a new mode of approximation***. The theorem of Morse and Tompkins on the relative continuity of A covers the most difficult point in the analysis. (Received April 17, 1942.)

241. M. E. Munroe: *On the finite dimensionality of certain Banach spaces.*

It is shown that if B is a Banach space, a necessary and sufficient condition that every unconditionally convergent series of elements of B be absolutely convergent is that B be finite-dimensional. On the basis of this result it is shown that finite dimensionality of the range space is necessary and sufficient that (1) all additive absolutely continuous functions of measurable sets be of bounded variation, (2) all Pettis integrals be of bounded variation, (3) all additive absolutely continuous functions of measurable sets be Bochner-Dunford integrals, and (4) the Pettis and Bochner-Dunford integrals be equivalent. (Received May 13, 1942.)

242. Mary De Pazzi Rochford: *Completely non-integrable pfaffians.*

The pfaffian $\sum P_i dx_i$ is called completely non-integrable if there does not exist even a continuous function λ which is an integrating factor, that is, makes $\int \sum P_i dx_i$ independent of the path. If (1) $P_i(x_1, \cdots, x_n)$ is continuous for $i=1, \cdots, n-1$, (2) $P_n=0$, (3) P_1 actually depends upon x_n, that is, $P_1(x_1, \cdots, x_{n-1}, x_n') \neq P_1(x_1, \cdots, x_{n-1}, x_n'')$ for a system $x_1, \cdots, x_{n-1}$, (4) $P_2, \cdots, P_{n-1}$ do not depend upon x_n, then $P_1 dx_1 + \cdots + P_{n-1} dx_{n-1}$ is completely non-integrable. (Received May 20, 1942.)

243. Mary De Pazzi Rochford: *Differentiability properties of certain functions of two variables.*

Let P and Q be two continuous functions of x,y. If $\int Pdx+Qdy$ is independent of the path, then for each y_0 and y_1 there exists a function $\eta(x,y_0,y_1)$ such that $Q(x, \eta(x,y_0,y_1))$ has a total derivative with respect to x which is equal to the difference quotient $[P(x,y_0)-P(x,y_1)]/(y_0-y_1)$. For each continuous function g of one variable, even if g is nowhere differentiable, there exists a function $\eta(x,y_0,y_1)$ such that $g(x + \eta(x,y_0,y_1))$ has a derivative with respect to x which is equal to $[g(x+y_0)-g(x$

$+y_1)]/(y_2-y_1)$. If $\partial(\int Qdy)/\partial x$ exists, then each integrating factor of $Pdx+Qdy=0$ has a similar differentiability property. (Received May 20, 1942.)

244. J. D. Rommel: *On conservative transformations of functions of two variables.*

Necessary and sufficient conditions are given in order that $y(s, t)=\int_0^\infty\int_0^\infty k(s, t, \sigma, \tau) x(\sigma, \tau)d\sigma d\tau$ shall transform $x(\sigma, \tau)$ such that $\lim_{\sigma,\tau\to\infty} x(\sigma, \tau)$ exists into $y(s, t)$ such that $\lim_{s,t} y(s, t)$ exists. From this result a characterization of regularity is obtained. By considering step-kernels and step-functions, regular transformations of double sequences are characterized. (Received April 18, 1942.)

245. Raphael Salem and D. C. Spencer: *The influence of gaps on density of integers*

An infinite sequence of integers is said to have the complete gap property with respect to $\omega(x)$ ($\omega(x)$ being a positive non-decreasing function for $x\geq 0$) if in every closed interval $(a,a+l)$ ($a\geq 0$, $l\geq l_0$) there exists an open interval of length not less than $\omega(l)$ which contains no point of the sequence. Such a sequence will be denoted by $S[\omega(x)]$. Let $\nu(m)$ be the number of terms of the sequence not greater than m. The following theorems are proved: (1) If the integral $\int_1^\infty(\omega(x)/x^2)dx$ diverges, $\lim \nu(m)/m=0$. (2) If $\omega(x)$ is such that $\omega(x)/x^2$ decreases and the integral converges, there exists an $S[\omega(x)]$ with $\liminf \nu(m)/m>0$. (3) If $\omega(x)=\theta x$ $(0>\theta>1/3)$, then $\nu(m)/m=O(m^{-\alpha})$ where $\alpha=[\log (1-2\theta)/(1-3\theta)]/[\log 2(1-2\theta)/(1-3\theta)]$. (4) If $0<\theta<1/3$, there exists a sequence $S[\theta x]$ with $m^{-\alpha}=O(\nu(m)/m)$. (5) If $\omega(x)=\theta x$ $(1/3\leq\theta<1/2)$, then $\nu(m)=O(\log m)$. (6) If $1/3\leq\theta<1/2$, there exists a sequence $S[\theta x]$ with $\log m=O(\nu(m))$. (7) If $\omega(x)\geq x/2$, there exists no infinite sequence $S[\omega(x)]$. (Received April 30, 1942.)

APPLIED MATHEMATICS

246. H. K. Brown: *The resolution of boundary value problems by means of the finite Fourier transformation.*

The finite sine transformation and the finite cosine transformation are defined as the linear functional operations $\mathfrak{S}\{G\}=\int_0^\pi G(x)\sin nx dx=g_s(n)$ and $\mathfrak{C}\{G\}=\int_0^\pi G(x)\cdot\cos nx dx=g_c(n)$, respectively. The inversion of the product of the transforms of G and H can be made by means of four Faltung theorems. The finite sine transformation was applied to a problem in general heat flow in one dimension in which the nonhomogeneous linear partial differential equation has coefficients which may be functions of the time. It was proved in detail that this problem can be resolved into a standard heat flow problem which has a differential equation of simpler type and constant coefficients. The Faltung theorems permitted an inversion in closed form of the solution of the transformed boundary value problem. A general heat flow problem of similar type in three dimensions was resolved into the same standard problem in one dimension. By the introduction of a fundamental set of solutions of the transformed problem it was possible to make a resolution of the boundary value problems of a general vibrating string, membrane and beam. (Received April 11, 1942.)

GEOMETRY

247. C. J. Everett: *Affine geometry of vector spaces over rings.*

Methods of E. Artin's *Coordinates in affine geometry* (Publication of the University of Notre Dame, Reports of a Mathematical Colloquium, (2), Issue 2) are used to show that a "plane geometry" subject to a weaker set of axioms on "points" and "lines" permits a point coordinatization $P=(x,y)$ with x, y in a ring R with unit, without zero-divisors, and with every two elements $(a,b)\neq(0,0)$ having a g.c.r.d. $=d$ (d right divisor of a, b divisible by every right divisor of a,b); set of all points on a line: $(x+ak,\ y+bk)$, x y fixed; a, b fixed, not 0, 0, with g.c.r.d. 1; k arbitrary in R. The intuitive ideas in the "geometry" of the lattice points (integer coordinates) of the affine real plane are thus abstracted in equivalent geometric and algebraic forms. (Received May 20, 1942.)

STATISTICS AND PROBABILITY

248. Henry Scheffé: *On the theory of testing composite hypotheses with one constraint.*

Suppose that one has a sample from a population whose distribution function depends on l parameters θ_1, θ_2, $\cdots$, θ_l, and that on the evidence of the sample it is wished to test the hypothesis H_0: $\theta_1=\theta_1^{\circ}$. Not only are many statistical problems of practical interest precisely of this type, but the theory of testing such hypotheses with one constraint is intimately related to Neyman's theory of confidence intervals (Proceedings of the Royal Society, 1937). Type B regions for testing H_0 were defined by Neyman (Bulletin Société Mathématique de France, vol. 64, (1935)) who gave a method of finding such regions in the case $l=2$. His theorem is generalized to the case $l\geqq 2$ in the present paper. Type B$_1$ regions are defined as the natural generalization of the type A$_1$ regions of Neyman and Pearson (Statistical Research Memoirs, 1936) to permit the presence of the nuisance parameters θ_2, $\cdots$, θ_l in the distribution. Sufficient conditions are given that a type B region be also of type B$_1$. The paper will appear in the Annals of Mathematical Statistics. (Received May 29, 1942.)

TOPOLOGY

249. W. H. Gottschalk: *On k-to-1 mappings.*

Let A and B be compact metric spaces and let the mapping $T(A)=B$ be continuous and at most k-to-1, where k is an integer. Let $o(x)$ denote the Urysohn-Menger order of the point x. It is shown that if $y\in B$ has exactly n inverse points $x_1, \cdots, x_n$, then $\sum_{i=1}^{n} o(x_i)\leqq k\ o(y)$. Suppose A and B are continua and T is (exactly) k-to-1. It follows from this inequality that if y is an end point of B, then each inverse point of y is an end point of A. Let P denote the property of being a subcontinuum of A on which T is k-to-1. It is proved that if A has (the inducible property) P irreducibly, then B contains no end point; and, in case $k=2$, B has the stronger property of containing no cut point. It follows that no dendrite is a continuous k-to-1, $k>1$, image of a continuum; or, equivalently, every continuous k-to-1, $k>1$, image of a continuum contains a non-cut non-end point. (Received April 11, 1942.)

250. P. R. Halmos and Hans Samelson: *On monothetic groups.*

A topological group is *monothetic* (van Dantzig) if it contains an element (called a *generator*) whose powers are dense in the group. (I) A compact abelian group is monothetic if and only if its character group is algebraically isomorphic to a subgroup of the circle. (II) A compact separable abelian group is monothetic if and only if the quotient group with respect to the component of the identity is such. (Here separability means the existence of a countable dense set.) In particular a compact separable and connected abelian group is always monothetic. (III) Almost every element (Haar measure) of a compact abelian connected group, satisfying the second countability axiom, is a generator (Schreier and Ulam). The countability restriction is essential in (III); the set of generators of the toral group of dimension continuum has inner measure 0 and outer measure 1. Similarly, without the hypothesis of connectedness the set of generators can have any measure between 0 and 1. It is an easy application of (I) to describe completely the structure of totally disconnected monothetic groups. (Received April 6, 1942.)

251. A. D. Wallace: *Separation spaces*. II.

A definition of compactness is given for a separation space. The use of closure is avoided. It is shown that a compact connected nondegenerate separation space contains at least two non-cut points. For metric spaces this result is well known. This note will appear in Annaes da Academia Brasileira de Sciencias. (Received May 21, 1942.)

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TRAVELING WAVES ON ELECTRIC POWER SYSTEMS

L. V. BEWLEY

The problems of traveling waves on the transmission lines of a power system[1] differ considerably from those of traveling waves on telephone or telegraph circuits. The primary object in the case of the former is to know how to protect the system from abnormal voltage disturbances which might damage apparatus or cause discontinuity of service; whereas the object in the case of the latter is the transmission of signals. Attenuation, distortion, wave shape modification, and successive reflections are deliberately sought after on the power system as a means of rendering the surges innocuous, but these effects must be carefully avoided or nullified on the communication circuits so as to preserve the wave shape and transmit the signal with strength, fidelity, and without interference. On the power lines, the surges often originate from unknown causes, or at the point of origin are of unknown magnitude and shape (except from a statistical point of view); while on the communication circuits the initial shape and magnitude of the wave train are known with exactness. External fields (due to charged clouds), corona, flashovers, faults, and so on are of great importance with respect to surges; but are of no concern in the normal functioning of a telephone or telegraph line. Thus on the power lines surges originate from external or undesirable causes and every effort must be made to withstand or control them; while on communication circuits the transients are the direct means to the end.

These differences have led to corresponding differences in the mathematical approach. The power engineer is satisfied with approximations which would be intolerable to the communications engineer, and he is willing to take a license with mathematical rigor which would make any self-respecting mathematician groan. Higher mathematics has found little or no application in the study of surges on power systems. This has been due primarily to the fact that the boundary conditions are not definite enough to justify purely mathematical refinements; particularly since engineering results must be obtained in a short time by men who are not mathematicians. However, there are numerous aspects of the problem which lend themselves to mathematical excursions. It is my purpose in this lecture to

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An address delivered before the meeting of the Society in Bethlehem, Pa., on December 30, 1941, by invitation of the Program Committee under the title *The mathematical theory of traveling waves*; received by the editors January 10, 1942.

[1] *Traveling Waves on Transmission Systems*, L. V. Bewley, New York, Wiley.

give you the engineering methods, and perhaps, here and there, to suggest how advanced mathematics might be applied.

The general problem. Figure 1 indicates a multi-conductor transmission line, protected by a ground wire[1] at the peak of the towers, terminating at a power station containing a lightning arrester, a circuit breaker, a transformer, and a generator. A cloud, floating over the line, has been charged by the action of the rising air currents on the falling raindrops, and this cloud charge has induced a stationary counter-charge on the conductors of the transmission line. As the potential of the cloud increases, local breakdowns occur throughout its mass, thereby uniting some of its regions and making available, through partially ionized paths, a reservoir of charge. Eventually the field gradient reaches an intensity sufficient to initiate a leader stroke or dart, which starts towards ground. The progress of this dart is not continuous, but by jerks, each jerk depending upon the supply of additional charge to the head of the dart. It is like the armored force in a blitzkrieg break-through—it strikes to the limit of its capabilities and must then wait for tactical reinforcements and logistical support before renewing its attack. As the dart approaches earth, the field

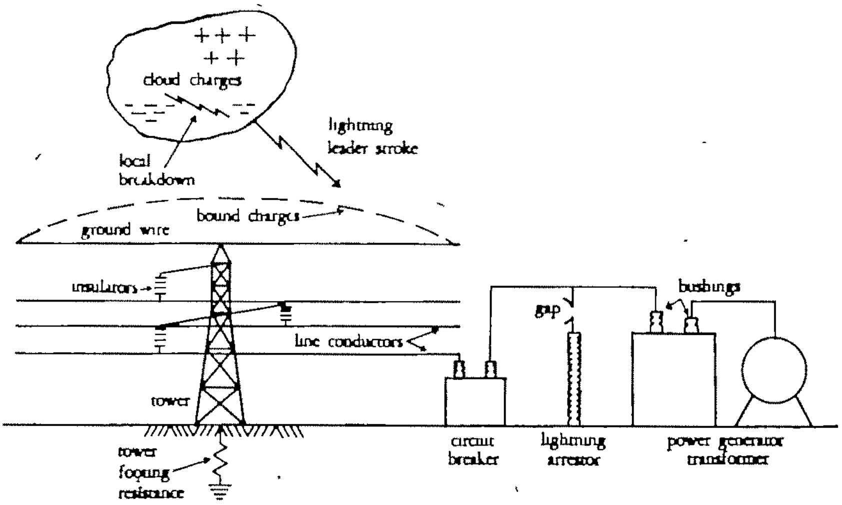

FIG. 1

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[1] The purpose of a ground wire is to intercept a lightning stroke and thereby protect the power conductors from destructive surges. Those surges which do come in to the station are absorbed by the lightning arrestor.

gradient at the transmission line increases and this causes a migration of charge up through the towers onto the ground wire, and from the remote parts on the line conductors towards the region of field concentration. The dart finally makes contact—say with the ground wire at the tower—and a lightning surge moves out in both directions on the ground wire, inducing waves on the line conductors. But when these waves reach the next tower, reflections occur, and very soon all the neighboring spans are filled by numerous waves reflecting back and forth, and perhaps flashovers have taken place to the line conductors. These waves are rushing, with the speed of light, towards the power station, where they may enter the windings of transformers and generators, causing steep gradients which may breakdown the turn-to-turn insulation, and oscillations which may develop destructive voltages on the major insulation to ground. Perhaps a bushing flashover, or an insulation failure will cause the circuit-breaker to function, interrupting the normal 60 cycle power current, and this operation will initiate a new transient which is called a "switching surge." And perhaps an insulator flashover out on the line will culminate in intermittent arcing which may result in a cumulative building-up of dangerous voltages—the so-called "arcing ground."

The engineer does not tackle this problem in its entirety by attempting to include all the terminal apparatus as boundary conditions. Rather is he compelled to make a piecemeal attack, by handling each part of the problem as a separate and independent proposition, thereby isolating and defeating them in detail. To this end, certain aspects of the problem will be considered under four main headings.

Multi-velocity waves (tensor notation). Consider a system of n overhead transmission line conductors with voltages e_r currents i^r, charges Q^r, and fluxes ϕ_r. Then in terms of Maxwell's electrostatic potential coefficients p_{rs}

$$e_r = p_{rs}Q^s \tag{1}$$

from which

$$Q^r = K^{rs}e_s \tag{2}$$

where K^{rs} is the inverse of p_{rs} in the matrix $[p_{rs}]$. These coefficients are calculated for parallel cylindrical conductors in the presence of ground by including the images of the conductors in the ground surface.

The magnetic flux linkages are given in terms of the inductance coefficients by

$$(3) \qquad \phi_r = L_{rs} i^s.$$

There are leakage currents flowing to ground and between conductors of amount $G^{rs} e_s$. The G^{rs} coefficients are supposed to include the effects of both leakage and corona.

And finally, there are resistance drops $R_{rs} i^s$ in the conductors due to the flow of currents.

Herefrom, the differential equations for the multi-conductor transmission system become (putting $p = \partial/\partial t$, in the Heaviside sense)

$$(4) \qquad -\frac{\partial e_r}{\partial x} = p\phi_r + R_{rt} i^t = (L_{rt} p + R_{rt}) i^t,$$

$$(5) \qquad -\frac{\partial i^t}{\partial x} = pQ^t + G^{ts} e_s = (K^{ts} p + G^{ts}) e_s.$$

Eliminating i^t there results

$$(6) \qquad \left(J_r^s - \delta_r^s \frac{\partial^2}{\partial x^2}\right) e_s = 0$$

in which δ_r^s is the Kronecker delta and

$$(7) \qquad J_r^s = (L_{rt} p + R_{rt})(K^{ts} p + G^{ts}).$$

Now if the losses are ignored ($R_{rs} = 0$, $G^{rs} = 0$), (6) is satisfied by the traveling wave[3]

$$(8) \qquad e_s = \overset{(\alpha)}{a_s} f_{(\alpha)}(x - v_{(\alpha)} t)$$

which substituted in (6) gives

$$(9) \qquad \left(L_{rt} K^{ts} p^2 - \delta_r^s \frac{\partial^2}{\partial x^2}\right) \overset{(\alpha)}{a_s} f_{(\alpha)}(x - v_{(\alpha)} t) = 0.$$

Since this equation must be satisfied for waves of the same velocity

$$(10) \qquad (L_{rt} K^{ts} v_{(\alpha)}^2 - \delta_r^s) \overset{(\alpha)}{a_s} = 0.$$

The velocities are given by the roots of the determinant

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[3] An enclosed index is used here to suspend the summation convention.

$$\left| L_{rt} K^{ts} v^{2}_{(\alpha)} - \delta^{s}_{r} \right| = \left| c^{s}_{r} \right| = 0. \tag{11}$$

To each root of (11) there correspond n values of $a_s^{(\alpha)}$, and any $(n-1)$ of them may be determined in terms of one value taken arbitrarily. Let this one value be $a_1^{(\alpha)}=1$. Solving (10)

$$a_s^{(\alpha)} = \frac{-A^{r}_{s}}{\left| b^{s}_{r} \right|} \left(L_{rt} K^{t1} v^{2}_{(\alpha)} \right) \quad \text{where } s \neq 1,\ r \neq 1, \tag{12}$$

in which $\left| b^s_r \right|$ is $\left| c^s_r \right|$ with the $s=1$ row and $r=1$ column deleted, and A^r_s is the cofactor of b^s_r in $\left| b^s_r \right|$.

The complete solution then becomes

$$e_r = a_r^{\alpha}\left[f_\alpha(x - v_{(\alpha)}t) + F_\alpha(x + v_{(\alpha)}t) \right]. \tag{13}$$

By (5) the corresponding currents are

$$i^r = -p \int K^{rs} e_s dx = K^{rs} a_s^{\alpha} (f_\alpha - F_\alpha) v_{(\alpha)} = Y^{r\alpha}(f_\alpha - F_\alpha). \tag{14}$$

Transition points. Consider the general case of Figure 2 in which any number of incoming lines terminate at a transition point consisting of an inter-connected network and any number of outgoing lines. When the incident waves on the incoming lines reach the transition point, currents will flow into the network, transmitted waves will move out on the outgoing lines, and reflected waves will start back on the incoming lines. In tensor notation let:

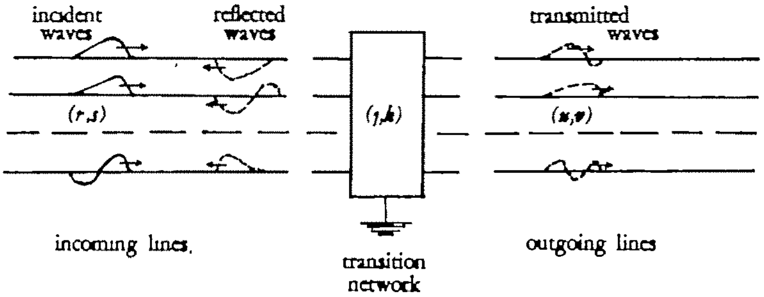

FIG. 2

z_{rs} = surge impedances of incoming lines,[4]

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[4] The surge impedance of a line is the coefficient of proportionality between its voltage and current $(e=zi)$. Its reciprocal is called surge admittance. For surges on transmission lines, these parameters are essentially constant, and are an indication of the current associated with a surge of given voltage.

y^{rs} = surge admittances of incoming lines (inverse of z^{rs}),
z_{uv} = surge impedances of outgoing lines,
y^{uv} = surge impedances of outgoing lines (inverse of z_{uv}),
$Z_{\rho\sigma}$ = branch impedances of the network,
$C^{\alpha}_{\alpha'}$ = transformation tensor specifying the total interconnections of the network and outgoing lines.

Then

(1) $Z_{\alpha\beta} = (Z_{\rho\sigma} + z_{uv})$ = impedance of network and outgoing lines before interconnection,
(2) $Z_{\alpha'\beta'} = C^{\alpha}_{\alpha'} C^{\beta}_{\beta'} Z_{\alpha\beta}$ = impedance after interconnection of the network and outgoing lines.

Now $Z_{\alpha'\beta'}$ may include branches other than those connected to the incoming lines. The open circuit branches will have been eliminated by $C^{\alpha}_{\alpha'}$, but the branches other than those connected to the incoming lines will have to be eliminated by the substitutions:

$$Z_{\alpha'\beta'} = Z_{rs} + Z_{rq} + Z_{ps} + Z_{pq}, \tag{3}$$

$$E_r = Z_{rs} I^s + Z_{rq} I^q, \tag{4}$$

$$0 = Z_{ps} I^s + Z_{pq} I^q, \tag{5}$$

from which

$$E_r = (Z_{rs} - Z_{rq} Y^{qp} Z_{ps}) I^s = Z'_{rs} I^s. \tag{6}$$

Now let (e_r, i^r) and (e'_r, i'^r) be the incident and reflected waves, respectively, on the incoming line. Then at the transition point

$$e_r + e'_r = E_r = Z'_{rs} I^s, \tag{7}$$

$$i^r + i'^r = I^r, \tag{8}$$

so that

$$e_r + e'_r = Z'_{rs}(i^r + i'^r) = Z'_{rs} y^{st}(e_t - e'_t) \tag{9}$$

from which

$$(\delta^t_r + Z'_{rs} y^{st}) e'_t = (-\delta^t_r + Z'_{rs} y^{st}) e_t. \tag{10}$$

This system of equations defines the reflected voltage waves e'_r.

The total voltage at the transition point then follows by (7), the total current by (8), the remaining network currents by (5) and the network voltages by (4), and so on.

Successive reflections. The calculation of successive reflections is oftentimes a long and involved process; particularly in those cases where reflections may occur from a whole series of neighboring junctions. An example is the case of lightning striking the ground wire at midspan. The incident waves move out in opposite directions until they reach the nearest towers, where they reflect as a consequence of the surge impedance of the continuing ground wire in parallel with that of the tower. Transmitted waves quickly reach the foot of the tower from which they reflect as a result of the ground resistance. Other waves reflect from the next tower, and from the next after that, and so on. Thus within a few microseconds the system is alive with a whole series of waves moving in different directions, arriving at different times, of different magnitudes and polarity, and having experienced different attenuations and distortions.

In order to keep track of all these components a *lattice diagram* has been devised, such as shown in Figure 3 for the case of lightning striking a ground wire at midspan. The progress of each wave component is easily followed as it slides downhill along its zig-zag path, giving rise to reflections at each junction. Thus at any instant of time the waves at all points on the line can be identified; or at any point on the line the time of arrival of each wave can be seen.

To construct such a reflection lattice it is first necessary to determined the reflection and refraction coefficients[8] at each junction, and to post these on the sketch of the system being studied, as has been done on Figure 3. The coefficients are, in general, Heaviside operators, such that when operating on an incident wave (regarded as a time function reckoned from its instant of arrival at the particular junction) they give the reflected or transmitted wave. Those shown in Figure 3 are A, B, C, D, B', C', D'. The initial wave coming down the lightning stroke of surge impedance $2Z_0$ (the 2 is occasioned by the condition of symmetry which permits the amputation of the system to the left of the stroke) refracts onto the ground wire a wave $A \cdot f(t)$ which moves on to the top of Tower 1, where it reflects a portion $AB' \cdot f(t)$ back towards its origin, and transmits a portion $AB \cdot f(t)$ to the next ground wire section and also down the tower. When the wave reaches the foot of the tower it reflects a portion $ABD' \cdot f(t)$ back up the tower. Likewise reflections return from the top and foot of Tower 2 and from the towers beyond. Now each of these reflections could be traced out independently on the lattice and all waves fully accounted for. But the labor is great. The work can be simplified by introducing the concept of "wave trains," and "retarder operators."

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[8] A "reflection coefficient" is the operator determined from equation (10) of the previous section on transition points, which permits the calculation of a reflected wave in terms of the incident wave. Likewise, the "refraction coefficient" gives the transmitted wave.

The system of waves reflected back on the ground wire due to the arrival at a tower top of a wave $f(t')$ and the successive reflections

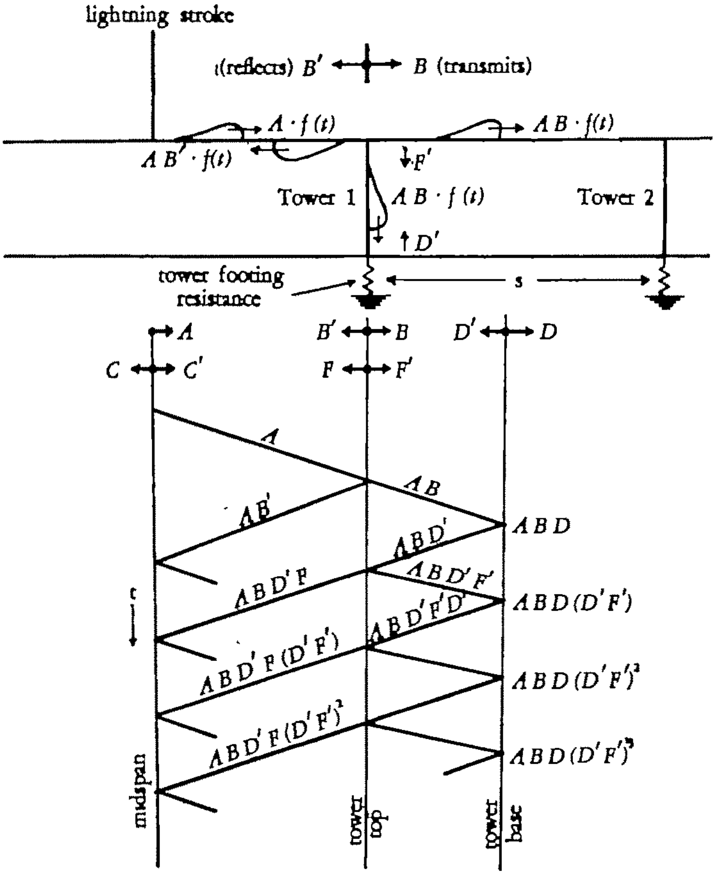

FIG. 3

up and down the tower is seen from the lattice to be a "wave train" of *Type* I:

$$
\begin{aligned}
e' &= B'\cdot f(t') + BD'F\cdot f(t' - 2h) + BD'F(D'F')\cdot f(t' - 4h) + \cdots \\
&= B'\left[f(t') + \frac{BD'F}{B'}\sum (D'F')^{m-1} f(t' - 2hm)\right] \qquad (1)\\
&= B'[1 + a\sum b^{m-1}\cdot\phi(m)]f(t') \\
&= B'\alpha_m f(t)
\end{aligned}
$$

in which $\phi(m)$ is a "retarder operator" such that the time of arrival of a wave to which it is applied is retarded by $(2km)$, thus

$$\phi(m)\cdot f(t') = f(t' - 2km). \tag{2}$$

The wave train transmitted on to the next tower by the incident wave and its tower reflection is the *Type* II wave train

$$\begin{aligned} e'' = e + e' &= B\left[f(t') + \frac{D'F}{B}\sum (D'F')^{m-1}\cdot f(t' - 2km)\right] \\ &= B[1 + c\sum b^{m-1}\phi(m)]f(t') \\ &= B\beta_m f(t'). \end{aligned} \tag{3}$$

In terms of the wave trains of Types I and II the complete history of the reflections can now be written down.

The initial wave transmitted by a lightning surge to midspan is $Af(t)$ which arrives at the first tower at time $0.5s$ and gives rise to the Type I wave train of the first order.

$$e_1 = AB'\alpha_m f(t - 0.5s). \tag{4}$$

This wave train arrives at midspan at time s and reflects therefrom as

$$e_2 = AB'C'\alpha_m f(t - s). \tag{5}$$

When the reflected wave train arrives at the tower at time $(1.5s)$ it generates a new Type I wave train of the second order

$$e_3 = AB'C'\alpha_m\alpha_n f(t - 1.5s). \tag{6}$$

Continuing this process we find combinations of the form $\alpha_m\alpha_n\alpha_p \cdots$ in which products are to be interpreted as

$$\begin{aligned} \alpha_m\alpha_n\alpha_p &= [1 + a\sum b^{m-1}\phi(m)][1 + a\sum b^{n-1}\phi(n)][1 + a\sum b^{p-1}\phi(p)] \\ &= [1 + 3a\sum b^{m-1}\phi(m) + 3a^2\sum\sum b^{m+n-2}\phi(m + n) \\ &\quad + a^3\sum\sum\sum b^{m+n+p-3}\phi(m + n + p)]. \end{aligned} \tag{7}$$

Now in addition to those wave trains operating between midspan and Tower 1, contributions eventually arrive from neighboring towers. Thus at time $(0.5s)$ there is transmitted beyond Tower 1, owing to the initial wave $Af(t)$, the Type II wave train of the first order

$$E_1 = AB\beta_m f(t - 0.5s). \tag{8}$$

This wave train reaches Tower 2 at time $(1.5s)$ where it generates a Type I wave train of the second order

$$E_2 = ABB'\beta_m\alpha_n f(t - 1.5s). \tag{9}$$

This wave train arrives at Tower 1 at time (2.5s) and generates a third order wave train of Type I

$$E_3 = AB^2B'\beta_m\alpha_n\beta_n f(t - 2.5s), \tag{10}$$

and this wave train, reaching midspan at (3s), reflects therefrom as

$$E_4 = AB^2B'C'\beta_m\alpha_n\beta_n f(t - 3s). \tag{11}$$

With the assistance of the lattice diagram and retarder operators, the potential at any point can be written down. For example, at the top of Tower 1

$$\begin{aligned} V = {} & AB\beta_m f(t - 0.5s) + ABB'C'\alpha_n\beta_n f(t - 1.5s) \\ & + AB(BC)^2\alpha_m\alpha_n\beta_n f(t - 2.5s) + \cdots \\ & + AB^2B'\beta_m\alpha_n\beta_n f(t - 2.5s) + \cdots \\ & + \cdots . \end{aligned} \tag{12}$$

Induced lightning surges. Suppose a cloud bearing a charge Q_0 is over a transmission line, Figure 4, and is discharging either to ground or to another cloud according to some function of time $\psi(t)$; that is, the charge remaining in the cloud at any instant is $Q_0[1-\psi(t)]$.

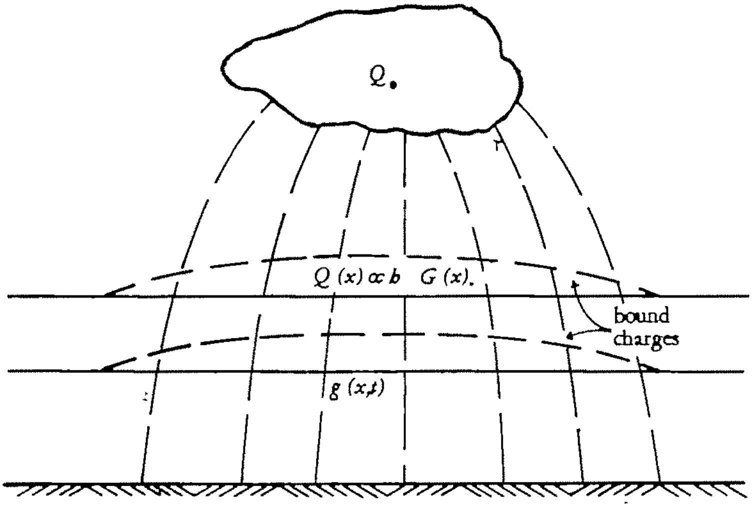

FIG. 4

Depending upon the shape and size of the cloud charge, its height above ground, and its position with respect to the transmission line, the line will experience a gradient

$$g(x, t) = G(x) \cdot [1 - \psi(t)] \tag{1}$$

in which $G(x)$ represents the initial distribution of gradient (at the beginning of cloud discharge) as a function of distance along the line. Under the influence of this field, charges of opposite sign to that of the cloud will leak over the insulators, or migrate from the remote parts of the system, and collect on the line conductors as bound charges. The density of bound charge at any point x will be proportional to the gradient and to the height h of the conductor above ground (since the field is substantially uniform for a hundred feet or so above ground). These bound charges nullify the potential due to the external field, so that initially the line charges are given by

$$V_r = 0 = G(x) \cdot h_r + p_{rs}Q^s, \qquad r = 1, 2, \cdots, n. \tag{2}$$

Suppose then m of the n wires are ground wires perfectly grounded throughout their length, and let these ideal ground wires be represented by (j, k) indices. The remaining $(n-m)$ wires are power conductors, and will be represented by (u, v) indices.

Now if the field gradient $G(x)$ is suddenly removed, the bound charges on the line wires will not change at the first instant, but those on the ground wires are instantly replaced by new charges Q' since the ground wires must remain at zero potential. Therefore

$$0 = p_{jk}Q'^k + p_{jv}Q^v, \tag{3}$$

$$V_u = p_{uk}Q'^k + p_{uv}Q^v. \tag{4}$$

From (2) and (3) all the charges Q^s and Q'^k may be found; and from (4) the potentials V_u may be determined. These potentials immediately move out as pairs of traveling waves (in opposite directions). At the first instant, however, the forward and backward waves $f_u(x-vt)$ and $F_u(x+vt)$ add up to the voltages given by (4), and the resultant current flow must be zero in the isolated power conductors. Thus

$$f_u + F_u = V_u, \tag{5}$$

$$Y^{uv}(f_v - F_v) = 0. \tag{6}$$

Hence

$$F_u = f_u = V_u/2, \tag{7}$$

that is, the forward and backward waves on a conductor are the same shape and magnitude.

However, the release of the bound charge is not instantaneous, but according to the law of cloud discharge $\psi(t)$. The corresponding traveling waves are then given by Duhamel's theorem

$$(8) \quad e = \int_0^t \{f[x + v(t - \tau)] + f[x - v(t - \tau)]\} \frac{\partial\psi(\tau)}{\partial\tau} d\tau$$

$$(9) \quad = \lim_{\Delta t \to 0} \sum_0^n \{f[x + v(n - k)\Delta t] + f[x - v(n - k)\Delta t]\} \Delta\psi_k$$

where $n\ \Delta t = t$, $k\ \Delta t = \tau$ and $\Delta\psi_k = \psi[(k+1)\Delta t] - \psi[k \cdot \Delta t]$. The application of the integral is limited to relatively simple expressions for f and ψ, but the summation can be used for any functions whose graphs are known or assumed. Ultimately, since both f and ψ derive from experimental data, it is best to use the summation expression. Both graphical and tabular methods have been devised for its ready application, and engineering solutions are quickly arrived at.

Equation (8) can also be derived by setting up the conditions in terms of retarded potentials. Very few engineers deal with retarded potentials, whereas a considerable number of them are familiar with Duhamel's theorem through Heaviside's operational calculus.

LEHIGH UNIVERSITY

SOME NEW METHODS OF SOLUTION OF TWO-DIMENSIONAL PROBLEMS IN ELASTICITY

IVAN S. SOKOLNIKOFF

1. **Introduction.** The main purpose of this address is to bring to the attention of the workers in the theory of elasticity and related branches of applied mathematics a simple general method of solution of several important classes of two-dimensional boundary value problems. I use the term "two-dimensional" or "plane" boundary value problems in the sense that their mathematical formulation requires the introduction of only two independent variables. In this sense the problems of St. Venant on torsion and flexure of cylinders, and the problems on deflection and buckling of elastic plates, which have a three-dimensional physical aspect, are two-dimensional.

The method which I intend to discuss was developed mainly by a group of Russian mathematicians, and despite the fact that it has been utilized extensively in Russia for more than a decade, it is virtually unknown in this country. A great variety of problems to which it has been applied to obtain useful solutions includes an investigation of flexure and torsion of beams, a study of thermo-elastic stresses in composite cylinders, an analysis of deflection of anisotropic plates, and a multitude of problems characterized by the states of plane stress and plane strain.

Inasmuch as familiarity with the concepts of applied mathematics is a rare virtue, I shall reduce the use of the technical language to a minimum, and shall ask you to take for granted certain basic equations of the theory of elasticity. Failure to comprehend the origin of these equations will not impair the understanding of the general method of their solution.

We shall suppose that a two-dimensional region R, occupied by an elastic medium, is referred to a system of cartesian axes (x, y). To fix the ideas we can think of the region R as representing the cross-section of a long cylinder whose elements are parallel to the z-axis, and whose lateral surface is subjected to a distribution of external forces that is independent of the z-coordinate. Under the action of such forces, the medium, in general, will be distorted, and the displacement of the points of the region R in the directions of the x- and y-axes will be denoted by $u(x, y)$ and $v(x, y)$, respectively. If the medium

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An address delivered before the meeting of the Society in Bethlehem, Pa., on December 30, 1941, by invitation of the Program Committee; received by the editors January 21, 1942.

is assumed to be isotropic, then the stresses produced in the medium are connected with the derivatives of u and v by the formulas (Hooke's law)

$$(1.1)\quad \tau_{xx} = \lambda\Delta + 2\mu\frac{\partial u}{\partial x}, \quad \tau_{yy} = \lambda\Delta + 2\mu\frac{\partial u}{\partial y}, \quad \tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

where the τ's are the components of the stress tensor, λ and μ are elastic constants introduced by Lamé, and $\Delta \equiv \partial u/\partial x + \partial v/\partial y$.

If the medium is in equilibrium (in the absence of body forces) then the components of the stress tensor τ satisfy throughout the interior of the region the differential equations

$$(1.2)\qquad \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0, \qquad \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} = 0,$$

while on the boundary C of the region R they fulfill the conditions

$$(1.3)\qquad \begin{aligned} \tau_{xx}\cos(x, \nu) + \tau_{xy}\cos(y, \nu) &= X(s) \\ \tau_{xy}\cos(x, \nu) + \tau_{yy}\cos(y, \nu) &= Y(s), \end{aligned}$$

where ν denotes the exterior normal to the contour C, and X and Y are the x- and y-components of the prescribed external force estimated per unit length s of the contour C.

I shall call the problem of determining the solution of the system of equations (1.1), (1.2), (1.3), for the five unknown functions τ_{xx}, τ_{xy}, τ_{yy}, u, v, the first boundary value problem of the plane theory of elasticity. Instead of specifying the distribution of external force on the boundary C of the region R, we can impose the requirement that the displacements u and v assume prescribed values on the contour C. This latter problem will be referred to as the second boundary value problem.

The great majority of the two-dimensional boundary value problems in the theory of elasticity is reducible to the solution of these boundary value problems. We shall be concerned only with those solutions in which the components of the stress tensor are continuous and single-valued throughout the region R together with their first and second partial derivatives with respect to x and y. The displacements u an v will be assumed to be continuous and single-valued in the region R together with their partial derivatives including those of the third order. These restrictions arise essentially from physical considerations, and it is a consequence of the restrictions imposed on the displacements u and v that the components of the stress tensor determined from equations (1.2) satisfy in the region R the Beltrami-Michell equation of compatibility, namely

$$\nabla^2(\tau_{xx}+\tau_{yy})=0, \tag{1.4}$$

where $\nabla^2 \equiv \partial^2/\partial x^2+\partial^2/\partial y^2$.

It is clear that equations (1.2) represent a necessary and sufficient condition for the existence of a function $U(x, y)$, such that

$$\tau_{xx}=\frac{\partial^2 U}{\partial y^2}, \qquad \tau_{yy}=\frac{\partial^2 U}{\partial x^2}, \qquad \tau_{xy}=-\frac{\partial^2 U}{\partial x \partial y}, \tag{1.5}$$

and it follows at once from (1.4) that $U(x, y)$ satisfies the biharmonic equation

$$\nabla^2\nabla^2 U=0 \tag{1.6}$$

throughout the region R. The function U is commonly known as Airy's stress function.

Substituting the relations (1.5) in the boundary conditions (1.3) and combining results lead to a compact boundary condition in the form

$$\frac{d}{ds}\left(\frac{\partial U}{\partial y}-i\frac{\partial U}{\partial x}\right)=X+iY, \qquad \text{on } C, \tag{1.7}$$

where d/ds represents differentiation along the arc length of the contour C and $i^2=-1$. Inasmuch as the right-hand member of (1.7) is a known function of the points of the contour C, it is clear that the solution of the first boundary value problem is reduced to the determination of the biharmonic function $U(x, y)$ whose derivatives, $\partial U/\partial x$ and $\partial U/\partial y$, are specified functions of the points of C. This latter problem was subject to a prize offered by the Paris Academy, and its complete theoretical solution for the case of a finite simply-connected region R, bounded by a contour C satisfying certain general conditions, was obtained by J. Hadamard and G. Lauricella.[1] They solved the problem by reducing it to the solution of a Fredholm's integral equation.

A significant step in the direction of the development of the theory was made by G. V. Kolossoff, whose Dorpat dissertation[2] (1909) was concerned with the application of the theory of functions of a complex variable to the solution of plane problems of the theory of elasticity. Further important results were obtained by N. I. Muschelišvili, who made use of conformal mapping and of integrals analogous to those of Cauchy to obtain an elegant solution of the first and second boundary value problems for an arbitrary simply-connected domain (finite or infinite).

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[1] Among the recipients of the prize were A. Korn, J. Hadamard, and G. Lauricella. See references [1], [2], and [3] at the end of this paper.

[2] See [4–8].

In a series of papers dating from 1919, Muschelišvili gave a detailed analysis of the character of solution in simply- and multiply-connected domains and illustrated his method of solution by providing practically useful solutions of a number of specific problems [9–21].

The questions pertaining to the existence of solution in multiply-connected domains were dealt with extensively by S. G. Michlin, who gave, in the period from 1933 to 1935, several methods of theoretical solution of the first and second boundary value problems [22–26].

In the following section I shall limit myself to a discussion of Muschelišvili's method of solution of the boundary value problems of plane elasticity for simply-connected domains. While the application of this method to multiply-connected domains is important from a technical point of view (see §3) the essence of the method is adequately illustrated by a consideration of the simpler case.

2. Reduction of the problem to a functional equation. It is clear from equation (1.6) that the function $\nabla^2 U = P(x, y)$ is harmonic in R. Let $Q(x, y)$ denote the conjugate harmonic function, and introduce an analytic function $\phi(z)$ of a complex variable $z = x+iy$ defined by the formula

$$P + iQ = 4\phi'(z),$$

where prime denotes the derivative with respect to z. If the real and imaginary parts of the function $\phi(z)$ are denoted by p and q, respectively, then it is easily verified that

$$\nabla^2(U - px - qy) = 0.$$

Accordingly, the function

$$p_1 = U - px - qy$$

can be considered as the real part of some analytic function $\chi(z)$, and since

$$2(px + qy) = \bar{z}\,\phi(z) + z\,\overline{\phi(z)},$$

one can write the biharmonic function U in the form[3]

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[3] The possibility of representing a biharmonic function with the aid of two analytic functions of a complex variable was noted first by É. Goursat, Bulletin de la

$$2U = \bar{z}\,\phi(z) + z\,\overline{\phi(z)} + \chi(z) + \overline{\chi(z)}, \tag{2.1}$$

where bars denote the conjugate complex values.

Inserting the value of U given by (2.1) in the left-hand member of equation (1.7) yields the boundary condition in the form

$$(X + iY)ds = -\,id\left[\phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)}\right], \tag{2.2}$$

where $\psi(z) = \chi'(z)$.

If both members of equation (2.2) are integrated along the contour C from some arbitrary point $s = s_0$, there results

$$\int_{s_0}^{s} (X + iY)ds = -\,i\left[\phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)}\right] + \text{const.},$$

so that the boundary condition can be written as

$$\phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)} = f_1(s) + if_2(s), \qquad \text{on } C, \tag{2.3}$$

where

$$i\int_{s_0}^{s} (X + iY)ds = f_1(s) + if_2(s).$$

The constant of integration has been omitted in the formula (2.3) since it can be fixed in an arbitrary manner without affecting the state of stress.[4]

A reference to the formulas (1.5) and (2.1) shows that the components of the stress tensor τ are easily calculable once the functions $\phi(z)$ and $\psi(z)$ are known, and it is readily checked by integrating equations (1.1) that

$$2\mu(u + iv) = -\left(\frac{\partial U}{\partial x} + i\,\frac{\partial U}{\partial y}\right) + \frac{2(\lambda + 2\mu)}{\lambda + \mu}\,\phi(z). \tag{2.4}$$

Thus the complete solution of the first boundary value problem is made to depend on the solution of the functional equation (2.3).

The formulation of the second boundary value problem in the form of a functional equation likewise presents no difficulty. In fact, the

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Société Mathématique de France, vol. 26 (1898), p. 236. The derivation of Goursat's formula given above is due to N. I. Muschelišvili, Bulletin de l'Académie des Sciences de l' URSS, 1919, pp. 663–686. The analytic character of solutions of a biharmonic equation follows from this derivation, while the derivation of Goursat depends on the assumption that every biharmonic function is analytic.

[4] This statement follows from the fact that the state of stress, in a simply-connected domain, is not altered when the function $\phi(z)$ is replaced by $\phi(z)+ciz+\alpha+i\beta$ and $\psi(z)$ by $\psi(z)+\alpha'+i\beta'$, where c, α, β, α', and β' are arbitrary real constants. See, for example, reference [15] listed at the end of this paper.

desired functional equation, in this case, follows almost at once from equation (2.4).

The limitations of time do not permit me to enter upon the discussion of the interesting questions connected with the existence and uniqueness of solution of the problem, and I shall confine myself to a remark that the proofs of uniqueness and existence of solution present no serious difficulties if one assumes the functions $\phi(z)$, $\phi'(z)$, and $\psi(z)$ to be continuous in the closed region R, the prescribed functions $X(s)$ and $Y(s)$ to satisfy Hölder's condition, and the contour C to have a continuously turning tanget.[5] It may be remarked that these restrictions are dictated by the behavior of stresses and displacement occurring normally in physical problems, and that they can be considerably relaxed by an analytically inclined mathematician.

I shall indicate in the remainder of this section some modes of attack on the problems that lead to the determination of the functions $\phi(z)$ and $\psi(z)$ in practically useful forms.

The solution of the functional equation (2.3) is almost immediate when the region R is that bounded by a unit circle $|z| \leq 1$. Since the functions $\phi(z)$ and $\psi(z)$ are analytic in R, one can write

$$(2.5) \qquad \phi(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \psi(z) = \sum_{n=0}^{\infty} b_n z^n, \qquad |z| < 1.$$

The functions X and Y, characterizing the prescribed distribution of stress on the boundary of the unit circle, can be regarded as functions of the polar angle θ defined by the relation $z = e^{i\theta}$. Then the right-hand member of (2.3) can be expressed in the form

$$f_1 + if_2 = i \int_{s_0}^{s} (X + iY)ds = \sum_{-\infty}^{\infty} c_n e^{in\theta},$$

if one assumes that the functions f_1 and f_2 are of bounded variation. The Fourier coefficients c_n being known, one can insert the foregoing expansion in the right-hand member of (2.3) and the series (2.5) in the left-hand member. A comparison of like powers of $e^{i\theta}$ on both sides of the resulting equation will yield expressions for the unknown coefficients a_n and b_n in terms of the known values c_n. The justification of the validity of the formal solution obtained in this manner presents no difficulty in most problems of practical interest, since the behavior of functions describing the specified distribution of stress on the boundary cannot be too pathological.

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[5] See, for example, [26].

If the contour C bounding the region R is not a circle, one can introduce a function

$$z = \omega(\zeta)$$

that maps the region R conformally on the unit circle $|\zeta| < 1$. Let the transforms of the functions $\phi(z)$ and $\psi(z)$ be denoted, respectively, by ϕ_1 and ψ_1, so that $\phi_1(\zeta) = \phi[\omega(\zeta)]$ and $\psi_1(\zeta) = \psi[\omega(\zeta)]$. Then a simple calculation shows that the functional equation (2.3) goes over into

$$\phi_1(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)}\overline{\phi_1'(\sigma)} + \overline{\psi_1(\sigma)} = f_1 + if_2, \tag{2.6}$$

where $\sigma = e^{i\theta}$ represents a point on the boundary γ of the circle $|\zeta| = 1$, and $f_1 + if_2$ is the transform of the right-hand member of (2.3) and hence can be regarded as a known function of θ.

The method of solution in series indicated above can be applied to determine the analytic functions

$$\phi_1(\zeta) = \sum_{n=0}^{\infty} a_n\zeta^n, \qquad \psi_1(\zeta) = \sum_{n=0}^{\infty} b_n\zeta^n, \qquad |\zeta| < 1,$$

and it is not difficult to verify that the coefficients a_n and b_n can be determined explicitly in terms of the Fourier coefficients of the functions $f_1 + if_2$ and $\omega(\sigma)/\overline{\omega'(\sigma)}$, whenever the mapping function $\omega(\zeta)$ is a polynomial.[6]

If it is recalled that the mapping function for a suitably restricted simply-connected region can be approximated by a polynomial $z = \omega_n(\zeta)$ of sufficiently high degree (in the sense that the polynomial $z = \omega_n(\zeta)$ maps the region $|\zeta| < 1$ on some region R_n which can be made to approximate the region R as closely as desired) then the practical usefulness of the method of solution outlined above becomes immediately obvious.

In a number of important problems the determination of the functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ can be greatly simplified by transforming the functional equation (2.6) into an equivalent pair of integral equations. Thus, consider the equation (2.6) and the equation

$$\overline{\phi_1(\sigma)} + \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\phi_1'(\sigma) + \psi_1(\sigma) = f_1 - if_2 \tag{2.7}$$

which is obtained from (2.6) by forming the conjugate expressions. If both members of these equations are multiplied by $(1/2\pi i)d\sigma/(\sigma-\zeta)$ and the resulting expressions integrated over the contour γ of the unit circle, one obtains

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[6] For further particulars see [13, pp. 287–296]. A detailed exposition and some specific calculations pertaining to several problems of physical interest are contained in [30, pp. 266–311].

$$
(2.8)\quad
\begin{aligned}
&\frac{1}{2\pi i}\int_\gamma \frac{\phi_1(\sigma)\,d\sigma}{\sigma-\zeta} + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\phi_1'(\sigma)}}{\sigma-\zeta}\,d\sigma \\
&\qquad + \frac{1}{2\pi i}\int_\gamma \frac{\overline{\psi_1(\sigma)}}{\sigma-\zeta}\,d\sigma = A(\zeta),\\
&\frac{1}{2\pi i}\int_\gamma \frac{\overline{\phi_1(\sigma)}}{\sigma-\zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\phi_1'(\sigma)}{\sigma-\zeta}\,d\sigma \\
&\qquad + \frac{1}{2\pi i}\int_\gamma \frac{\psi_1(\sigma)}{\sigma-\zeta}\,d\sigma = B(\zeta),
\end{aligned}
$$

where

$$A(\zeta)=\frac{1}{2\pi i}\int_\gamma[(f_1+if_2)/(\sigma-\zeta)]d\sigma,\quad B(\zeta)=\frac{1}{2\pi i}\int_\gamma[(f_1-if_2)/(\sigma-\zeta)]d\sigma.$$

The complete equivalence of the simultaneous equations (2.8) with (2.6) follows at once from the theorem of Harnack. The equations (2.8) can be simplified by applying Cauchy's integral formula, and by observing that

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{f(\sigma)}}{\sigma-\zeta}\,d\sigma = \overline{f(0)},$$

where $f(\zeta)$ is any function continuous in the closed region $|\zeta|\leq 1$ and analytic in the interior of the unit circle γ. Making use of these simplifications leads to a pair of integral equations

$$
(2.9)\quad
\begin{aligned}
&\phi_1(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\frac{\overline{\phi_1'(\sigma)}}{\sigma-\zeta}\,d\sigma + \overline{\psi_1(0)} = A(\zeta),\\
&\psi_1(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\phi_1'(\sigma)}{\sigma-\zeta}\,d\sigma + \overline{\phi_1(0)} = B(\zeta).
\end{aligned}
$$

It was demonstrated by Muschelišvili that the solution of these equations for the unknown functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ can be obtained in an elementary way in terms of the integrals of Cauchy's type whenever the mapping function $\omega(\zeta)$ is rational.[7]

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[7] See, for example, [15, 16].

The system of integral equations (2.9) is not of the standard type, but it is possible to reduce the first of equations (2.9) to an ordinary integral equation of Fredholm, of the second kind.[8] Once the function $\phi_1(\zeta)$ is determined from this equation, the second of equations (2.9) permits one to calculate the function $\psi_1(\zeta)$ by quadratures. Some effective methods of numerical solution of such equations were given recently by E. I. Nyström, D. I. Šerman, A. Gorgidze, A. Rukhadze and others [27–29].

3. **Thermal stresses.** One of the technically important problems is that of determining the state of stress produced in an elastic body by heating. Consider a body in the shape of a right cylinder whose length is large compared with the linear dimensions of the cross-section, and let the xy-plane of the cartesian coordinate system lie in one of the cross-sections of the cylinder. If such a body is subjected to a change in temperature which is a function of the x- and y-coordinates alone, then the change of temperature will cause the body to expand. In general, the expansion of an elastic body cannot proceed freely and the body becomes stressed. The problem of determining the distribution of thermal stresses assumes particular importance in a study of compound columns. A concrete beam reenforced by steel and subjected to varying temperatures is a specific example.

It is not difficult to see that the determination of stresses in this case essentially reduces to the solution of a system of equations which differ from the system (1.1), (1.2), (1.3) only in that Hooke's law (1.1) is replaced by a more general one that takes into account the effect of thermal expansion on stresses. This generalized law has the form

$$\tau_{xx} = \lambda\Delta + 2\mu\frac{\partial u}{\partial x} - (3\lambda + 2\mu)\alpha T(x, y),$$

$$\tau_{yy} = \lambda\Delta + 2\mu\frac{\partial v}{\partial y} - (3\lambda + 2\mu)\alpha T(x, y),$$

$$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

---

[8] This reduction is accomplished essentially by differentiating both members of the equation with respect to ζ and by allowing ζ to approach σ_0, where σ_0 is an arbitrary point of the contour γ. The resulting equation has the form $\phi'(\sigma_0)+(1/2\pi i)\int_\gamma K(\sigma_0, \sigma)\overline{\phi'(\sigma)}d\sigma = F(\sigma_0)$, where $F(\sigma_0)$ is a known function and the kernel $K(\sigma_0, \sigma)$ is a continuous function provided that suitable restrictions are imposed on the mapping function $\omega(\zeta)$. See, for example, [13, 16].

where α is the coefficient of the linear thermal expansion and $T(x, y)$ is the known change in temperature of the body.

If we introduce a stress function $U(x, y)$ defined by the formulas (1.5) and perform calculations analogous to those that led to the biharmonic equation (1.6), there will result the equation[9]

$$\nabla^2\nabla^2 U + k\nabla^2 T = 0, \tag{3.1}$$

where $k = 2\mu(3\lambda + 2\mu)(\lambda + 2\mu)^{-1}\alpha$.

Setting $U = \Phi - V$, where V is a suitable solution of the Poisson equation

$$\nabla^2 V = kT,$$

shows that equation (3.1) is equivalent to a system of two equations

$$\nabla^2\nabla^2\Phi = 0, \ \nabla^2 V = kT.$$

A calculation in every respect parallel to the one that led to the boundary condition (1.7) in this case yields

$$\frac{d}{ds}\left(\frac{\partial\Phi}{\partial y} - i\frac{\partial\Phi}{\partial x}\right) = X + iY + \frac{d}{ds}\left(\frac{\partial V}{\partial y} - i\frac{\partial V}{\partial x}\right), \quad \text{on } C, \tag{3.2}$$

and it follows that the thermo-elastic problem is identical with the first boundary value problem discussed in §§1 and 2.

An assertion of the mathematical equivalence of the two problems does not settle many specific questions regarding the behavior of cylinders when subjected to heat. The implications of the connection of the derivatives of the function Φ with those of the function V as expressed in formula (3.2) were analyzed by B. E. Gatewood in a doctoral dissertation.[10] Gatewood also studied the thermo-elastic problem for cylinders with longitudinal cavities and obtained some interesting results for cylinders composed of different materials. These results, however, are too specialized to be entered upon in this address.

It is obvious that a similar mode of attack can be expected to succeed in a study of thermal stresses in elastic plates made up of several layers of different materials.[11]

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[9] Of course, the equation of compatibility, in this case, assumes a form different from (1.4).

[10] *Thermal stresses in long cylindrical bodies*, dissertation, Wisconsin, 1939, 96 pp. See, also, B. E. Gatewood, this Bulletin, abstract 45-9-314. A paper based on this dissertation appears in Philosophical Magazine, (7), vol. 32 (1941), pp. 282–301.

[11] Some references to recent work on thermal stresses are given at the end of this paper. See [31–42].

4. Deflection of non-isotropic plates. The industrial demands for lighter and stronger materials have given a new impetus to a study of the behavior of various structural members made of non-isotropic materials. In particular, the need of an accurate knowledge of the performance of plates and shells made of plywood is strongly felt, and in this connection some extensions of the method outlined in §2 offer new possibilities. I shall indicate only one such extension which yields fruitful results in a study of small deflections of a thin non-isotropic plate subjected to a load $q(x, y)$ distributed normally over one of the faces of the plate. A very general type of anisotropy is assumed, and it will be supposed that the xy-plane of the coordinate system is the only plane of elastic symmetry.

It can be shown,[12] by utilizing the usual assumptions of the thin plate theory, that the differential equation governing small deflection w has the form:

$$
\begin{aligned}
(4.1)\qquad & b_{11}\frac{\partial^4 w}{\partial x^4} + 3b_{16}\frac{\partial^4 w}{\partial x^3\partial y} + 2(b_{12} + b_{66})\frac{\partial^4 w}{\partial x^2\partial y^2} \\
& \qquad + 3b_{26}\frac{\partial^4 w}{\partial x\partial y^3} + b_{22}\frac{\partial^4 w}{\partial y^4} = \frac{3q(x, y)}{2h^3},
\end{aligned}
$$

where the b_{ij} are the elastic constants and $2h$ is the thickness of the plate.

The equation (4.1) can be written in a symbolic form, with the aid of four linear operators, as

$$D_4 D_3 D_2 D_1 w = \frac{3q}{2h^3},$$

where $D_i = \partial/\partial y - \mu_i \partial/\partial x$, and the constants μ_i are the roots of the characteristic equation

$$(4.2)\qquad b_{22}\mu^4 + 3b_{26}\mu^3 + 2(b_{12} + b_{66})\mu^2 + 3b_{16}\mu + b_{11} = 0.$$

It follows from the fact that the potential energy of any physically realizable state of stress is positive that the roots μ_i of the characteristic equation (4.2) are complex, and since the coefficients b_{ij} are real, they must have the forms:[13]

$$\mu_1 = \alpha + \beta i, \qquad \mu_2 = \gamma + \delta i, \qquad \mu_3 = \bar{\mu}_1, \qquad \mu_4 = \bar{\mu}_2,$$

where α, β, γ, δ are real and $\beta \neq 0$, $\delta \neq 0$.

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[12] See [43, 44]

[13] A proof of this was given by S. N. Lechnitzky [43].

Accordingly, the general solution of equation (4.1), for the case of distinct roots,[14] can be written as

$$w = \sum_{k=1}^{4} F_k(x + \mu_k y) + w_0(x, y), \tag{4.3}$$

where $w_0(x, y)$ is any particular solution of (4.1), and the F_k are arbitrary functions. Since the deflection $w(x, y)$ is real, it follows that the functions $F_k(x+\mu_k y)$ must be conjugate in pairs, so that one can write the solution (4.3) in the form:

$$w = \phi(z_1) + \psi(z_2) + \overline{\phi(z_1)} + \overline{\psi(z_2)} + w_0(x, y), \tag{4.4}$$

where $\phi(z_1)$ and $\psi(z_2)$ are arbitrary functions of the complex variables

$$z_1 = x + \mu_1 y, \qquad z_2 = x + \mu_2 y. \tag{4.5}$$

Let the region corresponding to the middle surface of the unstrained plate be denoted by R and introduce two new systems of cartesian coordinates (x_1, y_1) and (x_2, y_2) related to the coordinates (x, y) by the formulas

$$z_1 = x_1 + iy_1, \qquad z_2 = x_2 + iy_2. \tag{4.6}$$

It follows from inspection of (4.5) and (4.6) that the three systems of coordinates (x, y), (x_1, y_1), and (x_2, y_2) are connected with one another by the equations

$$x_1 = x + \alpha y, \qquad y_1 = \beta y,$$
$$x_2 = x + \gamma y, \qquad y_2 = \delta y.$$

Hence to any point P in the region R there correspond points P_1 and P_2 in some regions R_1 and R_2 of the z_1- and z_2-planes, respectively, and it is obvious that the regions R_1 and R_2 are obtainable from the region R by a homogeneous deformation of the xy-plane. Thus, when the complex variable $z = x + iy$ varies in the region R, the variables z_1 and z_2 vary in the corresponding regions R_1 and R_2. Consequently, the functions $\phi(z_1)$ and $\psi(z_2)$ entering in the general solution (4.4) can be regarded as functions of the ordinary complex variables z_1 and z_2 defined in two different planes.

Ordinarily one is interested in determining the deflection w when the edges of the plate are subjected to a prescribed distribution of moments and shearing forces. Inasmuch as the moments and shearing forces are easily expressible in terms of the derivatives of the deflection function $w(x, y)$, one is led to a set of functional equations (appropriate to various modes of fixing plates at the edges) for the determination of the functions $\phi(z_1)$ and $\psi(z_2)$. In general, the functional equations are of the type

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[14] In case of equal roots ($\mu_1 = \mu_2$, $\mu_3 = \mu_4$) the general solution assumes the form $w = (x+\mu_3 y) \; F_1(x+\mu_1 y) + (x+\mu_1 y) \; F_2(x+\mu_3 y) + F_3(x+\mu_1 y) + F_4(x+\mu_3 y) + w_0(x, y)$ and a similar discussion can be carried out for this special case. If the plate is isotropic the equation (4.1) assumes the form $\nabla^2\nabla^2 w = q/D$, where D is the flexural rigidity, and the general solution can be written as $w = \phi(z) + \bar{z}\psi(z) + \overline{\phi(z)} + z\overline{\psi(z)} + w_0(x, y)$.

$$a_{1j}\phi'(z_1) + a_{2j}\psi'(z_2) + \overline{a_{1j}\phi'(z_1)} + \overline{a_{2j}\psi'(z_2)} = A_j(s), \quad j = 1, 2, \tag{4.7}$$

where the a_{ij} are known constants and $A_j(s)$ are known functions prescribed on the contour C bounding the region R.

The problem of determining the functions $\phi(z_1)$ and $\psi(z_2)$ in this case is naturally more involved than in the corresponding isotropic problem, but the complications are not of a basic sort. One practical mode of solving a system of equations (4.7) is to introduce a pair of suitable mapping functions $z_1 = \omega_1(\zeta)$ and $z_2 = \omega_2(\zeta)$ transforming the regions R_1 and R_2 into the unit circle $|\zeta| \leq 1$ in the auxiliary ζ-plane, and thereafter proceed in the same manner as indicated in §2.

Some interesting results pertaining to the problem of deflection of a clamped non-isotropic plate, subjected to a quite general distribution of normal load, were obtained recently by V. Morkovin in a partially completed doctoral dissertation. By utilizing the method outlined above, Morkovin obtained in closed form the solution of the problem of deflection of a clamped elliptical plate when the loading function $q(x, y)$ is a polynomial in x and y. The important case of a clamped circular plate follows directly from it by specializing the values of the parameters appearing in the solution.

5. **Concluding remarks.** The foregoing account of a somewhat novel use of the theory of functions of a complex variable was concerned mainly with the applications to two-dimensional problems of elasticity. However, the central idea of replacing the differential equation and boundary conditions by an equivalent functional equation is obviously capable of extension to problems in other branches of applied mathematics.

A consideration of the problems of equilibrium of cylindrical bodies, and especially those relating to a study of initially curved and twisted beams, shows that occasionally there are advantages in utilizing similar notions in some classes of three-dimensional problems [48–56].

The importance of the theory of functions of a complex variable in connection with the solution of the boundary value problems involving Laplace's equation is, of course, recognized by every applied mathematician. However, even in the consideration of such problems as those of Neumann and Dirichlet, the engineers appear to be so absorbed in special devices and artifices that a simple formulation of such problems is often overlooked.

In order to set in relief the basic notions that underlie the foregoing formulation of the two-dimensional problems of elasticity, it may not be out of place to recall a simple approach to the problem of Dirichlet.

Let the function $U(x, y)$, continuous in the closed simply-connected region R and harmonic in the interior of R, assume continuous values $f(s)$ on the boundary C of R. Then the boundary condition can be written in the form

$$\phi(z) + \overline{\phi(z)} = 2f(s) \qquad \text{on } C,$$

where $\operatorname{Re}\phi(z) = U(x, y)$. Now, if the region R is mapped conformally on the unit circle $|\zeta| \leq 1$ with the aid of the function $z = \omega(\zeta)$, then the foregoing boundary condition assumes the form

$$\phi_1(\sigma) + \overline{\phi_1(\sigma)} = 2f_1(\theta),$$

where $\sigma = e^{i\theta}$, and $\phi_1(\zeta)$ and $f_1(\theta)$ are the transforms of $\phi(z)$ and $f(s)$. Multiplying by $(1/2\pi i)d\sigma/(\sigma - \zeta)$ and integrating over the contour γ, yields

$$\frac{1}{2\pi i}\int_\gamma \frac{\phi_1(\sigma)}{\sigma - \zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\overline{\phi_1(\sigma)}}{\sigma - \zeta}\,d\sigma = \frac{1}{\pi i}\int_\gamma \frac{f_1(\theta)}{\sigma - \zeta}\,d\sigma,$$

and the evaluation of integrals gives at once

$$\phi_1(\zeta) = \frac{1}{\pi i}\int_\gamma \frac{f_1(\theta)}{\sigma - \zeta}\,d\sigma + \text{const.},$$

which is merely the formula of Schwarz.

The remarkable simplicity of the formulation of the two-dimensional problems of elasticity with the aid of functional equations is likely to produce an impression that the task of obtaining practically useful solutions is a straightforward matter. While it is true that in some cases the solution of rather difficult problems is obtained with remarkable ease and rapidity, in many instances severe calculational difficulties are present. These difficulties are connected mainly with the determination of the mapping function. However, recent advances in the development of practical methods of constructing mapping functions for simply- and multiply-connected domains justify some degree of optimism.

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UNIVERSITY OF WISCONSIN

ON EQUIVALENCE OF CERTAIN TYPES OF SERIES OF ORTHONORMAL FUNCTIONS

G. H. PEEBLES

If one has a set of "properly" independent functions $u_1(x), u_2(x), \cdots, u_n(x), \cdots$ and a function $p(x)$ which on an interval (a, b) is non-negative, integrable and such that $\int_a^b p(x)dx > 0$, one can construct a second set $v_1(x), v_2(x), \cdots, v_n(x), \cdots$ $[v_n(x) = a_{n1}u_1(x) + \cdots + a_{nn}u_n(x)$, where a_{nk} is a constant$]$ whose members satisfy the relations

$$\int_a^b p(x)v_n(x)v_m(x)dx = \delta_{nm}.$$

Associated with such sets of orthonormal functions is the problem of expanding an "arbitrary" function $f(x)$ in a convergent series of the members of a particular set, that is, the problem of determining conditions sufficient for

$$\lim_{n\to\infty} \sum_{k=1}^{n} v_k(x) \int_a^b p(t)f(t)v_k(t)dt = f(x). \tag{1}$$

One approach to the problem lies in showing that expansion (1) is equivalent to another expansion, usually a Fourier series, whose behavior is known.[1] In the case of orthogonal polynomials, where $u_n(x) = x^{n-1}$, the writer was able to show, under conditions not too restrictive, that the expansions of a function in terms of the two sets of orthogonal polynomials corresponding, respectively, to different weight functions converge to the same value or diverge together.[2] It is the purpose of this note to point out that similar results can be obtained for systems of orthonormal functions constructed from a set $[u_n(x)]$ which has the property that the product of any two members

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Presented to the Society, September 7, 1939; received by the editors August 7, 1941. Results presented to the Society, April 14, 1939 under the title *The boundedness of certain systems of orthogonal functions*, are implicit in the brief discussion on boundedness of orthonormal functions which appears at the end of this paper.

[1] Theorems and references on equiconvergence for orthogonal polynomials are given by G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, 1939, chaps. 9 and 13. For equivalence for another type of orthogonal functions, see J. Mercer, *Sturm-Liouville series of normal functions in the theory of integral equations*, Philosophical Transactions of the Royal Society of London, (A), vol. 211 (1912), pp. 111–198; pp. 174–177.

[2] *An equivalence theorem for series of orthogonal polynomials*, Proceedings of the National Academy of Sciences, vol. 3 (1939), pp. 97–104.

$u_n(x)u_m(x)$ is a linear combination of the first $n+m+\beta$ members, where β may depend on n and m, but never exceeds some integer, say, j. A set with this property, for instance, underlies a set of polynomials in two variables orthogonal on an arc of an algebraic curve.[3] If the curve is the hyperbola $y^2=x^2-1$, the set $[u_n(x)]$ is $1,\ x,\ (x^2-1)^{1/2},\ x^2,\ x(x^2-1)^{1/2},\ \cdots$. The system of trigonometric functions $1, \sin x, \cos x, \sin 2x, \cos 2x, \cdots$, which is essentially a set of polynomials orthogonal on a circle, will be used later for the purposes of illustration.

The proof of the equivalence of expansions is usually made to depend on asymptotic formulas for the orthogonal functions.[4] The method used here, however, rests chiefly on the familiar closure formula of Parseval and a fundamental property of orthonormal functions, namely, that $\int_a^b p(x)\,[\phi(x)-\sum_1^n c_k v_k(x)]^2 dx$ is a minimum with respect to the choice of the values of the c's when $c_k=\int_a^b p(x)\phi(x)v_k(x)dx$. In addition to the assumption that $u_n(x)u_m(x)$ is a linear combination of not more than the first $n+m+j$ members, it is convenient to suppose at the outset that the u's and all functions to be introduced are such that Parseval's closure formula holds and the necessary integrals exist whenever the argument so requires.

Let $[v_n(x)]$ and $[w_n(x)]$ be the sets of orthonormal functions constructed from the u's on the interval (a, b) and associated, respectively, with the weight functions $p(x)$ and $q(x)=r(x)p(x)$. Let

$$s_n(x) = \sum_{i=1}^{n} b_i v_i(x) = \sum_{i=1}^{n} v_i(x) \int_a^b p(t)f(t)v_i(t)dt$$

and

$$S_n(x) = \sum_{i=1}^{n} w_i(x) \int_a^b q(t)f(t)w_i(t)dt$$

be the corresponding nth order partial sums of the expansions of a function $f(x)$. The difference between the two sums is a linear combination of the u's, and so may be expressed in terms of the w's:

$$S_n(x) - s_n(x) = \sum_{k=1}^{n} r_{kn} w_k(x). \tag{2}$$

The coefficient

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[3] See Dunham Jackson, *Orthogonal polynomials on a plane curve*, Duke Mathematical Journal, vol. 3 (1937), pp. 228–236; pp. 232–235; *Orthogonal polynomials in three variables*, ibid., vol. 4 (1938), pp. 441–454.

[4] See Footnote 1.

$$r_{kn} = \int_a^b q(x) w_k(x) [S_n(x) - s_n(x)] dx$$

$$= \int_a^b p(x) [r(x) w_k(x)] [f(x) - s_n(x)] dx.$$

Hence, if one sets $B_{ik} = \int_a^b p(x) r(x) w_k(x) v_i(x) dx$, the assumption of the validity of Parseval's formula allows one to write

$$r_{kn} = \sum_{i=n+1}^{\infty} b_i B_{ik}$$

which with Cauchy's inequality yields the relation

$$|r_{kn}| \leqq \left\{ \sum_{i=n+1}^{\infty} b_i^2 \right\}^{1/2} \left\{ \sum_{i=n+1}^{\infty} B_{ik}^2 \right\}^{1/2}.$$

Since $\lim_{n\to\infty} \sum_{n+1}^{\infty} b_i^2 = 0$, it is clear from the last inequality and relation (2) that, if one writes $l_{nk} = \{\sum_{n+1}^{\infty} B_{ik}^2\}^{1/2}$, the existence of $L = \lim_{n\to\infty} \sum_{k=1}^{n} l_{nk}$ is sufficient for

$$\lim_{n\to\infty} [S_n(x) - s_n(x)] = 0$$

for any value of x for which $|w_k(x)|$ is bounded for all values of k.

The preceding result depends on Parseval's formula and not on the rule of combination assumed for the u's. A more serviceable form stating that L surely exists, if the ratio function $r(x)$ belongs to a suitable class of functions, is obtained when the special rule of combination holds.

The integral formula

$$(3) \qquad l_{nk}^2 = \sum_{i=n+1}^{\infty} B_{ik}^2 = \int_a^b p(x) \left[r(x) w_k(x) - \sum_{i=1}^{n} B_{ik} v_i(x) \right]^2 dx$$

provides a means for getting at conditions sufficient for the existence of L. An immediate consequence of (3) is that l_{nk} is bounded for all values of n and k, if $r(x)$ is bounded on (a, b); for, if C is an upper bound of $r(x)$, by Bessel's inequality for orthogonal functions,

$$l_{nk}^2 \leqq \int_a^b p(x) [r(x) w_k(x)]^2 dx = \int_a^b C r(x) p(x) [w_k(x)]^2 dx = C.$$

This result will be needed immediately.

Suppose $C \geqq r(x)$ and $r(x)$ is a linear combination of the u's of order m. Then $r(x) w_k(x)$ is a linear combination of the u's and consequently of the v's, of order not more than $m+k+j$. The integral of formula (3) vanishes for $n \geqq m+k+j$; therefore $l_{nk}=0$ for $k \leqq n-(m+j)$, and L exists since always $\sum_{k=1}^{n} l_{nk} \leqq (m+j)C$.

Suppose that on (a, b) $C \geqq r(x) \geqq c > 0$ and that $r(x) = \lim_{n\to\infty} G_n(x)$, where $G_n(x)$ is a linear combination of the u's with ϵ_n, the maximum or least upper bound of $|r(x)-G_n(x)|$ on (a, b), such that $\sum_1^\infty \epsilon_n$ is convergent. Since $w_k(x)G_{n-k-j}(x)$ is a linear combination of the v's of order n at most, one has for $k \leqq n-j-1$

$$l_{nk} \leqq \left\{ \int_a^b p(x)\left[w_k(x)r(x) - w_k(x)G_{n-k-j}(x)\right]^2 dx \right\}^{1/2}$$

$$\leqq \epsilon_{n-k-j} \left\{ \int_a^b p(x)\left[w_k(x)\right]^2 dx \right\}^{1/2}$$

$$\leqq \epsilon_{n-k-j} \left\{ c^{-1} \int_a^b p(x)r(x)\left[w_k(x)\right]^2 dx \right\}^{1/2} = c^{-1/2}\epsilon_{n-k-j}.$$

For the remaining $j+1$ values of k $(n-j, n-j+1, \cdots, n)$, $l_{nk} \leqq C^{1/2}$. Hence L exists.

The next assumption concerning the nature of $r(x)$ is very specific, but the result, when combined with that obtained in the case where $r(x)$ is a linear combination of the u's, may increase the generality of the results as an example will show later. On (a, b) suppose, unlike the last case, the lower bound of $r(x)$ is zero, but there exists an mth order linear combination of the u's $h_m(x)$ such that $C \geqq r(x) \geqq [h_m(x)]^2$ and such that $r(x)/h_m(x)$, like $r(x)$ in the preceding case, can be approximated by a set of G's. Then for $k \leqq n-m-2j-1$

$$l_{nk} = \left\{ \int_a^b p(x)\left[h_m(x)w_k(x)\right]^2\left[r(x)/h_m(x) - G_{n-m-k-2j}(x)\right]^2 dx \right\}^{1/2}$$

$$\leqq \epsilon_{n-m-k-2j} \left\{ \int_a^b p(x)r(x)\left[w_k(x)\right]^2 dx \right\}^{1/2} = \epsilon_{n-m-k-2j}.$$

As before, if $\sum_1^\infty \epsilon_n$ is convergent, L exists.

A review of the conditions imposed on $r(x)$ discloses that in no case is $r(x)$ allowed to become infinite and that only under very restrictive hypotheses is $r(x)$ allowed to vanish. If $r(x)$ were discontinuous, one would expect, in general, that approximating sums of the sort required would not exist. An example in trigonometric functions on the interval $(-\pi, \pi)$ will indicate briefly how and when the results just obtained may be used to circumvent difficulties presented by such exceptional points.

In the case of trigonometric functions, theorems due to Dunham Jackson provide the desired information about the approximating sums.[5] For present purposes it suffices to know that a function of period 2π which possesses everywhere a kth order derivative satisfying a Lipschitz condition of order α can be approximated by a trigonometric sum of order n so closely that the absolute value of the error nowhere exceeds $K/n^{k+\alpha}$, K a constant. Hence, if $r(x) \geqq c > 0$ and has a derivative which satisfies a Lipschitz condition of order α and if $r(x)$ and $f(x)$ are of classes $L_p(-\pi, \pi)$ and $L_p^2(-\pi, \pi)$ so that Parseval's formula holds, one has equivalence between the expansions of $f(x)$ in the two sets of trigonometric sums corresponding to $p(x)$ and $r(x)p(x)$ wherever the sums associated with $r(x)p(x)$ are bounded. Suppose, however, that $r(x)$, a function of period 2π, is bounded, stays away from zero and has a second derivative everywhere in $(-\pi, \pi)$ except at $x=0$. At the origin, suppose $r(x)$ tends to vanish on the left and to become infinite on the right, but in such a way that $r(x) \geqq \sin^2 x$ everywhere and $\lim_{x\to 0+} \sin x\, p(x)$ exists. The function described has no particular merit except that it serves to illustrate how exceptional points may be handled. Let it again be assumed that $f(x)$ and $r(x)$ are of classes $L_p(-\pi, \pi)$ and $L_p^2(-\pi, \pi)$. It follows from the first case considered, where $r(x)$ is a linear combination of the u's, that L corresponding to the weight functions $\sin^{10} x\, r(x)p(x)$ and $r(x)p(x)$ exists. On the other hand $\sin^{10} x\, r(x) \geqq (\sin^6 x)^2$ and $\sin^{10} x\, r(x)/\sin^6 x$, on removing the discontinuity at the origin, has a second derivative everywhere and so can be approximated by an nth order trigonometric sum with the absolute value of the error not exceeding K/n^2. Hence, by virtue of the third case, L corresponding to the weight functions $\sin^{10} x\, r(x)p(x)$ and $p(x)$ also exists. Therefore the expansions of $f(x)$ corresponding to $r(x)p(x)$ and $p(x)$ are each equivalent to the expansion corresponding to $\sin^{10} x\, r(x)p(x)$, wherever the sums orthonormal with respect to the last weight function are bounded. It follows that the expansions associated with $r(x)p(x)$ and $p(x)$ are equivalent, wherever the sums orthonormal with respect to $\sin^{10} x\, r(x)p(x)$ are bounded.

The results on equivalence given here depend, in part, on the boundedness of the functions orthonormal with respect to $r(x)p(x)$. There is at hand in the foregoing discussion a result which may be useful in this connection. Referring to the definition of B_{ik}, one sees that formally

$$r(x)w_k(x) = \sum_{i=k-j}^{\infty} B_{ik}v_i(x) \tag{4}$$

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[5] Dunham Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. 11, 1930, chap. 1.

and

$$v_i(x) = \sum_{k=1}^{i} B_{ik} w_k(x). \tag{5}$$

Since $B_{ik} \leqq \{\sum_{j=1}^{\infty} B_{jk}^2\}^{1/2} = l_{i-1,k}$, in all the cases of the ratio function $r(x)$ considered, the right-hand members of (4) and (5) are absolutely convergent and bounded, wherever, respectively, the v's and w's are bounded. Hence, if conditions are such that the right-hand member of (4) converges to the value of the left-hand member and if a set of points is known for which the v's are bounded, then the w's are bounded on the same set except where $r(x) = 0$. Similarly, boundedness of the w's leads through (5) to results on the boundedness of the v's.

UNIVERSITY OF MINNESOTA

A MAPPING CHARACTERIZATION OF PEANO SPACES

O. G. HARROLD, JR.

The Hahn-Mazurkiewicz theorem states that any Peano space (compact, connected, locally connected, metric space) is a continuous image of the interval $0 \leqq t \leqq 1$, and conversely. Clearly, the mapping function is not uniquely determined. If the Peano space $\mathfrak{M}$ has special topological properties, the mapping may be selected in a simpler fashion than might be expected generally. On the other hand, special properties of $\mathfrak{M}$ may impose certain necessary restrictions on the mapping. For example, if $\mathfrak{M}$ is a regular continuum in the sense of Menger, then, by a theorem due to Nöbeling,[1] there is a continuous mapping f of the circle[2] onto $\mathfrak{M}$ such that each point of finite order is covered by the mapping a number of times which does not exceed the order of the point. That is, if $o(x)$ is the order of the point x and $m(x)$ is the number of points in $f^{-1}(x)$, then $m(x) \leqq o(x)$ for each point for which $o(x)$ is finite. On the other hand, if $\mathfrak{M}$ is of dimension n, then *any* continuous mapping of a 1-dimensional compact set onto $\mathfrak{M}$,

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Presented to the Society, September 5, 1941; received by the editors October 21, 1941.

[1] G. Nobeling, *Reguläre Kurven als Bilder der Kreislinie*, Fundamenta Mathematicae, vol. 20 (1933), pp. 30–46.

[2] The interval may be used instead of the circle if we make $f(0) = f(1)$ and count inverses on $0 \leqq t < 1$.

in particular, of an interval or circle, is such that there is a dense set of points in the n-dimensional part of $\mathcal{M}$ each of which has at least n inverse points in the original set.[3]

Denote the set of local separating points of the Peano space $\mathcal{M}$ by $\mathcal{L}$.[4] If $\mathcal{M}\subset\overline{\mathcal{M}-\mathcal{L}}$, that is, if $\mathcal{M}$ contains no free arcs,[5] there is a strongly irreducible mapping of the interval $\mathfrak{J}$ or circle $\mathcal{C}$ onto $\mathcal{M}$.[6] That is, for such spaces there exist continuous mappings of $\mathfrak{J}$ or $\mathcal{C}$ onto $\mathcal{M}$ such that no proper closed subset maps onto the whole space. Thus if $\mathcal{M}$ is an n-dimensional sphere and f is a strongly irreducible mapping of $\mathfrak{J}$ onto $\mathcal{M}$, there is a dense set of points each covered at least $n+1$ times and also a dense set of points each covered just once.

In addition to the symbols f, $\mathcal{M}$, $\mathcal{L}$, $m(x)$ and $o(x)$ used above, the following notations will be observed. Let ψ denote the aggregate of points $x\in\mathcal{M}$ lying in an open free arc of $\mathcal{M}-a-b$. If for a (continuous) mapping of $\mathfrak{J}$ into a subset of $\mathcal{M}$, $y\in\psi$ implies $m(y)\leq 2$, the mapping will be said to be of type $\mathfrak{M}$.

THEOREM 1. *Let a and b be points of the Peano space $\mathcal{X}$. There is a continuous mapping of the interval $0\leq t\leq 1$ onto $\mathcal{X}$ of type $\mathfrak{M}$ such that $f(0)=a$, $f(1)=b$.*

The theorem asserts, essentially, that there is a mapping of $\mathfrak{J}$ onto $\mathcal{X}$ such that every free arc is swept through at most twice.

The following lemmas will be useful in the proof of Theorem 1.

LEMMA 1.1. *If D is a subcontinuum of the dendrite D^0, to $\epsilon>0$ there is a finite collection $D^1, D^2, \cdots, D^n$ of dendrites in D^0 such that $D=D^1\subset D^2\subset\cdots\subset D^n=D^0$ and each component of $D^{i+1}-D^i$ has a diameter less than or equal to ϵ.*

Let ρ be a convex metric[7] on D^0. Let $d=$glb of numbers r such that

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[3] If the original set is locally euclidean, the phrase *at least n* may be replaced by *at least $n+1$*. See W. Hurewicz, *Über dimensionserhohende stetige Abbildungen*, Journal für die reine und angewandte Mathematik, vol. 169 (1933), pp. 71–78.

[4] The point p is called a local separating point of M provided that to every neighborhood U of p there is some pair of points of the component of U containing p which is separated in $U-p$.

[5] The set A is called a free arc of M provided A is an arc and the interior of A is open in M. An *open* free arc is an open subset of M which is homeomorphic to $0<x<1$. A point is said to lie in an open free arc provided there is a neighborhood of the point in M which is an open free arc. It is to be noted that if M is an arc, neither end point lies in an open free arc.

[6] O. G. Harrold, Jr., *A note on strongly irreducible maps of an interval*, Duke Mathematical Journal, vol. 6 (1940), pp. 750–752.

[7] The metric ρ is called *convex* after Menger provided that to each pair of distinct points x and y in M there is a point of $M-x-y$ such that $\rho(x, z)+\rho(z, y)=\rho(x, y)$.

$D^e \subset S(D, r)$. Each of the sets $D^i = S[D, (i-1)e]$, $i = 1, 2, \cdots, n$ is a subcontinuum of D^e. The sets D^i satisfy our requirements, where n is the smallest integer such that $(n-1) \geqq d$.

LEMMA 1.2. *Let* $\mathcal{X}$ *be a Peano space. There is a sequence* (T_i) *of dendritic graphs in* $\mathcal{X}$ *such that* (a) $\lim T_i = \mathcal{X}$, (b) $T_{i+1} \supset T_i$, *and* (c) *each component of* $T_{i+1} - T_i$ *has a diameter less than or equal to* $1/2^{i+1}$.

That a sequence of dendrites exists in $\mathcal{X}$ satisfying (a) and (b) is well known. Application of Lemma 1.1 to the successive terms of this sequence gives the desired result.

Proof of Theorem 1. The theorem is true for a connected dendritic graph with n end points, as a simple induction shows. Suppose, temporarily, that neither a nor b lies in an open free arc. Let T_1 be a dendritic graph in $\mathcal{X}$ containing a and b:[8] Let f_1 denote a continuous mapping of type $\mathfrak{M}$ of $\mathfrak{J}$ onto T_1 with $f_1(0) = a$, $f_1(1) = b$. Let (T_i) be a sequence of dendritic graphs satisfying Lemma 1.2. Since T_2 is a graph, $T_2 - T_1$ has but a finite number of components which may be denoted by $C_1^1, C_2^1, \cdots, C_{n_1}^1$. Each $\overline{C}_i^1 \cdot T_1$ is a point c_i. Let $x_i \in f_1^{-1}(c_i)$. By a rearrangement of notation it may be supposed that $0 \leqq x_1 < x_2 < \cdots < x_{p_1} \leqq 1$, $p_1 \leqq n_1$, where each x_i corresponds to a distinct c_i. Set $d_1 = \min |x_i - x_j|$, $i \neq j$, $|x_i|$, $x_i \neq 0$, $|1 - x_i|$, $x_i \neq 1$. To $\epsilon = 1/2$ there is a $d_2 > 0$ such that $|x - y| < d_2$ implies $\rho[f_1(x), f_1(y)] < \epsilon/2 = 1/2^2$, where ρ denotes the metric of $\mathcal{X}$. Put $W = S(x_1 + x_2 + \cdots + x_{p_1}, d/3)$, where $d = \min(d_1, d_2)$. Let J_i be the component of W containing x_i. Let $I_1, I_2, \cdots, I_{p_1+1}$ be the intervals on $\mathfrak{J}$ complementary to W, where I_1 becomes degenerate if $x_1 = 0$ and I_{p_1+1} degenerate if $x_{p_1} = 1$. The interval $\mathfrak{J}$ is now subdivided into the intervals (in order) $I_1, \mathfrak{J}_1, I_2, \mathfrak{J}_2, \cdots, I_{p_1+1}$. Let t denote the piecewise linear map obtained by sending I_1 onto $(0x_1)$ with order preserved, I_2 onto $(x_1x_2), \cdots, I_{p_1+1}$ onto $(x_{p_1}1)$. For $x \in \sum I_i$, put $f_2(x) = f_1[t(x)]$. On $\mathfrak{J}_i$ define a map g_i of the desired type so that $g_i(\mathfrak{J}_i) = D_i$. where D_i is the enclosure of all components C_i^1 having c_i as a limit point. The set D_i is a dendrite of diameter less than or equal to $1/2^2$. The map g_i may be so selected that for the end points of J_i, $g_i = f_2$.

See K. Menger, *Untersuchungen uber allgemeine Metrik*, Mathematische Annalen, vol. 100 (1928), pp. 81 ff. For the existence of the metric assumed here, see C. Kuratowski and Whyburn, *Sur les elements cycliques et leurs applications*, Fundamenta Mathematicae, vol. 16 (1930), pp. 305–331.

[8] T_1 could be taken to be an arc but in order that the discussion to follow be general it is assumed only to be a connected linear graph containing no simple closed curve.

The definition of f_2 is now completed: if $x\in\mathfrak{J}_i$, $f_2(x)=g_i(x)$. Clearly, $f_2(\mathfrak{J})=T_2$ and f_2 is continuous. For $x\in\sum I_i$, $|x-t(x)|<d$, hence $\rho[f_1(x), f_2(x)]<1/2^2$. If $x\in\mathfrak{J}_i$, $\rho[f_1(x); f_1(x_i)]<1/2^2$, and since the diameter of D_i is less than or equal to $1/2^2$, $\rho[f_2(x), f_2(t^{-1}(x_i))]<1/2^2$. But $f_2(t^{-1}(x_i))=f_1(x_i)$, hence by the triangle inequality $\rho[f_1(x), f_2(x)]$ $<1/2$. Thus if σ denotes the usual metric of the function space $\mathfrak{X}^{\mathfrak{J}}$, $\sigma(f_1, f_2)\leqq 1/2$.[9]

To show that f_2 is of type $\mathfrak{M}$ consider a point $y\in\psi\cdot T_2$. It is to be shown that $f_2^{-1}(y)$ contains at most two points, that is, $m(y, f_2)\leqq 2$. If $y\in T_2-T_1$, y lies in an unique D_i, hence $m(y,f_2)\leqq 2$. If $y\in T_1-\sum D_i$, $f_2^{-1}(y)=t^{-1}f_1^{-1}(y)$. Since f_1 has the desired property and t is 1-1 on the set composed of the interiors of the intervals I_i, $m(y, f_2)\leqq 2$. Consider the remaining case $y=c_i$. Here $m(y, f_1)=1$, for suppose, on the contrary, $f_1^{-1}(y)\supset q_1+q_2$. Since $y\in\psi$, $y\neq a, b$, hence q_1 and q_2 divide $\mathfrak{J}$ into three subintervals A, B and C. But each of these subintervals has as an image under f_1 a nondegenerate continuum containing y. Hence points of T_1 near y have three inverses on $\mathfrak{J}$, which denies the property of f_1. Since $y\in\psi$, only one component $C_i^{1\prime}=D_i$ of T_2-T_1 can have y as a limit point. The mapping g_i has the desired property, thus $f_2^{-1}(y)=g_i^{-1}(y)+f_1^{-1}[t^{-1}(y)]$ is precisely a pair of points. Hence for $y\in\psi\cdot T_2$, $m(y, f_2)\leqq 2$, and f_2 is of type $\mathfrak{M}$.

The general inductive hypothesis is now clear.

To the dendrite T_n there is a continuous mapping f_n, $f_n(\mathfrak{J})=T_n$, of type $\mathfrak{M}$ and such that $f_n(0)=a$, $f_n(1)=b$. Further, $\sigma(f_{i-1}, f_i)\leqq 1/2^{i-1}$, $i=2, 3, \cdots, n$. The construction of f_{n+1} from f_n is accomplished precisely as above.

There is thus determined a sequence of points (f_n) of the space $\mathfrak{X}^{\mathfrak{J}}$ such that to $\epsilon>0$ there is an index N such that for $i, j>N$, $\sigma(f_i, f_j)<\epsilon$. The space $\mathfrak{X}^{\mathfrak{J}}$ being complete, let $\lim f_n=f$. Clearly, $f(\mathfrak{J})=\mathfrak{X}$. To complete the proof of Theorem 1 it will suffice to show that $y\in\psi$ implies $f^{-1}(y)$ has at most two components. For, if we grant this, the factorization $f=k[h(x)]$, where h is the monotone transformation obtained by shrinking the components of $f^{-1}(y)$ into points and k is the corresponding light transformation, yields $k(\mathfrak{J})=\mathfrak{X}$, $m(y\in\psi, k)\leqq 2$, hence k is of type $\mathfrak{M}$.[10]

Suppose, on the contrary, $y\in\psi$ and X_1, X_2 and X_3 are three components of $f^{-1}(y)$. Since $y\in\psi$, $y\neq a$, b, hence $\mathfrak{J}-\sum X_i$ has precisely 4 components R_i, $i=1, 2, 3$ and 4. Suppose w_i and t_i are the left and right end points of X_i, respectively. (If X_i is a point, $w_i=t_i$.) Let the notation be arranged so that $w_i = \bar{R}_i \cdot X_i$, $i=1$, 2 and 3. The points w_i and w_j, $i \neq j$, are, of course, distinct. Let A_i, $i=1$, 2 and 3 be a subinterval of $\bar{R}_i$ containing w_i such that $f(A_i) \subset U$, where U is any fixed open free arc of $\mathcal{X}$ containing y. Let B_{i+1} $i=1, 2$ and 3 be a proper subinterval of $R_{i+1} - A_{i+1}$ containing t_i such that $f(B_i) \subset U$, where $A_4 = 0$ by definition. Since X_i is a component of $f^{-1}(y)$, the sets of $f(A_i), f(B_i)$ are nondegenerate subarcs of U with at least the point y common. Some point y^0 of $U-y$ must be covered by at least three of these six sets. Denote three of the corresponding sets A_i (B_i), $i=1$, 2 and 3 by C_1, C_2 and C_3. One end point of C_i, say a_i, maps into y. As C_i is traversed from a_i let b_i be the first point in $f^{-1}(y^0)$. Let G denote any subarc of $U-y-y^0$ which lies between y and y^0. Set $d = \rho(G, y+y^0) > 0$. Then for n large enough $\sigma(f_n, f) < d/3$. But $f_n(a_i b_i)$ is a connected subset of U which contains a point from each component of $U-G$, hence $f_n(a_i b_i) \supset G$. This denies that f_n is of type $\mathfrak{M}$. The proof of Theorem 1 under the special restriction that neither a nor b lies in a free arc has been completed.

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[9] If $f, g\in\mathfrak{X}^{\mathfrak{J}}$, $\sigma(f, g)=\text{lub}\ \rho[f(x), g(x)]$, $x\in\mathfrak{J}$.

[10] This is an application of a factor theorem for continuous transformations due to Eilenberg and Whyburn.

To remove the restriction suppose first that only a lies in a free arc of $\mathcal{X}$. Imagine that $\mathcal{X}$ is situated in the Hilbert cube and let $\mathcal{A}$ be an arc which is joined onto $\mathcal{X}$ at a and has no other point in $\mathcal{X}$. Let a^1 be the other end point of $\mathcal{A}$. Construct a mapping as above with $f(0) = a^1$, $f(1) = b$, $f(\mathfrak{J}) = \mathcal{X} + \mathcal{A}$. Since neither a^1 nor b lies in an open free arc of $\mathcal{X}+\mathcal{A}$, such a mapping will exist. Let x^1 be the least x for which $f(x) = a$. Then the mapping f on the interval $x^1 \leqq t \leqq 1$ satisfies our requirements. A similar modification suffices to treat the case in which b is an open free arc and also the case in which both a and b have this property.

Set $\mathcal{W} = \mathcal{X} - \bar{\psi}$. The set $\mathcal{W}$ is open. Put $\mathfrak{Q} = \psi + (\mathcal{W} \cdot \mathcal{N})$, where $\mathcal{N}$ is the set of nonlocal separating points of $\mathcal{X}$. We come now to the principal result.

THEOREM 2. *Let a and b be points of the metric space $\mathcal{X}$. In order that $\mathcal{X}$ be a Peano space it is necessary and sufficient that for any countable subset $\mathfrak{P}$ of $\mathfrak{Q}-a-b$ there be a continuous mapping f of $0 \leqq t \leqq 1$ onto $\mathcal{X}$ such that $f(0) = a$, $f(1) = b$ and $y \in \mathfrak{P}$ implies $m(y) \leqq 2$.*

PROOF. The sufficiency is clear. If $\psi = 0$, the result is known, in fact, in this case a mapping of the described type exists such that for $y \in \mathfrak{P}$, $m(y) = 1$.[8] It is supposed, then, that $\psi \neq 0$. By application of Theorem 1, there is a mapping of type $\mathfrak{M}$ of $\mathfrak{J}$ onto $\mathcal{X}$ with $f(0) = a$, $f(1) = b$. The desired map will be obtained by a modification of f.

To facilitate the discussion it will be supposed that $\mathcal{X}$ has an S-metric, that is, a metric ρ such that for each $r > 0$ and $x \in \mathcal{X}$, $\bar{S}(x, r)$ is a locally connected continuum.[11] Let $A = a_1 + a_2 + \cdots = \mathcal{P} \cdot \mathcal{W} = \mathcal{P} \cdot \mathcal{W} \cdot \mathcal{N}$. Set $d_1 = \rho(a_1, \mathcal{X} - \mathcal{W})$. Choose a number e_1 such that $d_1/2 < e_1 < d_1$ and $A \cdot \{\overline{S}(a_1, e_1) - S(a_1, e_1)\} = 0$. This is possible since A is countable. Let a_{k_2} be the first point of A in $\mathcal{W} - \overline{S}_1$, where $S_1 = S(a_1, e_1)$. Let $d_2 = \rho[a_{k_2}, (\mathcal{X} - \mathcal{W}) + S_1]$. The number e_2 is chosen so that $d_2/2 < e_2 < d_2$ and $A \cdot \{\overline{S}(a_{k_2}, e_2) - S(a_{k_2}, e_2)\} = 0$. Continuing in this way a sequence of spheres (S_i) is determined such that (a) $\overline{S}_i$ is a Peano space, (b) $\overline{S}_i \cdot \overline{S}_j = 0$, $i \neq j$, (c) $\delta(S_i) \rightarrow 0$, (d) $\overline{\sum S_i} = \overline{\mathcal{W}}$ and (e) $A \cdot (\overline{S}_i - S_i) = 0$.

Set $V_i = f^{-1}(S_i)$. Let V_{ij}, $j = 1, 2, \cdots$ be the components of V_i. Let V_{i1} be a component of V_i such that $V_{i1} f^{-1}(a_{k_i}) \neq 0$. Every point of $\overline{S}_i$ is either a nonlocal separating point of $\overline{S}_i$ or a limit point of such points. This is clear if $x \in S_i$, for $S_i \subset \mathcal{W}$. If $x \in \overline{S}_i - S_i$, x is a limit point of points of S_i and hence a limit point of nonlocal separating points of $\overline{S}_i$. Thus, having shown that $\overline{S}_i$ is a Peano space with no free arcs, there is a strongly irreducible mapping, $f_{i1}(V_{i1}) = \overline{S}_i$, such that $f = f_{i1}$ on $\overline{V}_{i1} - V_{i1}$ and $y \in \mathcal{P}\ \overline{S}_i$ implies $f_{i1}^{-1}(y)$ is a single point.

On V_{ij}, $j > 1$, two cases are distinguished according as f maps the end points of V_{ij} into the same point or not. If f carries the end points of V_{ij} into x, define $f_{ij} = x$ on $\overline{V}_{ij}$. If f carries the end points of V_{ij} into distinct points x and y, proceed as follows. The set $\overline{S}_i - A$ is a connected and locally connected G_δ set[12] in a complete space, hence there is an arc $R_{ij} \subset \overline{S}_i - A$ which joins x and y.[13] On $\overline{V}_{ij}$ define f_{ij} to be a homeomorphism into R_{ij} such that f_{ij} agrees with f on $\overline{V}_{ij} - V_{ij}$.

The new mapping g will now be defined. On $\mathfrak{J} - \sum V_{ij}$, set $g(x) = f(x)$. On V_{ij}, set $g(x) = f_{ij}(x)$. Since f agrees with g on the end points of V_{ij} and each f_{ij} is continuous, g is continuous (we use here the condition (c) on the spheres (S_i)). Clearly, $g(\mathfrak{J}) = \mathcal{X}$. If $y \in \mathcal{P} \cdot \psi$, $m(y, f) = m(y, g) \leqq 2$, by virtue of the fact that f is of type $\mathfrak{M}$. If $y \in \mathcal{P} \cdot \mathcal{W}$, y lies in an unique S_i and $g^{-1}(y) = f_{i1}^{-1}(y)$, hence $m(y, g) = 1$.

NORTHWESTERN UNIVERSITY

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[11] J. L. Kelley, *A metric connected with property S*, American Journal of Mathematics, vol. 61 (1939), pp. 764–768.

[12] The complement of a countable set of nonlocal separating points in a Peano space is connected and locally connected, see G. T. Whyburn, *Semi-closed sets and collections*, Duke Mathematical Journal, vol. 2 (1936), pp. 685–690.

[13] This is the well known Moore-Menger generalization of the arcwise connectivity theorem for regions in a Peano space.

GENERALIZATIONS OF THE BERNOULLI POLYNOMIALS AND NUMBERS AND CORRESPONDING SUMMATION FORMULAS

TOMLINSON FORT

The Bernoulli polynomials and numbers have been generalized by Nörlund[1] to the Bernoulli polynomials and numbers of higher order. The Bernoulli numbers have been generalized by Vandiver. Analogous polynomials and sets of numbers have been defined from time to time, witness the Euler polynomials and numbers and the so-called Bernoulli polynomials of the second kind.[2]

In the present paper a generalization is made which includes all the above and many other interesting classes of polynomials and corresponding sets of numbers. As a matter of fact the definition of new classes of polynomials by the processes of this paper is a simple matter. In this connection particular attention is called to 2, (d), (h), (i), (j), (k), (l).

An important part of the paper is the development of a whole category of summation formulas related to the studied polynomials as the classical Euler-Maclaurin[3] formula is related to the Bernoulli polynomials or as Taylor's formula is related to $(x-a)^n$.

It will be observed that the work could be varied in detail resulting in closely related polynomials and numbers to those which are obtained. The particular procedure adopted is chosen so as to generalize the Bernoulli polynomials and numbers as now usually defined.[4]

1. **Definition of the polynomials and numbers.** Let us be given two linear operators P and Q with their inverses P^{-1} and Q^{-1}. We shall assume that P reduces the degree of any polynomial by 1 and that Q reduces the degree of any polynomial by $k \geqq 0$, that P operating on a constant gives zero and that Q operating on any polynomial of lesser degree than k gives zero. We assume that P, P^{-1}, Q, Q^{-1} each, where

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Presented to the Society, December 29, 1941; received by the editors of the Transactions of this Society October 1, 1941; accepted by them, and later transferred to this Bulletin.

[1] N. E. Nörlund, *Differenzenrechnung*, p. 119. For other generalizations see H. S. Vandiver, Proceedings of the National Academy of Sciences, vol. 23, p. 555. See also Leonard Carlitz, this Bulletin, abstract 47-7-296, also abstract 47-9-358.

[2] See C. Jordan, *Calculus of Finite Differences*, p. 265.

[3] For usage of the name Euler-Maclaurin see footnote of a paper by the author this Bulletin, vol. 45 (1939), p. 748. Usage in the present instance is that of Nörlund, loc. cit., p. 29.

[4] See, for example, Nörlund, loc. cit., p. 17.

applicable, gives a unique result except that it is permitted that the result of operating with P^{-1} lack in uniqueness by an arbitrary additive constant and by Q^{-1} by an additive arbitrary polynomial of degree less than k.

We assume, moreover, that we are given a set of polynomials $f_n(x)$ where $f_n(x)$ is of degree n, such that

$$f_0(x) \equiv 1, \tag{1}$$

$$Pf_n(x) = nf_{n-1}(x), \tag{2}$$

$$\begin{aligned} &Q^{-1}n(n-1)\cdots(n-k+1)f_{n-k}(x) \\ &= {}_nL_0 f_n(x) + n\,{}_nL_1 f_{n-1}(x) + \frac{n(n-1)}{2}\,{}_nL_2 f_{n-2}(x) + \cdots \\ &\quad + C_{n,k}\,{}_nL_{n-k} f_k(x) + c_1 x^{k-1} + c_2 x^{k-2} + \cdots + c_k, \quad 0 \leqq k \leqq n, \end{aligned} \tag{3}$$

where ${}_nL_0, \cdots, {}_nL_{n-k}$ are independent of x but uniquely determined. The c's may be determined constants or may be arbitrary depending upon the nature of the operator Q^{-1}.

From (3)

$$\begin{aligned} &\frac{1}{n} PQ^{-1}n(n-1)\cdots(n-k+1)f_{n-k}(x) \\ &= {}_nL_0 f_{n-1}(x) + (n-1)\,{}_nL_1 f_{n-2}(x) + \frac{(n-1)(n-2)}{2!}\,{}_nL_2 f_{n-3}(x) \\ &\quad + \cdots + C_{n-1,k}\,{}_nL_{n-k} f_{k-1}(x) \\ &\quad + [\text{terms of degree less than } k-1]. \end{aligned} \tag{4}$$

However,

$$\begin{aligned} &\frac{1}{n} Q^{-1}Pn(n-1)\cdots(n-k+1)f_{n-k}(x) \\ &= Q^{-1}(n-1)(n-2)\cdots(n-k)f_{n-k-1}(x) \\ &= {}_{n-1}L_0 f_{n-1}(x) + (n-1)\,{}_{n-1}L_1 f_{n-2}(x) \\ &\quad + \frac{(n-1)(n-2)}{2!}\,{}_{n-1}L_2 f_{n-3}(x) + \cdots + C_{n-1,k}\,{}_{n-1}L_{n-k} f_{k-1}(x) \\ &\quad + [\text{terms of degree less than } k]. \end{aligned} \tag{5}$$

We next assume

$$PQ^{-1}f_{n-k}(x) \equiv Q^{-1}Pf_{n-k}(x) + [\text{terms of degree less than } k].$$

Under this assumption

$$
\begin{aligned}
&{}_nL_0 f_{n-1}(x) + (n-1)\,{}_nL_1 f_{n-2}(x) + \frac{(n-1)(n-2)}{2!}\,{}_nL_2 f_{n-3}(x) \\
&\quad + \cdots + C_{n-1,k}\,{}_nL_{n-k} f_{k-1}(x) \\
&= {}_{n-1}L_0 f_{n-1}(x) + (n-1)\,{}_{n-1}L_1 f_{n-2}(x) \\
&\quad + \frac{(n-1)(n-2)}{2!}\,{}_{n-1}L_2 f_{n-3}(x) + \cdots + C_{n-1,k}\,{}_{n-1}L_{n-k} f_{k-1}(x) \\
&\quad + [\text{terms of degree less than } k].
\end{aligned}
\tag{6}
$$

Equating coefficients we find

$$
{}_nL_0 = {}_{n-1}L_0,\ {}_nL_1 = {}_{n-1}L_1,\ {}_nL_2 = {}_{n-1}L_2,\ \cdots .
$$

In other words, the L's are independent of n. Henceforth we write simply $L_0, L_1, L_2, \cdots$. We let

$$
\begin{aligned}
F_n(x) &= L_0 f_n(x) + nL_1 f_{n-1}(x) \\
&\quad + \frac{n(n-1)}{2!} L_2 f_{n-2}(x) + \cdots + L_n f_0(x), \qquad n \geqq 0.
\end{aligned}
\tag{7}
$$

By (3)

$$
QF_n(x) = n(n-1)\cdots(n-k+1) f_{n-k}(x), \qquad n \geqq k. \tag{8}
$$

For reasons of symmetry and consequent simplification of the sequel we do not take the L_n's just defined as our fundamental sequence of numbers but the set g_n determined from the following equations $f_n(g) = L_n$ where subscripts are applied to g rather than exponents in the expansion of $f_n(g)$. Such a determination is always possible and unique. We then write

$$
\begin{aligned}
F_n(x) &= f_0(g) f_n(x) + n f_1(g) f_{n-1}(x) \\
&\quad + \frac{n(n-1)}{2!} f_2(g) f_{n-2}(x) + \cdots + f_n(g) f_0(x), \qquad n \geqq 0.
\end{aligned}
\tag{9}
$$

We choose to write this $f_n(x+g)$ of which more will be said later. We then have

$$
F_n(x) = f_n(x+g), \tag{10}
$$

$$
PF_n(x) = nF_{n-1}(x), \qquad n \geqq 1. \tag{11}
$$

The polynomials $F_n(x)$ are the polynomials[3] in which we are interested and the numbers g_n constitute the corresponding sequence of numbers.

---

[3] If we replace (2) by $Pf_n(x) = n(n-1)\cdots(n-k+1)f_{n-k}(x)$ and add the requirement that the result of operating with P on any polynomial of degree less than k is zero, we are led to sequences of polynomials each of degree differing from that of the previous by k, a somewhat more general situation than that treated in the text.

2. **Special cases.**[6] We now consider these special cases.

(a) Bernoulli polynomials and numbers:

$$P = \frac{d}{dx}, \qquad Q = \Delta, \qquad f_n(x) = x^n.$$

(b) Bernoulli polynomials and numbers of the second kind:

$$P = \underset{\omega}{\Delta}, \quad Q = \frac{d}{dx}, \quad f_n(x) = x^{(n)} = x(x - \omega) \cdots (x - (n - 1)\omega).$$

(c) Bernoulli polynomials and numbers of higher order as defined by Nörlund:

$$P = \frac{d}{dx}, \qquad Q = \underset{\omega_1 \cdots \omega_k}{\Delta^k}, \qquad f_n(x) = x^n, \qquad k > 1.$$

(d) Bernoulli polynomials and numbers of the second kind of higher order:[7]

$$P = \underset{\omega}{\Delta}, \qquad Q = \frac{d^k}{dx^k}, \qquad f_n(x) = x^{(n)}, \qquad k > 1.$$

(e) Bernoulli polynomials with Bernoulli numbers of higher order as defined by Vandiver:

$$P = \frac{d}{dx}, \qquad Q = \underset{\omega_1, \cdots, \omega_k}{\Delta^k}, \qquad f_n(x) = (x + l)^n.$$

Vandiver's numbers are the L_n's of the text rather than the g_n's.

(f) Euler polynomials:

$$P = \frac{d}{dx}, \qquad Q = M, \qquad f(x) = x^n.$$

---

[6] Certain notation in this section is as follows:

$$\Delta f(x) = f(x + 1) - f(x); \qquad \underset{\omega}{\Delta} f(x) = \frac{1}{\omega}(f(x + \omega) - f(x));$$

$$\underset{\omega_1, \cdots, \omega_k}{\Delta} f(x) = \underset{\omega_1}{\Delta}\, \underset{\omega_2}{\Delta} \cdots \underset{\omega_k}{\Delta} f(x); \qquad Mf(x) = \tfrac{1}{2}(f(x + \omega) + f(x));$$

$$Tf(x) = \frac{f(qx) - f(x)}{qx - x}, \qquad q \neq 1.$$

[7] This name is applied through analogy with the Bernoulli polynomials of higher order as defined by Nörlund and as a natural extension of the Bernoulli polynomials of the second kind already referred to.

(g) Euler polynomials of higher order:

$$P = \frac{d}{dx}, \qquad Q = M^k, \qquad f(x) = x^n.$$

(h)[8] $$P = \frac{d}{dx}, \qquad Q = T^k, \qquad f(x) = x^{(n)}, \qquad k > 0.$$

(i) $$P = \frac{d}{dx}, \qquad Q = \underset{\omega_1, \cdots, \omega_k}{\Delta^k},$$

$f_n(x)$ any set of Appell[9] polynomials.

(j) $$P = \Delta, \qquad Q = \frac{d^k}{dx^k},$$

$f_n(x)$ any set of Appell polynomials of the second kind.

(k) $$P = a_0 D^r + a_1 D^{r-1} + \cdots + a_{r-1} D \equiv R(D),$$

where $D = d/dx$ and $a_0, a_1, \cdots, a_{r-1}$ are constants; $a_{r-1} \neq 0$, $Q = b_0 \Delta^r + b_1 \Delta^{r-1} + \cdots + b_s$, $b_0, b_1, \cdots, b_s$ being constants. $f_n(x)$ are polynomials obtained by successive solutions of the equation

$$R(D) f_n(x) = n f_{n-1}(x).$$

(l) Interchange D and Δ in (j).

3. **Summation formulas.** In the remainder of this paper we shall assume $f_n(0) = 0$.

We first prove a lemma with reference to $f_n(x)$. We shall prove that

$$\begin{aligned} f_n(x + k) = {} & f_n(x) + n f_{n-1}(x) f_1(k) \\ & + \frac{n(n-1)}{2!} f_{n-2}(x) f_2(k) + \cdots + f_n(k). \end{aligned} \tag{12}$$

To do this we write

$$f_n(x + k) = b_0(x) + b_1(x) f_1(k) + b_2(x) f_2(k) + \cdots + b_n(x) f_n(k), \tag{13}$$

where the b's are as yet undetermined. Such an expression is possible

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[8] F. H. Jackson, Quarterly Journal of Mathematics, vol. 41 (1909–1910), p. 195.

[9] We understand by a set of Appell polynomials a set of polynomials such that $(d/dx) f_n(x) = n f_{n-1}(x)$ and of the second kind such that

$$\underset{\omega}{\Delta} f_n(x) = n f_{n-1}(x).$$

since $f_n(x+h)$ is a polynomial of the nth degree in h. We determine the b's by successively applying the operator P to (13) as a function of h and then letting $h=0$ remembering that $f_n(0)=0$.

We have attached a meaning to $f_n(x+g)$. With this meaning in mind we prove that

$$f_n((x+h)+g) = f_n(x+(h+g)). \tag{14}$$

Exponents are applied to x and h, subscripts to g. Expansion of the left-hand member of (15) is the same as that given in (9) with $(x+h)$ replacing x. The whole expansion carried out on both sides of (12) is exactly the same as the expansions of $((x+h)+g)^n$ and $(x+(h+g))^n$ with the exponents replaced by the f-function. Thus $f_k(x)$ replaces x^k. Inasmuch as the binomial expansion yields an identity so does the expansion in terms of f that we are considering.

Now consider any polynomial of degree $m+k$ which we write

$$\begin{aligned} \psi(g) &= a_0 + a_1 f_1(x) + \cdots + a_{m+k} f_{m+k}(x), \\ \psi(x+g) &= a_0 + a_1 f_1(x+g) + \cdots + a_{m+k} f_{m+k}(x+g) \\ &\equiv a_0 + a_1 F_1(x) + \cdots + a_{m+k} F_{m+k}(x). \end{aligned}$$

Hence by (8) $Q\psi(x+g) = a_k(k!) + a_{k+1}(k+1)\cdots 2f_1(x) + \cdots + a_{m+k}(m+k)(m+k-1)\cdots(m+1)f_m(x)$. That is, $Q\psi(x+g) = P^k\psi(x)$, $m+k \geqq 0$. Similarly

$$Q\psi(x+h+g) = P^k\psi(x+h). \tag{15}$$

We apply Taylor's formula symbolically to this, that is, we write

$$Q\psi(x+h+g) = c_0(x) + c_1(x)F_1(h) + \cdots + c_m(x)F_m(h)$$

and determine the c's by first letting $h=-g$ and then successively applying P, remembering (10) and letting $h=-g$ after each application.[10] We get

$$Q\psi(x+h+g) = Q\psi(x) + \sum_{v=1}^{m} \frac{F_v(h)}{v!} P^v Q\psi(x).$$

Hence by (15)

$$P^k\psi(x+h) = Q\psi(x) + \sum_{v=1}^{m} \frac{F_v(h)}{v!} P^v Q\psi(x). \tag{16}$$

This is a polynomial identity. Let $\psi(x) = P^{-k}\chi(x)$ where some particular determination is chosen in case P^{-k} is not unique. Then

$$\chi(x+h) = \sum_{v=0}^{m} \frac{F_v(h)}{v!} P^v Q P^{-k}\chi(x). \tag{17}$$

---

[10] Here, of course, when h is replaced by g an exponent is changed to a subscript.

Like (16) this is a polynomial identity. It includes as special cases Taylor's formula, the Euler-Maclaurin formula[11] of classical mathematics, the Boole formula, and so on.

Formula (17) will be written out in detail for two interesting special cases.

Let

$$P = \frac{d}{dx}, \qquad Q = \underset{\omega}{\Delta}^{k}, \qquad P^{-1} = \int_0^x dx, \qquad\qquad k = 1, 2, \cdots;$$

$$\begin{aligned}
\chi(x + h) = {} & \underset{\omega}{\Delta}^{k} \int_0^x \int_0^{x_1} \cdots \int_0^{x_{k-1}} \chi(x_k)\,dx_1 dx_2 \cdots dx_k \\
& + B_1^{(k)}(h) \underset{\omega}{\Delta}^{k} \int_0^x \int_0^{x_1} \cdots \int_0^{x_{k-1}} \chi(x_{k-1})\,dx_1 dx_2 \cdots dx_{k-1} \\
& + \cdots + \frac{1}{(k-1)!} B_{k-1}^{(k)}(h) \underset{\omega}{\Delta}^{k} \int_0^x \chi(x_1)\,dx_1 \\
& + \frac{1}{k!} B_k^{(k)}(h) \underset{\omega}{\Delta}^{k} \chi(x) + \cdots + \frac{1}{m!} B_m^{(k)}(h) \underset{\omega}{\Delta}^{k} \frac{d^{m-k}}{dx^{m-k}} \chi(x).
\end{aligned} \tag{18}$$

Secondly we let $P = \Delta$, $Q = d^k/dx^k$, $P^{-1} = \sum_0^{x-1} = \sum]_x - \sum]_0$ where $\sum = \Delta^{-1}$;

$$\begin{aligned}
\chi(x + h) = {} & \frac{d^k}{dx^k} \sum_{x_1=0}^{x-1} \sum_{x_2=0}^{x_1-1} \cdots \sum_{x_k=0}^{x_{k-1}-1} \chi(x_k) \\
& + b_1^{(k)}(h) \frac{d^k}{dx^k} \sum_{x_1=0}^{x-1} \sum_{x_2=0}^{x_1-1} \cdots \sum_{x_{k-1}=0}^{x_{k-2}-1} \chi(x_{k-1}) + \cdots \\
& + \frac{1}{(k-1)!} b_{k-1}^{(k)}(h) \frac{d^k}{dx^k} \sum_0^{x-1} \chi(x_1) \\
& + \frac{1}{k!} b_k^{(k)}(h) \frac{d^k}{dx^k} \chi(x) + \cdots \\
& + \frac{1}{m!} b_m^{(k)}(h) \frac{d^k}{dx^k} \Delta^{m-k} \chi(x).
\end{aligned} \tag{19}$$

In the above formulas $B_j^{(k)}(h)$ is the Bernoulli polynomial of order k and degree j, similarly $b_j^{(k)}(h)$ is the analogous polynomial which we designate as the Bernoulli polynomial of the second kind of order k and degree j.

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[11] Norlund, p. 29.

Formula (18) should be compared with a related formula of Nörlund.[12] If the operator $\sum_{x_1=0}^{x-1}\sum_{x_2=0}^{x_1-1}\cdots\sum_{x_k=0}^{x_{k-1}-1}$ is applied to both sides of (18) we get an interesting result which should be compared with a formula studied by the author in an earlier paper.[13]

4. **A remainder formula.** Let us assume that $\psi(x)$ is no longer a polynomial but that it is such a function that all operations applied to it in the sequel are meaningful. Wè consider formula (16) as our fundamental form.

We assume that $Q\psi(x)+\sum_{v=0}^{m}(F_v(t)/v!)P^vQ\psi(x)$ involves $m+1$ points, dependent upon x alone, which we call set 1. Some or all of these may coincide. For example, $\Delta^m Q\psi(x)$ involves $m+1$ distinct points and $(d^m/dx^m)Q\psi(x)$ $m+1$ coincident points. Let x be a constant and t a variable. When t is such that $x+t$ coincides with any one of the points of set 1

$$P^k\psi(x+t)-\left[Q\psi(x)+\sum_{v=1}^{m}\frac{F_v(t)}{v!}P^vQ\psi(x)\right]$$

vanishes, inasmuch as every term is identical with the like term which is built for the polynomial of degree m which coincides with $\psi(x)$ at the points of set 1. It has been remarked that (16) is an identity for a polynomial. Now denote by $p(t)$ a polynomial of degree $m+1$ which vanishes when $x+t$ coincides with any one of the $m+1$ points of set 1. Let the coefficient of t^{m+1} in $p(t)$ be 1. Choose a constant T such that

$$P^k\psi(x+t)-\left[Q\psi(x)+\sum_{v=1}^{m}\frac{F_v(t)}{v!}P^vQ\psi(x)\right]-Tp(t)$$

vanishes at the additional point where $t=h$. Then by repeated application of Rolle's theorem $(m+1)!T=d^{m+1}(P^k\psi(\xi))/dx^{m+1}$ where ξ is somewhere between the extreme values of the $(m+2)$ points composed of the points of set 1 and the additional point $x+h$. We have a final form for the remainder to formula (16),

$$R=\frac{1}{(m+1)!}p(h)\frac{d^{m+1}}{dx^{m+1}}(P^k\psi(\xi)). \tag{20}$$

The corresponding formula for (17) is

$$R=\frac{1}{(m+1)!}p(h)\frac{d^{m+1}}{dx^{m+1}}\chi(\xi). \tag{21}$$

Lehigh University

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[12] Loc. cit., p. 160.

[13] Loc. cit.

FACTORIZATION OF DIFFERENTIAL IDEALS

HENRY J. RIBLET

1. **Introduction.** With the ultimate objective of introducing number theoretical considerations into the theory of algebraic differential fields,[1] conditions on a differential ring $\mathfrak{R}$ are given which are sufficient to insure that any differential ideal in it may be uniquely expressed as the product of a finite number of prime differential ideals. This program is justified by its intrinsic interest and the fact that the existence of such conditions suggests the possibility of obtaining a number theoretical complement for the theory of differential ideals in the ring of differential forms as developed by Ritt, Raudenbush and others,[2] paralleling the interdependent, classical theories of polynomial and number ideals.

With an additional axiom and certain alterations of the definitions, the proof of the basic theorems follows the pattern used by van der Waerden[3] in presenting the method of Krull.[4] Our procedure will be to discuss in detail only those questions raised by the altering of definitions.

2. **Definitions.** By a *differential ring* $\mathfrak{R}$, we shall mean a commutative ring with a unit element but no divisors of zero which is closed under differentiation. Its quotient field will be denoted by $\mathfrak{K}$. A *differential ideal* $\mathfrak{a}$ in $\mathfrak{R}$ is an ordinary ideal closed with respect to differentiation. $\mathfrak{R}$ will satisfy the *ascending chain condition* for differential ideals if every sequence of differential ideals, each properly containing the previous one, is finite in length. A differential ideal is said to be *divisorless* if the only differential ideal which properly contains it is the unit ideal $\mathfrak{R}$. A *principal* differential ideal is a differential ideal which is principal in the usual sense. The unit ideal is an example. $\mathfrak{R}$ will be said to be *integrally closed* if each element in $\mathfrak{K}$, having the property that all its integral powers have a fixed denominator, is in $\mathfrak{R}$. A differential ideal $\mathfrak{p}$ in $\mathfrak{R}$ will be called

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Presented to the Society, April 26, 1940 under the title *Factorization of differential ideals in an algebraic field*; received by the editors November 6, 1941.

[1] H. J. Riblet, *Algebraic differential fields*, American Journal of Mathematics, vol. 53 (1941), pp. 339–446.

[2] J. F. Ritt, *Algebraic aspects of the theory of differential equations*, American Mathematical Society Semicentennial Publications, vol. 2, bibliography.

[3] B. L. van der Waerden, *Moderne Algebra*, vol. 2, p. 85.

[4] W. Krull, *Zur Theorie der allgemeinen Zahlringe*, Mathematische Annalen, vol. 99 (1928), pp. 51–70.

prime if the existence of two differential ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a}\cdot\mathfrak{b}\equiv 0\ (\mathfrak{p})$ implies that $\mathfrak{a}\equiv 0\ (\mathfrak{p})$ or $\mathfrak{b}\equiv 0\ (\mathfrak{p})$.

It may be commented that the definition of an integrally closed differential ring is taken from Krull.[5] The more familiar definition[6] is a consequence of this plus the stronger assumption that an ascending chain condition holds for all ideals in $\mathfrak{R}$. It should be observed also that this definition of primeness for differential ideals is not equivalent to that of Raudenbush.[7] If we restrict ourselves to the perfect differential ideals[8] of $\mathfrak{R}$, however, the two definitions coincide.

3. **Axioms.** Let $\mathfrak{R}$ be a differential ring in which the following axioms are satisfied:

I. The ascending chain condition for differential ideals in $\mathfrak{R}$ holds.

II. Every prime differential ideal in $\mathfrak{R}$ is divisorless.

III. Every prime differential ideal in $\mathfrak{R}$ contains a principal differential ideal.

IV. $\mathfrak{R}$ is integrally closed.

With these axioms, we then prove the two basic theorems on prime factorization:

THEOREM 1. *Every differential ideal in $\mathfrak{R}$ is a product of a finite number of prime differential ideals.*

THEOREM 2. *If $\mathfrak{a}\equiv 0\ (\mathfrak{b})$, then each prime factor of $\mathfrak{a}$ always occurs at least as often as a prime factor of $\mathfrak{b}$.*

The proofs are identical with those given in Van der Waerden.[9] The only question occurs in using the symbol $\mathfrak{p}^{-1}$ to denote the set of elements in $\mathfrak{R}$ such that if a is one of them, $a\cdot\mathfrak{p}$ is in $\mathfrak{R}$. It is clear that $\mathfrak{p}^{-1}$ is closed with respect to addition and with respect to multiplication by elements of $\mathfrak{R}$. It is also true that it is closed with respect to differentiation for, if $a\in\mathfrak{p}^{-1}$ and $p\in\mathfrak{p}$, then $a\cdot p\in\mathfrak{R}$. Hence $a\cdot p^{(1)}-a^{(1)}\cdot p\in\mathfrak{R}$, where (1) denotes differentiation. Now $a\cdot p^{(1)}\in\mathfrak{R}$, so $a^{(1)}\cdot p\in\mathfrak{R}$; but p is any member of $\mathfrak{p}$ and accordingly $a^{(1)}\in\mathfrak{p}^{-1}$. Note that Axioms III and IV allow us to prove that $\mathfrak{p}\cdot\mathfrak{p}^{-1}=\mathfrak{R}$.

From the two theorems, it follows that divisibility implies factorization and that each differential ideal in $\mathfrak{R}$ may be expressed uniquely as a product of a finite number of prime differential ideals. The consistency of the axioms is easily shown. If we define differentiation in the ring $C(x)$, obtained by adjoining x to the field of the rational numbers, in *any* way so as to leave it closed, it may be shown that Axioms I–IV are always satisfied. In $C(x, y)$ differentiation *may* be defined in such way that the statement as above still holds. This is of interest because every ordinary ideal in $C(x, y)$ may not be expressed uniquely as a product of a finite number of prime ideals.

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[5] W. Krull, ibid., p. 60.

[6] B. L. van der Waerden, loc. cit., p. 90.

[7] H. W. Raudenbush, Jr., *Ideal theory and algebraic differential equations*, Transactions of this Society, vol. 36 (1934), p. 365.

[8] H. W. Raudenbush, Jr., ibid., p. 362.

[9] B. L. van der Waerden, loc. cit., pp. 97–100.

HOFSTRA COLLEGE

ON THE ITERATION OF LINEAR HOMOGENEOUS TRANSFORMATIONS

ARNOLD DRESDEN

1. **Statement of problem.** The question which this note tries to answer is that of determining under what conditions on the matrix (a_{ij}), $(i, j=1, \cdots, n)$, the n-fold multiple sequence of complex numbers $x_k, x_k', \cdots, x_k^{(m)}, \cdots$ $(k=1, 2, \cdots, n)$ obtained by iteration of the linear homogeneous transformation $x_k' = a_{kj}x_j$ will converge for every initial set $x=(x_1, x_2, \cdots, x_n)$. Convergence is to be understood in the sense that there exists a set $X_1, X_2, \cdots, X_n$ such that, for $k=1, 2, \cdots, n$; $x_k^{(m)} \rightarrow X_k$, as $m \rightarrow \infty$.

2. **Jordan normal form.** We begin by recalling that a matrix $A=(a_{ij})$ with complex elements is similar to its Jordan normal form J_0. This means that there exists a unimodular matrix P, such that $A=P^{-1}J_0P$ and $J_0=PAP^{-1}$, where J_0 is the direct sum of Jordan matrices $J_1, \cdots, J_N$. To each elementary divisor $(\lambda-\lambda_\rho)^{e_\rho}$ of the characteristic matrix $\lambda I-A$ $(\rho=1, 2, \cdots, N)$ and $e_1+e_2+\cdots+e_N=n$, corresponds a Jordan matrix J_ρ. If $e_\rho>1$, then J_ρ has zero elements everywhere, except in the principal diagonal, all of whose elements are λ_ρ, and in the diagonal immediately below the principal diagonal, all of whose elements are 1.[1] If $e_\rho=1$, then J_ρ consists of the single element λ_ρ.

It follows that any integral power of J_0 is the direct sum of the same powers of the Jordan matrices J_ρ. Let us now denote by J an arbitrary Jordan matrix of order $n>1$,

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Received by the editors August 25, 1941.

[1] See, for example, MacDuffee, *Introduction to Abstract Algebra*, p. 241.

$$J = \begin{pmatrix} \lambda & 0 & 0 & 0 \cdots & \cdot & 0 \\ 1 & \lambda & 0 & 0 \cdots & \cdot & 0 \\ 0 & 1 & \lambda & 0 \cdots & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdots 0 & \lambda & 0 \\ 0 & \cdot & \cdot & \cdots 0 & 1 & \lambda \end{pmatrix}.$$

It is then easily seen that J^2 is a matrix of order n, whose first column is $\lambda^2, 2\lambda, 1, 0, \cdots, 0$, and whose second column is obtained from the first column by shifting the elements one place downward, as far as the length permits, and placing 0 in the first place; the third column is obtained from the second in similar manner; the last column is $0\ 0 \cdots 0\ \lambda^2$. A simple induction enables us to prove that, for any positive integer h, the matrix J^h is of order n, that its first column consists of the successive terms in the binomial expansion of $(\lambda+1)^h$, followed by zeros, provided $n \geqq h+1$ and otherwise of as many of these terms as the length of the column permits, that each of its columns after the first is obtained from the one to the left of it by shifting the elements one place downward and filling the open position above with a zero, and that its last column is $0\ 0\ 0 \cdots 0\ \lambda^h$. If J is a Jordan matrix of order 1, consisting of the single element λ then J^n consists of the single element λ^n.

3. **Reduction of problem; solution.** Returning now to our linear homogeneous transformation, we write it in the form

$$x' = xA, \tag{1}$$

where x and x' stand for the row-vectors $(x_1, x_2, \cdots, x_n)$ and $(x_1', x_2', \cdots, x_n')$, respectively. By means of the unimodular matrix P, we map the space of the row-vectors x on the space of the row-vectors y: $x = yP, \quad y = xP^{-1}$. This mapping carries the transformation (1) into a transformation in the y-space:

$$y' = x'P^{-1} = xAP^{-1} = yPAP^{-1} = yJ_0. \tag{2}$$

It follows from the mapping that the vectors $x^{(m)}$ will converge, in the sense described in §1, if and only if the vectors $y^{(m)}$ converge. But $y^{(m)} = y^{(m-1)}J_0 = \cdots = y_0J_0^m = y(J_1^m + J_2^m + \cdots + J_N^m)$. We obtain therefore the following form for the mth iterate of the transformation (2) in terms of the roots λ_p of the characteristic equation $|a_{ij} - \lambda\delta_{ij}| = 0$ and the exponents e_p of the corresponding elementary divisors $(\lambda - \lambda_p)^{e_p}$ of $\lambda I - A$, provided every $e_p > 1$.

$$\begin{aligned} y_1^{(m)} &= \lambda_1^m y_1 + C_{m,1}\lambda_1^{m-1} y_2 + \cdots + C_{m,e_1-1}\lambda_1^{m-e_1+1} y_{e_1}, \\ y_2^{(m)} &= \lambda_1^m y_2 + C_{m,1}\lambda_1^{m-1} y_3 + \cdots + C_{m,e_1-2}\lambda_1^{m-e_1+2} y_{e_1}, \end{aligned}$$

$$\begin{aligned}
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
y_{e_1}^{(m)} &= \lambda_1^{m} y_{e_1},\\
y_{e_1+1}^{(m)} &= \lambda_2^{m} y_{e_1+1} + C_{m,1}\lambda_2^{m-1} y_{e_1+2} + \cdots + C_{m,e_2-1}\lambda_2^{m-e_2+1} y_{e_1+e_2},\\
y_{e_1+1}^{(m)} &= \qquad\qquad\qquad \lambda_2^{m} y_{e_1+2} + \cdots + C_{m,e_2-2}\lambda_2^{m-e_2+2} y_{e_1+e_2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
y_{e_1+e_2}^{(m)} &= \lambda_2^{m} y_{e_1+e_2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
y_{n}^{(m)} &= \lambda_N^{m} y_{n}.
\end{aligned}$$

This form makes it evident that the set $y_k^{(m)}$ will converge as m increases indefinitely, if and only if $|\lambda_\rho| < 1$, for $\rho = 1, \cdots, N$. If, on the other hand, all the elementary divisors involving a root λ_ρ are linear, then convergence will occur also if $\lambda_\rho = 1$. We have therefore obtained the following answer to our question:

THEOREM. *The iteration of the linear homogeneous transformation* $x_k' = a_{kj}x_j$ $(k, j = 1, 2, \cdots, n)$ *will converge for every initial set* x_k *if and only if all the roots of the characteristic equation of the matrix* (a_{ij}) *are less than unity in absolute value, except that the roots, which are involved only in linear elementary divisors, may be equal to unity.*[2]

4. **Remark.** It is clear that it is possible for some of the variables $y_k^{(m)}$ to converge to limits while others do not converge; for example, the iteration of the transformation $y_1' = 1/2y_1$, $y_2' = y_2 + y_3$, $y_3' = y_3 + y_4$, $y_4' = y_4$ will obviously yield a convergent sequence for $y_1^{(m)}$ and for $y_4^{(m)}$, but not for $y_2^{(m)}$, if we start with the initial set $y_1 = a$, $y_2 = b$, $y_3 = y_4 = 0$. In this case the roots of the characteristic equation are $1/2$ and 1, with multiplicities 1 and 3, respectively. It may be worth while to determine, in the general case of the linear homogeneous transformation, which variables do and which do not converge upon iteration, in case some, but not all the roots of the characteristic roots of the matrix of the transformation are less than unity in absolute value.

SWARTHMORE COLLEGE

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[2] In an earlier incomplete formulation of this theorem, the exceptional character of the linear elementary divisors was overlooked. The author is indebted for the detection of this error to Professor W. T. Reid, who called attention to the fact that convergence would take place if A were an idempotent matrix. In this connection it may be of interest to observe that for a matrix which satisfies an equation of the form $A^{r+1} = A^r$, $r \geq 1$, *all* the elementary divisors of $\lambda I - A$ are linear. For, on the one hand, its Jordan normal form would have to satisfy the same relation, while it follows from our discussion that this can not be the case if $\lambda I - A$ has a non-linear elementary divisor.

A SUFFICIENT CONDITION FOR CESÀRO SUMMABILITY

G. D. NICHOLS

S. Chapman[1] has proved the summability (C, k) of the series $\sum r^{s}e^{r\theta i}$ where $(k-1)\leqq s<k$, and $0<\theta<2\pi$. More recently M. S. MacPhail[2] has proved the exact summability (C, k) of the series $\sum P(r)\cdot f(r)$, where $P(r)$ is a polynomial of degree $k-1$ and $f(r)$ is a periodic function of mean value zero. He also gave a closed expression for the Cesàro "sum" of such a series. H. L. Garabedian[3] earlier published a special case of this result.

The purpose of the present paper is to prove the following closely related theorem in which the condition on the coefficients is somewhat more general. Also "sum" formulas are given for each of the sine and cosine series separately.

THEOREM. *The series* $(1/2)P(0)+\sum P(r)\cos rx$ *and* $\sum P(r)\sin rx$ *are summable* (C, k) *provided* $\{\Delta^{k}P(r)\}$ *is a monotone null sequence, and* $P(r)$ *together with its first* $k+1$ *derivatives each exist and are continuous for positive values of* r ($x\neq 2n\pi$ *in the cosine series*).

The proofs for the sine and cosine series are almost identical so only the proof for the latter will be given. Consider the series

$$S_n = (1/2)P(0) + \sum_{r=1}^{n} P(r)\cos rx. \tag{1}$$

If we multiply both sides of (1) successively k times by $2 \sin x/2$ there results, for k even,

$$\begin{aligned}(2^{k}\sin^{k} x/2)S_n = (1/2)D_{0k} &+ \sum_{r=1}^{(k-2)/2} D_{rk}\cos rx \\ &+ (-1)^{k/2}\sum_{r=0}^{n-k}\Delta^{k}P(r)\cos(r+(k/2))\,x + (\mathrm{Cos})_k,\end{aligned} \tag{2}$$

where the symbol $(\mathrm{Cos})_k$ represents k cosine terms, the highest order of any of the coefficients being the same as the order of $P(n)$.

For k odd, we have

Presented to the Society, January 1, 1941; received by the editors October 20, 1941.

[1] S. Chapman, Proceedings of the London Mathematical Society, (2), vol. 9 (1911), p. 398.

[2] M. S. MacPhail, this Bulletin, vol. 47 (1941), p. 483.

[3] H. L. Garabedian, this Bulletin, vol. 45 (1939), p. 592.

$$(3)\qquad \begin{aligned}(2^k \sin^k x/2)S_n &= \sum_{r=1}^{(k-1)/2} C_{rk} \sin (2r-1)x/2 \\ &\quad + (-1)^{(k+1)/2}\sum_{r=0}^{n-k} \Delta^k P(r) \sin (r+(k/2))x + (\mathrm{Sin})_k,\end{aligned}$$

where $(\mathrm{Sin})_k$ has the same significance as $(\mathrm{Cos})_k$ above. For $k=1$ the first summation in (3) is to be omitted. The expressions for D_{rk} and C_{rk} in (2) and (3), respectively, are given in the sequel.

If we now apply the Cesàro definition for the "sum," σ, of a non-convergent series, there results from (2),

$$(4)\qquad \begin{aligned}(2^k \sin^k x/2)\sigma &= (1/2)D_{0k} + \sum_{r=1}^{(k-2)/2} D_{rk} \cos rx \\ &\quad + \lim_{n\to\infty} (-1)^{k/2}\sum_{r=0}^{n-k} \frac{C(n-r+(k/2)+1,\, k)}{C(n+k,\, k)} \Delta^k P(r) \cos (r+(k/2))x \\ &\quad + \lim_{n\to\infty} \frac{(\mathrm{Cos})_k}{(n+1)(n+2)\cdots(n+k)},\end{aligned}$$

where the C's following the second summation are the binomial coefficients.

To prove the existence of the first limit in (4) we first note that since there are the same number of factors in numerator and denominator of the fraction preceding $\Delta^k P(r)$, the fraction is positive and never greater than one for the values of n, r, and k under consideration. Since $\{\Delta^k P(r)\}$ is a monotone null sequence by hypothesis, the existence of the first limit (except for $x = 2n\pi$) follows by Dirichlet's test.[4]

We wish to show that the second limit in (4) is equal to zero. Since the order of $(\mathrm{Cos})_k$ is the same as $P(n)$, this can be done by showing that if $\{\Delta^k P(n)\}$ is a monotone null sequence then the kth derivative of $P(n)$ approaches zero as n becomes infinite.

For this purpose we have

$$(5)\qquad P^{(k)}(n) + (k/2)P^{(k+1)}(\xi) = \Delta^k P(n),$$

where $\xi \to \infty$ as $n \to \infty$. This is a special case of a formula due to Markoff,[5] and depends on the continuity of $P(n)$ and its first $k+1$ derivatives.

Since, by hypothesis, $\Delta^k P(n) \to 0$ as $n \to \infty$, it is only necessary that if $(y + (k/2)y') \to 0$ as $n \to \infty$, y and y' separately do so, where $y = P^{(k)}(n)$, and y and y' are continuous. A proof of this fact is given by Bromwich.[6] Hence the summability of the original series is established.

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[4] Bromwich, *Theory of Infinite Series*, p. 49.

[5] Markoff, *Differenzenrechnung*, p. 21.

In case $P(r)$ is a polynomial of degree $k-1$, the first limit in (4) is also zero and σ is obtained in closed form from the first summation. Formulas for D_{rk} and C_{rk} are now given for this case. They can be established by induction, but the details are rather tedious and will be omitted

$$D_{rk} = \sum_{s=k/2+r-1}^{k-2} (-1)^{s+k+r+1}\Delta^{k-1-s}P(0)\left[C(x,\,(k/2)+r-1) + C(x,\,(k/2)-r-1)\right]. \tag{6}$$

For k odd, and $P(r)$ a polynomial of degree $k-1$, there results from (3),

$$(2^k \sin^k x/2)\sigma = \sum_{r=1}^{(k-1)/2} C_{rk} \sin (2r-1)x/2, \tag{7}$$

$$C_{rk} = \sum_{s=(k-2r-1)/2}^{k-2} (-1)^{s+k+r}\Delta^{k-1-s}P(0)\left[C(x,\,(k/2)+r-(3/2)) + C(x,\,(k/2)-r-(1/2))\right]. \tag{8}$$

For the sine series, $\sum P(r) \sin rx$, the results for $P(r)$, a polynomial of degree $k-1$, follow. For k odd,

$$(2^k \sin^k x/2)\sigma = \sum_{r=1}^{(k-1)/2} A_{rk} \cos (2r-1)x/2, \tag{9}$$

and for k even,

$$(2^k \sin^k x/2)\sigma = \sum_{r=1}^{(k-2)/2} B_{rk} \sin rx, \tag{10}$$

where A_{rk} and B_{rk} are given by

$$A_{rk} = \sum_{s=(k-2r-1)/2}^{k-2} (-1)^{s+k+r-1}\Delta^{k-1-s}P(0)\left[C(x,\,(k/2)-r-(1/2)) - C(x,\,(k/2)+r-(3/2))\right], \tag{11}$$

$$B_{rk} = \sum_{s=(k-2r-2)/2}^{k-2} (-1)^{s+k+r-1}\Delta^{k-1-s}P(0)\left[C(x,\,(k/2)-r-1) - C(x,\,(k/2)+r-1)\right]. \tag{12}$$

If the upper limit on $\sum$ in (4), (7), (9), and (10) is less than the lower limit, the value of the summation is to be taken as zero. Also $C(n,\,0)=1$ for all n including zero, and $C(n,\,m)=0$ for $n<m$.

UNIVERSITY OF ARKANSAS

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[6] Bromwich, *Theory of Infinite Series*, p. 272.

ON THE THEORY OF THE TETRAHEDRON

N. A. COURT

I. DEFINITION. *We associate with the general tetrahedron* $(T) = ABCD$ *a sphere* (Q) *whose center is the Monge point* M *of* (T) *and the square of whose radius is*

$$\text{(a)} \qquad q^2 = (MO^2 - R^2)/3,$$

where O *and* R *are the center and the radius of the circumsphere* (O) *of* (T).

In what follows, a number of propositions regarding the sphere (Q) will be established and it will be shown that from the properties of (Q) may be derived, as special cases, properties of the polar sphere (H) of the orthocentric tetrahedron (T_h).

For want of a better name we shall refer to (Q) as the "quasi-polar" sphere of the general tetrahedron (T).

The expression $MO^2 - R^2$ is the power of the Monge point M of (T) for the sphere (O).

THEOREM 1. *The square of the radius of the quasi-polar sphere of the general tetrahedron is equal to one-third of the power of the Monge point of the tetrahedron for its circumsphere.*

The sphere (Q) is real, a point sphere, or imaginary according as MO is greater than, equal to, or smaller than R. Moreover, we have $MO < 2R$, for the mid-point of MO is the centroid G of (T), and G necessarily lies within the sphere (O).

COROLLARY. *In an orthocentric tetrahedron* (T_h) *the Monge point coincides with the orthocenter* H, *and the above properties of* (Q) *are valid for the polar sphere* (H) *of* (T_h).[1]

The Monge point M of (T) is a center of similitude of the circumsphere (O) and the twelve point sphere (L) of (T),[2] hence M is the center of a sphere of antisimilitude of (O) and (L), that is, a sphere with respect to which the spheres (O) and (L) are inverse of one another.

Presented to the Society, December 31, 1941; received by the editors November 22, 1941.

[1] Nathan Altshiller-Court, *Modern Pure Solid Geometry*, New York, 1935, p. 265, §813. This book will be referred to as MPSG.

[2] MPSG, p. 251, §764.

The line of centers LO of the spheres (L), (O) meets (O) in the points whose distances from M are $MO-R$ and $MO+R$. Again, the line LO meets (L) in two points whose distances from M are $ML-R/3$ and $ML+R/3$, for the radius of (L) is equal to $R/3$. Now the points $MO+R$, $ML-R/3$ correspond to each other in the inversion considered (and so do the points $MO-R$, $ML+R/3$), hence if x^2 is the constant of inversion, we have

$$\begin{aligned} x^2 &= (MO+R)(ML-R/3) = (MO+R)(3ML-R)/3 \\ &= (MO^2-R^2)/3 = q^2. \end{aligned}$$

But the constant of inversion is the square of the radius of the sphere of inversion, and the sphere of inversion is coaxial with the two inverse spheres.

THEOREM 2. *The quasi-polar sphere of the general tetrahedron is coaxial with the circumsphere and the general twelve point sphere of the tetrahedron.*

COROLLARY. *To the square* (L) *of* (T) *corresponds in the orthocentric tetrahedron* (T_h) *the second twelve point sphere, and from the above property of* (Q) *we obtain the known property of the polar sphere of* (T_h).[3]

THEOREM 3. *The sum of the powers of the vertices of a general tetrahedron with respect to the quasi-polar sphere of that tetrahedron is equal to one-third of the sum of the squares of the edges of the tetrahedron.*

The power D_q of the vertex D of (T) for the sphere (Q) is

$$D_q = MD^2 - q^2 = MD^2 - (MO^2 - R^2)/3,$$

and from the triangle DMO we have

$$DM^2 + DO^2 = 2DG^2 + MO^2/2 = 2DG^2 + 2OG^2,$$

hence

$$D_q = 2DG^2 + 2(OG^2 - R^2)/3.$$

Adding to D_q the analogous formulas relative to the vertices A, B, C of (T) we obtain

$$A_q + B_q + C_q + D_q = 2(AG^2 + BG^2 + CG^2 + DG^2) + 8(OG^2 - R^2)/3.$$

Now if we denote by k^2 the sum of the squares of the edges of (T), we have

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[3] MPSG, p. 264, §805.

$$AG^2 + BG^2 + CG^2 + DG^2 = k^2/4,$$[4]

(b) $$OG^2 - R^2 = - k^2/16,$$[5]

hence

$$A_q + B_q + C_q + D_q = k^2/2 - k^2/6 = k^2/3.$$

COROLLARY. *In an orthocentric tetrahedron the sum of the squares of a pair of opposite edges is constant, hence the sum of the powers of the vertices of the tetrahedron for the polar sphere is equal to the sum of the squares of a pair of opposite edges.*[6]

THEOREM 4. *The sum of the squares of the distances of the Monge point of a tetrahedron from the mid-points of the edges is equal to three times the sum of the squares of the radii of the circumsphere and the quasi-polar sphere of the tetrahedron.*

Let E, F be the mid-points of the edges DA, BC of (T). The mid-point of the bimedian $EF = m_a$ of (T) is the centroid G and therefore we have, from the triangle MEF,

$$ME^2 + MF^2 = 2MG^2 + EF^2/2 = 2OG^2 + m_a^2/2.$$

We have two analogous formulas relative to the other two pairs of opposite edges of (T). Adding the three relations we have

$$\sum ME^2 = 6OG^2 + (m_a^2 + m_b^2 + m_c^2)/2.$$

Now the expression in the parenthesis is equal to[7] $k^2/4$, hence, making use of the formula (b), we have

$$\sum ME^2 = 6(R^2 - k^2/16) + k^2/8 = 3R^2 + 3(R^2 - k^2/12).$$

But, eliminating OG between the formulas (a) and (b) we have

(c) $$R^2 - k^2/12 = q^2,$$

hence

$$\sum ME^2 = 3(R^2 + q^2).$$

COROLLARY. *The proposition is valid for the orthocentric tetrahedron.*[8]

THEOREM 5. *The sum of the powers of the Monge point of a tetra-*

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[4] National Mathematics Magazine, vol. 15 (1941), p. 273, §3.

[5] V. Thebault, Nouvelles Annales de Mathématiques, (4), vol. 19 (1919), p. 425.

[6] Mathesis, vol. 42 (1928), p. 338, §2.

[7] MPSG, p. 56, §186.

[8] MPSG, p. 275, Example 4.

hedron with respect to the six spheres having for diameters the edges of the tetrahedron is equal to six times the square of the radius of the quasi-polar sphere of the tetrahedron.

The power of the Monge point M for the sphere (BC) having for diameter the edge $BC=a$ of (T) is equal to $MF^2-a^2/4$. We have five analogous expressions relative to the other five edges of (T). Adding the six expressions we have

$$\sum MF^2 - k^2/4,$$

or, using the value of $\sum MF^2$ from the preceding article,

$$6(R^2 - k^2/16) + k^2/8 - k^2/4 = 6q^2.$$

In an orthocentric tetrahedron the spheres having for diameters the edges of the tetrahedron are orthogonal to the polar sphere.

COROLLARY. *The power of the orthocenter for the six spheres is equal to the square of the radius of the polar sphere.*

The powers of the Monge point of (T) for the two spheres having for diameters a pair of opposite edges of (T) are equal.[3]

THEOREM 6. *The sum of the powers of the Monge point of a tetrahedron with respect to three spheres having for diameters three concurrent (or coplanar) edges of the tetrahedron is equal to three times the square of the radius of the quasi-polar sphere.*

The centroid G of (T) bisects the segment AA' joining A to the corresponding vertex A' of the tetrahedron (T') twin to (T), hence $AMA'O$ is a parallelogram.

Let A_1 be the diametric opposite of A on the circumsphere (O) of (T). The quadrilateral $MA'A_1O$ is a parallelogram, for OA_1 and MA' are equal and parallel. Hence the diagonal MA_1 bisects the diagonal OA', and therefore meets the median $A'G$ of the triangle $A'OM$ in the centroid G_a of that triangle. Thus

$$GG_a = A'G_a/2 = AG/3,$$

and G_a is therefore the centroid of the face BCD of (T). Moreover $MG_a = MA_1/3$.

Let A_2 be the second point of intersection, besides A, of the line AM with the sphere (O). The line A_1A_2 is perpendicular to the line AMA_2, hence if F is the foot of the perpendicular from G_a upon AMA_2 we have

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[3] G. Gallucci, Nouvelles Annales de Mathématiques, (3), vol. 16 (1897), p. 17, §5.

$$MF:MA_2 = MG_a:MA_1 = 1:3,$$

and therefore

$$MA\cdot MA_2 = 3MA\cdot MF.$$

But $MA\cdot MA_2$ is the power of M for the sphere (O), hence by Theorem 1, $MA\cdot MF$ is equal to the square of the radius of the quasi-polar sphere (Q), that is, the points A, F are inverse with respect to (Q).

The median AG_a of (T) subtends a right angle at F, hence the sphere (AG_a) having AG_a for diameter is orthogonal to (Q), similarly for the three analogous spheres (BG_b), (CG_c), (DG_d). Thus this theorem follows:

THEOREM 7. *The quasi-polar sphere of a tetrahedron is orthogonal to the four spheres having for diameters the four medians of the tetrahedron.*[10]

Since the points A, F are inverse for (Q) (Theorem 7), the polar plane α of A for (Q) is perpendicular to AM at F; hence the line FG_a is the trace of α in the plane $AMGO$.

THEOREM 8. *The polar reciprocal tetrahedron of a given tetrahedron* (T) *with respect to the quasi-polar sphere of* (T) *is circumscribed about the medial tetrahedron of* (T).

THEOREM 9. *The faces of the polar reciprocal tetrahedron of a given tetrahedron* (T) *with respect to the quasi-polar sphere of* (T) *cut the spheres having for diameters the corresponding medians of* (T) *along four circles lying on the same sphere, namely, the twelve point sphere of* (T).

The point G_a lies on the twelve point sphere (L) of (T), and the diametric opposite of G_a on (L) lies on the line AM;[11] hence the point F also belongs to the sphere (L). Thus the plane α (Theorem 8) cuts the two spheres (AG_a) and (L) along the same circle, and FG_a is a diameter of that circle.

THEOREM 10. *The faces of the polar reciprocal tetrahedron of the medial tetrahedron of a given tetrahedron* (T) *with respect to the quasi-polar sphere of* (T) *cuts the spheres having for diameters the respective medians of* (T) *along four circles lying on the same sphere, namely the circumsphere of* (T).

The polar plane of G_a with respect to the quasi-polar sphere (Q) passes through A and is perpendicular to MG_a, hence the trace K

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[10] American Mathematical Monthly, vol. 39 (1932), pp. 196, 197, §§11, 13.

[11] MPSG, p. 251.

of this plane on MG_a lies on the sphere (AG_a) having for diameter the median AG_a of (T).

On the other hand, the line MG_a passes through the diametric opposite A_1 of A on the circumsphere (O) of (T), hence K lies on (O). Thus the line AK is a common diameter of the two circles along which the polar plane considered cuts the two spheres (AG_a) and (O).

II. DEFINITIONS. *We associate with the general tetrahedron (T) the sphere having for center the centroid G of (T) and for the square of its diameter one-third of the sum of the squares of the bimedians of (T). The sphere will be denoted by (G) and referred to as the G-sphere of (T). The sphere (G) is concentric with the three spheres having for diameters the bimedians of (T).*

In an orthocentric tetrahedron the bimedians are equal, so that the three spheres having these bimedians for diameters coincide, and the G-sphere coincides with them, in the first twelve point sphere of the orthocentric tetrahedron, the three bimedians being diameters of that sphere.

THEOREM 1. *The quasi-polar sphere and the (G)-sphere of the general tetrahedron are orthogonal.*

The square of the radius, g^2, of the sphere (G) is equal to[7] (§I, Theorem 4)

$$(m_a^2 + m_b^2 + m_c^2)/3:4 = k^2/48,$$

hence

$$q^2 + g^2 = (R^2 - k^2/12) + k^2/48 = R^2 - k^2/16 = OG^2 = MG^2,$$

that is, the square of the line of centers of the spheres (Q), (G) is equal to the sum of the squares of their radii.

COROLLARY. *The polar sphere and the first twelve point sphere of the orthocentric tetrahedron are orthogonal.*[12]

THEOREM 2. *The sphere (G) belongs to the coaxial pencil formed by the circumsphere, the twelve point sphere, and the quasi-polar sphere of the tetrahedron.*

The centroid G is the second center of similitude, besides the point M, of the circumsphere (O) and the twelve point sphere (L) of (T). Now the quasi-polar sphere (Q) being a sphere of antisimilitude of

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[12] MPSG, p. 262, §799.

(O) and (L) (§I, Theorem 2), the sphere (G) having G for center and orthogonal to (Q) (§II, Theorem 1) is the second sphere of antisimilitude of (O) and (L); hence (G) is coaxial with these spheres.

THEOREM 3. *The four spheres having for centers the vertices of a tetrahedron and orthogonal to the quasi-polar sphere cut the spheres having for diameters the respective medians of the tetrahedron along four circles belonging to the same sphere, namely, the* (G)*-sphere of the tetrahedron.*

The sphere (A) having A for center and orthogonal to the sphere (Q) is coaxial with the spheres (G) and (AG_a), for the centers of these three spheres are collinear and all three are orthogonal to (Q). Similarly for the vertices B, C, D of (T).

UNIVERSITY OF OKLAHOMA

EUCLIDEAN CONCOMITANTS OF THE TERNARY CUBIC

T. L. WADE

1. Introduction; construction of concomitants. In this paper we use the results of Cramlet [1] and the writer [2] to study the euclidean concomitants of the ternary cubic curve

$$T_{abc}X^aX^bX^c = 0,$$

where $a, b, c = 1, 2, 3$ and T_{abc} is symmetric. With tensor algebra as the medium of investigation all types of concomitants are readily constructed, and their geometric interpretations are also readily made in most cases. As is conventional in classical invariant theory, the word concomitant will be used as meaning rational integral concomitant unless stated to the contrary.

As a consequence of Theorem 3 in [2], we have the following theorem.

THEOREM I. *Every euclidean concomitant of the ground form* $T_{abc}X^aX^bX^c$ $(a, b, c = 1, 2, 3)$ *is expressible by composition as a tensor of order zero with the use of the coefficient tensor* T_{abc}, *the variable coordinate tensors* X^a *and* U_a, *and the numerical tensors* ϵ^{abc}, L_a, *and* E^{ab}.

Presented to the Society, September 5, 1941; received by the editors of the Transactions of this Society, September 25, 1941; accepted by them and later transferred to this Bulletin.

The tensor ϵ_{abc} will not appear here since we are dealing with a ground form in X^a. In detail,

$$L_a = [0,\ 0,\ 1],\quad E^{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is convenient to introduce the tensor A^{ab}, where

$$A^{ab} = \epsilon^{abc}L_c = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem 3 of [2] is the first fundamental theorem of euclidean invariant theory in tensor form, and Theorem I above is the particular form of it which we need here; this theorem constitutes a basis for the construction of concomitants.

2. Reduction of concomitants. In [2] an algebraically complete system of euclidean invariants for the cubic $T_{abc}X^aX^bX^c=0$ is given in tensor form. For the conic $C_{ab}X^aX^b=0$ the familiar algebraically complete system of three invariants is also an irreducibly complete system. But this is not true for the cubic. In this paper we shall find an irreducible system, complete through the fifth degree, of euclidean invariants for the cubic, and shall investigate geometric interpretation of these invariants in connection with other concomitants of the ground form. It is generally recognized that the problem of reduction of concomitants is more difficult than the problem of construction. See H. Weyl [3].

It is of interest to note that between the simple case of the conic and the situation for the general cubic is that of the degenerate cubic considered as the line $B_aX^a=0$ and the conic $C_{ab}X^aX^b=0$. They have the irreducibly complete system of eight invariants

$$I_1 = B_aB_bE^{ab},\qquad I_2 = C_{ab}C_{cd}C_{ef}\epsilon^{ace}\epsilon^{bdf},\qquad I_3 = C_{ab}C_{cd}A^{ac}A^{bd},$$
$$I_4 = C_{ab}E^{ab},\qquad I_5 = C_{ab}E^{ac}E^{bd}B_cB_d,\qquad I_6 = C_{ac}C_{bd}\epsilon^{abe}\epsilon^{cdf}B_eB_f,$$
$$I_7 = C_{ab}C_{cd}\epsilon^{ace}A^{bd}B_e,\qquad I_8 = C_{ab}A^{ac}E^{bd}B_cB_d.$$

Six of these are algebraically independent; they may be chosen as $I_1, \cdots, I_6$; then I_7 and I_8 are expressible in terms of them by means of the syzygies

$$3(I_7)^2 = 3I_3I_6 - 2I_1I_2I_4 - 2I_2I_5,$$
$$(I_8)^2 = I_1I_4I_5 - (I_5)^2 - \tfrac{1}{2}(I_1)^2I_3.$$

The five invariants I_2, I_3, I_6, I_7, and $C_{ab}A^{ac}A^{bd}B_cB_d$ form an irreducibly complete system of affine invariants for the line and conic. They were studied recently by Weitzenböck [4], who used the symbolic notation of his text [5].

We now state the second fundamental theorem in the detailed and explicit tensor form needed to investigate the relations on the concomitants constructed in accordance with the first fundamental theorem.

THEOREM II. *Every identity satisfied by the concomitants constructed by the first fundamental theorem for a set of ground forms in* X^a *can be established with the basic identities:*

Iden 1. $\epsilon^{abc}E^{de} \equiv \epsilon^{dbc}E^{ae}+\epsilon^{adc}E^{be}+\epsilon^{abd}E^{ce}$,
Iden 2. $\epsilon^{abc}X^{d} \equiv \epsilon^{dbc}X^{a}+\epsilon^{adc}X^{b}+\epsilon^{abd}X^{c}$,
Iden 3. $\epsilon^{abc}\epsilon^{def} \equiv \epsilon^{dbc}\epsilon^{aef}+\epsilon^{adc}\epsilon^{bef}+\epsilon^{abd}\epsilon^{cef}$,
Iden 4. $A^{ab}A^{cd} \equiv E^{ac}E^{bd}-E^{ad}E^{bc}$,
Iden 4′. $E^{ac}E^{bd} \equiv A^{ab}A^{cd}+E^{ad}E^{bc}$,
Iden 5. $A^{ab}E^{cd} \equiv A^{cb}E^{ad}+A^{ac}E^{bd}$,
Iden 6. $A^{ab}X^{c} \equiv A^{cb}X^{a}+A^{ac}X^{b}+\epsilon^{abc}L_dX^{d}$,
Iden 7. $A^{ab}\epsilon^{cde} \equiv A^{cb}\epsilon^{ade}+A^{ac}\epsilon^{bde}+\epsilon^{abc}A^{de}$,
Iden 8. $\epsilon^{abc}A^{de} \equiv \epsilon^{dbc}A^{ae}+\epsilon^{adc}A^{be}+\epsilon^{abd}A^{ce}$,
Iden 9. $A^{ab}A^{cd} \equiv A^{cb}A^{ad}+A^{ac}A^{bd}$,
Iden 10. $E^{ab}E^{cd}E^{ef} \equiv E^{ab}E^{cf}E^{ed}+E^{ad}E^{eb}E^{cf}-E^{ad}E^{cf}E^{eb}-E^{af}E^{cb}E^{ed}$
$+E^{ad}E^{cf}E^{eb}$,
Iden 11. $X^{a}E^{bc}E^{de} \equiv X^{c}E^{ba}E^{de}+E^{ac}X^{b}E^{de}-E^{ac}E^{be}X^{d}-E^{ae}X^{b}E^{dc}$
$+E^{ae}E^{bc}X^{d}+\epsilon^{abd}A^{ce}L_fX^{f}$.

Iden 2 may be established by expanding

$$\begin{vmatrix} \delta_1^a & \delta_2^a & \delta_3^a & X^a \\ \delta_1^b & \delta_2^b & \delta_3^b & X^b \\ \delta_1^c & \delta_2^c & \delta_3^c & X^c \\ \delta_1^d & \delta_2^d & \delta_3^d & X^d \end{vmatrix} \equiv 0,$$

and using

$$\epsilon^{abc} = \begin{vmatrix} \delta_1^a & \delta_2^a & \delta_3^a \\ \delta_1^b & \delta_2^b & \delta_3^b \\ \delta_1^c & \delta_2^c & \delta_3^c \end{vmatrix}.$$

Most of the other identities are consequences of Iden 2 in rather obvious ways. We remark briefly about some not of this class. To obtain Iden 4 consider

$$A^{ab}A^{cd} \equiv \epsilon^{ab3}\epsilon^{cd3} = \begin{vmatrix} \delta_1^a & \delta_2^a \\ \delta_1^b & \delta_2^b \end{vmatrix} \cdot \begin{vmatrix} \delta_1^c & \delta_1^d \\ \delta_2^c & \delta_2^d \end{vmatrix} \equiv \begin{vmatrix} E^{ac} & E^{ad} \\ E^{bc} & E^{bd} \end{vmatrix}.$$

Iden 10 may be obtained from the determinantal product

$$\begin{vmatrix} \delta_1^a & \delta_2^a & \delta_3^a \\ \delta_1^b & \delta_2^b & \delta_3^b \\ \delta_1^c & \delta_2^c & \delta_3^c \end{vmatrix} \cdot \begin{vmatrix} \delta_1^d & \delta_1^e & \delta_1^f \\ \delta_2^d & \delta_2^e & \delta_2^f \\ 0 & 0 & 0 \end{vmatrix} \equiv 0,$$

and Iden 11 from

$$\begin{vmatrix} \delta_1^a & \delta_2^a & \delta_3^a \\ \delta_1^b & \delta_2^b & \delta_3^b \\ \delta_1^c & \delta_2^c & \delta_3^c \end{vmatrix} \cdot \begin{vmatrix} \delta_1^d & \delta_1^e & X^1 \\ \delta_2^d & \delta_2^e & X^2 \\ 0 & 0 & X^3 \end{vmatrix} \equiv \epsilon^{abc}A^{de}L_fX^f.$$

That these identities constitute a reduction basis for the concomitants constructed by the first fundamental theorem may be seen by observing that they give the alternate ways of writing all types of products which arise.

3. **An irreducible system, complete through the fifth degree, of euclidean invariants for the cubic $T_{abc}X^aX^bX^c$.** We note that the cubic has only one "formal" invariant of the first degree, $T_{abc}\epsilon^{abc}$, and this vanishes identically due to the skew-symmetry of ϵ^{abc}; so (i) for the cubic there is no invariant of the first degree.

We can construct three (and only three) invariants of the second degree:

$$A = T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1a_2}E^{b_1b_2}E^{a_3b_3}, \qquad B = T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1b_1}E^{a_2b_2}E^{a_3b_3},$$
$$C = T_{a_1a_2a_3}T_{b_1b_2b_3}A^{a_1b_1}A^{a_2b_2}E^{a_3b_3}.$$

Invariants as $T_{a_1a_2a_3}T_{b_1b_2b_3}A^{a_1b_1}A^{a_2b_2}A^{a_3b_3}$ and $T_{a_1a_2a_3}T_{b_1b_2b_3}A^{a_1a_2}E^{b_1b_2}E^{a_3b_3}$ which vanish identically on the interchange of equivalent indices are not listed here, nor in similar circumstances in the future. In this connection one should keep in mind the skew-symmetry of ϵ^{abc} and of A^{ab}. The only identities applicable to A and B are Idens 4′ and 10. The application of the latter to each of these merely gives $A \equiv A$ and $B \equiv B$. The application of Iden 4′ to either of them results in $C \equiv A - B$. The only identities applicable to C are Idens 4 and 5. The first results in the relation just given, and the latter in $C \equiv C$. Thus $C \equiv A - B$ is the only relation on the invariants A, B, C. Therefore, (ii) for the cubic there are two irreducible invariants of the second degree, and these may be chosen as

$$II_1 = T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1a_2}E^{b_1b_2}E^{a_3b_3}, \qquad II_2 = T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1b_1}E^{a_2b_2}E^{a_3b_3}.$$

Often we shall use the contracted notation

$${}_{(k)}T = T_{a_{11}a_{12}a_{13}}T_{a_{21}a_{22}a_{23}} \cdots T_{a_{k1}a_{k2}a_{k3}}.$$

To illustrate, $II_1 = {}_{(2)}T \cdot E^{a_1a_2}E^{b_1b_2}E^{a_3b_3}$.

There are three invariants of the third degree to consider:

$$A = {}_{(3)}T \cdot A^{a_1b_1}E^{b_2c_1}A^{c_2a_2}\epsilon^{a_3b_3c_3}, \qquad B = {}_{(3)}T \cdot E^{a_1c_1}E^{b_1b_2}A^{c_2a_2}\epsilon^{a_3b_3c_3},$$
$$C = {}_{(3)}T \cdot A^{a_1b_1}A^{b_2c_1}A^{c_2a_2}\epsilon^{a_3b_3c_3}.$$

Applying Iden 5 to $A^{c_2a_2}E^{b_1b_2}$ of B, we get $B = -A - A$, or $B = -2A$. Applying Iden 4 to $A^{a_1b_1}A^{b_2c_1}$ of C, we obtain $C = A - B$. On application of the appropriate reduction formulas in all possible ways to A, B, C, it is found that there is no relation on these expressions other than the two given. Hence: (iii) For the cubic there is one irreducible invariant of the third degree, and this may be chosen as

$$III = (1/6)T_{a_1a_2a_3}T_{b_1b_2b_3}T_{c_1c_2c_3}A^{a_1b_1}A^{b_2c_1}A^{c_2a_2}\epsilon^{a_3b_3c_3}.$$

The number of formal invariants which one can construct increases rapidly as the degree goes beyond three. Their consideration is facilitated by considering all invariants of a certain type together. (iv-1) For the cubic there is only one invariant of the fourth degree which contains the numerical tensor ϵ^{abc} four times, and this is irreducible, it being the well known projective invariant of the fourth degree .

$$IV_1 \doteq {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2d_1}\epsilon^{a_3c_2d_2}\epsilon^{b_3c_3d_3}.$$

There are six invariants of the fourth degree which contain factors like ϵ^{abc} twice, the remaining factors being like E^{ab}:

$$A = {}_{(4)}T\ \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{a_3d_1}E^{b_3d_2}E^{c_3d_3}, \quad B = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{a_3b_3}E^{c_3d_1}E^{d_2d_3},$$
$$C = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2a_3}E^{b_3c_3}E^{d_2d_3}, \quad D = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2a_3}E^{b_3d_2}E^{c_3d_3},$$
$$E = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2d_2}E^{a_3d_3}E^{b_3c_3}, \quad F = {}_{(4)}T \cdot \epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2b_3}E^{c_3d_2}E^{a_3d_3}.$$

Applying the basic identities in all possible ways, we find that these invariants are connected by the relations

$$A \equiv D + 2F; \quad B \equiv E + 2F; \quad B \equiv C + 2D; \quad C \equiv 2D + E - 2F;$$

and only these. Therefore: (iv-2) For the cubic there are two irreducible invariants of the fourth degree which contain factors like ϵ^{abc} twice, the remaining factors being like E^{ab}, and these may be chosen as

$$IV_2 = {}_{(4)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2a_3}E^{b_3c_3}E^{d_2d_3},$$

$$IV_3 = {}_{(4)}T\ \epsilon^{a_1b_1c_1}\epsilon^{b_2c_2d_1}E^{a_2b_3}E^{c_3d_2}E^{a_3d_3}.$$

Like considerations lead us to the conclusions: (iv-3) For the cubic there is one irreducible invariant of the fourth degree which contains factors like ϵ^{abc} twice, the other factors being like E^{ab}, and this may be chosen as

$$IV_4 = {}_{(4)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{b_3c_3}E^{d_1d_2}A^{a_3d_3}.$$

(iv-4) For the cubic there is one irreducible invariant of the fourth degree whose numerical tensor product is composed entirely of factors like E^{ab}, except for one factor like A^{ab}, and this may be chosen as

$$IV_5 = {}_{(4)}T\cdot E^{a_1a_2}A^{a_3b_1}E^{b_2c_2}E^{b_3c_1}E^{c_3d_1}E^{d_2d_3}.$$

For invariants of the fourth degree constructed wholly with the aid of E^{ab}, there are five possibilities:

$$A = {}_{(4)}T\cdot E^{a_1a_2}E^{a_3b_1}E^{b_2c_1}E^{b_3c_2}E^{c_3d_1}E^{d_2d_3},$$

$$B = {}_{(4)}T\cdot E^{a_1b_1}E^{a_2b_2}E^{a_3c_1}E^{b_3d_1}E^{c_2d_2}E^{c_3d_3},$$

$$C = {}_{(4)}T\cdot E^{a_1b_1}E^{a_2c_1}E^{a_3d_1}E^{b_2c_2}E^{b_3d_2}E^{c_3d_3},$$

$$D = {}_{(4)}T\cdot E^{a_1a_2}E^{a_3b_1}E^{b_2c_1}E^{b_3d_1}E^{c_2d_2}E^{c_3d_3},$$

$$E = {}_{(4)}T\cdot E^{a_1a_2}E^{a_3b_1}E^{b_2c_1}E^{b_3d_1}E^{c_2c_3}E^{d_2d_3}.$$

Applying the identities 1–11 in all possible ways it is found that there are four and only four relations on these expressions; so: (iv-5) For the cubic there is only one irreducible invariant constructed wholly with the aid of E^{ab}, of the fourth degree, and this may be chosen as

$$IV_6 = {}_{(4)}T\cdot E^{a_1a_2}E^{a_3b_1}E^{b_2c_1}E^{b_3c_2}E^{c_3d_1}E^{d_2d_3}.$$

Then

$$B = IV_6 + (1/2)(II_2)^2 - (1/2)(II_1)^2,$$

$$C = IV_6 + II_1\cdot II_2 - (II_1)^2,$$

$$D = IV_6 + (1/2)II_1\cdot II_2 - (1/2)(II_1)^2,$$

$$E = IV_6 + (1/2)(II_1)^2 - (1/2)II_1\cdot II_2.$$

All other invariants of the fourth degree are found to be reducible; so we may summarize as follows: (iv) For the cubic there are six irreducible invariants of the fourth degree, and these may be chosen as IV_1, IV_2, IV_3, IV_4, IV_5, IV_6.

In a similar manner it is found that: (v) For the cubic there are two irreducible invariants of the fifth degree, and these may be chosen as

$$V_1 = {}_{(5)}T\cdot \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}\epsilon^{a_3d_1e_1}A^{d_2e_2}E^{b_3c_3}E^{d_3e_3},$$

$$V_2 = {}_{(5)}T\cdot \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}\epsilon^{a_3d_1e_1}A^{b_3c_3}E^{d_2d_3}A^{e_2e_3}.$$

A typical reduction is that of $A \equiv {}_{(5)}T\cdot \epsilon^{d_1b_1c_1}A^{a_1b_2}E^{a_2d_2}E^{c_2a_3}E^{b_3d_3}E^{e_1e_2}E^{c_3e_3}$. Iden 5 applied to $A^{a_1b_2}E^{d_2e_2}$ of A gives $A \equiv 4II_1\cdot III - A$, or $A \equiv 2II_1\cdot III$.

We may combine (i), (ii), (iii), (iv), (v) in the following theorem.

THEOREM III. *For the cubic* $T_{abc}X^aX^bX^c$ $(a, b, c = 1, 2, 3)$ *there are eleven irreducible euclidean invariants of degree less than six, and these may be chosen as* II_1, II_2, III, IV_1, IV_2, IV_3, IV_4, IV_5, IV_6, V_1, V_2.

4. **An irreducible system, complete through the fourth degree, of euclidean covariants for the cubic.** There seems to have been no systematic study of euclidean covariants for the cubic curve of the third order. The best known are the Hessian, the polar conic of the line at infinity, and the Laplacian. We shall often speak of a covariant of degree i in T_{abc} and order j in X^a as a (i, j) covariant. The line at infinity is a (0, 1) covariant, having for its equation $\mathfrak{L} = L_aX^a = X^3 = 0$.

4.1. *$(i$, 1) covariants.* For the cubic $T_{abc}X^aX^bX^c$ there is only one (1, 1) covariant, and this is $L_1 = T_{abc}E^{ab}X^c$. There is no (2, 1) covariant. There are four irreducible (3, 1) covariants, and these may be chosen as

$$L_{31} = {}_{(3)}T\cdot \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{b_3c_3}X^{a_3}, \qquad L_{32} = {}_{(3)}T\cdot E^{a_1b_1}E^{a_2c_1}E^{c_2c_3}E^{b_2b_3}X^{a_3},$$

$$L_{33} = {}_{(3)}T\cdot A^{a_1b_1}E^{a_2c_1}E^{b_2c_2}E^{b_3c_3}X^{a_3}, \qquad L_{34} = {}_{(3)}T\cdot A^{a_1b_1}E^{a_2c_1}E^{b_2b_3}E^{c_2c_3}X^{a_3}.$$

The other (3, 1) covariants

$$A = {}_{(3)}T\cdot E^{a_1b_1}E^{a_2c_1}E^{b_2c_2}E^{b_3c_3}X^{a_3}, \qquad B = {}_{(3)}T\cdot E^{a_1b_1}E^{a_2b_2}E^{b_3c_1}E^{c_2c_3}X^{a_3},$$

$$C = {}_{(3)}T\cdot A^{a_1b_1}E^{a_2b_2}E^{b_3c_1}E^{c_2c_3}X^{a_3}, \qquad D = {}_{(3)}T\cdot E^{a_1b_1}E^{a_2b_2}A^{b_3c_1}E^{c_2c_3}X^{a_3},$$

are expressible in terms of the irreducible ones by the relations:

$$A \equiv L_{32} + (II_2 - II_1) + 2III\cdot\mathfrak{L},$$

$$B \equiv L_{32} + (1/2)(II_2 - II_1)L_1 + 2III\cdot\mathfrak{L},$$

$$C \equiv L_{33},$$

$$D \equiv L_{34} - L_{33}.$$

In like manner it is found that there is only one irreducible (4, 1) covariant, and this may be chosen as

$$L_4 \equiv {}_{(4)}T\cdot \epsilon^{a_1b_1c_1}E^{a_2b_2}E^{d_1d_2}E^{b_3c_2}E^{c_3d_3}X^{a_3}.$$

Therefore we have this theorem.

THEOREM IV. *For the cubic there are seven irreducible* $(i, 1)$ *euclidean covariants* $(i=0, 1, 2, 3, 4)$, *and these may be chosen as* $\mathfrak{L}$, L_1, L_{31}, L_{32}, L_{33}, L_{34}, *and* L_4.

4.2. $(i, 2)$ *covariants*. Obviously there is no $(1, 2)$ covariant. There are four $(2, 2)$ covariants:

$$A = {}_{(2)}T\cdot E^{a_1b_1}E^{a_2b_2}X^{a_3}X^{b_3}, \qquad B = {}_{(2)}T\cdot A^{a_1b_1}A^{a_2b_2}X^{a_3}X^{b_3},$$

$$C = {}_{(2)}T\cdot A^{a_1b_1}E^{b_2b_3}X^{a_2}X^{a_3}, \qquad D = {}_{(2)}T\cdot E^{a_1b_1}E^{b_2b_3}X^{a_2}X^{a_3}.$$

The expressions C and D contain no significant reducible factor, and consequently are irreducible. Iden 4 applied to $A^{a_1b_1}A^{a_2b_2}$ gives $B=(L_1)^2-A$. This is the only relation that exists on A and B; so (i) there are three irreducible $(2, 2)$ covariants, and these may be chosen as

$$C_{21} = {}_{(2)}T\cdot A^{a_1b_1}A^{a_2b_2}X^{a_3}X^{b_3}, \qquad C_{22} = {}_{(2)}T\cdot A^{a_1b_1}E^{b_2b_3}X^{a_2}X^{a_3},$$

$$C_{23} = {}_{(2)}T\cdot E^{a_1b_1}E^{b_2b_3}X^{a_2}X^{a_3}.$$

One may construct three $(3, 2)$ covariants

$$A = {}_{(3)}T\cdot \epsilon^{a_1b_1c_1}A^{a_2b_2}E^{c_2c_3}X^{a_3}X^{b_3}, \qquad B = {}_{(3)}T\cdot \epsilon^{a_1b_1c_1}A^{a_2c_2}E^{b_2c_3}X^{a_3}X^{b_3},$$

$$C = {}_{(3)}T\cdot \epsilon^{a_1b_1c_1}A^{b_2c_2}E^{b_3c_3}X^{a_2}X^{a_3}.$$

Applying Iden 5 to $A^{a_2b_2}C^{c_2c_3}$ of A we obtain $A=2B$. Iden 6 applied to $A^{a_2c_2}X^{b_3}$ of B gives $B=C-B-L_{31}\cdot\mathfrak{L}$, or $2B=C-L_{31}\cdot\mathfrak{L}$. There is no other relation on these expressions. Therefore (ii) there is one irreducible $(3, 2)$ covariant, and this may be chosen as

$$C_3 = {}_{(3)}T\cdot \epsilon^{a_1b_1c_1}A^{b_2c_2}E^{b_3c_3}X^{a_2}X^{a_3}.$$

(iii) By a similar procedure it is found that there are five irreducible $(4, 2)$ covariants, and these may be chosen as

$$C_{41} = {}_{(4)}T\ \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{c_3d_1}E^{d_2d_3}X^{a_3}X^{b_3},$$

$$C_{42} = {}_{(4)}T\cdot \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2d_1}E^{c_2c_3}E^{d_2d_3}X^{a_3}X^{b_3},$$

$$C_{43} = {}_{(4)}T\cdot \epsilon^{b_1c_1d_1}\epsilon^{b_2c_2d_2}E^{a_1b_3}E^{c_3d_3}X^{a_2}X^{a_3},$$

$$C_{44} = {}_{(4)}T\cdot \epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{d_1d_2}A^{c_3d_3}X^{a_3}X^{b_3},$$

$$C_{45} = {}_{(4)}T\cdot \epsilon^{b_1c_1d_1}\epsilon^{b_2c_2d_2}E^{c_3d_3}A^{b_3a_1}X^{a_2}X^{a_3}.$$

Combining (i), (ii), and (iii) we have this theorem

THEOREM V. *For the cubic there are nine irreducible* $(i, 2)$ *euclidean covariants* $(i=1, 2, 3, 4)$, *and these may be chosen as* C_{21}, C_{22}, C_{23}, C_3, C_{41}, C_{42}, C_{43}, C_{44}, C_{45}.

Using the general methods of construction and reduction as we have above, we find that this theorem follows.

THEOREM VI. *For the cubic* ($i=1, 2, 3, 4$): (i) *there are six euclidean irreducible* (i, 3) *covariants, and these may be chosen as*

$$\begin{aligned} T_{31} &= {}_{(3)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}X^{a_3}X^{b_3}X^{c_3},\\ T_{32} &= {}_{(3)}T\cdot A^{a_1b_1}E^{b_2c_1}E^{c_2a_2}X^{b_3}X^{a_3}X^{c_3},\\ T_{33} &= {}_{(3)}T\cdot A^{b_1c_1}E^{b_2c_2}E^{a_1c_3}X^{b_3}X^{a_2}X^{a_3},\\ T_{41} &= {}_{(4)}T\cdot\epsilon^{a_1b_1c_1}E^{b_2d_1}E^{c_2d_2}E^{a_2d_3}X^{b_3}X^{a_3}X^{c_3},\\ T_{42} &= {}_{(4)}T\cdot\epsilon^{d_1b_1c_1}E^{b_2c_2}E^{a_1d_2}E^{a_2d_3}X^{b_3}X^{a_3}X^{c_3},\\ T &= T_{abc}X^aX^bX^c. \end{aligned}$$

(ii) *There are three irreducible* (i, 4) *covariants, and these may be chosen as*

$$Q_2 = {}_{(2)}T\cdot E^{a_1b_1}X^{a_2}X^{a_3}X^{b_2}X^{b_3},\quad Q_{41} = {}_{(4)}T\cdot\epsilon^{b_1c_1d_1}\epsilon^{b_2c_2d_2}E^{a_1d_3}X^{a_2}X^{b_3}X^{c_3},$$

$$Q_{42} = {}_{(4)}T\cdot\epsilon^{b_1c_1d_1}\epsilon^{b_2c_2d_2}A^{a_1d_3}X^{a_2}X^{a_3}X^{b_3}X^{c_3}.$$

(iii) *There are two irreducible quintics, and these may be chosen as*

$$\begin{aligned} R_3 &= {}_{(3)}T\cdot E^{a_1c_1}A^{b_1c_2}X^{a_2}X^{a_3}X^{b_2}X^{b_3}X^{c_3},\\ R_4 &= {}_{(4)}T\cdot\epsilon^{a_1c_1d_1}E^{b_1c_2}E^{d_2d_3}X^{a_2}X^{a_3}X^{b_2}X^{b_3}X^{c_3}. \end{aligned}$$

(iv) *There is no irreducible covariant for* $i<5, j>6$.

5. **Geometric interpretations.** It should be understood that the general cubic curve given by the general equation of the third degree in X^a,

$$T = T_{abc}X^aX^bX^c = 0,$$

($a, b, c=1, 2, 3$ and T_{abc} symmetric) is under consideration here. For such cubic no invariant is zero, and no covariant vanishing identically, unless specifically stated so. The point $P_2 = T_{a_1a_2a_3}T_{b_1b_2b_3}\cdot\epsilon^{a_1b_1r}A^{a_2b_2}E^{a_3b_3}U_r=0$ in expanded form is

$$\begin{gathered} (T_{112}T_{133}+T_{122}T_{223}-T_{123}T_{113}-T_{123}T_{222})U_1\\ (T_{122}T_{133}+T_{112}T_{113}-T_{112}T_{223}-T_{111}T_{123})U_2\\ (T_{111}T_{112}-(T_{113})^2-(T_{133})^2+T_{112}T_{222})U_3=0. \end{gathered}$$

Keeping this expanded form in mind, and examining the contents of the paper by Thomae' [6], or that by Stuyvaert [7], we observe the truth of this theorem.

THEOREM VII. *P_2 is the unique point whose polar conic is a circle.*

When we speak of the polar of a point (or of the pole of a line), unless specified to the contrary, we mean with respect to the fundamental cubic T.

THEOREM VIII. (a) *The polar conic of* P_3 *is* C_3, *and* (b) C_3 *is a circle.*

Obviously the polar conic of Y^a with respect to T has the equation $T_{abc}Y^aX^bX^c=0$. Let $Y^rU_r=T_{a_1a_2a_3}T_{b_1b_2b_3}\epsilon^{a_1b_1r}A^{a_2b_2}E^{a_3b_3}U_r$. Then (a) is evident, and (b) follows.

THEOREM IX. *Every covariant conic of the cubic T whose coefficients are of the third degree in T_{abc} is a circle.*

In §4.2 it was shown that one can construct three (3, 2) covariants, there designated by A, B, C, and that these are connected by the relations $A=2B$, and $2B=C-L_{31}\cdot\mathfrak{L}$. By Theorem VIII, C_3 is a circle, and as a consequence of the relations just given A and B are also circles. Recall from §4 that there are no covariant conics of degree less than two, that there are covariant lines of the first and third degree, but none of the second, and further from §3 there is no invariant of degree less than two. As a consequence of these facts we conclude that every covariant conic Q of the third degree is of the form $Q=k_1C_3+k_2R$, where k_1 and k_2 are constants, and R is at most linear in X^a. But since C_3 is a circle, every such conic Q is a circle.

The condition that the polar conic of the point Y^a, $T_{abc}X^aX^bY^c$, be a rectangular hyperbola is that $(T_{11c}+T_{22c})Y^c=T_{abc}E^{ab}Y^c=0$.

THEOREM X. *The line $L_1=0$ is the locus of points whose polar conics are rectangular hyperbolas.*

This line is called the Laplacian of the cubic. The cubic for which $L_1=0$ has been studied by Brooks [8].

THEOREM X′. *The Laplacian of the Hessian* $H={}_{(3)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}\cdot X^{a_3}X^{b_3}X^{c_3}=0$ *is* $L_{31}=0$.

Adapting the argument of White [9] to tensor notation, we find that the equation of the polar conic of the line $V_aX^a=0$ with respect to T is

$$T_{a_1a_2a_3}T_{b_1b_2b_3}\epsilon^{a_1b_1p}\epsilon^{a_2b_2q}X^{a_3}X^{b_3}U_pU_q = 0.$$

If in this we replace U_p by $L_p=[0, 0, 1]$ and use $A^{ab}=\epsilon^{abc}L_c$, we have this next theorem.

THEOREM XI. *The polar conic of the line at infinity is* $C_{31}=0$.

In like manner, we have this theorem.

THEOREM XI′. *The polar conic of the Laplacian is* $C_{42}=0$.

The polar line of the point $Y^rU_r=0$ with respect to $C_{21}=0$ is

$$T_{a_1a_2a_3}T_{b_1b_2b_3}A^{a_1b_1}A^{a_2b_2}X^{a_3}X^{a_3}Y^{b_3} = 0.$$

In particular let $Y^rU_r=0$ be P_2. We have for the polar of P_2 with respect to C_{21}

$$\begin{aligned} K &= T_{a_1a_2a_3}T_{b_1b_2b_3}T_{c_1c_2c_3}T_{d_1d_2d_3}A^{a_1b_1}A^{a_2b_2}\epsilon^{c_1d_1b_3}A^{c_2d_2}E^{c_3d_3}X^{a_3} \\ &= 4III\cdot L_1 - {}_{(4)}T\cdot\epsilon^{d_1b_1c_1}A^{d_2b_2}E^{d_3b_3}E^{c_2c_3}E^{a_2a_3}X^{a_1} \text{ (by Iden 4).} \end{aligned}$$

Using other identities we find that the last term is $2III\cdot L_1$; so $K=2III\cdot L_1$.

THEOREM XII. *The Laplacian of a cubic is the polar with respect to the polar conic of the line at infinity of the unique point whose polar conic is a circle.*

Rather evident are these theorems and corollary.

THEOREM XIII. *The polar conic of* $T_{abc}E^{ab}A^{cr}U_r=0$, *the point at infinity on the Laplacian, is* $C_{22}=0$.

COROLLARY. $C_{22}=0$ *is a rectangular hyperbola.*

THEOREM XIV. *The polar conic of* $T_{abc}E^{ab}E^{cr}U_r=0$, *the point at infinity in the direction perpendicular to the Laplacian, is* $C_{22}=0$.

Taking the polar of $T_{abc}E^{ab}E^{cr}U_r=0$ and $T_{abc}E^{ab}A^{cr}U_r=0$, respectively, with respect to $C_{22}=0$, we get these theorems.

THEOREM XV. *The diameter of the conic* C_{22} *conjugate to the direction perpendicular to the Laplacian is* $L_{22}=0$.

THEOREM XVI. *The diameter of the conic* C_{22} *conjugate to the direction of the Laplacian is* $L_{24}=0$.

The condition for two lines V_a and W_a to be perpendicular is that $V_aW_bE^{ab}=0$, and the condition for them to be parallel is that $V_aW_bA^{ab}=0$. From these a number of geometric facts follow quite directly:

A.1. The locus of points whose linear polars are parallel to the Laplacian is the conic $C_{22}=0$.

2. The locus of points whose linear polars are perpendicular to the Laplacian is the conic $C_{22}=0$.

3. The locus of points whose linear polars with respect to the Hessian are perpendicular to the Laplacian is the conic $C_{41}=0$.

4. The locus of points whose linear polars with respect to the Hessian are parallel to the Laplacian is the conic $C_{44}=0$.

5. The locus of points whose linear polars are perpendicular to the Laplacian of the Hessian is the conic $C_{43}=0$.

6. The locus of points whose linear polars are parallel to the Laplacian of the Hessian is the conic $C_{44}=0$.

7. The locus of points whose linear polars with respect to the fundamental cubic and the Hessian are perpendicular is the quartic $Q_{41}=0$.

8. The locus of points whose linear polars with respect to the fundamental cubic and the Hessian are parallel is the quartic $Q_{42}=0$.

9. The locus of points whose linear polars with respect to fundamental cubic and the quartic Q_2 are parallel is the quintic $R_2=0$.

10. The Laplacian of the fundamental cubic and the Laplacian of the Hessian are parallel if (and only if) $IV_4=0$, and they are perpendicular if

$$ {}^{(4)}T\cdot\epsilon^{a_1b_1c_1}\epsilon^{a_2b_2c_2}E^{b_3c_3}E^{a_3d_1}E^{d_2d_3}\equiv IV_2+2IV_3=0. $$

11. The Laplacian and $L_{32}=0$ are parallel if $IV_5=0$, and they are perpendicular if

$$ {}^{(4)}T\cdot E^{a_1a_2}E^{a_3b_1}E^{b_2c_2}E^{b_3c_3}E^{c_1d_1}E^{d_2d_3}\equiv IV_6+(1/2)(II_1)^2-(1/2)II_1\cdot II_2=0. $$

The line $V_aX^a=0$ is said to be minimal if $V_aV_bE^{ab}=0$. Hence we have the following facts.

B.1. The Laplacian is a minimal line if (and only if) $II_1=0$.

2. The locus of points whose linear polars are minimal lines is the quartic $Q_2=0$.

Using the condition for incidence of the point V_a and the line Y^a, namely that $V_aY^a=0$, we get these statements.

C.1. P_2, the unique point whose polar conic is a circle, lies on the Laplacian if $III=0$.

2. P_2 lies on the Laplacian of the Hessian if $V_1=0$.

Not so direct is the next fact.

3. The line through P_2 and perpendicular to the Laplacian is $L_{32}=0$.

6. **Some concomitants in expanded form.** We list a few typical concomitants in expanded form. These are obtained by carrying out the indicated summations in the tensor-invariant forms of the concomitants, using the defined values of the components of ϵ^{abc}, A^{ab}, and E^{ab}.

$$ \begin{aligned} II_1 &= T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1a_2}E^{b_1b_2}E^{a_3b_3}=(T_{111}+T_{122})^2+(T_{112}+T_{222})^2,\\ II_2 &= T_{a_1a_2a_3}T_{b_1b_2b_3}E^{a_1b_1}E^{a_2b_2}E^{a_3b_3}\\ &= (T_{111})^2+3(T_{112})^2+3(T_{122})^2+(T_{222})^2, \end{aligned} $$

$$III = (1/6)T_{a_1a_2a_3}T_{b_1b_2b_3}T_{c_1c_2c_3}A^{a_1b_1}A^{b_2c_1}A^{a_2c_2}\epsilon^{a_3b_3c_3}$$

$$= \begin{vmatrix} T_{111} & T_{112} & T_{113} \\ T_{211} & T_{212} & T_{213} \\ T_{221} & T_{222} & T_{223} \end{vmatrix},$$

$$IV_5 = {}_{(4)}T \cdot E^{a_1a_2}A^{a_3b_1}E^{b_2c_1}E^{b_3c_2}E^{c_3d_1}E^{d_2d_3}$$

$$= 3(T_{111})^2T_{112}T_{122} + 2(T_{122})^3T_{222} + 3T_{111}(T_{122})^2T_{222}$$
$$+ 6T_{111}T_{112}(T_{122})^2 + T_{111}(T_{222})^3 + 3T_{112}(T_{122})^3$$
$$- 3T_{112}T_{122}(T_{222})^2 - 2T_{111}(T_{112})^3 - 3T_{111}(T_{112})^2T_{222}$$
$$- 6(T_{112})^2T_{122}T_{222} - (T_{111})^3T_{222} - 3(T_{112})^3T_{122},$$

$$L_1 = T_{abc}E^{ab}X^c = (T_{111} + T_{122})X^1 + (T_{112} + T_{222})X^2$$
$$+ (T_{113} + T_{223})X^3,$$

$$(1/2)C_{21} = T_{a_1a_2a_3}T_{b_1b_2b_3}A^{a_1b_1}A^{a_2b_2}X^{a_3}X^{b_3}$$

$$= [T_{111}T_{122} - (T_{112})^2](X^1)^2 + [T_{111}T_{222} - T_{112}T_{122}]X^1X^2$$
$$+ [T_{112}T_{222} - (T_{122})^2](X^2)^2$$
$$+ [T_{122}T_{113} + T_{111}T_{223} - 2T_{112}T_{123}]X^1X^3$$
$$+ [T_{112}T_{223} + T_{113}T_{222} - 2T_{122}T_{123}]X^2X^3$$
$$+ [T_{113}T_{223} - (T_{123})^2](X^3)^2.$$

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UNIVERSITY OF ALABAMA

GENERALIZATION OF A THEOREM OF KOROUS ON THE BOUNDS OF ORTHONORMAL POLYNOMIALS

DUNHAM JACKSON

1. **Introduction.** An elementary treatment of the convergence of series of orthogonal polynomials is greatly facilitated if the polynomials of the orthonormal set are known to be uniformly bounded on the domain of orthogonality, or on a part of it where convergence is to be proved.[1] A demonstration due to J. Korous[2] shows in a few lines that the orthonormal polynomials corresponding to a weight function $\rho\sigma$ on a finite interval are thus bounded, if the polynomials for weight ρ have the desired property, and if the factor σ satisfies a Lipschitz condition and has a positive lower bound on the entire domain of orthogonality. The purpose of this note is to show that the argument of Korous can be extended so as to apply under fairly general conditions to orthogonal polynomials in two real variables on an algebraic curve[3] and in particular to orthogonal trigonometric sums,[4] which can be regarded as orthogonal polynomials on a circle.[5] A problem of the same category has been discussed by Peebles[6] with less simple hypotheses on the factor σ.

In the case of trigonometric sums it is known in advance that the orthonormal functions for weight $\rho\equiv 1$, namely $(2\pi)^{-1/2}$, $\pi^{-1/2}\cos kx$, $\pi^{-1/2}\sin kx$, $k=1, 2, \cdots$, are uniformly bounded, and a theory of the convergence of developments in series of orthogonal trigonometric sums is opened up immediately. For other algebraic curves the question of the existence of a weight function which gives rise to a bounded system of orthonormal polynomials is one requiring separate investigation, and the answer to this question is known at present only in particular instances.[7] When the existence of a single such

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Presented to the Society April 12, 1941 under the title *Generalization of a theorem of Korous*; received by the editors November 17, 1941.

[1] See, for example, D. Jackson, *Series of orthogonal polynomials*, Annals of Mathematics, (2), vol. 34 (1933), pp. 527–545; pp. 531–538.

[2] See G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, 1939, p. 157; D. Jackson, *Fourier Series and Orthogonal Polynomials*, Carus Mathematical Monographs, no. 6, 1941, pp. 205–208.

[3] See D. Jackson, *Orthogonal polynomials on a plane curve*, Duke Mathematical Journal, vol. 3 (1937), pp. 228–236. This paper will be cited by the letter A.

[4] See, for example, D. Jackson, *Orthogonal trigonometric sums*, Annals of Mathematics, (2), vol. 34 (1933), pp. 799–814. This paper will be cited as B.

[5] [A, p. 234.]

[6] G. H. Peebles, this Bulletin, abstract 45-5-219.

[7] See, for example, D. Jackson, this Bulletin, abstract 45-5-192; Fulton Koehler,

ρ has been established for a specified domain of integration of the type to be discussed below, the conclusions of the present note lead to a wide generalization of the convergence theory for series of orthogonal polynomials associated with the curve in question.

2. **Statement of the problem.** Let C be an algebraic curve or a portion of an algebraic curve, degenerate or nondegenerate, consisting of a single connected piece of finite extent or a finite number of such pieces. Let C be of the mth degree, in the sense that there is a polynomial $\Pi(x, y)$ of the mth degree in x and y, but no polynomial of lower degree, which vanishes identically on C, while each irreducible factor of Π vanishes on a portion of C of positive length. The expression "connected pieces" will be understood to mean "pieces each of which is connected," not "pieces connected with each other." A single "piece," on the other hand, may be made up of parts corresponding to different irreducible factors of Π; it may be, for example, the perimeter of a square, or a set of line segments radiating from a common point. Let $\rho(x, y)$ be a non-negative weight function which is integrable over C with respect to arc length, and for each irreducible factor of Π is positive on a set of positive measure on the corresponding part of C.

Under these circumstances[3] there is a set of polynomials in x and y orthonormal on C with arc length as variable of integration and ρ as weight function, which includes just m polynomials of the nth degree for each value of $n \geqq m$, and $n+1$ polynomials of the nth degree for $n<m$. Let $p_{ni}(x, y)$, $n=0, 1, 2, \cdots$; $i=1, 2, \cdots, m$, be such an orthonormal set, with the supplementary convention that when $n+1<m$, $p_{ni}\equiv 0$ for $i>n+1$, and also, on occasion, that $p_{-1,i}\equiv 0$ for $1\leqq i\leqq m$. The value of $\int_C \rho p_{ni}^2 ds$ then is in every case either 1 or 0.

Let $\sigma(x, y)$ be a function defined and having a positive lower bound h on C, and satisfying the condition that if P_1 and P_2 are any two points on a connected portion of C, with coordinates (x_1, y_1) and (x_2, y_2),

$$\left| \sigma(x_2, y_2) - \sigma(x_1, y_1) \right| \leqq \lambda \Delta s,$$

where λ is constant (for simplicity of notation, a single constant for the whole of C), and Δs is the distance from P_1 to P_2 along C, or the shortest such distance, if P_1 and P_2 are connected by C in more than one way.

Systems of orthogonal polynomials on certain algebraic curves, this Bulletin, vol. 46 (1940), pp. 345–351.

[3] See [A, pp. 232–234]. The present hypotheses are somewhat more general than those which were explicitly formulated in the earlier paper, but no essential change is required in the reasoning which leads to recognition of the form of the orthonormal system.

Let $q_{ni}(x, y)$, $n=0, 1, 2, \cdots; i=1, 2, \cdots, n+1$ for $n<m$, $i=1, 2, \cdots, m$ for $n\geqq m$, be a set of orthonormal polynomials on C with $\rho\sigma$ as weight function; there is no occasion to introduce identically vanishing q's.

The question at issue is that of inferring boundedness of the polynomials q_{ni} *from that of the polynomials* p_{ni}. It will be found that such an inference is possible, under conditions to be specified.

3. **Proof of the theorem for an arbitrary algebraic curve.** Let

$$K_n(x, y, u, v) = \sum_{k=0}^{n}\sum_{i=1}^{m} p_{ki}(x, y)p_{ki}(u, v).$$

Then any one of the polynomials $q_{nj}(x, y)$, as a polynomial of the nth degree, has on C the representation

$$\begin{aligned} q_{nj}(x, y) &= \int_C \rho(u, v)K_n(x, y, u, v)q_{nj}(u, v)ds \\ &= \sum_{i=1}^{m} p_{ni}(x, y)\int_C \rho(u, v)p_{ni}(u, v)q_{nj}(u, v)ds \qquad (1) \\ &\quad + \int_C \rho(u, v)K_{n-1}(x, y, u, v)q_{nj}(u, v)ds; \end{aligned}$$

the integration is extended over the curve C in the (u, v)-plane. By application of Schwarz's inequality when $p_{ni}\neq 0$,

$$\begin{aligned} \left[\int_C \rho p_{ni}q_{nj}ds\right]^2 &\leqq \int_C \rho p_{ni}^2 ds\int_C \rho q_{nj}^2 ds = \int_C \rho q_{nj}^2 ds \qquad (2) \\ &\leqq \frac{1}{h}\int_C \rho\sigma q_{nj}^2 ds = \frac{1}{h}, \end{aligned}$$

the p's being normalized for weight ρ and the q's for weight $\rho\sigma$. It remains to examine in some detail the integral involving K_{n-1}. Let this integral be denoted by $I(x, y)$ or I.

The sum $K_{n-1}(x, y, u, v)$ has alternative representations of the form[9]

$$(3)\quad K_{n-1} = \frac{1}{u-x}\sum_{i=1}^{m}\sum_{l=1}^{m} c_{il}\left[p_{ni}(u, v)p_{n-1,l}(x, y) - p_{n-1,l}(u, v)p_{ni}(x, y)\right],$$

$$(4)\quad K_{n-1} = \frac{1}{v-y}\sum_{i=1}^{m}\sum_{l=1}^{m} d_{il}\left[p_{ni}(u, v)p_{n-1,l}(x, y) - p_{n-1,l}(u, v)p_{ni}(x, y)\right],$$

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[9] [A, pp. 235–236.]

as special cases of a more general expression having an arbitrary linear combination of $u-x$ and $v-y$ in the denominator. If G is a number such that $|x| \leqq G$, $|y| \leqq G$ on C, then

$$|c_{il}| \leqq G, \qquad |d_{il}| \leqq G$$

for all values of the subscripts. For

$$c_{il} = \int_C \rho u p_{ni} p_{n-1,l} ds,$$

and by another application of Schwarz's inequality

$$c_{il}^2 \leqq \int_C \rho u^2 p_{ni}^2 ds \int_C \rho p_{n-1,l}^2 ds \leqq G^2 \int_C \rho p_{ni}^2 ds = G^2;$$

a similar calculation applies to d_{il}.

Since $K_{n-1}(x, y, u, v)$ is for fixed (x, y) a polynomial in (u, v) of degree lower than the nth, it follows from the property of orthogonality of q_{nj} that

$$\int_C \rho(u, v)\sigma(u, v)K_{n-1}(x, y, u, v)q_{nj}(u, v)ds = 0.$$

Consequently $\sigma(x, y)I$ can be represented in the form

$$\sigma(x, y)I = \int_C \rho(u, v)[\sigma(x, y) - \sigma(u, v)]K_{n-1}(x, y, u, v)q_{nj}(u, v)ds. \tag{5}$$

Let C_1 be a portion of C forming a closed point set, consisting of a finite number of connected pieces, and containing no singular point, that is, neither a singular point of any nondegenerate component of C nor any point common to two such components. Then each connected part of C_1 is a straight line segment, a smooth arc, or a smooth closed loop.

The essential ideas of the argument are illustrated with a simpler initial formulation, and in a form adequate for some of the most interesting applications, if it is assumed that C itself has the characteristics ascribed to C_1, and that C_1 is the whole of C; the method is however of wider applicability as indicated. On the other hand, the reasoning could be made still more general, at the expense of some further attention to details, but the usefulness of the additional generality would not be so immediately apparent. The need for explicit consideration of certain wholly elementary items of detail arises from the fact that for fixed (x, y) either of the denominators $u-x$, $v-y$

in the expressions for K_{n-1} may vanish not only at the point $u=x$, $v=y$, but also at a number of other points of the curve.

At any point of C_1 the derivatives dy/dx and dx/dy have definite values, or else one of them has the value 0. Except on straight line segments, dy/dx can take on the values ± 1 at only a finite number of points. The locus C_1 is made up of a finite number of arcs or segments, on any one of which $|dy/dx|$ is everywhere greater than or equal to 1 or else everywhere less than or equal to 1. Any such arc C' (the word *arc* being regarded now and henceforth as applicable in particular to a line segment) is included in or coextensive with an arc C'', also belonging to C, on which $|dy/dx|$ is everywhere greater than or equal to 1/2, if it is greater than or equal to 1 on C', or everywhere less than or equal to 2, if it is less than or equal to 1 on C', and such that each end of C', if not a terminal point of C, is an interior point of C''. Let a definite arc C'' be associated in this way with each C'. The distance from any point of an arc C' to any point of C not belonging to the corresponding arc C'' has a positive lower bound δ for the whole of the locus C_1 which constitutes the aggregate of the arcs C'. It is to be noted further that $|ds/dx| \leqq 5^{1/2}$ on an arc C'' where $|dy/dx| \leqq 2$, and $|ds/dy| \leqq 5^{1/2}$ on the other type of arc C''.

Let P, with coordinates (x, y), be an arbitrary point of C_1. Let C'_P be the arc C' to which P belongs, or, if P is a common end point of two such arcs, let C'_P be either of them. Let K be the corresponding arc C''. Let the rest of C be subdivided into two parts K', K'', so that if P', with coordinates (u, v), is any point of K', the line PP' makes an angle not greater than $\pi/4$ with the x-axis, and if P' is on K'', PP' makes an angle less than $\pi/4$ with the y-axis. With the value of δ defined in the preceding paragraph, $|u-x| \geqq \delta/2^{1/2}$ for any point P' on K', and $|v-y| \geqq \delta/2^{1/2}$ if P' is on K''.

Let it be supposed for definiteness that C_P' is an arc of slope numerically less than or equal to 1; the alternative case would be treated in the same way, with the obvious interchanges of variables. For integration in (5), the point (x, y) being regarded as fixed, let $K_{n-1}(x, y, u, v)$ be represented by (3) when (u, v) is on K or K', and by (4) when (u, v) is on K''.

On K, since $\Delta s \leqq 5^{1/2}|\Delta x|$,

$$|\sigma(x, y) - \sigma(u, v)| \leqq 5^{1/2}\lambda |u - x|.$$

Let μ be an upper bound for σ on the whole of C. Then on K',

$$|[\sigma(x, y) - \sigma(u, v)]/(u - x)| \leqq 2^{3/2}\mu/\delta,$$

and there is a similar inequality on K'' with $u-x$ replaced by $v-y$.

By application of Schwarz's inequality, as in (2),

$$\int_C \rho(u, v) \left| p_{\alpha\beta}(u, v) q_{nj}(u, v) \right| ds \leq k^{-1/2},$$

and the same upper bound is valid if the integration is extended only over K, K', or K''.

Let it be supposed that $|p_{\alpha\beta}(x, y)| \leq H$ *on a point set* C_2 *contained in* C_1, *uniformly for all values of the subscripts* α *and* β. (In particular, C_2 may be the whole of C_1.) Then, by combination of the inequalities that have been obtained,

$$\sigma(x, y) \left| I(x, y) \right| \leqq (5^{1/2}\lambda + 2^{5/2}\mu/\delta) 2m^2Gk^{-1/2}H$$

if (x, y) is on C_2. (It is readily seen that the coefficient $2^{5/2} = 2 \cdot 2^{3/2}$ could be replaced by $2^{3/2}$, but the difference is immaterial for the purpose in hand.) The absolute value of the sum in the third member of (1) does not exceed $mk^{-1/2}H$. Consequently, for (x, y) on C_2,

$$\left| q_{nj}(x, y) \right| \leqq H\left[mk^{-1/2} + (5^{1/2}\lambda + 2^{5/2}\mu/\delta) 2m^2Gk^{-3/2}\right].$$

Since the right-hand member is independent of x, y, j, and n, it follows that *if the p's are uniformly bounded on* C_2, *the same is true of the q's.*

4. **Orthogonal trigonometric sums.** In a particular case the results can be interpreted as relating to orthogonal trigonometric sums, the unit circle or a finite set of arcs on the unit circle being taken for C and C_1. With $x = \cos \theta$, $y = \sin \theta$, the polynomials $p_{ni}(x, y)$, $q_{ni}(x, y)$ are trigonometric sums in θ. The weight function $\rho(x, y)$ and the factor $\sigma(x, y)$ will be represented alternatively by $\rho(\theta)$ and $\sigma(\theta)$. As far as the definition of the orthonormal system is concerned, the case of a domain of orthogonality consisting of detached intervals in a period is of course equivalent to that of a domain consisting of the entire period, with a weight function which vanishes identically outside the intervals in question. The conclusion can be stated as follows:

Let $u_0(\theta)$, $u_1(\theta)$, $v_1(\theta)$, $\cdots$ *constitute a set of orthonormal trigonometric sums with respect to a weight function* $\rho(\theta)$ *on a domain* D_1 *consisting of a period interval, regarded as closed, or of a finite number of closed intervals contained in a period; let* $\sigma(\theta)$ *have a positive lower bound on* D_1, *and satisfy a condition of the form*

$$\left| \sigma(\theta_2) - \sigma(\theta_1) \right| \leqq \lambda \left| \theta_2 - \theta_1 \right|$$

on each interval of D_1, *taking on the same value at both ends of the period interval if these end points belong to* D_1; *and let* $U_0(\theta)$, $U_1(\theta)$, $V_1(\theta)$, $\cdots$

be the orthonormal trigonometric sums on D_1 *for weight* $\rho(\theta)\sigma(\theta)$; *if the u's and v's are uniformly bounded on a point set* D_2 *contained in* D_1, *the same is true of the U's and V's.*

For this case the proof admits a materially simpler formulation than when geometric configurations are contemplated having the degree of generality previously considered. The details relating to the loci C', C'', K, K', K'', can be dispensed with for the most part; with θ replacing the pair of coordinates (x, y), and ϕ replacing the pair (u, v), it is sufficient, for any particular value of θ, to consider separately the intervals $(\theta-\pi/2, \theta+\pi/2)$ and $(\theta+\pi/2, \theta+3\pi/2)$, and in the integral corresponding to the right-hand member of (5) to represent $K_{n-1}(\theta, \phi)$ in the former interval by an expression with denominator $\sin(\theta-\phi)$, and in the latter interval by an alternative expression with $1-\cos(\theta-\phi)$ in the denominator.[10]

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[10] [B, pp. 808–809.]

APPROXIMATION OF CONTINUOUS FUNCTIONS BY MEANS OF LACUNARY POLYNOMIALS

BERNARD DIMSDALE

1. Introduction. All rational integral polynomials are linear combinations of members of the complete set of powers whose exponents are the non-negative integers. If certain members of this set are deleted, the linear combinations formed from the resulting set are, in the strict sense of the term, "lacunary polynomials." In a large part of this paper, however, methods of reasoning designed for the treatment of such polynomials are applicable to combinations from much more general sets of powers whose exponents are non-negative but not in general integral. The term "polynomial in x^{μ} of degree μ_n" will be applied to combinations from the set $1, x^{\mu_1}, x^{\mu_2}, \cdots$ where $\mu_1, \mu_2, \cdots$ form an arbitrarily preassigned set of real numbers such that $0<\mu_1<\mu_2<\cdots$, and μ_n is the largest exponent.

This paper started out as an investigation of lacunary orthogonal polynomials, and although this aspect of it became subordinate to the

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Presented to the Society, September 7, 1939 under the title *Degree of approximation by linear combinations of powers*; received by the editors March 20, 1941, and, in revised form, October 29, 1941.

problem of approximation, the relation of the latter to the theory of convergence of series of the orthogonal polynomials is pointed out at the end.

2. **Preliminary results.** It has been shown[1] that for all positive integers m the trigonometric sum

$$I_m(t) = h_m \int_{-\pi/2}^{\pi/2} g(t + 2u)F_m(u)du$$

with

$$F_m(u) = \left[\frac{\sin mu}{m \sin u}\right]^4, \qquad \frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} F_m(u)du,$$

of order $2m-2$, satisfies $|I_m(t)-g(t)| \leqq D\lambda/m$, for all t, provided that $g(t)$ has period 2π and satisfies a Lipschitz condition, constant λ, for all real t, where D is an absolute constant; and, furthermore, that if $g(t)$ is an even function, $I_m(t)$ will be a cosine sum, $I_m(t) = \alpha_0^{(m)} + \alpha_1^{(m)} \cos t + \cdots + \alpha_{2m-2}^{(m)} \cos (2m-2)t$.

Now

$$\alpha_k^{(m)} = \frac{1}{\pi} \int_0^{2\pi} I_m(t) \cos kt\, dt,$$

and when $I_m(t)$ is replaced by its integral form, order of integration reversed, and the substitution $t+2u=y$ made,

$$\alpha_k^{(m)} = \frac{h_m}{\pi} \int_{-\pi/2}^{\pi/2} F_m(u)\left[\int_0^{2\pi} \cos k(y - 2u)g(y)dy\right] du,$$

since $g(t)$ has period 2π. Since[2]

$$\left|\frac{1}{\pi}\int_0^{2\pi} g(t) \begin{matrix}\cos\\ \sin\end{matrix}\, kt\, dt\right| \leqq \frac{\pi\lambda}{k}$$

it follows on expanding $\cos k(y-2u)$ that[3]

$$|\alpha_k^{(m)}| \leqq \frac{h_m}{\pi}\int_{-\pi/2}^{\pi/2} F_m(u)\, \frac{\pi^2\lambda}{k}\, du + \frac{h_m}{\pi}\int_{-\pi/2}^{\pi/2} F_m(u)\, \frac{\pi^2\lambda}{k}\, du,$$

$$|\alpha_k^{(m)}| \leqq \frac{2\pi\lambda}{k}, \qquad k = 1, 2, \cdots, m - 1;\ m = 2, 3, \cdots.$$

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[1] D. Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. 11, 1930, pp. 2–6.

[2] See, for example, Titchmarsh, *The Theory of Functions*, p. 426.

[3] The following shorter proof has been given $|\alpha_k^{(m)}| = |(1/\pi)\int_0^{2\pi}[I_m(t)-g(t)+g(t)] \cos kt\, dt| \leqq 2D\lambda/m+\pi\lambda/k<\lambda(2D+\pi)/k$. The different value of the constant here would change subsequent computations slightly.

If $g(x)$ satisfies a Lipschitz condition, constant λ, on the closed interval[4] $[-1, 1]$, then for each positive integer m there exists a polynomial $P_m(y)$ in $1, y, y^2, \cdots$, of degree $2m-2$, such that on $[-1, 1]$

$$|g(y) - P_m(y)| \leqq \frac{D\lambda}{m}.$$

Here[5] $P_m(y) = I_m(t)$, where $I_m(t)$ is defined for $g(\cos t)$ and $y = \cos t$. Let $P_m(y) = a_0^{(m)} + a_1^{(m)} y + \cdots + a_{2m-2}^{(m)} y^{2m-2}$. The following inequalities result directly from the fact that $P_m(y) = I_m(t)$;

$$|a_{2k}^{(m)}| \leqq \frac{2^{2k}\pi\lambda}{2k} C_{2k}(m + k - 1),$$

$$|a_{2k-1}^{(m)}| \leqq \frac{2^{2k-1}\pi\lambda}{2k - 1} C_{2k-1}(m + k - 2),$$

where

$$C_r(s) = \frac{s!}{(s - r)!r!}, \qquad k = 1, 2, \cdots, m - 1; m = 2, 3, \cdots.$$

Let $y = x^H$, $g(x^H) = f(x)$, $H > 0$. Then[6] $|f(x_2) - f(x_1)| \leqq \lambda |x_2^H - x_1^H|$ on $[0, 1]$, implies that for every positive integer m there exists a polynomial

$$P_{mH}(x) = a_0^{(m)} + a_1^{(m)} x^H + \cdots + a_{2m-2}^{(m)} x^{(2m-2)H}$$

such that on $[0, 1]$,

$$|f(x) - P_{mH}(x)| \leqq \frac{D\lambda}{m}.$$

It will be observed that in this transformation the coefficients $a_k^{(m)}$ remain unchanged.

It has been shown by Szász[7] that for a sequence $\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_n$, where $0 < \gamma_0$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n$ and $\gamma_0 \neq \gamma_i$, $i = 1, 2, \cdots, n$, there exists a set of constants $u_1, u_2, \cdots, u_n$ such that

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[4] Hereafter the bracket notation for an interval implies closure at both ends.

[5] See the argument in D. Jackson, loc. cit., pp. 13–14.

[6] Hereafter this condition will be called a "Lipschitz condition H." It has been pointed out that this condition implies that $f(x)$ satisfies a Lipschitz condition of order α, $\alpha = H$ if $H \leqq 1$, $\alpha = 1$ if $H \geqq 1$. However, the converse is not true, since $x = 0$ belongs to the interval in question. Hence this condition is more restrictive than a Lipschitz condition of order α. On any closed interval not including 0 the two conditions are, however, equivalent.

[7] O. Szász, *Über die Approximation Stetiger Funktionen durch lineare Aggregate von Potenzen*, Mathematische Annalen, vol. 77 (1916), pp. 485–487.

$$\int_0^1 [x^{\gamma_0-1/2}+u_1x^{\gamma_1-1/2}+\cdots+u_nx^{\gamma_n-1/2}]^2dx \leqq \frac{1}{2\gamma_0}\prod_{k=1}^{n}\left[\frac{\gamma_k-\gamma_0}{\gamma_k+\gamma_0}\right]^2.$$

Let

$$A(x)=\frac{x^{\gamma_0+1/2}}{\gamma_0+1/2}+\frac{u_1x^{\gamma_1+1/2}}{\gamma_1+1/2}+\cdots+\frac{u_nx^{\gamma_n+1/2}}{\gamma_n+1/2},$$

$$B(x)=\frac{(\gamma_0+1/2)A(x)}{x^{1/2}}=x^{\gamma_0}+c_1x^{\gamma_1}+\cdots+c_nx^{\gamma_n},$$

$$C(x)=x^{\gamma_0-1/2}+u_1x^{\gamma_1-1/2}+\cdots+u_nx^{\gamma_n-1/2}.$$

Then

$$A(x)=\int_0^x C(x)dx,$$

and

$$\begin{aligned}[A(x)]^2&=\left[\int_0^x C(x)dx\right]^2\leqq\int_0^x dx\cdot\int_0^x [C(x)]^2dx\\&=x\cdot\int_0^x [C(x)]^2dx\leqq x\cdot\int_0^1 [C(x)]^2dx\\&\leqq\frac{x}{2\gamma_0}\prod_{k=1}^{n}\left[\frac{\gamma_k-\gamma_0}{\gamma_k+\gamma_0}\right]^2 \qquad \text{on } [0,1].\end{aligned}$$

Therefore on $[0, 1]$

$$|A(x)|\leqq\frac{x^{1/2}}{(2\gamma_0)^{1/2}}\prod_{k=1}^{n}\frac{|\gamma_k-\gamma_0|}{\gamma_k+\gamma_0},$$

$$|B(x)|\leqq\frac{2\gamma_0+1}{2(2\gamma_0)^{1/2}}\prod_{k=1}^{n}\frac{|\gamma_k-\gamma_0|}{\gamma_k+\gamma_0}.$$

If $\gamma_0=\gamma_i$ for some i, then the u's, and hence the c's, may obviously be chosen so that $|B(x)|=0$, and the inequality still holds.

It may now be shown that if $\gamma_0=rH$, $(i-1)H<\gamma_i\leqq iH$, then

$$L_{r,n}(x)=|B(x)|\leqq\frac{2rH+1}{2(2rH)^{1/2}}\cdot\frac{1}{C_r(2r-1)\cdot C_n(n+r)} \qquad \text{on } [0,1],$$

for $1\leqq r\leqq n$. The proof depends upon the fact that for $\gamma_0\geqq\gamma_k$,

$$\frac{\gamma_0-\gamma_k}{\gamma_0+\gamma_k}\leqq\frac{rH-(k-1)H}{rH+(k-1)H}=\frac{r-k+1}{r+k-1},$$

with a similar situation for $\gamma_0 \leqq \gamma_k$. If, in particular $\gamma_i = iH$ for some i, $\gamma_0 = iH$, then $L_{r,n}(x) = 0$, and the inequality is still true.

3. Degree of approximation.

This theorem will now be proved.

THEOREM I. *Consider the sequence* $\mu_0, \mu_1, \mu_2, \cdots$, *for which* $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$, *where* $\mu_{n+1} - \mu_n \leqq H$ *for all* n, *and* $\lim_{n\to\infty} \mu_n = \infty$. *If* $f(x)$ *satisfies a Lipschitz condition* H, *constant* λ, *on* $[0, 1]$, *then for every integer* $n > 0$ *there exists* $P_{\mu_n}(x)$, *a polynomial in* x^{μ} *of degree* μ_n, *such that on* $[0, 1]$ $|f(x) - P_{\mu_n}(x)| \leqq K\lambda/\mu_n$, *where* K *depends on* H *only.*

To prove this theorem, a polynomial $P_{mH}(x) = a_0^{(m)} + a_1^{(m)}x^H + \cdots + a_{2m-2}^{(m)}x^{(2m-2)H}$ is constructed so that on $[0, 1]$ $|f(x) - P_{mH}(x)| \leqq D\lambda/m$, with bounds on the a's as given in the last section. Unless $rH = \gamma_k$, each x^{rH} is then approximated by a polynomial $P^{(r)}_{\gamma_{4m-4}}(x) = -c_1x^{\gamma_1} - \cdots - c_{4m-4}x^{\gamma_{4m-4}}$, where the set of γ's is a subsequence of the set of μ's, $\gamma_0 = \mu_0$, γ_i is the smallest μ_n such that $(i-1)H < \gamma_i \leqq iH$, $i \geqq 1$; and $|x^{rH} - P^{(r)}_{\gamma_{4m-4}}(x)| = L_{r,4m-4}(x)$. Let

$$P_{\gamma_{4m-4}}(x) = a_0^{(m)} + \sum_{k=1}^{2m-2} a_k^{(m)} P^{(k)}_{\gamma_{4m-4}}(x).$$

Then

$$\begin{aligned} |f(x) - P_{\gamma_{4m-4}}(x)| &\leqq \frac{D\lambda}{m}, && \text{if } m = 1, \\ &\leqq \frac{D\lambda}{m} + \sum_{k=1}^{m-1} P_k + \sum_{k=1}^{m-1} Q_k, && \text{if } m = 2, 3, \cdots, \end{aligned}$$

where P_k and Q_k are the respective bounds of $|a_{2k}^{(m)}| \cdot |L_{2k,4m-4}(x)|$ and $|a_{2k-1}^{(m)}| \cdot |L_{2k-1,4m-4}(x)|$, as previously obtained. On calculation it is found that

$$\begin{aligned} \frac{P_{k+1}}{P_k} &= \frac{4(k+1)H+1}{4kH+1} \cdot \frac{k^{1/2}}{(k+1)^{1/2}} \cdot \frac{2k+1}{4m+2k-3} \\ &\quad \cdot \frac{m+k}{2m+k-1} \cdot \frac{2k}{2m-k-2} \cdot \frac{m-k-1}{4m-2k-5}, \\ \frac{Q_{k+1}}{Q_k} &= \frac{2(2k+1)H+1}{2(2k-1)H+1} \cdot \frac{(2k-1)^{1/2}}{(2k+1)^{1/2}} \cdot \frac{2k}{4m+2k-3} \\ &\quad \cdot \frac{m+k-1}{2m+k-2} \cdot \frac{2k-1}{2m-k-2} \cdot \frac{m-k-1}{4m-2k-3}, \end{aligned}$$

for $k = 1, 2, \cdots, m-2$; $m = 3, 4, 5, \cdots$. If now each fraction is replaced by its maximum for the given range of k with m fixed, and then by its maximum for the given range of m, it follows that, since $(8H+1)/(4H+1)\leqq 2$, $(6H+1)/(2H+1)\leqq 3$, $P_{k+1}/P_k\leqq 2/9$, $Q_{k+1}/Q_k \leqq 1/3$; hence $P_k\leqq(2/9)^{k-1}P_1$, $Q_k\leqq(1/3)^{k-1}Q_1$. Finally

$$\left| f(x) - P_{\gamma_{4m-4}}(x) \right| \leqq \frac{D\lambda}{m} + \frac{9}{7} P_1 + \frac{3}{2} Q_1, \qquad m = 1, 2, \cdots .$$

Now direct calculation shows that

$$P_1 \leqq \frac{(4H+1)\pi\lambda}{45H^{1/2}m}, \qquad Q_1 \leqq \frac{(2H+1)\pi\lambda}{5(2H)^{1/2}m}, \qquad m = 2, 3, \cdots ,$$

the equalities holding for $m=2$. Hence it appears that

$$\left| f(x) - P_{\gamma_{4m-4}}(x) \right| \leqq \frac{L\lambda}{m}, \qquad m = 1, 2, 3, \cdots ,$$

where L depends on H only.

Now let n be an arbitrary integer, and let $m=(n+4)/4$, $(n+3)/4$, $(n+2)/4$, $(n+1)/4$, respectively, according as n has the form $4k$, $4k-1$, $4k-2$, $4k-3$. In any event $4m-4\leqq n<4m$, $\gamma_n\leqq nH$. Let the corresponding $P_{\gamma_{4m-4}}(x)$ be denoted by $P_{\gamma_n}(x)$, a polynomial of degree γ_n. Since $1/m\leqq 4/n\leqq 4H/\gamma_n$ it follows that on $[0, 1]$

$$\left| f(x) - P_{\gamma_n}(x) \right| \leqq \frac{4HL\lambda}{\gamma_n}.$$

Consider the original sequence x^{μ_0}, x^{μ_1}, x^{μ_2}, $\cdots$. Let $P_{\mu_n}(x)$ be a polynomial of degree μ_n formed from this sequence, $P_{\mu_n}(x)=P_{\gamma_k}(x)$ where γ_k is the largest of the γ's in $\mu_0, \mu_1, \cdots, \mu_n$. Now $\mu_n\leqq kH$, since otherwise γ_k would not be the largest γ in the given set of μ's. If $k\geqq 2$, then $2\gamma_k>(2k-2)H\geqq kH\geqq\mu_n$, $1/\gamma_k<2/\mu_n$, and $\left|f(x)-P_{\mu_n}(x)\right| \leqq 8HL\lambda/\mu_n$. If $k=0$, 1 then $P_{\mu_n}(x)=P_{\gamma_0}(x)$, $\mu_n\leqq H$, $1\leqq H/\mu_n$, and $\left|f(x)-P_{\mu_n}(x)\right|\leqq L\lambda/1\leqq HL\lambda/\mu_n$. Hence the first of these two inequalities on $f(x)$ serves for all n, and the theorem is established.

This result may be extended to $[0, b]$, $b>0$, by assuming that $f(x)$ satisfies a Lipschitz condition H, constant λ, on $[0, b]$ and substituting $x=by$. Then there exists $\overline{P}_{\mu_n}(y)$ of degree μ_n such that on $[0, 1]$

$$\left| f(by) - \overline{P}_{\mu_n}(y) \right| \leqq \frac{Kb^H\lambda}{\mu_n},$$

whence on $[0, b]$

$$\left| f(x) - P_{\mu_n}(x) \right| \leqq \frac{Kb^H\lambda}{\mu_n},$$

with $P_{\mu_n}(x)=\overline{P}_{\mu_n}(y)$.

The result may also be extended to $[a, b]$, where $a > 0$, by defining $f(x) \equiv f(a)$ on $[0, a]$. In this case the condition on $f(x)$ may be replaced by one of the two simpler conditions.[5]

THEOREM II. *Suppose $f(x)$ is continuous on $[0, 1]$ and let*[6] *$\omega_H(\delta) = \max |f(x_2) - f(x_1)|$ for all x_1, x_2 on $[0, 1]$ satisfying the condition $|x_2^H - x_1^H| \leqq \delta$, and $\omega_H(\delta) \equiv \omega_H(1)$ if $\delta > 1$. Then for each $\mu_n > 0$ there exists a polynomial $P_{\mu_n}(x)$ such that on $[0, 1]$*

$$|f(x) - P_{\mu_n}(x)| \leqq \bar{K}\omega_H\left(\frac{1}{\mu_n}\right)$$

with $\bar{K} = K + 2$.

Let

$$\phi(x) = a_i x^H + b_i, \qquad \left(\frac{i}{\mu_n}\right)^{1/H} \leqq x \leqq \left(\frac{i+1}{\mu_n}\right)^{1/H}, \quad i = 0, 1, \cdots,$$

$$a_i = \mu_n\left\{f_1\left[\left(\frac{i+1}{\mu_n}\right)^{1/H}\right] - f_1\left[\left(\frac{i}{\mu_n}\right)^{1/H}\right]\right\},$$

$$b_i = (i+1)f_1\left[\left(\frac{i}{\mu_n}\right)^{1/H}\right] - i f_1\left[\left(\frac{i+1}{\mu_n}\right)^{1/H}\right],$$

where $f_1(x) \equiv f(x)$, $[0, 1]$, $f_1(x) \equiv f(1)$, $x \geqq 1$. Then

$$|\phi(x_2) - \phi(x_1)| \leqq \mu_n\omega_H\left(\frac{1}{\mu_n}\right)|x_2^H - x_1^H|, \qquad [0, 1],$$

$$|f(x) - \phi(x)| \leqq 2\omega_H\left(\frac{1}{\mu_n}\right), \qquad [0, 1].$$

It follows from the first of these two inequalities that a $P_{\mu_n}(x)$ exists such that $|\phi(x) - P_{\mu_n}(x)| \leqq K\omega_H(1/\mu_n)$ on $[0, 1]$, and the theorem follows. This theorem may also be extended to $[a, b]$, $0 \leqq a \leqq b$, in which case

$$|f(x) - P_{\mu_n}(x)| \leqq K'\omega_H\left(\frac{b^H - a^H}{\mu_n}\right).$$

THEOREM III. *Let μ_n be given as before and let $\mu_\alpha = H$ be the smallest of the μ's for which $\mu_{n+1} - \mu_n \leqq \mu_\alpha$ for all n. Suppose $f(x)$ has a derivative for every x on $[a, b]$, and that $x^{1-H}f'(x)$ is continuous on $[a, b]$. Let $\eta_r = \mu_{r+\alpha} - H$ for all $r \geqq 0$. Suppose that there is given a sequence of positive numbers $\{\epsilon_n\}$, and suppose that for each ϵ_n there exists $P_n(x)$, a*

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[6] Hereafter the function $\omega_H(\delta)$ will be called a "modulus of continuity H."

polynomial in x^{μ} *of degree* $\eta_{n-\alpha}$, *such that on* $[a, b]$ $|x^{1-H}f'(x)-P_n(x)| \leq \epsilon_n$. *Then, for every* $n>\alpha$ *there exists* $P_{\mu_n}(x)$, *a polynomial in* x^{μ} *of degree* μ_n, *such that on* $[a, b]$

$$|f(x) - P_{\mu_n}(x)| \leq \frac{K''\epsilon_n}{\mu_n}$$

where K'' *depends on* H, a, *and* b.

Let

$$R_n(x) = \int_a^x x^{H-1}P_n(x)dx,$$

and let $r_n(x) = f(x) - R_n(x)$. Then, with the possible exception of $x = a$, $r_n'(x)$ may be found; hence $x^{1-H}r_n'(x)$, from which it follows that $|r_n'(x)| \leq x^{H-1}\epsilon_n$. Since

$$r_n(x_2) - r_n(x_1) = \int_{x_1}^{x_2} r_n'(x)dx,$$

it follows that on $[a, b]$

$$|r_n(x_2) - r_n(x_1)| \leq \frac{\epsilon_n}{H}|x_2^H - x_1^H|.$$

Therefore there exists $\bar{R}_n(x)$ in x^{μ} of degree μ_n which approximates $r_n(x)$ with error at most $K\epsilon_n/H\mu_n$ and, with $P_{\mu_n}(x) = R_n(x) + \bar{R}_n(x)$,

$$|f(x) - P_{\mu_n}(x)| \leq \frac{K\epsilon_n}{H\mu_n}.$$

From Theorem III it follows that if $x^{1-H}f'(x)$ satisfies a Lipschitz condition H, constant λ, on $[a, b]$, then for every $n>\alpha$ there exists $P_{\mu_n}(x)$ such that

$$|f(x) - P_{\mu_n}(x)| \leq \frac{K_1\lambda}{\mu_n(\mu_n - H)}.$$

If $x^{1-H}f'(x)$ has a modulus of continuity H on $[a, b]$, then for every $n>\alpha$ there exists a polynomial $P_{\mu_n}(x)$ of degree μ_n for which

$$|f(x) - P_{\mu_n}(x)| \leq \frac{K_2}{\mu_n}\,\omega_H\left(\frac{b^H - a^H}{\mu_n - H}\right).$$

4. Uniform convergence. With the sequence $\{x^{\mu_i}\}$ and a non-negative function $\rho(x)$, integrable on $[a, b]$ with positive integral, polynomials $p_{\mu_n}(x)$ in x^μ of degree μ_n can be constructed for every μ_n by well known methods so that these polynomials will be orthonormal with respect to $\rho(x)$ on $[a, b]$. If $f(x)$ is an arbitrary continuous function, it follows as a formal result

$$f(x) \sim \sum_{n=0}^{\infty} a_n p_{\mu_n}(x), \tag{1}$$

where

$$a_n = \int_a^b f(t) p_{\mu_n}(t) \rho(t) dt.$$

In the following the results of the previous section are combined with certain results from the theory of orthogonal polynomials to give theorems on convergence for the above series expansion.[9] It will be necessary at this point to suppose that the μ's are integers, since Bernstein's theorem is implicitly involved in the following. It is to be observed that this restriction admits sequences so irregular that the results obtained are by no means trivial, and further that the admission of any set of commensurable exponents would not be a material generalization.

In the following statement of those results from orthogonal polynomial theory[10] which are applied to the present problem of convergence, all polynomials are polynomials in x^μ, of degree given by their subscript, and the use of ϵ_n implies that there exists a polynomial $P_{\mu_n}(x)$ such that on $[a, b]$, $|f(x) - P_{\mu_n}(x)| \leq \epsilon_n$. If $\bar{Q}_{\mu_n}(x)$ is a polynomial minimizing[11]

$$\int_a^b \rho(x)[f(x) - Q_{\mu_n}(x)]^2 dx,$$

if $\rho(x)$ has a positive lower bound on $[a, b]$, then there are constants C_1, C_2, independent of x and n, such that

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[9] For the sequence $1, x^{\mu_1}, \cdots, x^{\mu_{N-1}}; x^{\mu}, x^{\mu+\mu_1}, \cdots, x^{\mu+\mu_{N-1}}; \cdots; x^{m\mu}, x^{m\mu+\mu_1}, \cdots, x^{m\mu+\mu_{N-1}}, \cdots$, where $0<\mu_1<\mu_2< \cdots <\mu_{N-1}<\mu$, a recursion formula and hence a Christoffel-Darboux formula can be given for the orthonormal sequence. It is then possible to construct a theory for point-by-point convergence in the usual way. Since, on the one hand, there are no non-trivial results on boundedness of $p_{\mu_n}(x)$; and, on the other hand, no very interesting extensions of Bernstein's and Markoff's theorems are available, the results which are obtainable are omitted.

[10] D. Jackson, *Certain problems of closest approximation*, this Bulletin, vol. 39 (1933), pp. 903–906, Theorems 11, 15.

[11] It is a well known fact that $\bar{Q}_{\mu_n}(x) = \sum_{r=0}^{n} a_r p_{\mu_r}(x)$. Statements (A) and (B) therefore refer to the error of approximation by partial sums of the series for $f(x)$.

$$(A)\qquad \left| f(x) - \bar{Q}_{\mu_n}(x) \right| \leqq C_1 \mu_n \epsilon_n, \qquad a \leqq x \leqq b,$$

$$(B)\qquad \left| f(x) - \bar{Q}_{\mu_n}(x) \right| \leqq C_2 \mu_n^{1/2} \epsilon_n, \qquad c \leqq x \leqq d,$$

where $a<c<d<b$.

The problem of convergence then reduces to one of placing sufficient conditions on $f(x)$ so that $\mu_n \epsilon_n \to 0$ or $\mu_n^{1/2} \epsilon_n \to 0$. It is assumed in the following theorems that $\rho(x)$ is bounded from zero and integrable. From the results of the preceding sections values of ϵ_n are obtained which lead to the following theorems.

THEOREM IV. *If $f(x)$ has a derivative for which $x^{1-H} f'(x)$ satisfies a Lipschitz condition H on $[a, b]$, then the series* (1) *converges uniformly to $f(x)$ throughout $[a, b]$, and the magnitude of the remainder after n terms does not exceed $O[1/(\mu_n - H)]$.*

THEOREM V. *If $f(x)$ has a derivative for which $x^{1-H} f'(x)$ is continuous on $[a, b]$ and has a modulus of continuity H, $\omega_H(\delta)$,* (1) *converges uniformly to $f(x)$ throughout $[a, b]$, and the magnitude of the remainder after n terms does not exceed $O[\omega_H((b^H - a^H)/(\mu_n - H))]$.*

THEOREM VI. *If $f(x)$ satisfies a Lipschitz condition H on $[a, b]$,* (1) *converges uniformly to $f(x)$ throughout $[c, d]$, where $a<c<d<b$, and the magnitude of the remainder after n terms does not exceed $O(1/\mu_n^{1/2})$.*

THEOREM VII. *If $f(x)$ has a modulus of continuity H, $\omega_H(\delta)$, for which $\lim_{\delta \to 0} \omega_H(\delta)/\delta^{1/2} = 0$,* (1) *converges uniformly to $f(x)$ on $[c, d]$, where c and d have the same meaning as before.*

UNIVERSITY OF IDAHO

GENERALIZED FISCHER GROUPS AND ALGEBRAS[1]

RICHARD H. BRUCK

Introduction. In this paper we shall be concerned with the structure of the rational representation of certain sets of matrices, to which we give the name *generalized Fischer sets*. If K is any field, ϕ any fixed automorphism of K, and A any matrix with elements in K, we use the notation A^ϕ for the ϕ-automorph of A; that is, the matrix obtained from A by subjecting each of its elements to the automorphism ϕ. Again, we denote by A^* the transposed ϕ-automorph A'^ϕ of A.

DEFINITION. *A set* $\mathfrak{M}$ *of n-rowed square matrices which contains* A^* *if it contains* A *is defined to be a generalized Fischer set.*

Generalized Fischer groups, semi-groups and algebras are similarly defined.

In either of the special cases (1) $K=k$, a real field, ϕ is the identity automorphism; (2) $K=k^+=k(-1)^{1/2}$, ϕ is the operation of taking the conjugate complex, the set $\mathfrak{M}$ will be called a *Fischer set*. Fischer groups were probably named[2] by M. Schiffer, who, in 1933, proved in an unpublished work that every such group is completely reducible. This result has also been given by Specht [3], and will again be derived for all Kronecker product representations in the present paper (Theorem I, §4). In §1 we give a partial converse in the cases of the field of all reals (Example (6)), and the field of all complex numbers (Example (5)); this is summed up in Theorem II (§4).

Unlike Fischer sets, generalized Fischer sets and their rational representations are not always completely reducible; the regular representation of a finite group over a field of prime characteristic dividing the order of the group is a case in point (§1, Example (8)). When ϕ is non-involutary, the most we can give concerning the structure of g.F. sets is contained in Lemma II (§4) and Lemma IV (§5). But when ϕ is an involutary automorphism, a more satisfactory result is

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Presented to the Society, December 30, 1940, under the title *The structure of the rational representation of a wide class of linear groups*; received by the editors December 11, 1940, and, in revised form, October 27, 1941.

[1] The following is essentially contained in the author's doctorate thesis [1], written under the direction of Professor Richard Brauer. Professor Brauer has also offered many helpful suggestions in connection with the present paper. The thesis undertook a general study of $GL(n)$, and employed the results for specific calculation of the irreducible characters of $GL(4)$ over an infinite modular field.

[2] They were considered earlier by E. Fischer [2], who proved that the rational integral invariants of a Fischer group possessed a finite integrity basis.

given by Theorem III (§6); this states that each Kronecker product representation[3] of a g.F. set may be written in the form

$$\begin{pmatrix} \mathfrak{A} & & \\ & \mathfrak{B} & \\ * & & \mathfrak{C} \end{pmatrix}$$

where $\mathfrak{B}$ is in completely reduced form and $\mathfrak{C}$ is dual to $\mathfrak{A}$ in a sense which we will not describe at this point.

The main tool of the paper is the scalar product of two forms (or vectors) of the linear vector space S upon which the transformations of $\mathfrak{S}=\pi_f(\mathfrak{M})$ are performed. This is introduced in §2.

1. Examples of generalized Fischer sets. In this section proofs are omitted. The first two examples and the last hold true if the word set is replaced by group, semi-group or algebra. We use without specific mention material from [4], [5] and [6].

(1) *The direct sum of a finite number of g.F. sets (w.r.t. the same automorphism ϕ) is a g.F. set.*

(2) *The direct product of two g.F. sets (w.r.t. the same automorphism ϕ) is a g.F. set.*

(3) *A general linear group is a g.F. group w.r.t. every automorphsim ϕ of the underlying field.*

(4) *A total matric algebra is a g.F. algebra w.r.t. every ϕ.*

(5) *A semi-simple algebra of matrices over an algebraically closed field is similar to a g.F. algebra w.r.t. every ϕ.*

(6) *A semi-simple algebra of matrices over the field of all real numbers is similar to a Fischer algebra.*

(7) *Any group of matrices leaving invariant the bilinear form*

$$x^* y = \sum_{i=1}^{n} x_i^{\phi} y_i$$

is a g.F. group.

(8) *The right-hand (or first) regular representation $\mathfrak{R}$ of a finite group G is a g.F. group, and the linear closure of $\mathfrak{R}$ is a g.F. algebra, w.r.t. any ϕ, even if the underlying field is modular.*

(9) *If $\mathfrak{M}$ is a g.F. set w.r.t. the automorphism ϕ, its commutator algebra $\mathfrak{C}$ is a g.F. algebra w.r.t. the inverse automorphism $\theta=\phi^{-1}$.*

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[3] By the fth Kronecker product representation of a set $\mathfrak{M}$ we mean simply the set $\mathfrak{S}=\pi_f(\mathfrak{M})$ consisting of the fth Kronecker product [4] of each matrix of $\mathfrak{M}$. If $\mathfrak{M}$ is a (semi-) group, $\mathfrak{S}$ forms a representation in the usual sense. The corresponding statement in the case of an algebra is true only if $f=1$; thus the paper gives information only about the algebra itself.

2. Kronecker product representations. The matrices of a g.F. set $\mathfrak{A}$ may be considered as transformations of a contravariant vector x^i

$$A:\quad x^i = \sum_{j=1}^{n} a^i_j x'^j. \tag{1}$$

Then the fth Kronecker product $\mathfrak{S} = \pi_f(\mathfrak{A})$ may be regarded as a set of transformations of the linear vector space S of all forms

$$\begin{aligned} F(T) &= \sum c_{i_1 i_2 \cdots i_f} T^{i_1 i_2 \cdots i_f}, \\ G(T) &= \sum d_{i_1 i_2 \cdots i_f} T^{i_1 i_2 \cdots i_f}, \end{aligned} \tag{2}$$

where $T^{i_1 i_2 \cdots i_f}$ is an arbitrary contravariant tensor with f indices. If under the transformation (1), T becomes T', then $F(T)$ becomes a new form $F_{(A)}(T')$ defined by

$$F(T) = F_{(A)}(T'). \tag{3}$$

We define the form $F_{(A)}(T)$ (in T, not T'!) to be the *transform of F by A*. Since

$$F_{(I)} = F, \qquad F_{(A)(B)} = F_{(AB)}, \tag{4}$$

it follows that if $\mathfrak{A}$ is a (semi-) group the set of transformations under $\mathfrak{A}$ of a basis of forms of S is a representation of $\mathfrak{A}$ similar to $\pi_f(\mathfrak{A})$.

We define the *scalar product* of the two forms F, G of (2) by

$$F \circ G = \sum c^{\theta}_{i_1 i_2 \cdots i_f} d_{i_1 i_2 \cdots i_f}, \qquad \theta = \phi^{-1}. \tag{5}$$

Clearly $F \circ G$ is a nondegenerate bilinear form in the two n^f-dimensional vectors F^θ, G whose coordinates are, respectively, $c^{\theta}_{i_1 i_2 \cdots i_f}$ and $d_{i_1 i_2 \cdots i_f}$. It has all the properties of such a bilinear form; we single out

$$(cF) \circ G = c^{\theta}(F \circ G), \qquad F \circ (cG) = c(F \circ G), \tag{6}$$

where c is any element of the underlying field. Again, it is readily verified that

$$F_{(A^*)} \circ G = F \circ G_{(A)}, \qquad A^* = A'^{\phi}. \tag{7}$$

These remarks and equations are fundamental to the present paper. Equation (7) explains the necessity for restricting attention to g.F. sets. Although all subsequent results are stated for the above Kronecker product representation they may be applied to others as well;[4] for this reason there would be some value in adopting a more axiomatic treatment. The essentials are: a vector space S with a g.F.

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[4] In [1] the results were applied to Kronecker products of power products of alternating representations (that is, those afforded by skew-symmetric tensors) of $GL(n)$.

set $\mathfrak{A}$ of operators, as indicated by (4), and a nondegenerate scalar product $F\circ G$ bilinear in F^{ϕ} and G with the additional property (7).

We shall wish to refer to the following special properties which do not hold generally:

$$(8)\qquad (H)\colon\ G\circ F = (F\circ G)^{\phi},\qquad\qquad \phi = \theta.$$

This is true when ϕ is an involutary automorphism, and means that $F\circ G$ is a "hermitian" form.

$$(9)\qquad (F)\colon\ G\neq 0\ \text{ implies }\ G\circ G\neq 0.$$

This is certainly true in each of the following cases: (1) $K=k$, a real field, ϕ is the identity automorphism; (2) $K=k^{+}$, ϕ is the operation of taking the conjugate complex. Thus (F), and indeed also (H), may be assumed in dealing with Fischer sets.

3. **Modules.** Any subspace $S_1\subset S$ which is invariant under the g.F. set $\mathfrak{A}$ we will call a *module*. Let the module S_1 have basis F_i $(i=1, 2, \cdots, \sigma_1)$, and define the transpose of the matrix M_1 by

$$(10)\qquad M_1' = (F_i\circ F_j),$$

where i, j are, respectively, the row and column index of the σ_1-rowed square matrix on the right. M_1 will be referred to as *the matrix associated with the module* S_1. Its rank is invariant under change of basis of S_1. If M_1 is nonsingular, S_1 is a *nonsingular* module. If M_1 is the zero matrix, S_1 is a module *of rank zero*. Since S_1 is a module, $F_{i(A)}$ is in S_1:

$$(11)\qquad F_{i(A)} = \sum_{j=1}^{\sigma_1} s_{ij}^{(1)}(A)\cdot F_j.$$

If we set

$$(12)\qquad s_1(A) = (s_{ij}^{(1)}(A))$$

then the set $\mathfrak{S}_1$ of matrices $s_1(A)$ forms a representation of $\mathfrak{A}$.

DEFINITION. *Let $\mathfrak{A}$ be mapped upon a set $\mathfrak{U}$ of rectangular matrices: $A\to u(A)$. Then $\overline{\mathfrak{U}}$ denotes the set of matrices $\bar{u}(A)=[u(A^{*})]'^{\phi}$.*

Note that if $\phi=\theta$ is involutary, $\bar{u}(A)=u^{*}(A^{*})$. Moreover if $\mathfrak{R}$ is a representation of a semi-group $\mathfrak{G}$, then $\overline{\mathfrak{R}}$ is also a representation.

LEMMA I. *With the above notations*

$$(13)\qquad M_1\cdot\overline{\mathfrak{S}}_1 = \mathfrak{S}_1\cdot M_1.$$

PROOF. By (7)

$$F_{j(A^*)} \circ F_i = F_j \circ F_{i(A)},$$

and by (6)

$$\sum_k F_k \circ F_i \cdot s_{jk}^{(1)\prime}(A^*) = \sum_l s_{il}^{(1)}(A) \cdot F_j \circ F_l.$$

But in matrix notation, if i, j are taken as row and column index, respectively, the last equation is identical with (13).

LEMMA II. *An irreducible module is either nonsingular or of rank zero.*

PROOF. Let S_1, M_1, $\mathfrak{S}_1$ be, respectively, the irreducible module, its associated matrix and the corresponding (irreducible) set of matrices. From Schur's lemma [5] and equation (13), M_1 is either nonsingular or zero.

We restrict our attention to modules of the two types mentioned in Lemma II, without, however, requiring them to be irreducible. The full space S is itself a nonsingular module. In fact, since the bilinear form $F \circ G$ is nondegenerate, the equation $F \circ G = 0$ holding for a fixed G and all F of S implies $G = 0$; it follows readily that the matrix M associated with S is nonsingular.

4. **Nonsingular modules.** Let the module S_1 be a proper subspace of S, in the sense that not every form of S is in S_1.

LEMMA III. *If S_1 is a nonsingular module, then S is the direct sum of S_1 and another nonsingular module S_2, and $\mathfrak{S}$ is decomposable:*

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & 0 \\ 0 & \mathfrak{S}_2 \end{pmatrix}. \tag{14}$$

PROOF. Let F_i ($i = 1, 2, \cdots, \sigma_1$) be a basis of S_1, and let these with G_α ($\alpha = \sigma_1 + 1, \cdots, \sigma$) form a basis of S. Let G be any form of S. Then since the matrix M associated with S is nonsingular, we may uniquely determine elements $u_1, u_2, \cdots, u_{\sigma_1}$ such that

$$F = u_1 F_1 + u_2 F_2 + \cdots + u_{\sigma_1} F_{\sigma_1}$$

satisfies $F_i \circ F = F_i \circ G$, all i. Thus the form $G' = G - F$ satisfies $F_i \circ G' = 0$, for all i. Therefore, without loss of generality we may assume that the equations

$$F_i \circ G_\alpha = 0 \tag{15}$$

hold for all i, α. The matrix M associated with S then takes the form

$$M = \begin{pmatrix} M_1 & * \\ 0 & M_2 \end{pmatrix} \tag{16}$$

where $M_2 = (G_\alpha \circ G_\beta)'$ is, with M and M_1, nonsingular. Since S_1 is a module, $\mathfrak{S}$ and $\overline{\mathfrak{S}}$ are given by

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & 0 \\ \mathfrak{U} & \mathfrak{S}_2 \end{pmatrix}, \qquad \overline{\mathfrak{S}} = \begin{pmatrix} \overline{\mathfrak{S}}_1 & \overline{\mathfrak{U}} \\ 0 & \overline{\mathfrak{S}}_2 \end{pmatrix}. \tag{17}$$

But Lemma I, quoted for S, takes the form $M\overline{\mathfrak{S}} = \mathfrak{S}M$. From (16) and (17) follows $0 = \mathfrak{U}M_1$, $\mathfrak{U} = 0$. Thus $\mathfrak{S}$ assumes the form (14), and the set of all linear combinations of the G_α is a module; nonsingular, since M_2 is nonsingular.

When property (F) holds (see equation (9)), S clearly possesses no modules of rank zero. Hence we have this lemma.

LEMMA IIIa. *If (F) is true in S, $\mathfrak{S}$ is completely reducible.*

As an immediate consequence we have the following theorem.

THEOREM I. *The Kronecker product representations of a Fischer set are completely reducible.*

In particular the theorem states that Fischer groups and algebras are completely reducible. As a partial converse we have from examples (6) and (5) of §1 that a semi-simple algebra over the field of all real numbers or the field of all complex numbers is similar to a Fischer algebra.

THEOREM II. *Let $\mathfrak{G}$ be a (semi-) group of matrices over k or $k(-1)^{1/2}$, (k the field of all real numbers). Let $\mathfrak{A}$ be the linear closure of $\mathfrak{G}$. Then a necessary and sufficient condition that $\mathfrak{G}$ (and its Kronecker product representations) be completely reducible is that $\mathfrak{A}$ be similar to a Fischer algebra.*

5. **Modules of rank zero.** We now prove this lemma.

LEMMA IV. *If S_1 is a module of rank zero, then, in the sense of similarity,*

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & & \\ & \mathfrak{S}_2 & \\ * & & \overline{\mathfrak{S}}_1 \end{pmatrix}. \tag{18}$$

PROOF. Let F_i $(i = 1, 2, \cdots, \sigma_1)$, be a basis of S_1, and let G_α $(\alpha = \sigma_1 + 1, \cdots, \sigma)$ complete the basis of S. Consider the matrix

$$U = (F_i \circ G_\alpha) \tag{19}$$

and the form $F = v_1F_1 + v_2F_2 + \cdots + v_{\sigma_1}F_{\sigma_1}$. If the matrix U had rank less than σ_1, we could choose a set of v's, not all zero, so that $F \circ G_\alpha = 0$ for all α. This, coupled with the fact that $F \circ F_i = 0$ for all i (since S_1 is a module of rank zero), would imply $F \circ G = 0$ for every G of S, in contradiction to the fact that the bilinear form $F \circ G$ is nondegenerate. Thus U has rank σ_1, and there exist (cf. [7]), two nonsingular matrices $P = (p_{ij})$ and $Q = (q_{\alpha\beta})$ such that $PUQ = (0 \mid I_{\sigma_1})$. We make the transformation of basis given by

$$\bar{F}_i = \sum p_{ij}^{\phi} F_j, \qquad \bar{G}_\alpha = \sum q_{\beta\alpha} G_\beta \tag{20}$$

so that

$$\bar{U} = (\bar{F}_i \circ \bar{G}_\alpha) = PUQ = (0 \mid I_{\sigma_1}). \tag{21}$$

Without loss of generality we may assume that (21) holds for U. We rename the last basis elements, calling them H_i ($i = 1, 2, \cdots, \sigma_1$), so that our basis is now

$$F_1, F_2, \cdots, F_{\sigma_1}, G_{\sigma_1+1}, \cdots, G_{\sigma-\sigma_1}, H_1, H_2, H_{\sigma_1},$$

where the G's now appear only when $\sigma - 2\sigma_1 > 0$, but the H's necessarily appear. Calculation of M for this basis gives, in view of (21),

$$M = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ I_{\sigma_1} & * & * \end{pmatrix}. \tag{22}$$

We may assume

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{S}_1 & 0 & 0 \\ * & \mathfrak{S}_2 & \mathfrak{U} \\ * & * & \mathfrak{S}_3 \end{pmatrix}, \qquad \bar{\mathfrak{S}} = \begin{pmatrix} \bar{\mathfrak{S}}_1 & * & * \\ 0 & \bar{\mathfrak{S}}_2 & * \\ 0 & \bar{\mathfrak{U}} & \bar{\mathfrak{S}}_3 \end{pmatrix}. \tag{23}$$

Using Lemma I, or $M\bar{\mathfrak{S}} = \mathfrak{S}M$, we obtain: by comparing the blocks (2, 1), $\mathfrak{U} = 0$; by comparing the blocks (3, 1), $\mathfrak{S}_3 = \bar{\mathfrak{S}}_1$. This completes the proof.

Lemma IV seems to be the most we can say when ϕ is a non-involutary automorphism.

Assume that $\phi = \theta$ is involutary, so that property (H) (equation (8)) holds for the scalar product. In this case (22) may be replaced by

$$M = \begin{pmatrix} 0 & 0 & I_{\sigma_1} \\ 0 & N & * \\ I_{\sigma_1} & * & * \end{pmatrix}, \qquad N = (G_k \circ G_k)', \tag{22a}$$

where h and k have the range σ_1+1 to $\sigma-\sigma_1$. The set S_2 of all linear combinations of the G_k is a linear vector space. For S_2 we may obtain a useful scalar product of two forms

$$G = x_{\sigma_1}G_{\sigma_1} + \cdots + x_{\sigma-\sigma_1}G_{\sigma-\sigma_1}, \tag{24}$$
$$R = y_{\sigma_1}G_{\sigma_1} + \cdots + y_{\sigma-\sigma_1}G_{\sigma-\sigma_1},$$

by merely taking over the scalar product for S

$$G \circ R = \sum n_{hk}\overset{\phi}{x_h}y_k, \qquad n_{hk} = G_h \circ G_k. \tag{25}$$

From (22a), $\det M = \pm\det N \neq 0$, hence $N=(n_{hk})$ is nonsingular, $G \circ R$ is nondegenerate. In order to obtain a property comparable to (7) we must redefine the transform of a form of S_2 by a matrix of $\mathfrak{A}$. Since, by Lemma IV, S_2 is invariant (modulo S_1) under $\mathfrak{A}$,

$$G_{(A)} = G(A) + F(A) \tag{26}$$

where $G(A)$, $F(A)$ are uniquely determined forms of S_2, S_1. We define $G(A)$ to be the transform (*in* S_2) of G by A. Then, by virtue of (22a),

$$G(A^*) \circ R = G_{(A^*)} \circ R = G \circ R_{(A)} = G \circ R(A). \tag{27}$$

Formula (27) gives the desired property. There results this lemma.

LEMMA IVa. *If the automorphism ϕ is involutary, the representation $\mathfrak{S}_2$ of Lemma* IV *may be associated with a vector space S_2, with operator-set $\mathfrak{A}$, which possesses a nondegenerate* (*hermitian*) *scalar product satisfying* (7). *Thus the preceding methods and results may be applied to S_2 and $\mathfrak{S}_2$ just as they were to S and $\mathfrak{S}$.*

6. **The structure of** S. Lemmas III and IV may be combined in a number of ways to give information concerning the structure of S and $\mathfrak{S}$. This information is annoyingly limited, however, in case we cannot assume the conclusion of Lemma IVa. The following theorem therefore concerns only g.F. sets defined relative to an involutary automorphism.

THEOREM III. *Let $\mathfrak{A}$ be a set of n-rowed square matrices which contains $A^* = A'^{\theta}$ with every A, where θ is a fixed involutary automorphism of the underlying field. If $\mathfrak{R} = \{R(A)\}$ is any representation of $\mathfrak{A}$, define*

$$\overline{\mathfrak{R}} = \{\overline{R}(A)\},$$

where

$$\overline{R}(A) = [R(A^*)]^*.$$

Then the fth Kronecker product representation $\pi_f(\mathfrak{A})$ of $\mathfrak{A}$ is given in reduced form by

$$\pi_f(\mathfrak{A}) = \left(\begin{array}{c|c|c} \begin{matrix}\mathfrak{B}_1 & & \\ & \mathfrak{B}_2 & \\ * & \ddots & \\ & & \mathfrak{B}_\kappa\end{matrix} & & \\ \hline * & \begin{matrix} \mathfrak{C}_1 & & \\ & \mathfrak{C}_2 & \\ 0 & \ddots & \\ & & \mathfrak{C}_\lambda \end{matrix} & \\ \hline * & * & \begin{matrix}\bar{\mathfrak{B}}_\kappa & & & \\ & \ddots & & \\ & & \bar{\mathfrak{B}}_2 & \\ * & & & \bar{\mathfrak{B}}_1\end{matrix}\end{array}\right), \tag{28}$$

where the $\mathfrak{B}$*'s and* $\mathfrak{C}$*'s are irreducible representations, and each* $*$ *denotes elements not necessarily zero.*

The wording of the theorem is especially adapted to the case that $\mathfrak{A}$ is a (semi-) group. If $\mathfrak{A}$ is an algebra, the last sentence should be replaced by "Then $\mathfrak{A}$ is given in reduced form by $\cdots$."

PROOF. First we apply Lemma IV a number of times, assuming in each case that the S_1 is an irreducible module. After a finite number of steps, say κ, we must reach a projection space which contains no modules of rank zero. On applying Lemma III a number of times to this projection space we obtain the completely reduced block of $\mathfrak{C}$'s.

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UNIVERSITY OF TORONTO AND
UNIVERSITY OF ALABAMA

A NEW PROOF OF THE CYCLIC CONNECTIVITY THEOREM

W. L. AYRES

The cyclic connectivity theorem was first proved for the plane in 1927 by G. T. Whyburn [5]. The extension of this theorem to metric space afforded some difficulty and the first proof [1] was long and tedious and complicated with convergence difficulties. A second and simpler proof appeared in 1931 [6], but in this proof it is necessary that quite a number of properties of Peano spaces be proved in advance.

This note attempts to give a new proof in which convergence troubles are encountered at just one point (step (b)) and in which just three theorems about Peano space need be known in advance: (A) *Every component of an open set is open.* (B) *Open connected sets are arc-wise connected.* (C) *The space is arc-wise locally connected.* Actually just two properties need to be established before cyclic connectivity can be proved, for the third theorem (C) is a simple consequence of the first two.[1] Thus the cyclic connectivity theorem may be established at the very beginning of the theory of Peano spaces and is available for use in studying other properties.

CYCLIC CONNECTIVITY THEOREM. *If no single point of a locally compact, connected and locally connected metric space separates the space between the two given points, there is a simple closed curve containing the two points.*

Let p and q be the two points. There exists an arc α of the space S with end points p and q by (B). We shall say that an arc β *spans* the point v of α if β has only its end points on α and v lies between these end points. We shall say that a set of arcs C *spans* a subset K of α if each point of K is spanned by some arc of the set C.

If an arc β exists with end points r and q and such that $\alpha\cdot\beta = r+q$, then step (d) in the proof has been achieved. Hence we consider only the case where no such arc exists. This assumption is used in the proof of step (b).

Presented to the Society, September 7, 1939, under the title *Peano spaces as the theory of continuous images of intervals*; received by the editors November 12, 1941.

[1] Let G be the component of $S(p, \epsilon)$ containing the point p. By (A), G is open. Then for some δ, $S(p, \delta)\subset G$. By (B) G is arc-wise connected. Hence every point of $S(p, \delta)$ may be joined to p by an arc in G, and thus in $S(p, \epsilon)$, which proves arc-wise local connectivity.

(a) *If r is a point of $\alpha-p-q$, there is an arc β spanning r.*

As r does not cut S between p and q, by (A) p and q belong to the same component C of $S-r$. Then C contains an arc γ with end points p and q by (A) and (B). On γ in the order from p to q there is a first point y of the subarc rq of α. On γ in the order y to p there is a first point x of the subarc rp of α. Then the subarc xy of γ is the desired arc β.

For any point r, let $f(r)$ be the greatest lower bound of the diameters of β, for all arcs β spanning r.

(b) *$f(r)\to 0$ as $r\to q$.*

If this is false, there is an $\epsilon>0$ and a sequence of points of α converging to q so that no one of these points may be spanned by an arc of diameter less than ϵ. We may choose ϵ so small that $S(q, \epsilon)$ is compact. Let us select a spanning arc for each point of the sequence. Infinitely many of these arcs are distinct since we assumed in the beginning that no one of them has q as an end point. Hence there exists an infinite set of arcs $\beta_1, \beta_2, \beta_3, \cdots$ such that (i) $\alpha\cdot\beta_i=x_i+y_i$, the end points of β_i, and x_i precedes y_i on α, (ii) diam $\beta_i\geqq\epsilon$, (iii) β_i spans a point p_i of α which cannot be spanned by any arc of diameter less than ϵ, (iv) we have the order $pp_1y_1p_2y_2\cdots q$ on α, (v) $y_i\to q$ and $\rho(y_i, q)<\epsilon/6$. On β_i in the order y_i to x_i let z_i be the first point such that $\rho(z_i, q)=\epsilon/3$. The set of points $\{z_i\}$ has a limit point z and we may assume $z_i\to z$. By (C) there exist arcs z_iz of diameter less than $\epsilon/6$ for i sufficiently large. Then $y_iz_i+z_iz+z_{i+1}z+y_{i+1}z_{i+1}$ contains an arc of diameter less than ϵ spanning p_{i+1}, which is a contradiction.

(c) *If r is any point of $\alpha-p-q$, there exists a countable sequence of arcs $\{\beta_i\}$ spanning the set $rq-q$ and such that* diam $\beta_i\to 0$.

Let $p_i\to q$ with the order $prp_1p_2\cdots q$. Let $p_0=r$ and α_i be the subarc $p_{i-1}p_i$ of α. From (a) there exists a family of arcs spanning α_i. From (b) these may be chosen of arbitrarily small diameter for i large. Using the Borel theorem there is a finite subfamily spanning α_i. The set of all these finite subfamilies is a countable set with the desired properties.

(d) *There is a simple closed curve containing q.*

Let u_i and v_i be the end points of β_i and suppose the order pu_iv_iq on α. From the method in which the arcs β_i were chosen we see that only a finite number span any one point and no one intersects more than a finite number of others. Of the finite set spanning r, choose the one whose end point v_i is nearest q on α in the order p to q. We may assume this is β_1. Of the finite set, if nonvacuous, of arcs β_i which intersect $\beta_1-u_1-v_1$, choose the one whose end point v_i follows all others on α. We may assume this is β_2. Of the finite sets, if nonvacuous, which intersect $\beta_2 - u_2 - v_2$, we choose the one whose end point v_i follows all others on α, and we may suppose this is β_2. This process must terminate at some finite step for otherwise $\sum \beta_i$ would contain an arc having u_1 and q as end points and containing no other point of α.[2] This arc together with the subarc u_1q of α would give the desired simple closed curve.

If the process terminates at β_k, then $\sum_1^k \beta_i$ contains an arc γ_1 with end points $s_1 = u_1$ and $t_1 = v_k$ and containing no other point of α. We now define an arc γ_2 with end points s_2 and t_2 using exactly the process by which γ_1 was defined but starting with the arcs spanning v_k instead of r. We continue and define $\gamma_3, \gamma_4, \cdots$. The arcs γ_i are mutually exclusive except that t_i may coincide with s_{i+2}. On α we have the order $s_1 < s_2 < t_1 \leqq s_3 < t_2 \leqq s_4 < \cdots < q$. Then the desired two arcs forming a simple closed curve containing q are defined

$$\eta_1 = q + \sum \gamma_{2i-1} + \sum \text{ subarcs } t_{2i-1}s_{2i+1} \text{ of } \alpha,$$

$$\eta_2 = q + \text{ subarc } s_1s_2 \text{ of } \alpha + \sum \gamma_{2i} + \sum \text{ subarcs } t_{2i}s_{2i+2} \text{ of } \alpha.$$

From step (d) the cyclic connectivity theorem follows easily. We have a simple closed curve containing q. Similarly one contains p. If these simple closed curves have two points in common, their sum contains a simple closed curve containing p and q. If they have but one point in common, their sum plus an arc pq not containing this point will contain the desired simple closed curve. If they do not intersect, then their sum plus an arc joining them plus its finite spanning system enables one to choose the desired two arcs. In this case the selection of the two arcs follows the methods used in picking η_1 and η_2 but the whole process here is finite.

Remarks. The cyclic connectivity theorem is a special case of the n-Bogensatz where $n = 2$ and the two closed sets are single points. All proofs that have been given for the n-Bogensatz are extremely long and intricate [2, 3, 4, 7], and a simple proof of this important theorem would be a real contribution. Unfortunately the method used in this note does not appear to generalize to the higher values of n.

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[2] This arc would be given by q plus the subarc of β_1 from u_1 to the first point of β_2 on β_1, plus the subarc of β_2 from this point to the first point point of β_3, and so on. From diam $\beta_i \rightarrow 0$ it follows that this set is an arc with end points u_1 and q.

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PURDUE UNIVERSITY

INVERSES AND ZERO-DIVISORS

REINHOLD BAER

It may happen that an element in a ring is both a zero-divisor and an inverse, that it possesses a right-inverse though no left-inverse, and that it is neither a zero-divisor nor an inverse. Thus there arises the problem of finding conditions assuring the absence of these paradoxical phenomena; and it is the object of the present note to show that chain conditions on the ideals serve this purpose. At the same time we obtain criteria for the existence of unit-elements.

The following notations shall be used throughout. The element e in the ring R is a *left-unit for the element* u in R, if $eu = u$; and e is a *left-unit for* R, if it is a left-unit for every element in R. Right-units are defined in a like manner; and an element is a *universal unit for* R, if it is both a right- and a left-unit for R.

The element u is a *right-zero-divisor*, if there exists an element $v \neq 0$ in R such that $vu = 0$; and u is a *right-inverse in* R, if there exists an element w in R such that wu is a left-unit for u and a right-unit for R. Left-zero-divisors and left-inverses are defined in a like manner. Note that 0 is a zero-divisor, since we assume that the ring R is different from 0.

$L(u)$ denotes the set of all the elements x in R which satisfy $xu = 0$; clearly $L(u)$ is a left-ideal in the ring R and every left-ideal of the form $L(u)$ shall be termed a *zero-dividing left-ideal*. *Principal left-ideals*[1] are the ideals of the form Rv for v in R and the ideals vR are the principal right-ideals.

Presented to the Society December 31, 1941; received by the editors September 19, 1941, and, in revised form, October 13, 1941.

[1] This is a slight change from the customary terminology.

LEMMA 1. *If vu is a left-unit for u and a right-unit for R, then $uv=vu$, $Rv=R$ and $L(u)=0$ are equivalent properties.*

PROOF. If $uv=vu$, then $R=Rvu=Ruv\leq Rv\leq R$, or $R=Rv$. If $R=Rv$, and if $xu=0$, then there exists an element y in R such that $x=yv$; and $L(u)=0$ is a consequence of $y=yvu=xu=0$. If $L(u)=0$ then $(vu-uv)u=vuu-uvu=u-u=0$ implies that $vu-uv$ is in $L(u)$ and that therefore $vu=uv$.

EXAMPLE 1. Let G be the additive group of all the countably infinite sequences of numbers from the commutative field F, considered as an abelian operator group over F; and denote by R the ring of all the F-admissible automorphisms of G (that is, of all the linear transformations of G over F). Elements u, v, w in R are defined by

$$(a_1, \cdots)^u = (0, a_1, \cdots), \qquad (a_1, \cdots)^v = (a_1, 0, \cdots),$$
$$(a_1, \cdots)^w = (a_2, a_3, \cdots)$$

and they satisfy $uv=0$, $uw=1$, $wu\neq 1$. Thus a left-inverse may be a left-zero-divisor.

LEMMA 2. *Each of the following properties of the ring R implies all the others.*

(i) *There exists a universal unit in R.*

(ii) *There exists a right-unit for R, and $x=0$ is a consequence of $Rx=0$.*

(iii) *There exists one and only one right-unit for R.*

(iv) *There exist a right-unit for R and a left-unit for R.*

PROOF. (ii) is an obvious consequence of (i). If (ii) holds, and if u and v are right-units for R, then $x(u-v)=xu-xv=0$ for every element x in R; thus $u=v$ and (iii) is an implication of (ii). If (iii) holds, if e is the uniquely determined right-unit for R, and if x and y are elements in R, then $x(e+y-ey)=xe+xy-xey=x+xy-xy=x$ for every x in R. Thus $e+y-ey$ is a right-unit for R; and the equality of all right-units implies that $y=ey$, that is, that e is also a left-unit for R; and (iv) is therefore a consequence of (iii). If (iv) holds, then there exist a right-unit u and a left-unit v for R; and (i) may be inferred from $u=vu=v$.

The importance of condition (ii) may be seen from the fact that the elements x in R which satisfy $Rx=0$ form a two-sided ideal T whose square is 0.

LEMMA 3. *If the minimum condition is satisfied by the principal left-ideals in the ring R, if J is a principal left-ideal in R and if $Jv=R$ for v an element in R, then $J=R$.*

REMARK. Omitting the word "principal" throughout we obtain a true statement that may be proved in a similar fashion.

PROOF. If $R=Jv$ for J a principal left-ideal and v an element in R, then there exists a smallest principal left-ideal $S=Rs$ such that $R=St$ for some t in R. Hence $s=rst$ for r in R. Since $Rrs\leqq Rs=S$, and since $(Rrs)t^2=R(rst)t=Rst=St=R$, it follows from our choice of S that $Rrs=S$; and hence $S=Rs=Rrst=St=R$, a fact which proves our contention.

THEOREM 1. *The element z in the ring R with minimum condition for principal left-ideals is a right-inverse in R if, and only if, z is not a right-zero-divisor in R.*

PROOF. Assume first the existence of an element v in R such that vz is a left-unit for z and a right-unit for R. Then $J=Rv$ is a principal left-ideal and it follows from Lemma 3 and the equation $Jz=Rvz=R$ that $R=J=Rv$; and hence it follows from Lemma 1 that z is not a right-zero-divisor.

If there exist elements in R which are not right-zero-divisors, then among these elements there is one, say v, with minimal (principal left-ideal) Rv. Since $xv^2=0$ implies $xv=0$, and since this implies $x=0$, it follows that v^2 is not a right-zero-divisor. Thus $Rv^2\leqq Rv$ and the minimum property of Rv imply $Rv^2=Rv$. Hence for every element r in R there exists an element r' in R such that $rv=r'v^2$ or $(r-r'v)v=0$. Since v is not a right-zero-divisor, $r-r'v=0$ or $r=r'v$. Thus we have shown that $R=Rv$; and this fact makes it evident that $R=Rz$ for every z which is not a right-zero-divisor in R.

If the element z in R is not a right-zero-divisor in R, then $R=Rz$; and there exists therefore one (and only one) element e in R such that $ez=z$. Since $(r-re)z=rz-rz=0$ for every r in R, it follows that e is both a left-unit for z and a right-unit for R. Since the existence of an element w satisfying $wz=e$ is a consequence of $R=Rz$, it has been shown that z is a right-inverse in R.

An immediate consequence of this theorem is the following fact.

COROLLARY. *The ring R with minimum condition for principal left-ideals contains a right-unit for R if (and only if) at least one element in R is not a right-zero-divisor.*

The impossibility of substituting the maximum condition for the minimum condition in Theorem 1 and its corollary may be seen from the example of the even rational integers. The example of the ring of all the matrices

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$$

with coefficients from some commutative field shows that the conditions of the corollary are not sufficient for the existence of a universal unit.

THEOREM 2. *If there exists a universal unit* 1 *in the ring* R, *if the minimum condition is satisfied by both the principal right- and the principal left-ideals in* R, *then each of the following properties of the element* z *in* R *implies all the others*:

(a) z *is not a right-zero-divisor.*
(b) z *is a right-inverse.*
(c) z *is not a left-zero-divisor.*
(d) z *is a left-inverse.*

That (b) implies (c) and that (d) implies (a) are obvious consequences of Lemma 1; and that (a) implies (b) and that (c) implies (d) may be inferred from Theorem 1 and Lemma 2.

The impossibility of substituting maximum conditions for the minimum conditions in this theorem may be seen from the example of the ring of all the rational integers.

THEOREM 3. *If the minimum condition is satisfied by the principal left-ideals and by the zero-dividing left-ideals in the ring* R, *then* $R=Rz$ *is a necessary and sufficient condition for* z *to be a right-inverse in* R.

PROOF. If z is a right-inverse in R, then vz is for some v in R a right-unit for R and $R=Rvz\leqq Rz\leqq R$ or $R=Rz$.

If, conversely, $R=Rz$, then $z=ez$ for some e in R. The principal left-ideal $J=Re$ satisfies $Jz=Rez=Rz=R$. Hence it follows from Lemma 3 that $R=Re$ for every left-unit e of the element z.

Among the left-units for z there exists one, say f, with minimal $L(f)$. Then it is a consequence of $R=Rf$ that $f=gf$ for some g in R. Since $gz=g(fz)=(gf)z=fz=z$, and since $xg=0$ implies $0=xgf=xf$, it follows that g is a left-unit for z and that $L(g)\leqq L(f)$. Hence $L(g)=L(f)$ is a consequence of our choice of f. Since $(g^2-g)f=f-f=0$, the element g^2-g is in $L(f)=L(g)$ so that $(g^2-g)g=0$ or $g^3=g^2$. Consequently the element $e=g^2$ satisfies $e^2=g^4=g^3=g^2=e$ and $ez=g^2z=z$, that is, the element e is an idempotent and a left-unit for z. Thus e is an idempotent satisfying $R=Re$ (as has been shown in the second paragraph of the proof); and the left-unit e for z is therefore a right-unit for R. Finally there exists an element w in R such that $wz=e$, since $R=Rz$; and this completes the proof of the fact that z is a right-inverse in R.

The following fact is an immediate consequence of Theorem 3.

COROLLARY. *The ring R with minimum condition for principal and for zero-dividing left-ideals contains a right-unit for R if, and only if, R is a principal left-ideal in R.*

As an application of the preceding results we prove the following statement.

The ring R is a (not necessarily commutative) field if, and only if,

(a) *0 is the only right-zero-divisor in R,*

(b) *the minimum condition is satisfied by the principal left-ideals in R.*

It is obvious that the two conditions are necessary and that neither of them could be omitted. If the conditions (a) and (b) are satisfied by R, then every element $z \neq 0$ in R is a right-inverse by Theorem 1. Since R contains therefore a right-unit for R, it follows from Lemma 2 that R contains a universal unit 1; and it follows from Lemma 1 that every element $z \neq 0$ in R is both a right- and a left-inverse. Hence R is a field.

LEMMA 4.[3] *If the maximum condition is satisfied by the zero-dividing left-ideals in the ring R, if the left-ideal J and the element z in R satisfy $Jz = R$, then $L(z) = 0$ and $J = R$.*

PROOF. If $R = Jz$ for some left-ideal J and some element z in R, then among these elements z there exists one, say w, with maximal $L(w)$. If W is some left-ideal such that $R = Ww$, then $R = Ww \leq Rw = Ww^2 \leq R$ or $R = Ww^2$. Hence it follows from our choice of w and the obvious inequality $L(w) \leq L(w^2)$ that $L(w) = L(w^2)$. Every element in $L(w)$ has the form $x = yw$ for y in W, since $R = Ww$. If yw is in $L(w)$, then y is in $L(w^2) = L(w)$, $yw = 0$ and consequently $L(w) = 0$. Thus we have shown that $L(z) = 0$ whenever $R = Jz$ for some left-ideal J in R.

If $Jz = R$, then for every element r in R there exists an element j in the left-ideal J such that $jz = rz$. This implies $j = r$, since z is not a right-zero-divisor. This completes the proof of the fact that $J = R$ and $L(z) = 0$ are consequences of $Jz = R$.

THEOREM 4. *Suppose that the maximum condition is satisfied by the zero-dividing left-ideals in the ring R.*

(a) *uv is a right-unit for R if, and only if, vu is a right-unit for R.*

(b) *The right-unit vu for R is a left-unit for u if, and only if, $uv = vu$.*

(c) *R contains a right-unit (for R) if, and only if, R is a principal left-ideal (in R).*

PROOF. If vu is a right-unit for R, then $R = Rvu$; and it follows from

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[3] Cf. Lemma 3.

Lemma 4 that $L(u)=0$, $Rv=R$, $L(v)=0$, $Ru=R$. If r is any element in R, then $(r-ruv)u=ru-ru(vu)=ru-ru=0$. Since $L(u)=0$, we find that $r=ruv$ for every r in R. This proves (a); and (b) may be obtained as an immediate consequence of Lemma 1. If $R=Rz$ for some z in R, then it follows from Lemma 4 that $L(z)=0$. Furthermore, there exists an element e in $R=Rz$ such that $z=ez$. If r is any element in R, then $(r-re)z=rz-rz=0$. The element e is a right-unit for R, since $L(z)=0$ implies $r=re$ for every r. This completes the proof of the theorem.

THEOREM 5. *If there exists a universal unit* 1 *in R, if the minimum condition for principal left-ideals or the maximum condition for zero-dividing left-ideals is satisfied in R, then the relations $uv=1$ and $vu=1$ imply each other.*

PROOF. If the maximum condition is satisfied by the zero-dividing left-ideals in R, then our contention is an immediate consequence of Theorem 4, (a) and Lemma 2. If the minimum condition is satisfied by the principal left-ideals in R, then $uv=1$ implies $L(u)=0$. By Theorem 1 there exists an element w such that $wu=1$. But $w=wuv=v$ so that $vu=wu=1$.

The impossibility of omitting the chain conditions in Theorem 5 is a consequence of Example 1.

The following special case of Theorem 5 may be worth mentioning, since it solves a problem of importance in the theory of matrices.[3] If R is a ring with universal unit 1, then denote by R_n the ring of all square matrices with n rows and n columns whose coefficients are in R. If the maximum (minimum) condition is satisfied by the left-ideals in R, then the same condition holds in R_n; and hence it is a consequence of Theorem 5 that the matrices A and B in R_n satisfy either $AB=1$ *and* $BA=1$ or neither of these relations; in particular it is impossible that a matrix in R_n is both a zero-divisor and an inverse in R_n.

A right- or left-ideal J in the ring R is termed nilpotent, if $J^i=0$ for some positive integer i. The sum $P=P(R)$ of all the nilpotent right- and left-ideals in R is a two-sided ideal in R which is called the *radical* of R. Hopkins[4] has shown that the radical itself is nilpotent, if the minimum condition is satisfied by the left-ideals in R.

THEOREM 6. *There exists a left-unit*[5] *for the ring R with minimum condition for left-ideals if, and only if,*

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[3] Cf., for example, B. L. van der Waerden, *Moderne Algebra*, vol. 2, Berlin, 1927, pp. 114–115.

[4] C. Hopkins, Annals of Mathematics, (2), vol. 40 (1939), pp. 712–730; Theorem 1.4, p. 714.

[5] Note that the preceding criteria assured the existence of right-units for the ring R.

(i) $Rx=0$ *implies* $x=0$,

(ii) $P^iRx=xP=0$ *implies* $P^ix=0$ *for* $0<i<m$ (*where* m *is the smallest positive integer such that* $P^m=0$).

PROOF. If R contains a left-unit for every element then x is always an element of Rx; and this shows the necessity of conditions (i) and (ii).

Suppose that conditions (i) and (ii) are satisfied by the ring R. From the choice of the integer m it follows that $0=P^m<P^{m-1}$ (using the notation $P^0=R$). If x is an element not 0 in P^{m-1}, then $Px=0$ so that $P<R$ is a consequence of condition (i). Hence[6] there exists in R an idempotent $e\neq 0$ such that $R=Re+P$.

Denote by T the set of all the elements x in R which satisfy: $xP=0$; and denote by $S(i)$ (for positive i) the set of all those elements in T which satisfy: $P^ix=0$. Clearly T and therefore every $S(i)$ is a two-sided ideal in R. We are going to prove by complete induction that $eS(i)=S(i)$. This is true for $i=1$, since it follows from $Rx=Rex+Px$ and condition (i) that $x=0$ is the only element x in $S(1)$ which satisfies $ex=0$. Suppose now that $S(i)=eS(i)$ and that the element t in $S(i+1)$ satisfies $et=0$. Then $Rt=Ret+Pt=Pt\leqq PS(i+1)\leqq S(i)$, $P^iRt\leqq P^iS(i)=0$; and it follows from condition (ii) that $P^it=0$. The element t is therefore an element in $S(i)=eS(i)$, that is, $t=et=0$ or $S(i+1)=eS(i+1)$; and this completes the induction. In particular it follows that $T=eT$, since $T=S(m)$ is a consequence of $P^m=0$.

Denote by $T(i)$ the set of all the elements x in R which satisfy $xP^i=0$. Every $T(i)$ is a two-sided ideal in R. We are going to prove by complete induction that $T(i)=eT(i)$. This is true for $i=1$, since $T=T(1)$. Suppose that $T(i)=eT(i)$ and that the element v in $T(i+1)$ satisfies $ev=0$. Then $vP=0$, since $vP\leqq T(i)=eT(i)$ and $ev=0$. Thus v is an element in $T=eT$ and $v=ev=0$. This shows that $T(i+1)=eT(i+1)$; and this completes the induction. Since $P^m=0$ implies $T(m)=R$, we have proved in particular that $R=eR$; and the idempotent e is therefore a left-unit for R.

We note that we proved the following fact. If the conditions (i) and (ii) of Theorem 6 are satisfied by the ring R, if the radical of R is nilpotent, and if the idempotent e in R satisfies $R=Re+P$, then e is a left-unit for R; and we mention that the minimum condition for zero-dividing left-ideals would be sufficient to assure the existence of such an idempotent e.

One verifies readily that the conditions (i) and (ii) of Theorem 6 are consequences of each of the following (not necessary) conditions

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[6] Cf. Hopkins, loc. cit., Theorems 4.1, 4.2, p. 721.

(a) $P=0$; (b) condition (i), and $xP=0$ implies $Px=0$; (c) condition (i), and $RP^i \leqq P^iR$.

That condition (ii) is indispensable for the validity of Theorem 6 may be seen from the following example. Denote by R the ring of all the matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & d & e \end{pmatrix}$$

with coefficients from some (commutative) field F. It is readily verified that $Rx=0$ implies $x=0$; that $yR=0$ implies $y=0$; and that this ring R does not contain a universal unit 1. We note that R would be a finite ring, if F were a finite field.

COROLLARY. *There exists a left-unit for the ring R with minimum condition for left-ideals if (and only if) there exists a left-unit for every element in R.*

PROOF. The existence of a left-unit for the element x in R is equivalent to the fact that x is an element in the left-ideal Rx. This shows that the conditions (i), (ii) of Theorem 6 are consequences of the condition of the corollary.

THEOREM 7. *There exists a universal unit in the ring R with minimum condition for left-ideals if, and only if,*

(1) $Rx=0$ *implies* $x=0$,

(2) $yR=0$ *implies* $y=0$,

(3) $RP=PR$.

PROOF. The necessity of the conditions is obvious. If the conditions are satisfied by R, then it follows from (1), (3) that conditions (i), (ii) of Theorem 6 are satisfied. Hence there exists a left-unit e for R; and it follows from Lemma 2 and condition (2) that e is a universal unit for R.

COROLLARY. *There exists a universal unit in the commutative ring R with minimum condition for ideals if, and only if, $Rx=0$ implies $x=0$.*

This is an immediate consequence of Theorem 7.

The impossibility of substituting the maximum condition for the minimum condition in Theorems 6 or 7 may be seen from the example of the even rational integers.

Comparing the results of this investigation one sees that minimum conditions are more powerful than maximum conditions. This phenomenon is closely connected with Hopkins' theorem:[7] *If the ring R contains a left-unit or a right-unit for R, then the maximum condition for left-ideals in R is a consequence of the minimum condition for left-ideals in R.* We could not make use of this theorem, since we usually considered rings with minimum (maximum) condition for principal or zero-dividing left-ideals. On the other hand it is possible to improve slightly the theorem of Hopkins by applying our criteria for the existence of units.

It is well known that *the commutative law of addition* is a consequence of the customary postulates for a field, but not of the analogous postulates for a ring. To investigate this situation we consider a nonvacuous set R of elements which are connected by two operations: addition $u+v$ and multiplication uv, subject to the following rules:

I. R is a group under addition.

II. The product uv of the elements u and v in R is a uniquely determined element in R.

III. $u(v+w)=uv+uw$, $(u+v)w=uw+vw$.

We denote by R^2 the subgroup of the additive group R which is generated by all the products uv and by $C=C(R)$ the commutator subgroup of the additive group R. Both R^2 and C are two-sided ideals in R.

The addition is commutative in R^2, and $CR=RC=0$.

PROOF. If a, b, c, d are elements in R, then it follows from the distributive laws that

$$ad + ab + cd + cb = (a + c)(d + b) = ad + cd + ab + cb;$$

and hence it follows from the cancellation law of addition that

$$ab + cd = cd + ab.$$

Our contention is an immediate consequence of this equality.

It is now obvious that each of the following two conditions is sufficient to assure commutativity of addition.

(a) $R=R^2$.

(b) 0 is the only element in R which satisfies both $Rx=0$ and $xR=0$.

We note that each of these two conditions is necessary, but not sufficient for the existence of a universal unit in R.

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[7] Hopkins, loc. cit., Theorem 6.7, p. 728.

BOOK REVIEWS

On the Principles of Statistical Inference. By Abraham Wald. (Notre Dame Mathematical Lectures, No. 1.) Notre Dame, Indiana, 1942. 50 pp. $1.00.

The University of Notre Dame has started its new series of mathematical publications, the "Notre Dame Mathematical Lectures," by four lectures on modern theory of statistics, delivered by one of its most brilliant proponents, Dr. Abraham Wald. While wishing the new series the best of success one can somewhat regret that, probably, considerations of cost have prevented it from appearing in a printed form. The little book is lithoprinted and lithoprinted very well, with all the formulas perfectly legible. Still, in a printed form it would look much better. We may hope that in time a private benefactor or an institution will be found who could provide funds for giving good mathematical publications the external form that they deserve.

As mentioned, Dr. Wald's booklet contains four lectures. However it is divided into six chapters, I. Introduction, II. The Neyman-Pearson theory of testing of a statistical hypothesis, III. R. A. Fisher's theory of estimation, IV. The theory of confidence intervals, V. Asymptotically most powerful tests and asymptotically shortest confidence intervals, and VI. Outline of a general theory of statistical inference. Having in view a mathematical reader the booklet has nothing in common with numerous books "on statistical methods;" is not concerned with applied problems of a particular character but deals with fundamental conceptions, actually covering about all the most important that were achieved during the last 25 years or so. Compared with this scope the size of the booklet is very small and therefore the presentation of certain sections necessarily short. This applies especially to the first 28 pages, given to the description of what had been done before Dr. Wald himself appeared as an active contributor to the theory of statistics with his conceptions of asymptotically most powerful tests and of general statistical inference. In spite of the shortness of presentation the book is exceedingly informative. In fact, it could be used as an excellent source of information for all those mathematical readers who, without hunting for articles spread in a number of journals, would like to get a bird's eye view of what is being done in the field of mathematical statistics.

There are just two gaps that one can regret in the latter respect.

When speaking of the problems of estimation by intervals the author indicates the distinction between the theory of confidence intervals and that of fiducial argument as developed by R. A. Fisher. However, while giving an outline of the former the author does not present any information about the latter, except for references to several papers by Fisher.

Another gap is the omission of at least a few details concerning tests of composite hypotheses and problems of estimation of some but not all the parameters that may be involved in the probability law of observable random variables. As these statistical problems have created problems of pure analysis, those of the so-called "similar regions," not previously considered, their description in a book like that under review might have contributed to its value and increased the chances of the problems getting a speedy and more satisfactory solution than the one that is available now.

However, to say that a book meant to be short is actually short, should not be considered as a criticism.

JERZY NEYMAN

Les Fonctions Multivalentes. By M. Biernacki. (Actualités Scientifiques et Industrielles, no. 657.) Paris, Hermann, 1938. Fr. 66.

The notion of multivalent functions was first introduced and developed by Paul Montel in his book, *Leçons sur les Fonctions Univalentes ou Multivalentes*. An analytic function of a complex variable in a region is said to be multivalent of order p (or p-valent) in that region if it assumes no value more than p times and at least one value exactly p times. The case $p=1$, that of univalent functions, has been studied extensively and has yielded a unified and rather complete theory. The extension of this theory to any positive integral p is considerably more difficult and the resulting theory is far less complete.

The author collects these results with the intention of aiding future research in the field. Most of the results are stated without proof, whenever the known proofs are at all complicated. The first chapter deals for the most part with extensions of the theory of univalent analytic functions to the general multivalent case and to other related classes of functions. The second chapter deals with meromorphic multivalent functions. The last chapter takes up a few results connected with systems of functions.

It is unfortunate that the book was written before the appearance in 1940 of important papers of D. C. Spencer on the subject of finitely mean valent functions which have greatly clarified the whole notion of p-valence.

The book, however, should prove useful as a reference book on the literature of the subject prior to 1938.

W. SEIDEL

Dimension Theory. By Witold Hurewicz and Henry Wallman. (Princeton Mathematical Series, no. 4.) Princeton University Press, 1941. 156 pp. $3.00.

In contrast with Karl Menger's well known *Dimensionstheorie* which could claim a considerable measure of completeness at the time of its publication (1928) the present modest volume includes "only those topics . . . which are of interest to the general worker in mathematics as well as the specialist in topology." Despite this self-imposed limitation, the authors have presented a simple and connected account of the most essential parts of dimension theory. They have treated this branch of topology—hardly surpassed in the elegance of its results—with discrimination and technical skill (it should be pointed out that the senior author is one of the outstanding builders of the theory). The result is an unusually interesting book. It must have been a pleasure to write; it was certainly a pleasure to read.

The dimension of a space is the least integer n for which every point has arbitrarily small neighborhoods whose boundaries are of dimension less than n; empty sets are of dimension -1. This well known recursive definition is due independently to Menger and Urysohn although its intuitive content goes back to Poincaré. It would be natural to expect that a theory based on such a definition would be purely point-set theoretical in character. The remarkable thing is, however, that there exist a number of equivalent definitions of dimension, each belonging to a different domain of ideas and each "right" and natural in its domain. The dimension of X can be defined, for example, as the least n such that X can be approximated arbitrarily well (in a certain sense) by polytopes of dimension not exceeding n. Or, dim X can be defined as the least n for which every continuous mapping of an arbitrary closed subset of X into an n-sphere S_n can be extended to a mapping of the whole of X into S_n. The first of these definitions brings the concept of dimension into the realm of algebraic topology (complexes, homology theory, and so on) and the second gives it a place in the theory of continuous mappings. By a skillful interplay between these two settings for dimension theory, the authors obtain a simple characterization of dimension purely in terms of homology theory. The technique which leads so smoothly to this result includes the use of cohomologies and character groups,—an indication of the thoroughly contemporary quality of the book.

The reader—specialist in topology or not—will find particularly interesting the elementary proof of Brouwer's fixed-point theorem for closed n-cells and the use of this theorem in showing that the dimension of E_n (Euclidean n-space) is n; Hurewicz's beautiful proof that every n-dimensional space can be imbedded in E_{2n+1} and the use of the technical devices in that proof in establishing the relation between dimension and the orders of coverings; the systematic use of the notion of extension (of a mapping or homomorphism); a separation theorem for homeomorphs of $(n-1)$-spheres in E_n as a corollary of a theorem about mappings; a discussion of the relation between dimension and measure; a number of remarks about spaces of infinitely many dimensions; an exposition of Čech homology theory and the dual cohomology theory; the use of cohomologies in questions about mappings and their extensions. It is understood throughout the book that "space" means "separable metric space." A short appendix contains an account of the difficulties which arise in extending the theory to more general spaces.

P. A. Smith

The Laplace Transform. By David Vernon Widder. (Princeton Mathematical Series, no. 6.) Princeton University Press, 1941. 10+406 pp. $6.00.

The theory of integral operators has developed into a major branch of analysis. It has proved to be a valuable, in fact essential, complement to the theory of differential equations, while its logical structure is more satisfactory in the sense that one deals in general with equivalences, rather than the sufficient conditions of differential equations.

As one might expect, the theory of integral operators, even in the linear case, also has complementary subdivisions. There is the well known spectral theory, which while highly effective when applicable, seems to be closely confined to functions in L_2. For other classes of functions, there have been investigations of particular operators, for example the Fourier transform.

But besides the Fourier transform, there are others which have been investigated for more than a century and it is to certain of these that the present work is devoted. However, the author also presents various applications of integral transforms (including the Fourier) of great interest and importance so that this work attains a generality which makes its subject matter of basic importance for the analyst.

The book is concerned with the following transforms: the Laplace transform

$$(*)\qquad f(s) = \int_0^\infty e^{-st}d\alpha(t),$$

the bilateral Laplace transform (*′) in which the lower limit of integration is not zero but minus infinity, the Mellin transform which is obtainable by a change of variable from the bilateral transform, the Stieltjes transform

$$(**)\qquad f(s) = \int_0^\infty \frac{d\alpha(t)}{s+t},$$

which is, formally, the iterate of (*), and the Fourier transform in certain applications. $\alpha(t)$ is, at least, of bounded variation on every finite interval not containing the origin but during the course of the investigation, it is specialized in various ways, for instance, to be of bounded variation, or to be the integral of a function ϕ of a specified class, L, L_2, L_p, or M.

The integration in (*) or (**) is Riemann-Stieltjes and Chapter I contains a rather complete discussion of this type of integration and of functions of bounded variation, including Helly's theorem. If s is regarded as a complex variable, the domains of existence and analyticity of $f(s)$ may be specified by certain convergence considerations given in the first part of Chapter II for (*), the first part of Chapter VI for (*′) and the first part of Chapter VIII for (**).

Besides the important applications, which we will discuss below, there are two interrelated problems treated for these transforms. One of these is the problem of "inversion," that is, given f, to find α. There are, in general, for any such transform at least two such inversion theories. The other is the "representation problem" which may be described as the problem of finding the class of f's, which corresponds to a given class of α's. In general, one can readily guess at the class of f's, but the difficulty arises in trying to show that each f in the class has the specified representation.

The "complex" inversion theory is concerned with formulas which express α in terms of $f(s)$ regarded as a function of a complex variable s. For (*) and (*′), as given in Chapters II and VI, the formula involves integration along a line parallel to the axis of imaginaries and the theory has similarities with that of the Fourier transform. For (**), we have in Chapter VIII, a theory somewhat like that of the Poisson integral. For (*), this inversion formula yields a representation theory for those α's which are integrals of functions in L_2. In

Chapter II, one also finds certain other representation discussion for (*) in the case in which $f(s)$ is regarded as a function of a complex variable.

The "real" inversion theory is given in Chapter VII for (*) and in Chapter VIII for (**). It involves the values of all the derivitives of $f(s)$ along the positive real axis. For (*) in the case in which α is absolutely continuous the result is due to E. L. Post and has been extended by Widder. Formulas for the variation of α and for the saltus at a point are given. The associated representation theory is more highly developed and considers the class of α's of bounded variation, the class of non-decreasing α's, the bounded subclass of these, the class of integrals of a $\phi \in L_p$ for a fixed $p \geqq 1$, and the class of integrals of a $\phi \in M$. The relation between (*) and (**) is used in Chapter VIII, §25, to obtain a real inversion formula for (*), involving just the values of f itself.

The most interesting representation problem for (*) is undoubtedly that in which α is increasing. Under these circumstances, $f(s)$ is "completely monotonic," that is for $s \geqq 0$, $f^{(k)}(s)(-1)^k \geqq 0$. The converse, that every completely monotonic f is the Laplace transform of a monotonic bounded $\alpha(t)$, is called Bernstein's theorem. This result was also obtained in a different form by Hausdorff and independently by Widder.

In preparation for this result, three moment problems are discussed and solved in Chapter III. A moment problem is one in which, given a sequence $\{\mu_n\}$ one is required to find a monotonic $\alpha(t)$, such that

$$\mu_n = \int_a^b t^n d\alpha(t).$$

If $a = 0$, $b = 1$, we have the Hausdorff moment problem; if $a = 0$, $b = \infty$, the Stieljes moment problem and if $a = -\infty$, $b = \infty$, the Hamburger moment problem. In the Hausdorff case, one may even consider the more general case in which α is of bounded variation instead of monotonic, and the result is unique. In the other cases if α is not monotonic the answer is not unique. The Hausdorff problem is associated with the process of summation of series and permits one to solve equivalence and comparison questions for the usual methods of Cesàro and Hölder. Those series $\{\mu_n\}$ for which α is monotonic are "positive definite" and this is related to the corresponding notion for quadratic forms. The determinant criterion for positive definiteness appears and is used in a discussion of certain work of Boas on those cases of the moment problem in which the limits of integration are infinite.

In Chapter IV, three proofs of Bernstein's theorem are given. Two of these are based on the solution of the Hausdorff moment problem given in the preceding chapter, the remaining one involves the Laguerre polynomials. Since the set of completely monotonic functions is multiplicative, the same is true for the set of functions f in the form (*). The chapter concludes by discussing "completely convex" functions for which $(-1)^k f^{(2k)}(s) \geqq 0$ for $s \geqq 0$ and these are shown to be entire.

Chapter V discusses various limit questions for integrals and certain striking applications. Suppose $f(s)$ is defined by (*) for $s > 0$. Then if $\lim_{t\to\infty} \alpha(t)$ exists and equals A, $\lim_{s\to 0} f(s)$ exists and equals A. The converse is true only if α is suitably restricted and this converse is one of the Tauberian theorems treated in Chapter V and including results on (**). These results introduce two famous theorems. The first is Wiener's general Tauberian theorem on the equation

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} g(x-t)p(t)dt = A\int_{-\infty}^{\infty} g(t)dt$$

for a fixed bounded p. This theorem states that if this equation holds for a $g \in L$ whose Fourier transform does not vanish, it holds for all $g \in L$. A number of variants of this theorem are also shown. The second is the prime number theorem for which two proofs are given, one due to Wiener, the other due to Ikehara.

The notion of a resultant is treated for a number of transforms. For instance if we let

$$\gamma(x) = \int_{-\infty}^{\infty} \alpha(x-t)\beta(t)dt,$$

it is shown in Chapter VI, that for both the Fourier and bilateral Laplace transform, the product of the transforms of α and β is the transform of the result, γ. The book concludes with a discussion of the integral equation

$$f(s) = \lambda \int_0^{\infty} \frac{f(t)dt}{s+t}$$

and the corresponding equation for the Laplace case.

The book is certainly enjoyable and interesting. The style is clear, there are few typographical errors and the subject matter is increasingly impressive as one reads on. The applications are particularly striking. In some cases, it is not clear why so many proofs of the same theorem are given and a guide to a reader who might be interested in any one of the many specific results would be valuable. However, it would be quite easy to use various topics treated in the book in a course, whose main interest is not integral operators. One might mention, the Riemann-Stieltjes integral, functions of bounded variation, methods of summation of series, positive definite series, the moment problems, Bernstein's theorem, the Tauberian theorems, the prime number theorem, the Laguerre polynomials, the notion of a positive definite kernel of an integral equation, and the specific integral equations mentioned.

Thus the author has presented us with a treatise on a branch of analysis of great importance and whose applications are of wide interest. The book is extremely satisfactory, when concerned with either its principal topics or the other related developments and one is confident that it will have a most valuable effect both on research and graduate study.

F. J. MURRAY

Mathematical Methods in Engineering. By Theodore von Kármán and Maurice A. Biot. New York, McGraw-Hill, 1940. 12+505 pp. $4.00.

This book by two masters of applied mathematics selects certain representative groups of advanced engineering problems and presents the appropriate mathematical methods of solution, together with helpful excursions into purely mathematical topics. This is a very effective plan that might well be followed by future books in applied mathematics. The problems are largely in mechanics and are interesting, instructive and up-to-date, especially those on the airplane—as might be expected from the interests and accomplishments of the senior author.

Besides the problems worked out in the text, each chapter includes a set of problems of graded difficulty to be worked out by the student, the answers being given at the end of the book. Each chapter opens with a thought-provoking quotation from an authority in the field and ends with a well selected list of references, mostly standard texts.

The authors are as careful with their mathematics as with their physics and engineering, but have chosen to omit mathematical proofs in many cases where they felt that space was not available. It seems to the reviewer that in some of these cases at least a brief outline of the proof could have been given to the advantage of the reader.

Chapter I is a direct, clear introduction to ordinary differential equations. Numerical solutions are properly emphasized; but the step-by-step method given is not the most suitable for ordinary engineering purposes. This method involves the calculation of several initial values by a Taylor series, and so is not applicable to differential equations with empirical coefficients, which are very common in engineering. Furthermore, it is based on an elaborate difference table, which means tedious calculations and requires equal steps in the argument. The latter limitation makes it unsuitable for a function whose derivative varies over a wide range and inapplicable where the derivative becomes infinite. (Actually the example worked out on on p. 19 could have been done with less work by using the Taylor series throughout.) Numerical methods for systems of first-order differential equations (including single higher-order differential equations) are not given, although these are common in engineering.

Simpson's rule is given as the preferred method for numerical quadrature; and an example on p. 6 shows that it gives 5-digit accuracy as compared with 2-digit accuracy for the simple trapezoidal rule. However, the inaccuracy of the trapezoidal rule is mainly due to lack of end corrections: if the single correction term $(1/12)h^2f'(x)\big|_0^1$ is included, the trapezoidal rule here gives 7-digit accuracy. (This correction term is the first of a series in the Euler summation formula; it can also be derived geometrically from the assumption that the integrand $f(x)$ is sufficiently approximated by a parabolic arc.) In the reviewer's opinion, Simpson's rule should be abandoned as not giving the best approximation obtainable from a given set of values.

On p. 7, it is stated that by means of isoclines a step-by-step solution of $D_xy=f(x, y)$ can always be obtained "provided that $f(x, y)$ is a single-valued continuous function of x and y." The implication that a unique solution can be obtained is incorrect because the Lipschitz condition has been disregarded. For example, the right-hand member of the equation $D_xy=3y^{1/2}-x$, with $y^{1/2}$ restricted to positive values, is single-valued and continuous at (0, 0) but there is an infinity of solutions starting at this point, given by $y^{1/2}=x+C-(Cx+C^2)^{1/2}$.

Chapter II treats of Bessel functions as the solutions of Bessel's equation. The power series are derived by the method of undetermined coefficients, without a test for convergence nor a proof that the series actually satisfies the differential equation, although the method of successive substitution (or its operational equivalent) would both be shorter and would provide a remainder permitting rigorous treatment. Contour integrals are not employed; so no proof is given for the relations between the coefficients of the converging and the asymptotic series. However, the description of the properties of the Bessel functions is good and is illustrated by helpful graphs. Also there are given various common equivalent forms of Bessel's equations, with their solutions. The notation used is that of Watson.

Chapter III is devoted to the fundamental concepts of dynamics and includes Lagrange's equations in generalized coordinates. It starts with Newton's laws of motion and later derives the principle of conservation of energy for a mechanically conservative system. This is the classical approach; but, inasmuch as conservation of energy is the most basic principle of all physics, it would seem sounder philosophically to start with it as an experimental fact (as the reviewer has done for some years in his physics classes). Newton's third law of motion is, in fact, essentially a statement of conservation of energy as applied to mechanical systems—that when a body A exerts a force on a body B, the energy taken out of A in a joint displacement of the two bodies is equal to the energy put into B. (As has been recognized by Kelvin and Tait, Newton, in the Scholium following the enunciation of his laws, implicitly anticipated the concepts of energy and power.)

Newton's second law is given in the form $F=D_t mv$ and reference is made to the fact that m as well as v may be variable. This generalization, however, is treacherous: it applies, for example, to a falling raindrop whose mass is increasing by accretion from water vapor; but it does not apply to a leaking water bucket, where the correct form is $F=mD_t v$, even though m is varying.

Theorems are worked out for mass points (particles); and it is then stated without proof (p. 93) that they also apply to continuous fluid and elastic bodies. This is taking a big jump. Had energy been used from the start, all theorems could have been proved rigorously.

Much of the development in this chapter is in vector notation, including scalar $(\cdot)$ and vector $(\times)$ products.

Chapter IV discusses various problems in dynamics and includes two purely mathematical discussions: (1) elliptic integrals, introduced to express the motion of a pendulum; and (2) the singularities of first-order differential equations. Both of these discussions are clear and interesting and they gain by their association with important physical problems.

Chapter V is on the small oscillations of conservative systems and employs the Lagrange equations. In connection with the numerical calculation of natural frequencies, the following methods of finding the real roots of algebraic equations are given: Newton's, iteration, Graeffe's. Matrices are also introduced to find the frequencies and the normal modes by successive approximations. The only matrix manipulation used is the product of a square matrix by a column matrix; and no proof is given for the rule.

In Chapter VI, small oscillations of non-conservative systems are taken up; and dynamic stability is treated, calling for the calculation of the complex roots of algebraic equations with real coefficients.

Chapter VII discusses the differential equations of various continuous structures, such as strings and beams, with and without uniformly distributed elastic restraints. Harmonic vibration is shown to lead to the same forms of equation as these elastic restraints. Other problems considered are the buckling of uniform and tapered columns and the combined axial and lateral loading on the spar of an airplane wing.

In Chapter VIII, the Fourier series and integral are applied to continuous structures. It is shown that the Fourier coefficients give the best approximation to an arbitrary function, for a terminated trigonometric series, and that the mean-square error steadily decreases as terms are added. Sufficient conditions for convergence are stated without proof; and the Gibbs phenomenon is described. Numerical, graphical and mechanical methods for determining the coefficients are described. This chapter also discusses the Rayleigh-Ritz method of approximating natural frequencies and characteristic functions, applying it to trigonometric series.

Chapter IX applies complex numbers to periodic phenomena, both mechanical and electrical. The concept of impedance is properly emphasized, but unfortunately two kinds of mechanical impedance are used, force/displacement and force/velocity; and the former is preferred, although it is not the analogue of electrical impedance. This confusion is quite unnecessary, as the two impedances are related simply by the factor $i\omega$. For arbitrary impressed forces or voltages, the Fourier series is introduced in complex form.

Chapter X is devoted to the Heaviside operational calculus, which is applicable to linear differential equations (ordinary or partial) with constant coefficients. Various procedures have been used for establishing the methods of this calculus: (a) the direct operational approach; (b) Fourier integrals and transforms (or the equivalent Laplace transform); (c) the Bromwich contour integral; and (d) Carson's integral equation. Of these, procedure (a) is the simplest for ordinary differential equations, which is as far as this chapter goes: it involves only the separation of rational functions of the derivative operator D into partial fractions and the use of the single evaluating formula $(D-\alpha)^{-q-1}DH(t)=(t^q/q!)\epsilon^{\alpha t}H(t)$, where $H(t)$ is the Heaviside function or unit-step function. If the operand consists of terms of the form $t^q\epsilon^{\alpha t}H(t)$, it can be put back into operational form by using this formula backwards, after which partial fractions are applied to the resulting combined operator. If the operand $f(t)$ has another form, any rational operation $g(D)f(t)$ is evaluated as a single integral by the superposition formula $g(D)f(t)=\int_{-\infty}^{\infty}f(\tau)g(D)DH(t-\tau)d\tau$. The proofs of these formulas and of the legitimacy of the algebraic manipulation of the operators are simple by the direct operational approach, making use of the principle characteristic of the Heaviside operational calculus, that the operands $f(t)$ are zero for all sufficiently negative values of t (commonly for all $t<0$). This is a natural condition in many engineering problems and can be introduced artificially in other cases, for example where the physical operand is not defined for $t<0$.

The book under review unfortunately uses more difficult procedures, first employing the Fourier integral and leading up to the Bromwich integral and then proving the operational formulas by Carson's integral equation. The symbol p is confusingly used both as a complex variable and for the derivative operator D. [It is a common misconception that there is a distinction between inverse functions of Heaviside's p and of D, which arose from a failure to restrict operands as in the preceding paragraph. With such restriction and with D^{-1} defined as $\int_{-\infty}^{t}$, all functions of D become perfectly definite and commutative with one another; and no reason remains for a special symbol in place of the ordinary D.] The awkward italic $l(t)$ is used for the Heaviside function $H(t)$; and $S(t)$ is used for the unit impulse function, which is better represented as $DH(t)$, since $D^{-1}S(t)=H(t)$. These methods and usages, of course, are found commonly in the literature; but they seem to be delaying the incorporation of the very valuable Heaviside calculus into the current stream of mathematics.

A too little appreciated branch of applied mathematics is introduced in Chapter XI, the calculus of finite differences. However, the method used is that of assuming the form of a solution and finding the constants which satisfy the given difference equation and the given conditions. This is a field to which operational methods are particularly applicable, especially in a form analogous to the Heaviside operational calculus for differential equations, the evaluating formula being $(1-\alpha E^{-1})^{-q-1}(1-E^{-1})H(t)=[(t+1)\cdots(t+q)/q!]\alpha^{t}H(t)$, now where E is Boole's operator defined by $E^{-1}f(t)=f(t)-1$ and $H(t)$ equals 0 for negative integer t and equals 1 for zero and positive integer t. The problems include deflections of continuous beams supported at equal intervals, the buckling of a rectangular lattice truss, voltage distribution in a suspension insulator, critical speeds of a multicylinder engine, leading to a mechanical wave filter, and electric wave filters. (In the last, an attenuator is incorrectly called a "wave trap," which is a resonant element.)

Following the chapters appears a short section entitled "words and phrases" intended to give certain "strictly mathematical definitions" and starting with Kronecker's dictum, "God made the integers; all the rest is the work of man." It seems surprising to have this dictum endorsed by engineers and physicists: it might better be replaced by the statement, *God made both discrete and continuous physical quantities; man devised means (integers and real numbers) for representing them.* This section in general seems out of place and of little help to a reader who would be capable of using the book.

Besides a few obvious typographical errors, the reviewer noticed the following errors: last equation, p. 108, a minus sign should precede the first term in each member; the answer to Problem 12, p. 108, has the inequality reversed; the answer to Problem 16, p. 109, not only uses k for K but incorrectly gives a stable solution; the first answer to Problem 1, p. 210, incorrectly has the factor 1/2; the equation on p. 131 is written as if t were under the radical sign. (It is unfortunate that radical signs are used in place of fractional exponents throughout the book.)

This is a book which should be in the library of every engineer who is interested in the analytical development of his subject. It should be studied by mathematicians who are willing to admit that their place in society may need justification on other than purely intellectual grounds. It is well adapted as a text or for collateral study in an advanced course in applied mathematics or in theoretical mechanics. The authors are to be congratulated in so competently supplying a real need.

ALAN HAZELTINE

Development of the Minkowski Geometry of Numbers. By Harris Hancock. (Published with the aid of the Charles Phelps Taft Memorial Fund and of two Friends.) New York, Macmillan, 1939. 24+839 pp. $12.00.

Professor Hancock says in the introduction of his book: "In every subject that occupies the human mind, be it history, philosophy law, medicine, science, music, etc., there arise outstanding men who evince an innate genius in their special fields, an innateness that seems as it were of divine origin. Minkowski was one of the great mathematicians of all time." It is the aim of Hancock's book to make an important part of Minkowski's work, his geometry of numbers, more easily accessible to the student of advanced mathematics. The author has endeavored to give an exposition of the whole of Minkowski's investigations as far as they were concerned with the geometry of numbers, following for the most part rather closely Minkowski's own publications. For the convenience of the reader, additional explanations and computations are given, numerous figures and examples are added, and in some cases missing proofs are provided.

There seems to be no need today to describe the mathematical work of Minkowski in a review. None less than Hilbert has given a picture of Minkowski in his "Gedächtnisrede" which can be found at the beginning of the *Collected Works* of Minkowski. Most mathematicians today have at least some idea about Minkowski's geometry of numbers. So we may refrain from discussing the significance of the theories developed in Hancock's book.

The table of contents is as follows: I. Preliminary notions and historical notes, II. Surfaces that are nowhere concave, III. The volume of bodies, IV. Bodies which with respect to their volumes have more than one point with integral coordinates, V. Applications of the preceding investigations, VI. Algebraic numbers, VII. Arithmetical theory of a pair of lines; theory of continued fractions and of the real quadratic irrational numbers, VIII. Shorter papers by Minkowski. Historical, IX. A criterion for algebraic numbers, X. The theory of continued fractions, XI. Periodic approximation of algebraic numbers, XII. On the approximation of a real quantity through rational numbers, XIII. A further analytic-arithmetic inequality, XIV. The arithmetic of the ellipsoid, XV. Computation of a volume through successive integrations, XVI. Proof of the new analytic-arithmetic inequality, XVII. The extreme standard bodies, XVIII. Densest placement of congruent homologous bodies, XIX. Miscellany, XX. New theory of quadratic forms. Region of discontinuity for arithmetical equivalence.

Chapter I corresponds to the first part of the third chapter of Minkowski's *Diophantische Approximationen* while the second part is treated at the beginning of Chapter XVIII. In the Chapters II to VII, Hancock follows the first four chapters of Minkowski's *Geometrie der Zahlen*; the last chapter of Minkowski's book is the subject of Hancock's Chapters XIII to XVII. The remaining parts of Hancock's book discuss the various papers of Minkowski which are concerned with the geometry of numbers, and which can be found in the *Collected Works*. In Chapter X Hancock follows the thesis of his pupil P. Pepper who worked out proofs for theorems stated by Minkowski.

As will be seen from this description, the book restricts itself entirely to Minkowski's own work in the geometry of numbers. Professor Hancock explains in the introduction that this was done in order to limit the content of the book which otherwise would have been beyond bounds. Still, one may regret that the newer developments were not at least indicated. There is another factor which decreases the value of the book, this being that the representation often is not as good as one may wish (compare, for instance, Article 8 with the corresponding section of the *Diophantische Approximationen*). The different parts of the book could have been connected more closely. Finally, there are numerous misprints some of which are confusing. In the opinion of the reviewer, it would not be surprising, if many readers should prefer the original texts. On the other hand, there will be many mathematicians who will be very grâteful to Professor Hancock for facilitating for them access to Minkowski's beautiful investigations.

RICHARD BRAUER

Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band I, Teil 1, Heft 2, 114 pp.; Band I_1, Teil 1, Heft 4, 51 pp.; Band I_1, Heft 5. 54+54+28 pp. Leipzig and Berlin, Teubner, 1939.

This new edition of the Enzyklopädie der Mathematischen Wissenschaften appears exactly forty years after the publication of the first volume of the first edition in 1899. The original project of compiling and presenting a comprehensive review of the science of mathematics and its allied fields was considered a monumental and ambitious task which aroused great interest among contemporary mathematicians. The initiative to the Enzyklopädie was taken by Felix Klein, Heinrich Weber and Franz Meyer and a great number of other prominent mathematicians was gradually associated with this initial group. To begin with, the work had been planned in the form of a regular encyclopedia in which the material should appear as special articles for each mathematical term. After an early attempt along these lines it became clear that this method of presentation led to considerable overlapping and the artificial classification of the subjects according to the alphabet tended to make them incoherent and lacking in general views. This prompted the fortunate decision of giving a systematic account of the field of mathematics in which the various articles on the subdivisions of the science were fitted into their natural connections as far as it was possible. It may also be remarked that although the original encyclopedia was to a large extent created and dominated by German mathematicians, the plan was supposed to be at least partially international in character and certain volumes of a French edition, slightly different from the German version, were actually published.

It took nearly forty years to complete the first edition of the encyclopedia and many of its parts, particularly those which were printed in the earliest volumes, have long outlived the usefulness for which they were intended and they now appear more as scientific historic documents than reviews of their subjects. The new edition proposes to remedy this situation. Already from the few numbers which are available it is clear that the plan of the new edition is almost identical with the former one. The old articles have been thoroughly rewritten and brought up to date, and new articles have been put in whenever recent developments have warranted it. But the new edition, in spite of its many strong points also seems destined soon to develop the deficiencies of the first.

The intention of these remarks is of course not to criticize the general idea of a mathematical encyclopedia. Clearly a modern and authoritative review of the state of the various branches of mathematics and some of its most closely allied fields is highly desirable both as a tool for the working scientist and as a reference for information in general. Such a comprehensive presentation cannot be replaced by the numerous accounts of special fields which have recently become so popular, perhaps in some measure just because the encyclopedia has been so deficient.

With the previous experiences in mind it is evident that the project of an encyclopedia, if it is to fill the needs for which it is intended, must be on a permanently self-renewing basis. The best form of such an undertaking is of course wide open to debate and discussion. However it would appear absolutely necessary to base the work on an encyclopedia perhaps not so very different from the previous one, but with the strong proviso that the several volumes on the various branches of mathematics should be published simultaneously as far as possible so that the various parts would be approximately in the same state of actuality. Presumably one would have to have an editorial body of specialists for each field. This group should be responsible both for the initial volumes and also for the decision when they were to be revised. After a period of ten to fifteen years most accounts would probably have to be rewritten. But equally important is the problem of keeping the reports up to date in the meanwhile as far as possible, and this is the main point in which the suggested scheme would differ from the present encyclopedia. Supplemental reports would have to be issued at fairly regular intervals, for instance every second or third year. These bulletins need not give any new reviews, it would be sufficient if they contained the references to those new papers on the subject which had appeared in the intervening period.

It is not necessary to point out the advantages of such an encyclopedic institution could it be initiated. It is quite evident to anyone engaged in mathematical research and familiar with the bibliographical investigations which are necessary in order to establish the actual state of some mathematical domain. Anyone engaged in advanced instruction will have observed the notable difficulties many of the beginning, younger men have in gathering their material. Perhaps it is not too optimistic to believe that some efforts could be saved, some disappointments avoided or even some publication costs reduced by minimizing the duplication which does not occur too rarely in mathematical literature.

In spite of the present all-time lowest ebb in international scientific collaboration it is quite evident that for the publication of such a permanent mathematical encyclopedia international cooperation would be almost imperative and in any case extremely desirable. I do not believe it would be too great a task for the American Mathematical Society, which has already taken the lead in so many mathematical enterprises, to sponsor such a publication. The Society has among its members more qualified and outstanding mathematicians than any other similar body in the world so that it would have no dearth of competent collaborators even if it were to carry the burden alone. The cooperation of the Mathematical Reviews would also greatly facilitate the compiling of periodic bulletins.

The economic basis for such a project is another question which would have to be carefully studied. The first encyclopedia was extremely expensive and it received its main support through the subscription of the libraries. Relatively few individuals could afford to own the whole encyclopedia. Even separate volumes were not commonly sold and the purchase of such single volumes does not seem to have been particularly encouraged. For a new publication the support of the libraries could of course be counted upon to continue. But it appears probable that one could also successfully urge the subscription to separate parts and their supplements by individuals. The number of research workers in mathematics at the present time is many times larger than it was in 1900, and there is every reason to believe that this number will continue at the same high level or even that it will increase. If the encyclopedia could be established as an ever ready research tool it is not unreasonable to expect that a considerable group of mathematicians could be induced to take partial subscriptions in their own special fields. Such an arrangement would materially contribute to the economic foundation of the encyclopedia.

Let me conclude by saying these are of course thoughts for the future and for times of total peace. But when the present urgent tasks have been attended to, it may not prove to be a utopian scheme.

After this long preamble let us turn to the new edition of the encyclopedia which occasioned it. The three numbers at hand afford a typical example of the differences one is confronted with in the various topics. The first part on the number system and general set theory deals with a field which in many ways has acquired a settled form. The second part on group theory covers a well-developed domain still going strong and with an overwhelming number of detailed contributions. Somewhat similar is the situation in the ideal theory considered in the third part. But here in the final article on "Verbände" or lattices one encounters a new field which is in such a state of rapid growth that even a review written a couple of years ago cannot be considered to give a satisfactory account of its present status.

The article on the foundations of the number system by F. Bachmann gives in clear and quite broad form the axiomatic foundations for the concept of a number. The axioms given by Peano as well as Dedekind's introduction of the integers by means of chains and set-theoretical considerations serve as the foundations for the theory of natural numbers. From this starting point one finds a survey of the various paths which have been followed in order to obtain the negative, the rational, the real and the complex numbers. The Dedekind cut method and Cantor's fundamental sequences and papers on these subjects are discussed at some length. The formal introductions of the complex numbers and some of the characterizations of the real and the complex number fields form the concluding parts of the article. It may be observed in regard to the complex numbers that the historical side of their development is curiously lacking, the only names of early mathematicians dealing with them, which have been mentioned, are those of Gauss, Hamilton and Cauchy. Perhaps it is the intention to fill this lacuna in connection with the article on the theory of equations.

The related chapter on representation of real numbers by limiting processes is written by K. Knopp. Here one finds a discussion of the classical method of exhaustion and its more modern counterpart, the Cauchy sequences. The representations of real numbers by infinite series, infinite products and continued fractions are rather summarily treated since these topics will be considered in special articles later. Among the representations of real numbers which are not so commonly used one may mention the Cantor series and the representations by principal fractions studied particularly by Engel and Lüroth. The goodness of approximations is touched upon and in the last few pages one finds a discussion of the criteria for irrationality. The transcendency of real numbers will be dealt with in a separate article.

The general theory of sets, limited to those parts of set theory into which the topological concepts do not enter, is very ably presented by E. Kamke. After the basic operations of set theory, including limit sets and the Suslin scheme, have been introduced, one obtains a general view of the theory of cardinal numbers with their various formal properties. Of particular interest is the author's discussion of the questionable points in set theory. The difficulty in obtaining a suitable definition for sets is illustrated by the antinomies, which the author proposes to eliminate by a reference to the theory of types. The axiom of choice is considered at length with special mention of the numerous investigations by Sierpinski. The author prefers the acceptance of a definition of sets which actually includes the axiom of choice. Certain authors have also questioned the tacit assumption of the existence of the power set of a given set, but there does not seem to be any logically objectionable consequences to this assumption. The various axiomatic foundations for set theory, particularly those given by Zermelo, Brouwer and Russel-Whitehead are considered. The final and longest part of the exposition is devoted to ordinal numbers, well-ordering, transfinite induction and related questions. The tendency of the last few years to eliminate the use of transfinite induction by means of general principles, for instance Zorn's lemma, came too late to be included.

In the article by W. Magnus on general group theory the author has achieved a *tour de force* by condensing into 50 pages a review of practically all that has been done in group theory since this important branch of mathematics was created. A modern survey of group theory of this bibliographical kind did not exist till now and the article will certainly prove to be a valuable aid for anyone who wishes to establish what has been done in connection with any particular problem. It is also the kind of exposition which one should like to see kept up to date in the future.

On the other hand the extremely condensed presentation makes this chapter much less readable than the preceding ones. There does not seem to be any advantage in going into details of this chapter in this review. It falls in four main parts: (1) the general concepts of group theory, (2) structure of groups with finite chain conditions, (3) finite groups, (4) construction of groups, infinite groups. The fourth part contains a brief account of the topological groups, while Lie groups will be presented in some other connection. The references seem to be complete up to the year 1937 with a few later papers mentioned.

The abstract ideal theory and some of its most important applications are discussed by W. Krull in two consecutive articles. The first review is divided into two parts, one giving the general decomposition theory and dimension theory for ideals, the other containing the multiplicative ideal theory which is closer to the ordinary ideal theory for algebraic numbers. In this latter part one finds a discussion of such important concepts as integrally closed rings and evaluation theory. The final divisions of the article deal with the extension of norms, degrees, ramification theory including differents, conductors and discriminants to general rings. This article covers a field in which the author himself has made some of the main contributions.

The second article by Krull on ideal theory covers its applications to polynomial ideals and elimination theory. The nucleus of the polynomial ideal theory is to be found in the theorems of Lasker and Hilbert as well as the results of van der Waerden. The elimination theory contains the theory of resultants and the various elimination methods for homogeneous and nonhomogeneous systems. Of importance is also the general theory of multiplicities due to van der Waerden and its applications to the theory of singularities of algebraic manifolds.

In a last, joint article Hermes and Köthe review the theory of "Verbände" or lattices. One finds first an exposition of the fundamental properties of lattices and then in several divisions some of the most important branches of the theory. One chapter deals with the theory of Dedekind lattices, the theorem of Jordan-Hölder and its extensions and the main decomposition theorems of algebra. Next follows the theory of distributive lattices and Boolean algebras including Stone's representation theory and some of the applications of Boolean algebras to topology and logic. A final chapter contains the theory of complemented lattices with special accounts of the works of Birkhoff, Menger and von Neumann. The article is clear and the presentation covers the material capably and satisfactorily as far as it goes, but, as we have already indicated, this is a typical example of a field in which a review cannot remain up to date for any considerable period. The account seems to have been written in the beginning of the year 1938 and to supplement it a list of papers published in 1938 and 1939 has been appended. But even so, a great number of contributions have already been made since this time, some of them of such importance that they could not be omitted even in a general survey.

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NOTES

The journal Revista, Matematicas y Fisica Teorica, published at the University of Tucumán in Argentina invites contributions from mathematicians in North America.

Two new volumes of the American Mathematical Society Colloquium Series have just appeared. *Algebraic Topology* by Professor Solomon Lefschetz of Princeton University is Volume 27 of the Series. *Analytic Topology* by Professor G. T. Whyburn of the University of Virginia is Volume 28 of the Series. The price for Volume 27 is $6.00 and for Volume 28 $4.75. Members of the American Mathematical Society receive a discount of twenty-five per cent. Orders for either of these books should be addressed to the Society at 531 West 116th Street, New York City.

A new series of pocket-size books dealing with varied aspects of modern scientific knowledge is to be published under the title *The Treasury of Science* by the L. B. Fischer Publishing Corporation. The editorial board includes Dr. Alvin Johnson of the New School for Social Research, Professor Harlow Shapley of Harvard University, and Dr. A. E. Cohn of the Rockefeller Institute.

Professor W. B. Carver of Cornell University has been elected Secretary-Treasurer of the Mathematical Association of America to replace Professor W. D. Cairns who is retiring January 1, 1943.

It is reported in the June 5, 1942, issue of Science that the Russian mathematicians Mr. S. N. Bernstein and Mr. S. A. Khristianovich have been awarded Stalin prizes.

Since the beginning of 1942 Professor G. D. Birkhoff of Harvard University has been serving as an exchange professor with Latin American Universities. He has given a series of lectures in the universities at Mexico City, Lima, Areguipa, and Santiago. At a special session he was made an honorary member of the faculty of the Universidad Nacional Mayor de San Marcos de Lima.

The honorary degree of doctor of laws was conferred by Park College on Dean P. P. Boyd of the University of Kentucky.

Dr. Vannevar Bush, president of the Carnegie Institution of Washington and director of the Office of Scientific Research and Development, has received an honorary doctorate of laws from the University of Pennsylvania and from Yale University.

Professor Jekuthiel Ginsburg of Yeshiva College has been awarded an honorary doctorate by Columbia University.

Professor Harlow Shapley of Harvard University has received an honorary doctorate from Bates College.

Professor Emeritus Virgil Snyder of Cornell University has received the Chicago Merit Award as an alumnus of Iowa State College.

Dean R. C. Tolman of California Institute of Technology has received an honorary doctorate of science from Princeton University.

Professor H. B. Williams of Columbia University has retired with the title professor emeritus.

Dr. F. T. Adler of the Institute for Advanced Study has been appointed to a visiting assistant professorship at the University of Wisconsin.

Assistant Professor C. B. Allendoerfer of Haverford College has been promoted to an associate professorship. He has also been appointed associate physicist in the Navy Department.

Assistant Professor Frances E. Baker of Mt. Holyoke College has been appointed to an associate professorship at Vassar College.

Assistant Professor D. H. Ballou of Georgia School of Technology has been appointed to an assistant professorship at Middlebury College.

Dr. C. B. Barker of the University of New Mexico has been promoted to an assistant professorship.

Mr. E. M. Beesley of the University of Nevada has been promoted to an assistant professorship.

Dr. A. T. Brauer of the Institute for Advanced Study and New York University has been appointed to an assistant professorship at the University of North Carolina.

Dr. R. H. Cook of Pennsylvania State College has been appointed to an assistant professorship at the South Dakota State School of Mines.

Associate Professor C. C. Craig of the University of Michigan has been promoted to a professorship.

Assistant Professor P. D. Crout of Massachusetts Institute of Technology has been promoted to an associate professorship.

Assistant Professor W. M. Davis of Illinois Institute of Technology has been appointed to an assistant professorship at Cornell College, Mount Vernon, Iowa.

Assistant Professor P. S. Dwyer of the University of Michigan has been promoted to an associate professorship.

Assistant Professor C. H. Fischer of Wayne University has been appointed to an assistant professorship at the University of Michigan.

Professor J. G. Hardy of Williams College has retired.

Assistant Professor Ernest Hawkins of the United States Naval Academy has been promoted to an associate professorship.

Dr. E. R. Hedrick has retired as vice president and provost of the University of California at Los Angeles.

Dr. M. H. Heins of the Institute for Advanced Study has been appointed to an assistant professorship at the Illinois Institute of Technology.

Assistant Professor W. L. Hutchings of Rollins College, Winter Park, Florida, has been promoted to an associate professorship.

Professor M. H. Ingraham of the University of Wisconsin has been appointed dean of the College of Letters and Science.

Dr. H. T. Karnes of Louisiana State University has been promoted to an assistant professorship.

Dr. Cecilia Krieger of the University of Toronto has been promoted to an assistant professorship.

Associate Professor Lincoln LaPaz of Ohio State University has been promoted to a professorship.

Assistant Professor H. D. Larsen of the University of New Mexico has been promoted to an associate professorship.

Assistant Professor Will Lipscombe of the University of Akron has been promoted to an associate professorship.

Dr. C. T. McCormick of Fort Hays Kansas State College has been made head of the department of mathematics.

Associate Professor N. H. McCoy of Smith College has been promoted to a professorship.

Mr. M. L. Manning of the Westinghouse Electric and Manufacturing Company has been appointed to an associate professorship of electrical engineering at Illinois Institute of Technology.

Dr. M. G. Moore of Tri-State College, Angola, Indiana, has been promoted to a professorship.

Associate Professor Rufus Oldenburger of the Illinois Institute of Technology has been promoted to a professorship.

Assistant Professor E. R. Ott of the University of Buffalo has been promoted to an associate professorship.

Professor W. V. Parker of the Louisiana State University has been made head of the department of mathematics.

Associate Professor Maximilian Philip of the College of the City of New York has been named head of the department of mathematics.

Dr. Edmund Pinney of California Institute of Technology has been appointed research associate at Massachusetts Institute of Technology.

Dr. Arthur Rosenthal has been appointed to an assistant professorship at the University of New Mexico.

Associate Professor H. C. Shaub of Washington and Jefferson College has been promoted to a professorship and made head of the department of mathematics.

Associate Professor J. A. Shohat of the University of Pennsylvania has been promoted to a professorship.

Associate Professor Marian M. Torrey of Goucher College has been promoted to a professorship.

Dr. J. L. Vanderslice of Lehigh University has been appointed to an assistant professorship at the University of Maryland.

Dr. André Weil has been appointed to an assistant professorship at Lehigh University.

Professor Jacob Westlund of Purdue University has retired.

Mr. E. F. White of the University of Cincinnati has been promoted to an assistant professorship.

Dr. J. W. T. Youngs of Purdue University has been promoted to an assistant professorship.

Professor H. B. Curry of Pennsylvania State College is on leave of absence serving in the Fire Control Design Department at Frankford Arsenal, Philadelphia, Pennsylvania.

Assistant Professor C. H. Graves of Pennsylvania State College is on leave of absence serving as associate educational statistician in the Federal Security Agency, Office of Education, Washington, D. C.

Assistant Professor A. E. Heins of Purdue University will be on leave of absence for the academic year to work in the Radiation Laboratory at Massachusetts Institute of Technology.

Assistant Professor C. C. Hurd of Michigan State College will be on leave during this academic year to serve as instructor in mathematics at the United States Coast Guard Academy.

Assistant Professor C. V. L. Smith of Lafayette College is on leave of absence in the United States Naval Reserve.

The following appointments to instructorships are announced: Barnard College, Columbia University: Dr. Mary E. Ladue; University of Delaware: Dr. G. L. Walker; Drew University, Madison, New Jersey: Mr. E. A. Knobelauch; Fenn College: Mr. E. L. Godfrey; Harvard University: Dr. Edwin Hewitt; Hollins College, Hollins College, Virginia: Dr. Josephine M. Mitchell; Illinois Institute of Technology: Dr. G. W. Mackey; Louisiana Aircraft School, New Orleans, Louisiana: Mr. J. O. Evans; University of Louisville: Mr. S. T. Parker; University of Michigan: Dr. Paul Civin; University of Missouri, Rolla, Missouri: Mr. D. H. Erkiletian; University of Nebraska: Dr. Albert Neuhaus; New York University: Mr. C. F. Rehberg; North Carolina State College: Mr. H. S. Kieval; Northwestern University: Mr. R. S. Wolfe; Purdue University: Dr. B. H. Arnold, Mr. N. J. Fine, Dr. Ivan Niven, Dr. Maxwell Reade; Ground School, U. S. Naval Air Station, Corpus Christi, Texas: Mr. W. G. Leavitt; Washington and Jefferson College: Mr. W. G. Brady, Mr. D. T. Finkbeiner; Washington University: Dr. S. P. Avann; Williams College: Mr. R. R. Stoll; University of Wisconsin: Dr. C. J. Everett; Yale University: Dr. D. S. Miller, Dr. A. L. Putnam.

Sir James Larmor, formerly Lucasian professor of mathematics in the University of Cambridge, died on May 18, 1942, at the age of eighty-four years.

Professor G. A. Witherington, formerly professor of mathematics in the Royal Naval College, Greenwich, died on May 1, 1942, at the age of sixty-nine years.

Associate Professor E. W. Miller of the University of Michigan died on July 23, 1942 at the age of thirty-seven years. He had been a member of the Society since 1930.

Professor N. C. Riggs of Carnegie Institute of Technology died July 20, 1942. He had been a member of the Society since 1903.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

Algebra and Theory of Numbers

252. D. H. Lehmer: *Recurrence formulas for certain divisor function.*

Certain numerical functions giving the excess of the number of divisors of n of one sort over the number of divisors of n of a second sort have arisen from the theory of binary quadratic forms and from the theory of elliptic functions. A systematic discussion of these has been given in several papers by Glaisher. In this paper it is pointed out that these functions are sums over divisors of certain periodic Lucas' functions. A general recurrence formula embracing all Lucas' functions is derived in an elementary way from Jacobi's celebrated triple product identity. Special applications to the Pell equation and Fibonacci's series are given. (Received July 6, 1942.)

253. Dorothy M. Smiley: *A note on Burnside's problem.*

William Burnside (Proceedings of the Cambridge Philosophical Society, vol. 20 (1921), pp. 482–484), Dorothy Manning (Transactions of this Society, vol. 40 (1936), pp. 324–342), and Rudolf Kochendörffer (Schriften des Mathematischen Seminars und des Instituts fur angewandte Mathematik der Universität Berlin, vol. 3 (1937), pp. 155–180) have endeavored to prove that a simply transitive permutation group of degree p^{a+b} (p prime) which contains a transitive Abelian subgroup of type (p^a, p^b) with $a \neq b$ is neither primitive nor simple. All three papers contain mistakes. Those occurring in Kochendorffer's paper are corrected in this note. (Received July 20, 1942.)

254. H. S. Thurston: *The solution of p-adic equations.*

If the congruence $f(x) \equiv 0 \pmod{p}$ has a solution $x = a_0$ such that $f'(a_0) \not\equiv 0 \pmod{p}$, then the equation $f(x) = 0$ has a p-adic root $\alpha = a_0 + a_1 p + a_2 p^2 + \cdots$, the coefficients $a_0, a_1, a_2, \cdots$, being successively determined by a method analogous to Newton's method for approximating irrational roots. If, however, $f'(a_0) \equiv 0 \pmod{p}$, the method not only fails to determine α, but leaves in doubt the existence of such a root. The present paper introduces a sequence of functions $F_i(x)$, by means of which it is possible, in a finite number of steps, to ascertain the number of roots of $f(x) = 0$ in the p-adic field Ω_p. By a technique analogous to Horner's method for approximating irrational roots, $f(x) = 0$ may be solved for any root α whose existence has been demonstrated. (Received July 27, 1942.)

ANALYSIS

255. C. R. Adams and A. P. Morse: *On approximating certain integrals by sums.*

For $f \in L(E)$, B a measurable subset of E, $0<|B|=\text{measure }(B)<\infty$, let $\mathfrak{M}_B f=\int_B f/|B|$. As B varies, let $\mathfrak{R}(f)$ represent the set of values of $\mathfrak{M}_B f$; and let ϕ be a function whose domain includes $\mathfrak{R}(f)$. For $0<\delta\leqq\infty$ let F be an arbitrary set-partition of E into disjoint measurable subsets each with diameter less than δ; and let the aggregate of all such partitions be denoted by $\Gamma_\delta(E)$. What conditions on f and ϕ will insure the (finite) existence of $\int_E \phi[f(x)]dx$ and of $\lim_{\delta\to 0}\inf_{F\in\Gamma_\delta(E)}\sum_{B\in F}\phi[\mathfrak{M}_B f]|B|$, $\lim_{\delta\to 0}\sup_{F\in\Gamma_\delta(E)}\sum_{B\in F}\phi[M_B f]|B|$ and their equality? For ϕ continuous, a necessary and sufficient condition is found. The hypothesis of continuity on ϕ cannot be dispensed with. "Sampling" can be allowed in the sum (see Adams and Morse, *Random sampling in the evaluation of a Lebesgue integral*, this Bulletin, vol. 45 (1939), pp. 442–447). A sufficient condition, often useful for testing, is found in terms of the existence of a convex dominant for $|\phi|$; such a convex dominant need not exist, but a condition is determined under which it does. Applications are made to functions f which are of bounded variation or are absolutely continuous in a certain generalized sense involving ϕ. Some new results in the general theory of functions of sets are included. (Received July 14, 1942.)

256. G. E. Albert: *Criteria for the closure of systems of orthogonal functions.*

Let the system F of functions $f_n(x)$, $n=0, 1, 2, \cdots$, be orthonormal on the interval (a, b). For any fixed point t in (a, b) let $g_t(x)$ denote the function which is equal to unity on (a, t) and zero on (t, b). Let $s_n(x)$ denote the partial sum of the generalized Fourier series with respect to F for the function $g_t(x)$. Define the function $\sigma_n(t)$ which, for each t in (a, b), is equal to $s_n(t)$. A necessary and sufficient condition that the system F be closed in the class of functions having integrable (Riemann or Lebesgue) squares on (a, b) is: $\lim_n \int_a^b |1-2\sigma_n(t)|dt=0$. A sufficient condition is that $\lim_n \int_a^b \{1-2\sigma_n(t)\}^2dt=0$. The verification of the latter criterion for the trigonometric system F is a matter of elementary calculus. Both criteria are extended to systems F orthogonal with respect to a positive weight function; in such cases the interval (a, b) may be infinite. The criteria stated follow easily from a theorem due to Vitali (Rendiconti dei Lincei, (5), vol. 30 (1921)). (Received June 6, 1942.)

257. R. H. Cameron and W. T. Martin: *Infinite linear difference equations with arbitrary real spans and first degree coefficients.*

The authors investigate the equation $\int_{-\infty}^{\infty}(s-\lambda)f(s-\lambda)dp(\lambda)+\int_{-\infty}^{\infty}f(s-\lambda)dq(\lambda)=g(s)$ in a strip $a<lm s<b$. Under fairly weak conditions on p, q, and g it is shown that the equation has a unique analytic solution of a fairly general character. (Received June 24, 1942.)

258. J. A. Clarkson and Paul Erdös: *On the approximation of continuous functions by polynomials.*

Let x^{n_i} be a set of powers of x, $n_i\to\infty$. Then a well known theorem of Müntz and Szász states that the necessary and sufficient condition that the powers x^{n_i} and 1 shall span the whole space of continuous functions, in the interval (0, 1) is that

$\sum_{i=1}^{\infty} 1/n_i = \infty$. It is proved that if $\sum 1/n_i < \infty$ and the continuous function $f(x)$ can be uniformly approximated in (0, 1) by polynomials in the x^{n_i}, then $f(x)$ can be continued to be analytic in the unit circle. The following result is also shown: Given $0<a<b$; then the necessary and sufficient condition that the powers x^{n_i} shall span the space of all continuous functions in (a, b) is that $\sum 1/n_i < \infty$. (Received July 3, 1942.)

259. Mark Kac: *On the distribution of values of trigonometric sums with linearly independent frequencies.*

Let $f(t)=\sum_{k=1}^{n} a_k \cos 2\pi\lambda_k t$, where the a's are real and the λ's real and linearly independent. Denote by $N_T(a)$ the number of t's in $(-T, T)$ for which $f(t)=a$. It is proved that $E(a)=\lim N_T(a)/2T$ as $T\rightarrow\infty$ exists and that $E(a) = 2\pi^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos a\xi K(\xi,\eta)d\eta d\xi/\eta^2$, where $K(\xi,\eta) = \prod_{k=1}^{n} J_0(|a_k|\xi) - \prod_{k=1}^{n} J_0(|a_k|(\xi^2+4\pi^2\lambda_k^2\eta^2)^{1/2})$. This, in a way, completes an earlier investigation of E. R. van Kampen, A. Wintner, and the present author. (See American Journal of Mathematics, vol. 61 (1939), pp. 985–991, in particular section 3.) (Received July 23, 1942.)

260. Ella Marth: *On Garvin's F-series.* Preliminary report.

A generalized Lambert series of the form $F(z)=\sum a_n z^{\lambda^n}/(1-z^{\lambda^n})$ was defined and studied by Garvin (M. C. Garvin, *A generalised Lambert series*, American Journal of Mathematics, vol. 58 (1936), pp. 507–513). With certain restrictions on the coefficients the series was found to have the unit circle as a natural boundary. In obtaining this natural boundary Garvin used a radial approach to points on the boundary. Furthermore she found that for a rational point z_0 on the boundary (1) $\lim_{z\rightarrow z_0}\{(1-z/z_0)F(z)\}$ equals $(1/\mu)\sum a_{l^\nu}/l^\nu$ with certain reservations. In this paper the approach to the boundary is extended to complex approach for which (1) holds under specific conditions. For the case when z_0 is an irrational point (1) becomes zero for both radial and complex approach. (Received July 21, 1942.)

261. Josephine M. Mitchell: *On double Sturm-Liouville series.*

From the Sturm-Liouville system $\{\phi_m(x)\}$ $(m=0, 1, 2, \cdots; 0\leqq x\leqq\pi)$, the double Sturm-Liouville series of a function $f(x, y)$, integrable (L) over $Q(0\leqq x\leqq\pi, 0\leqq y\leqq\pi)$, namely: $\sum_{m,n=0}^{\infty} a_{mn}\phi_m(x)\phi_n(y)$, $a_{mn}=\int_0^\pi\int_0^\pi f(s, t)\phi_m(s)\phi_n(t)ds\,dt$, is formed and the equiconvergence and equisummability of this series with the Fourier cosine-cosine series of $f(x, y)$ is considered. Haar's theorem (1910) for the one variable case that the Sturm-Liouville series of an integrable function is uniformly equiconvergent on $(0, \pi)$ with its Fourier cosine series does not generalize; only a weaker theorem on equiconvergence is proved. The main result is, however, that the double Sturm-Liouville series of $f(x, y)$ is $(C, 1, 1)$ *equisummable with its Fourier cosine-cosine series at all points* (x, y) *in* Q *for which* $(1/hk)\int_0^h\int_0^k|f(x+s, y+t)|ds\,dt$ $(0<|h|\leqq\pi, 0<|k|\leqq\pi)$ *is bounded*. A summability theorem, similar to one for double Fourier series (B. Jessen, J. Marcinkiewicz and A. Zygmund, 1935) stating that *if* $f\log^+|f|$ *is integrable, the double Sturm-Liouville series of* $f(x, y)$ *is* $(C, 1, 1)$ *summable to* $f(x, y)$ *almost everywhere* follows readily. Finally applying to the Sturm-Liouville system a modification of the Poisson method of summation, introduced by S. Bergmann (1941), it is proved that the double Sturm-Liouville series of any integrable function is equisummable by this method with its Fourier cosine-cosine series. (Received July 23, 1942.)

262. K. L. Nielsen and B. P. Ramsay: *On particular solutions of linear partial differential equations.*

A method for the solution of boundary value and characteristic value problems consisting of approximations by expressions $W_n=\sum_{\nu=1}^{n}a_\nu^{(n)}\phi_\nu(x, y)$, where $\phi_\nu(x, y)$ are particular solutions of the considered differential equation, has been given by Bergman (Duke Mathematical Journal, vol. 6 (1940), pp. 537–561). In applying this method it is important for practical purposes to obtain a simple procedure for the construction of the particular solutions and in this connection Bergman (Matematicheskii Sbornik, vol. 44 (1937), pp. 1169–1197) has proved that to every equation $L(U)=U_{z\bar{z}}+aU_z+bU_{\bar{z}}+cU=0$, where a, b, c are functions of $z=x+iy$ and $\bar{z}=x-iy$ and the subscripts denote the partial derivatives, there exist functions $E(z, \bar{z}, t)$ such that $P(f)=\int_{-1}^{+1}E(z, \bar{z}, t)f(z(1-t^2)/2)\left[1-t^2\right]^{-1/2}dt$, where f is an arbitrary analytic function of one complex variable, will be a particular solution of $L(U)=0$. To expedite the numerical computation for practical problems it is further desired that E has a simple structure so that the integral $P(f)$ may be easily evaluated for arbitrary values of z and $\bar{z}$. In this note the authors determine certain types of equations, $L(U)=0$, for which E has the simple form $E(z, \bar{z}, t)=\exp\,(Nt^n+Mt^m)$, where $N=N(z, \bar{z})$ and $M=M(z, \bar{z})$. The authors further show that M and N may be determined from the coefficients a, b, c. (Received June 22, 1942.)

263. Raphael Salem: *On a theorem of Zygmund.*

The following theorem is due to Zygmund: If a continuous function $f(x)$ of bounded variation and of period 2π has a modulus of continuity $\omega(\delta)$ such that the series $\sum n^{-1}\left[\omega(n^{-1})\right]^{1/2}$ converges, then the Fourier series of $f(x)$ is absolutely convergent. The purpose of the present paper is to prove that the exponent 1/2 of this theorem is the best possible one: the theorem becomes false if the exponent 1/2 is replaced by $\epsilon+1/2$, ϵ being any positive number. (Received August 1, 1942.)

264. Otto Szász: *On the partial sums of Fourier series at points of discontinuity.*

In the first part of this paper the summability method $T_n=\sum_1^n a_{n\nu}r_\nu$ with $a_{n\nu}=\rho_n^\nu\nu^{-1}\sin\nu\theta_n$, $\nu=1, 2, \cdots, n$, $\rho_n\to1$, $\theta_n\to0$, is considered. Necessary and sufficient conditions are given for permanency of this transform relative to Cesàro summability of the sequence $\{r_n\}$, of some integral order. In the second part application to Fourier series yields a Gibbs' phenomenon for rather general classes of functions. To quote one such case: If $f(\theta)\sim\sum b_n\sin\theta$, $\sum_1^n\nu b_\nu>-pn$, for some $p>0$, and for all $n>0$, and $(2/\pi)\int_0^t\left|f(t)-j/2\right|dt\to0$, for some j, then for any positive integer λ: $\sum_1^n b_\nu\sin\nu\theta_n\to\int_0^{\lambda\pi}(\sin t)/t\,dt$, whenever $n\theta_n-\lambda\pi=O(n^{-1})$ as $n\to\infty$. (Received July 15, 1942.)

265. C. J. Thorne: *An Appell subset.*

Some of the properties and applications of the polynomials defined as follows are given: $\int_a^b\phi_n^{[r]}(x)dx=\delta_n^r$; where $\delta_n^r=0$ for $n\neq r$ and $=1$ for $n=r$, n-degree of polynomial, $[r]$—rth derivative. In particular this polynomial set is shown to be an Appell subset and to satisfy difference equations similar to those satisfied by the Bernoulli polynomials which lead to the Euler-MacLaurin expansion formula. Use of the Appell subset in the determination of functions defined in various ways is illustrated. (Received July 30, 1942.)

266. Hermann Weyl: *On Hodge's theory of harmonic integrals.*

Hodge's fundamental existence theorem for harmonic integrals on Riemannian manifolds of any dimensionality is proved by the parametrix method. (The proof incorporated in Hodge's recent book on *Harmonic Integrals*, Cambridge, 1941, is wrong.) (Received July 1, 1942.)

267. Hassler Whitney: *Differentiability of the remainder term in Taylor's formula.*

If $f(x)$ is of class C^m, and $1 \leq n \leq m$, then $f(x) = \sum_{i=0}^{n-1} f^{(i)}(0)x^i/i! + x^n f_n(x)/n!$. It is shown that $f_n(x)$ is of class C^{m-n}, but not necessarily of higher class, and $\lim_{x\to 0} x^k f^{(m-n+k)}(x) = 0$ $(k = 1, \cdots, n)$. A converse is true. A similar theorem holds in more dimensions. (Received July 28, 1942.)

268. Hassler Whitney: *Note on differentiable even functions.*

It is shown that an even function $f(x)$ of class C^{2n} (or class C^∞, or analytic) may be written as $g(x^2)$, with g of class C^n (or class C^∞, or analytic). (Received July 28, 1942.)

269. Hassler Whitney: *The general type of singularity of a set of $2n-1$ smooth functions of n variables.*

Let f be a mapping of class C^1 of an n-manifold M^n into an M^{2n-1}. Then arbitrarily near f is a mapping f', regular except at isolated singular points; at each of these, a certain condition (C) holds. (C) involves first and second derivatives, but is independent of the coordinate system employed. If (C) holds at p, and the mapping is of class C^{4r+4} (or class C^∞, or analytic), then coordinate systems about p and $f(p)$, of class C^r (or class C^∞, or analytic), exist such that the mapping is exactly $y_1 = x_1^2$, $y_i = x_i$, $y_{n+i-1} = x_1 x_i$ $(i = 2, \cdots, n)$. (Received July 28, 1942.)

Applied Mathematics

270. Stefan Bergman: *Operators in the theory of differential equations and their application.* I.

By introducing $u = x \cos\theta + y \sin\theta$, $v = -x \sin\theta + y \cos\theta$ and $\xi = (\sigma/2k) + \theta$, $\eta = (\sigma/2k) - \theta$, where $\sigma_x = \sigma + k \sin 2\theta$, $\sigma_y = \sigma - k \sin 2\theta$, $\tau_{xy} = -k \cos 2\theta$ the equations of the theory of plasticity can be written in the form $(\partial^2 u/\partial\xi\partial\eta) - u/4 = 0$, $(\partial^2 v/\partial\xi\partial\eta) - v/4 = 0$ (see Geiringer and Prager, Ergebnisse der exakten Naturwissenschaften, vol. 13, p. 350). Here σ_x, σ_y, τ_{xy} are stresses, x, y, cartesian coordinates. Particular solutions of these equations can be written in the form $u(\xi, \eta) = \int_{-1}^{1} \exp\,(t(\xi\eta)^{1/2}) \{f[\xi(1-t^2)/2] + g[\eta(1-t^2)/2]\} (1-t^2)^{1/2} dt$ where f and g are arbitrary twice continuously differentiable functions of one variable. (See Duke Mathematical Journal, vol. 6 (1940), pp. 538 and 557.) This class of functions possesses a base $\{u_\nu(\xi, \eta)\}$ such that each u_ν satisfies two (simple) ordinary linear differential equations of second order with rational coefficients. Entire solutions u_ν of the above partial differential equation are such that every u defined in a convex domain can be approximated by sums of the form $\sum_{\nu=1}^{n} a_\nu^{(n)} u_\nu$. The author indicates an approximation procedure of a function u given by its boundary values. These functions u possess singularities which can be characterized in a way analogous to that in Comptes Rendus de l'Académie des Sciences, vol. 205 (1937), pp. 1360–1362. (Received June 3, 1942.)

271. Stefan Bergman: *Operators in the theory of partial differential equations and their application.* II.

Let $v(x, y)e^{i\theta(x,y)}$ denote the velocity vector of an irrotational steady flow of compressible fluid. Let $\zeta=\Lambda(v)+i\theta$, $\bar{\zeta}=\Lambda(v)-i\theta$, where $d\Lambda(v)/dv=[1-M^2]^{1/2}/v$, and $M=v/[d_0^2-(1/2)(k-1)v^2]^{1/2}$, d_0 and k being constants. Finally: let $E^*=1+t\zeta^{1/2}Q(\zeta, \bar{\zeta}, t\zeta^{1/2})$ where $Q(\zeta, \bar{\zeta}, p)$ is an (arbitrary) solution of $Q_{p\bar{\zeta}}+2p(Q_{\zeta\bar{\zeta}}+FQ)+2F=0$, and Q is supposed to be an odd function of p. Then $\psi(v, \theta)=Re\{\int_{-1}^{1}T(\zeta+\bar{\zeta})E^*(\zeta, \bar{\zeta}, t)f[(1/2)\zeta(1-t^2)]dt/(1-t^2)^{1/2}\}$ where f is an arbitrary analytic function of one complex variable is the stream function of a suitable subsonic flow, and the stream function of every flow can be represented in the above form. T and F are suitable functions of $(\zeta+\bar{\zeta})$. Using this result the author proves that various sets of particular solutions $\{p_\nu(v, \theta)\}$ are complete. The author indicates a method of determining the constants $a_\nu^{(n)}$ in $\psi_n=\sum_{\nu=1}^{n}a_\nu^{(n)}p_\nu$ in such a way that ψ_n approximates the flow in a channel which is given in the xy-plane (physical plane), provided that the image of the flow in the hodograph plane is schlicht. The method is a generalization of one given in Duke Mathematical Journal, vol. 6 (1940), p. 537. A similar procedure can be applied in the case of a supersonic flow. (Received July 28, 1942.)

272. Vladimir Morkovin: *On the deflection of anisotropic thin plates.*

Deflections w of an anisotropic plate (with one plane of elastic symmetry) bounded by an analytic curve C_0 are considered. The general solution of the differential equation for w is known to be expressible in terms of two analytic functions $f_1(z_1)$ and $f_2(z_2)$, where the complex variables z_1 and z_2 are related to the variable z_0 of the original plane by $z_k=p_kz_0+q_k\bar{z}_0$, the constants p_k and q_k depending on the material of the plate. (See S. N. Lechnitzky, Journal of Applied Mathematics and Mechanics, (n. s.), vol. 2 (1939), pp. 181–210.) Transformations $z_k=\omega_k(\zeta_k)$ are found which make any point on C_0 correspond to points $e^{i\theta}$ on the circumferences γ_k of unit radii in new ζ_1 and ζ_2 planes having the same polar angle θ, and which are conformal in some neighborhoods of γ_k. Then the functions $\phi_k(\zeta_k)=f_k(z_k)$ can be determined from the two given boundary conditions if these are expressed in terms of $e^{i\theta}$. A detailed solution illustrating this general procedure is carried out in the case of a clamped elliptic plate with polynomial loading. (Received July 31, 1942.)

GEOMETRY

273. H. S. M. Coxeter: *A geometrical background for the description of de Sitter's world.*

This paper begins with an elementary treatment of the process by which an elliptic or hyperbolic metric in the plane at infinity of affine space induces a Euclidean or Minkowskian metric in the whole space. The various kinds of sphere are defined, and are seen to provide models for non-Euclidean planes, including the "exterior-hyperbolic" plane which is a two-dimensional de Sitter's world. (See Eddington, *The Mathematical Theory of Relativity*, 1924, p. 165.) Then comes a simple proof of Study's theorem to the effect that one side of a triangle is greater than the sum of the other two, and finally a discussion of some cosmological paradoxes. (Received July 31, 1942.)

274. J. J. DeCicco: *New proofs of the theorems of Beltrami and Kasner on linear families.*

Here new proofs of the theorems of Beltrami and Kasner on linear families of curves are submitted. Beltrami's result is that a surface S may be mapped upon a plane π so that its geodesics correspond to straight lines if and only if S is of constant Gaussian curvature. Kasner's result states that a complete system of isogonal trajectories of a simple family of curves F is linear if and only if F is isothermal. From this work is also deduced another theorem of Kasner which is that a surface S can possess exactly ∞^3 isothermal families of geodesics if and only if S is of constant Gaussian curvature. A simple uniform method is used in all the proofs. (Received July 30, 1942.)

275. J. J. DeCicco: *The application of turbine geometry to the inverse problem of dynamics.*

Kasner has discussed analytically the inverse problem of dynamics. (See *Differential-Geometric Aspects of Dynamics*, American Mathematical Society Colloquium Publications, vol. 3, 1913.) In this paper a synthetic solution of this is given by use of turbine geometry, and the whirl-motion group G_6. In the course of this development, a new type of element-series is introduced, which will be called a limaçon series of the second kind. This is different from the limaçon series of the first kind, which was discussed in *The geometry of fields of lineal elements*, Transactions of this Society, vol. 47 (1940), pp. 207–229. The indeterminate case is completely explained. (Received July 30, 1942.)

276. B. H. Gere and David Zupnik: *On the construction of curves of constant width.*

General properties of a curve of constant width, which is a curve such that the distance between parallel tangents is a constant, were given by Barbier (Journal de Mathématiques Pure et Appliquées, (2), vol. 5 (1860), pp. 273–286). The purpose of this paper is to give a general method for constructing such curves. An evolute star, which is a configuration of n curvilinear sides, n odd, is first defined. It is shown how, with any such star as evolute, one and only one family of parallel curves of constant width may be constructed. A necessary and sufficient condition in terms of the lengths of the sides of the star is found for the construction of a curve of specified constant width. Finally, the same method of construction is shown applicable to degenerate stars, giving curves which contain circular arcs. (Received July 20, 1942.)

277. Edward Kasner: *Differential equations of the type*: $y''' = Gy'' + Hy''^2$.

Certain problems in geometry and physics give rise to a general third order differential of the type (G): $y''' = Gy'' + Hy''^2$, where G and H are functions of (x, y, y') only. The family of ∞^3 integral curves of any such equation may be characterized as follows. For any curve of the family containing an arbitrary lineal element E, construct the osculating parabola at E. The foci of the ∞^1 parabolas so obtained will describe a circle through the point of E. An equivalent property is that the directrices will form a pencil of lines. The following eight problems are discussed: (1) the ∞^3 trajectories of a given field of force; (2) the complete system of velocity curves; (3) the systems S_k; (4) the general catenaries; (5) the ∞^3 brachistochrones in a conservative field; (6) sectional families; (7) curvature trajectories, and (8) additive-multiplicative trajectories. The type (G) was first encountered in the study of dynamical trajectories (*Differential-Geometric Aspects of Dynamics*, American Mathematical Society Colloquium Publications, vol. 3, 1913; *The trajectories of dynamics*, Transactions of this Society, vol. 7 (1906), pp. 401–424). All eight problems are distinct but lead to species of this type. Recently Terracini has given a new projective characterization of (G). (Received July 7, 1942.)

278. Edward Kasner: *Differential equations of the type:* $y^{iv}=Ay'''^2+By'''+C$.

In this paper are discussed the various problems connected with general fourth order equations of the type (Q): $y^{iv}=Ay'''^2+By'''+C$, where A, B, C are functions of (x, y, y', y'') only. The systems of ∞^4 integral curves of any such equation may be characterized as follows. For any curve of the system containing an arbitrary curvature element E_2, construct the osculating conic at E_2. The ∞^1 conics so obtained have their centers on a conic which contains the lineal element E_1 of E_2. The following seven problems are discussed: (1) the sets of ∞^4 curves characterized by linear families of osculating conics; (2) the systems of extremals in a calculus of variations problem; (3) the systems of curves in a field of force for which the pressure is proportional to the normal component of the force; (4) the system of curves in a field of force for which the pressure is constant; (5) the system of trajectories of a particle of special variable mass in a positional field of force; (6) the orthogonal trajectories of the space-time curves of a field of force which depends not only on the position but also upon the time, and (7) the complete set of the rate of variation of curvature trajectories. These seven species are distinct but all lead to (Q), which the author first encountered in a study of extremals (this Bulletin, vol. 14 (1908), pp. 461–465). (Received July 7, 1942.)

279. Edward Kasner: *The inverse problem of dynamics.*

In any given positional field of force of the plane, there are ∞^3 trajectories. Conversely, if the totality of trajectories is given, the field of force is determined except for a constant factor. See *Differential-Geometric Aspects of Dynamics*, American Mathematical Society Colloquium Publications, vol. 3, 1913, for a discussion of "the geometric exploration of a field." The essential problem discussed in this paper is to find the direction of the force at a given point 0 when only four trajectories through 0 are given. The complete analytic solution is treated. Usually the solution is unique but there arises an important indeterminate case. (Received July 30, 1942.)

280. Edward Kasner and J. J. DeCicco: *Transformation theory of isogonal trajectories of isothermal families.*

In the paper, *Lineal element transformations which preserve the isothermal character* (Proceedings of the National Academy of Sciences, vol. 27 (1941), pp. 406–409), Kasner determined the complete group of lineal element transformations of the plane which send any isothermal family of curves into an isothermal family. In this new work, it is the purpose of the authors to extend this result to transformations of second differential elements. The total group of curvature element transformations carrying any complete system of isogonal trajectories of isothermal families into any other such system is found in explicit form. The only contact transformations of Kasner's group and this new group are the conformal maps. The doubly-infinite systems of curves mentioned above are identical with the conformal images of the ∞^2 straight lines, and thus may properly be described as "conformal rectilinear wexes." The content of the new group is $2\infty^{4f(1)}$, as contrasted with the content $2\infty^{1+4f(1)}$ of the old group. (Received June 19, 1942.)

281. Domina E. Spencer: *Geometric figures in affine space.*

A study of the figures in affine space has been undertaken in order to establish the place of Study's "Geometrie der Dynamen" in the structure of geometry. Geometric figures have been considered whose tensor representations are: 1. contravariant alternating tensors of valence two; 2. covariant alternating tensors of valence two; 3. mixed tensors of valence two. Among this set are found the affine ancestors of all of the Study figures. Thus the foundation is laid for the tensor interpretation of all of the figures of the "Geometrie der Dynamen." (Received July 30, 1942.)

STATISTICS AND PROBABILITY

282. L. A. Aroian: *The relationship of Fisher's z distribution to Student's t distribution.*

For n_1 and n_2 sufficiently large, $W=(1/\beta)(N/(N+1))^{1/2}z$ is distributed as Student's t with N degrees of freedom, $N=n_1+n_2-1$, $\beta^2=(1/2)(1/n_1+1/n_2)$. If the level of significance is α for Student's distribution, the level of significance for z will be $(N/(N+1))^{1/2}\alpha e^{\beta^2/2-t/12N}<\alpha$. As a corollary it follows that the distribution of z approaches normality, n_1, $n_2\rightarrow\infty$, with mean zero and variance β^2. This simplifies a previous proof of the author. Application of this result is made to finding levels of significance of the z distribution. On the whole R. A. Fisher's formulas for finding such levels, n_1 and n_2 large as modified by W. G. Cochran, are superior. The formulas of Fisher-Cochran are compared with the recent formula of E. Paulson. (Received August 1, 1942.)

283. E. J. Gumbel: *Graphical controls based on serial numbers.*

The index m of the observed value x_m $(m=1, 2, \cdots, n)$ is called its serial number (or rank). A value x of the continuous statistical variable defined by a probability $W(x)=\lambda$ is called a grade (for example, the median). The coordination of serial numbers with grades furnishes two graphical methods for comparing the observations and the theory, namely the equiprobability test based on $m=n\lambda$, and the return periods based on $m=n\lambda+1/2$. From the distribution of the mth value, determine the most probable serial number $\tilde{m}=n\lambda+\Delta$, where Δ depends upon the distribution. For a symmetrical distribution, the corrections for two grades defined by λ and $1-\lambda$, are $\Delta(1-\lambda)=-\Delta(\lambda)$. For an asymmetrical distribution, calculate the most probable serial number of the mode considered as an mth value. Thus the mode is obtained from the observations, but it is not the most precise mth value. If m is of the order $n/2$ the distribution of the mth value converges towards a normal distribution with a standard deviation $s(x)=(W(x)(1-W(x)))^{1/2}/(w(x)n^{1/2})$. The intervals $x\mp s(x)$ give controls for the equiprobability test, the step function and the return periods. Besides, the standard error of the mth value leads to the precision of a constant obtained from a grade. (Received July 20, 1942.)

284. Mark Kac: *On the average number of roots of a random algebraic equation.*

Let $X_0+X_1x+\cdots+X_{n-1}x^{n-1}=0$ be an algebraic equation whose coefficients $X_0, \cdots, X_{n-1}$ are independent random variables having the same normal distribution with density $\pi^{-1/2}\exp(-x^2)$. If $N_n=N_n(X_0, \cdots, X_{n-1})$ denotes the number of real roots of the equation then the average number of roots (mathematical expectation of N_n) is asymptotically equal to $2\pi^{-1}\log n$. Moreover, for $n\geqq 2$ the mathematical expectation of N_n is not more than $2\pi^{-1}\log n+14\pi^{-1}$. This is an improvement of a result of Littlewood and Offord (Journal of the London Mathematical Society, vol. 13 (1938), pp. 288–295) who proved that for $n \geqq 2000$ the mathematical expectation of the N_n is not more than $25 (\log n)^2 + 12 \log n$. (Received July 13, 1942.)

285. H. B. Mann and Abraham Wald: *On the choice of the number of class intervals in the application of the chi-square test.*

The distance of two distribution functions is defined as the l.u.b. of the absolute value of the difference between the two cumulative distribution functions. Let $C(\Delta)$ be the class of alternatives with distance Δ from the null-hypothesis Let $f(N, k, \Delta)$ be the g.l.b. of the power with respect to alternatives in $C(\Delta)$ of the chi-square test with sample size N and k equally probable class intervals. A positive integer k is called best with respect to sample size N if there exists a Δ such that $f(N, k, \Delta) = 1/2$ and $f(N, k', \Delta) \leqq 1/2$ for every positive integer k'. The authors show that $k_N = 4(2(N-1)^2/C^2)^{1/5}$, where $1/(2\pi)^{1/2}\int_C^\infty e^{-x^2/2}dx$ is equal to the size of the critical region, fulfills approximately the conditions of a best k with $\Delta_N = 5/k_N - 4/k_N^2$ as the corresponding value of Δ The approximation is shown to be satisfactory for $N \geqq 450$ if the 5 per cent level of significance is used and for $N \geqq 300$ if the 1 per cent level is used. (Received July 31, 1942.)

286. F. E. Satterthwaite: *Generalized Poisson distribution.*

In this paper the Poisson distribution is generalized to allow for the assignment of varying weights to a set of events when the number of events follows the Poisson law. The development used brings out the fact that distributions falling in this class do not require that the underlying statistics be homogeneous. The only requirement is that they be independent. Formulas are given for the moments of the generalized distribution as functions of the moments of the underlying distribution of weights. The principles to be observed in the solution of practical problems are outlined. (Received July 30, 1942.)

287. Henry Scheffé: *On the ratio of the variances of two normal populations.*

Let θ be the above ratio. The two problems considered in this paper are the formulation and comparison of (i) significance tests for the hypothesis $\theta = \theta_0$, and (ii) confidence intervals for θ. The paper is divided into two parts, the first is kept on an elementary level and only solutions based on the F-distribution are considered. Following various approaches, six tests and corresponding sets of confidence intervals are introduced. It turns out that the limits on the F-distribution which yield an unbiased test are the same as those which yield confidence intervals optimum in a certain intuitive sense. The values of these limits are difficult to compute and some numerical data are given to indicate the loss of efficiency in using instead the easily obtained "equal tails" limits. The second part of the paper is concerned with the existence of common best critical regions and type B_1 regions, and the application of Neyman's theory of confidence intervals. No new tests or confidence intervals not already considered in Part I are obtained, but those previously judged best of a very narrow class are now shown to be best of all those based on similar regions of the same size. (Received July 16, 1942.)

288. Saly R. R. Struik and Miriam van Waters: *Modern statistical methods in penology.*

In applying statistical methods to penological problems, so far the best known studies have considered 100, 500, or once in England (to refute Lombroso's theory) 1500 cases. But from the correct statistical standpoint, far more cases are needed to establish a law. Over a period of years, an attempt has been made to use statistical methods in the study of penological problems in the Massachusetts Reformatory for Women, but the results will take on real significance and be conclusive only when similar investigations are made all over the United States. (Received August 1, 1942.)

289. D. S. Villars: *Significance tests for multivariate distributions.*

The observed mean of sets of m variates, each normally and independently distributed, is distributed around the population mean according to a χ^2 distribution with m degrees of freedom. The sum of squares of deviations of n observed points from the observed mean is distributed as χ^2 with $m(n-1)$ degrees of freedom (not with $n-1$). A much more powerful test for correlation than that by the correlation coefficient is described, which for bivariate distributions, involves comparisons between $n-1$ and $n-1$ degrees of freedom. This can be extended to $m-1$ tests with m variates. Distribution of distance between two means and distribution of fiducial radius is worked out in detail for two variates. (Received July 30, 1942.)

TOPOLOGY

290. D. W. Hall: *On a partial solution of a problem of J. R. Kline.*

As a partial solution of a problem of J. R. Kline, the following theorem is established. In order that a compact locally connected continuum M be homeomorphic with a sphere it is necessary and sufficient that it satisfy the following conditions: (a) no two points separate M, (b) for every simple closed curve J in M the set $M-J$ has at least two and at most a finite number of components. (Received June 22, 1942.)

291. W. M. Kincaid: *On non-cut sets of locally connected continua.*

This paper is concerned with certain generalizations of the well known result that corresponding to any non-cut point p of a space S which is a locally connected continuum, an arbitrarily small region having a connected complement and containing p can be found. It is shown that any closed non-cut set P of such a space S can be imbedded in the sum R of a finite number of regions (lying in a preassigned ϵ-neighborhood of P) so chosen that $S-R$ is a locally connected continuum. If, in addition, there exists a family of sets $\mathfrak{J}$ no element of which separates $S-P$, then another set R', contained in R and having the same properties, can be found such that no element of $\mathfrak{J}$ contained in $S-R$ separates $S-R'$. If the elements of $\mathfrak{J}$ are single points, the sets R and R' can be replaced by a single set having the properties of both. Further results are obtained in the special case where S is not separated by any m points. (Received July 24, 1942.)

292. R. G. Lubben: *Mappings of spaces H Fréchet on completely regular spaces.*

Let T be a space H Fréchet, K be the aggregate of all completely regular Hausdorff decompositions (Alexandroff and Hopf, *Topologie*, p. 70; the space of this decomposition is to be a completely regular Hausdorff space) of T into mutually exclusive point sets, and $\sum$ be the sum of the elements of K. If $T \supset M$ and for $G = \bar{G} \subset T - M$ there exists a function which is continuous over T, takes on values from zero to unity, and is identical to zero over M and to unity over G, then it is said that M is completely T-regular. (1) A set is T-regular if and only if it belongs to $\sum$. (2) K is the totality of upper semi-continuous decompositions of T into completely T-regular sets. (3) For A and B elements of K let "$A<B$" mean that each element of A is a subset of an element of B; this ordering of the elements of K defines a complete lattice, L, which has properties analogous to those of the "$\delta(S)$" studied by the author (Transactions of this Society, vol. 49 (1941), pp. 459–463). If S is a regular Hausdorff space and T is the space of its atomic decomposition points, then L and $\delta(S)$ are isomorphic. (Received August 1, 1942.)

293. Deane Montgomery and Hans Samelson: *Groups transitive on spheres.*

Let G be a compact connected Lie group which is transitive and effective on the n-sphere S^n. If n is even it is shown that G must be simple. If n is odd then G must be essentially the product of two simple groups one of which is either the identity, the circle group, or the group of quaternions of absolute value unity. There is given a fairly complete analysis of exactly what groups can be transitive on S^n. Speaking in general terms the results show that in the main only groups well known to be transitive on S^n can be transitive on S^n. There are one or two results on the structure of rotation groups. (Received July 9, 1942.)

294. Deane Montgomery and Hans Samelson: *Groups transitive on the n-dimensional torus.*

It is shown that if G is a compact connected Lie group which acts effectively and transitively on the n-dimensional torus, then G is the n-dimensional toral group. (Received July 9, 1942.)

295. Deane Montgomery and Leo Zippin: *A class of transformation groups in E_n.*

If a compact connected topological group acts on euclidean n-space and if it has at least one $(n-1)$-dimensional orbit, then it is known that all orbits except one are $(n-1)$-dimensional. It is shown here that this exceptional orbit must be a point and that there is a closed set which touches each orbit precisely once and is homeomorphic to a ray. The $(n-1)$-dimensional orbits are all homeomorphic and in many homology and homotopy properties they resemble $(n-1)$-spheres. (Received June 12, 1942.)

296. Hassler Whitney: *Complexes of manifolds.*

In geometric applications of combinatorial topology, it is very convenient to be able to realize chains and cochains with the help of geometric figures. These figures may not appear naturally subdivided into complexes; but they will in general be made up of manifolds, joined together like the cells of a complex. These "complexes of manifolds" are studied thoroughly here. The most general ones (which have fairly simple intersection properties) are more general than complexes, even in the neighborhood of a point. The "locally flat" ones are also studied. The most helpful tools are vectors and cones. (Received July 28, 1942.)

297. Hassler Whitney: *The self-intersections of an M^n in E^{2n}.*

Let the differentiable n-manifold M^n be mapped regularly by f into E^{2n}; by a slight deformation, suppose that the self-intersections are isolated. The space $\mathfrak{T}$ of ordered pairs of points of M may be considered as bounded by the space $\mathfrak{S}$ of directions in M at all points of M. Let n be even, and M, orientable. To f corresponds F, mapping $\mathfrak{T}$ into E^{2n}; the signed self-intersections of f give the signed zeros of F, and their algebraic number I_f is half d_F, the degree of F on $\mathfrak{T}$ over the origin, or on $\mathfrak{S}$ about the origin. If f_0 is deformed into f_1, each f_t being regular, then $I_{f_1}=I_{f_0}$. There exist mappings with any desired I_f. If n is odd or M is non-orientable, all holds if one reduces mod 2 and identifies "opposite points of $\mathfrak{T}$ and $\mathfrak{S}$." M^n may be imbedded (without self-intersections) in E^{2n}. (Received July 28, 1942.)

298. Hassler Whitney: *The singularities of an M^n in E^{2n-1}*.

Let M be a piece of an orientable n-manifold M^n, bounded by A. Let f be a mapping of M into E^{2n-1}, such that the singular points (points where f is not regular) are isolated and a certain condition (see this Bulletin, abstract 48-9-269) holds at each. Let f be one-one in A. Let $n\geqq 3$ be odd. Then if $\mathfrak{L}(M)$ is the looping coefficient $\mathrm{LC}(A, A')=$ Kronecker index $\mathrm{KI}(A, M)$ (A' being A pushed slightly into M), this measures the signed number of singularities of f in M. Hence if $\mathfrak{L}(M)\neq 0$, f cannot be regular. For M closed, of course $\mathfrak{L}(M)=0$. All holds for n even if reduced mod 2. For n even, there is a regular mapping f of an n-cube in $E^n\subset E^{2n-1}$ into E^{2n-1}, arbitrarily near (but not with derivatives) to the identity, equal to the identity on A, and with $\mathfrak{L}(M)=2$. Arbitrarily near any mapping f of M^n into E^{2n-1} there is a regular mapping; thus M may be immersed in E^{2n-1}. (Received July 28, 1942.)

NEW PUBLICATIONS

Bacon, H. M. Differential and Integral Calculus. New York, McGraw-Hill, 1942. 7+771 pp. \$3.75.

Brown, O. E. See Morris, M.

Chandrasekhar, S. Principles of Stellar Dynamics. University of Chicago Press, 1942. 10+251 pp. \$5.00.

Contributions to the Calculus of Variations 1938–1941. University of Chicago Press, 1942. 7+527 pp.

Dyess, W. B., and Gilmore, R. O. Mathematics of Business and Finance. Including Compound Interest and Annuity Tables by F. C. Kent and M. E. Kent. New York, McGraw-Hill, 1942. 18+435 pp. \$3.50.

Evans, P. L. Algebra. (Mathematics for Technical Training.) Boston, Ginn, 1941. 8+126 pp. \$1.25.

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Freeman, H. A. Industrial Statistics. New York, Wiley; London, Chapman and Hall, 1942. 9+178 pp. \$2.50.

Gilmore, R. O. See Dyess, W. B.

Granville, W. A., Smith, P. F., and Mikesh, J. S. Spherical Trigonometry and Tables. Boston, Ginn, 1942. 18+60+4+43 pp. \$1.25.

Keller, E. G. Mathematics of Modern Engineering. Vol. 2. Mathematical Engineering. New York, Wiley; London, Chapman and Hall, 1942. 12+309 pp. \$4.00.

Kent, F. C. See Dyess, W. B.

Kent, M. E. See Dyess, W. B.

Lefschetz, S. Algebraic Topology. (American Mathematical Society Colloquium Publications, vol. 27.) New York, American Mathematical Society, 1942. 6+389 pp. \$6.00.

Lehmann, C. H. Analytic Geometry. New York, Wiley; London, Chapman and Hall, 1942. 14+425 pp.

Mikesh, J. S. See Granville, W. A.

Mira, J. A. See Moore, J. H.

Moore, J. H., and Mira, J. A. The Gist of Mathematics. New York, Prentice-Hall, 1942. 12+726 pp. \$5.35.

Morris, M., and Brown, O. E. Differential Equations. Revised edition. New York, Prentice-Hall, 1942. 11+355 pp. \$3.00.

Munshower, C. W., and Wardwell, J. F. Basic College Mathematics. New York, Holt, 1942. 11+612 pp. \$3.20.

Phillips, H. B. Analytical Geometry and Calculus. Cambridge, Massachusetts, Addison-Wesley, 1942. 10+490 pp. \$5.00.

Smith, P. F. See Granville, W. A.

Wald, A. On the Principles of Statistical Inference. Notre Dame, Indiana, University of Notre Dame, 1942. 50 pp. \$1.00.

Wardwell, J. F. See Munshower, C. W.

Whyburn, G. T. Analytic Topology. (American Mathematical Society Colloquium Publications, vol. 28.) New York, American Mathematical Society, 1942. 10+278 pp. \$4.75.

BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

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1941–1942

SEPTEMBER, 1942

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK

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ENDOWMENT FUND

In 1923 an Endowment Fund was collected to meet the greater demands on the Society's publication program caused by the ever increasing number of important mathematical memoirs. Of this fund, which now amounts to some $60,000, a considerable proportion was contributed by members of the Society.

SPECIAL FUNDS

The Bôcher Memorial Prize.

This prize was founded in memory of Professor Maxime Bôcher. It is awarded every five years for a notable research memoir in analysis which has appeared during the preceding five years in a recognized journal published in the United States or Canada; the recipient must be a member of the Society, and not more than fifty years old at the time of publication of his memoir.

First (Preliminary) Award, 1923: To G. D. Birkhoff, for his memoir *Dynamical systems with two degrees of freedom.*

Second Award, 1924: To E. T. Bell, for his memoir *Arithmetical paraphrases,* and to Solomon Lefschetz, for his memoir *On certain numerical invariants with applications to abelian varieties.*

Third Award, 1928: To J. W. Alexander, for his memoir *Combinatorial analysis situs.*

Fourth Award, 1933: To Marston Morse, for his memoir *The foundations of a theory of the calculus of variations in the large in m-space,* and to Norbert Wiener, for his memoir *Tauberian theorems.*

Fifth Award, 1938: To John von Neumann, for his memoir *Almost periodic functions and groups.*

The Frank Nelson Cole Prize in Algebra.
The Frank Nelson Cole Prize in the Theory of Numbers.

These prizes were founded in honor of Professor Frank Nelson Cole, on the occasion of his retirement as Secretary of the American Mathematical Society and editor of the Bulletin, after twenty-five years of service; the fund was later doubled by his son, Charles A. Cole. They are awarded at five-year intervals for contributions to algebra and the theory of numbers, respectively, under restrictions similar to those for the Bôcher prize

First Award, 1928: To L. E. Dickson, for his book *Algebren und ihre Zahlentheorie,* Zurich, 1927.

Second Award, 1931: To H. S Vandiver, for his several papers on Fermat's last theorem published in the Transactions of the American Mathematical Society and the Annals of Mathematics during the last five years, with special reference to a paper entitled *On Fermat's last theorem,* which appeared in volume 31 of the Transactions.

Third Award, 1939: To A. A. Albert, for his papers on the construction of Riemann matrices, published in volumes 35 and 36 of the Annals of Mathematics

Fourth Award, 1941: To Claude Chevalley, for his paper entitled *La théorie du corps de classes,* published in volume 41 of the Annals of Mathematics.

The Eliakim Hastings Moore Fund.

This fund was founded in 1922 in honor of Professor Eliakim Hastings Moore, on the occasion of the twenty-fifth anniversary meeting of the Chicago Section of the American Mathematical Society. The income from the fund is to be used at the discretion of the Council of the Society, for the publication of important mathematical books or memoirs, or the award of prizes.

The Marion Reilly Fund.

Dean Marion Reilly willed to the American Mathematical Society a portion of her estate to be used for the advancement of research in pure mathematics The Society has received approximately $23,500.

The Ernest William Brown Fund.

From the estate of Professor E. W. Brown, a fund of $1000 is available, the interest on which, at the discretion of the Council, can be used for the furtherance of such mathematical interests as (a) the publication of important mathematical books, memoirs, and periodicals, and (b) lectures to be delivered on special occasions by invited guests of the Society.

The Robert Henderson Fund.

Upon his retirement from the Board of Trustees on December 31, 1940, Dr Robert Henderson presented to the Society an unrestricted gift of $1,000 which the Trustees have set aside as the Robert Henderson Fund.

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Metropolitan Life Insurance Company, New York, N.Y.
Michigan State College of Agriculture and Applied Science, East Lansing, Mich.
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University of Missouri, Columbia, Mo.

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Oregon State College, Corvallis, Ore.
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University of Pennsylvania, Philadelphia, Pa.
University of Pittsburgh, Pittsburgh, Pa.
Purdue University, Lafayette, Ind.
University of Rochester, Rochester, N.Y.
Rutgers University, New Brunswick, N.J.
The College of St. Thomas, St. Paul, Minn.
Smith College, Northampton, Mass.
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Sweet Briar College, Sweet Briar, Va.
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MEMBERS OF THE SOCIETY

September, 1942

This directory includes the name of every person who is a member of the Society as of September, 1942.

Personal information is as follows: (a) surname; (b) initials or one name with parentheses around parts usually omitted in correspondence; first names of women members are written out; (c) official title or position; (d) name and address of institution or organization; (e) mail address if different from the institution address; (f) in italic type, information regarding temporary or special assignments.

For many members in foreign countries, it has been necessary to repeat information which appeared in the previous directory.

The † before a name indicates life membership.

The § indicates membership as a nominee of an institutional member.

Ablow, C. M. 1943 N. Van Ness Ave., Los Angeles, Calif. *Ensign U.S.N.R., Naval Base, Long Beach, Calif.*

Abramowitz, Milton. Asso. Mathematician, Mathematical Tables Project, National Bureau of Standards, U.S. Dept of Commerce, 70 Columbus Ave., New York, N.Y. 2775 Kingsbridge Ter., Bronx.

Adams, C. R. Prof., Brown Univ., Providence, R.I. 60 Intervale Road.

Adams, E. P. Prof., Princeton Univ., Princeton, N.J. Edgerstoune.

Adams, G. J. Junior Civil Engineer, State Highway Dept., Linden, Ala. 2411 14th Ave., Sheffield, Ala.

Adams, Dr. **Rachel B.** (Mrs. C. R.). Radcliffe Coll., Cambridge, Mass. 60 Intervale Road, Providence, R.I.

Ader, O. B. Asst. Prof., Wofford Coll., Spartanburg, S.C. 492 N. Church St.

Adkisson, V. W. Prof., Univ. of Arkansas, Fayetteville, Ark. 236 Buchanan St.

Adler, Dr. **Claire F.** (Mrs. E. H.). 189-21 Tioga Dr., St. Albans, N.Y.

§**Adler, F. T.** Visiting Asst. Prof., Physics Dept., Univ. of Wisconsin, Madison, Wis.

Adshead, J. G. Asso. Prof., Dalhousie Univ., Halifax, N.S., Canada.

Agard, H. L. Prof., Williams Coll., Williamstown, Mass.

Agnew, R. P. Prof., Cornell Univ., Ithaca, N.Y. White Hall.

Alaoglu, Dr. **Leonidas.** Instr., Purdue Univ., Lafayette, Ind.

Albert, A. A. Prof., Univ. of Chicago, Chicago, Ill. Eckhart Hall.

Albert, Dr. **G. E.** Instr., Ohio State Univ., Columbus, Ohio. University Hall.

Albert, O. W. Prof., Univ. of Redlands, Redlands, Calif. 629 Buena Vista St.

Alden, H. H. Prof., New Mexico Military Inst., Roswell, N.Mex. Box 1237.

Alexander, J. W. Prof., Inst. for Advanced Study, Princeton, N.J. 29 Cleveland Lane.

Alfieri, F. A. Metropolitan Life Ins. Co., 1 Madison Ave., New York, N.Y. 9 Metropolitan Oval, Bronx.

Alger, P. L. Staff Asst., General Electric Co., Schenectady, N.Y. 1758 Wendell Ave.

Allen, E. B. Prof., Rensselaer Polytech. Inst., Troy, N.Y. 4 Sheldon Ave.

Allen, E. S. Asso. Prof., Iowa State Coll., Ames, Iowa.

Allen, Dr. **Florence E.** Instr., Univ. of Wisconsin, Madison, Wis. 219 Lathrop St.

Allen, Harriet W. Asst. Prof., Hollins Coll., Hollins College, Va. *Physicist, Bureau of Ordnance, U. S. Navy Dept., Washington, D.C.* 2141 Eye St. N.W.

Allendoerfer, C. B. Asso. Prof., Haverford Coll., Haverford, Pa. *Asso Physicist, U. S. Navy Dept., Washington, D.C 8603 Hempstead Ave., Bethesda, Md.*

Alsberg, Julius. Consulting Engineer. *Office of Chief of Chemical Warfare Service, U. S. War Dept., Washington, D.C. 2515 K St. N.W.*

§**Alton, T. P.** Asst., Univ. of Illinois, Urbana, Ill. 153 Math. Bldg.

Ambrose, Dr. **Warren.** Instr., Princeton Univ., Princeton, N.J. Fine Hall.

Ames, D. B. Asst. Prof., Rensselaer Polytech. Inst., Troy, N.Y. 117 Elmgrove Ave.

Ames, L. D. Prof., Univ. of Southern California, Los Angeles, Calif.

Amundson, N. R. Instr., Univ. of Minnesota, Minneapolis, Minn. 208 Main Engineering Bldg.
Ananda-Rau, K. Prof., Presidency Coll., Madras, India.
Anastasia Maria, Sister. Prof., Immaculata Coll., Immaculata, Pa.
Andersen, C. C. 1917 Warren Blvd., Chicago, Ill. *2d Lt., U.S.A., Air Corps, Morris Field, Charlotte, N.C.*
Anderson, Mae R. Prof., Concordia Coll., Moorhead, Minn. 507 10th St. S.
Anderson, Dr. **P. H.** Instr., Louisiana State Univ., University, La.
Anderson, R. Lucile. Asst. Prof., Hunter Coll., 695 Park Ave., New York, N.Y.
Anderson, T. W. Instr., Princeton Univ., Princeton, N.J. 2435 Laundale Ave., Evanston, Ill.
Anderson, W. E. Prof., Miami Univ., Oxford, Ohio. 112 E. Walnut St.
§**Annenberg, Lawrence.** 131 E. 21st St., Brooklyn, N.Y.
Anthony, Lucius. Woodinville, Wash.
Archibald, H. F. Asso. Prof., Keuka Coll., Keuka Park, N.Y.
†**Archibald, R. C.** Prof., Brown Univ., Providence, R.I.
Archibald, R. G. Asst. Prof., Queens Coll., Flushing, N.Y.
Arms, R. A. Prof., Gettysburg Coll., Gettysburg, Pa. 59 Lincoln.
Armstrong, Dr. **Beulah M.** Asso., Univ. of Illinois, Urbana, Ill. 364 Math. Bldg.
Arnold, Dr. **B. H.** Instr., Purdue Univ., Lafayette, Ind.
Arnold, Dr. **H. A.** 2480 Lake St., Lincoln, Neb. *Lt. (j.g.), U.S.N.R.*
Arnold, H. E. Asso. Prof., Wesleyan Univ., Middletown, Conn. Wesleyan Station.
†**Arnold, J. W.** *Lt. Col., U.S.A., Coast Artillery Corps, Camp Haan, Calif.*
§**Arnold,** Dr. **K. J.** Instr., Massachusetts Inst. of Tech., Cambridge, Mass.
Arnoldy, Sister M. Nicholas. Marymount Coll., Salina, Kan.
Aroian, Dr. **L. A.** Instr., Hunter Coll., Bedford Park Blvd. and Navy Ave., New York, N.Y.
Arrow, K. J. Fellow, Economics Dept., Columbia Univ., New York, N.Y. 749 West End Ave.
Arthur, R. S. Univ. of California at Los Angeles, Los Angeles, Calif. 314 Buena Vista St., Redlands, Calif.
Artin, Emil. Prof., Indiana Univ., Bloomington, Ind. 715 S. Fess Ave.
Ashman, E. T. Darien High School, Darien, Conn. 11 Fairview Ave.
Asadourian, Dr. **Fred.** Instr., New York Univ., University Heights, New York, N.Y.
Astrachan, Max. Asso. Prof., Antioch Coll., Yellow Springs, Ohio.
Atanasoff, J. V. Asso. Prof., Iowa State Coll., Ames, Iowa. Physics Bldg. *Official Investigator, National Defense Research Com.*
§**Atchison, W. F.** Asst., Univ. of Illinois, Urbana, Ill. 153 Math. Bldg.
§**Atkins, H. P.** Instr., Univ. of Rochester, Rochester, N.Y. River Campus.
§**Avann,** Dr. **S. P.** Instr., Washington Univ., St. Louis, Mo.
Ayres, Frank. Asso. Prof., Dickinson Coll., Carlisle, Pa. 426 S. Pitt St.
Ayres, H. C. Asst. Prof., U. S. Naval Acad., Annapolis, Md. *Lt., U.S.N.R., U.S.S. Texas, Care of Postmaster, New York, N.Y.*
Ayres, W. L. Prof., Purdue Univ., Lafayette, Ind.

Babcock, R. W. Dean, School of Arts and Sciences, Kansas State Coll. of Agric. and Applied Science, Manhattan, Kan.
Babcock, Wealthy. Asso. Prof., Univ. of Kansas, Lawrence, Kan.
†**Bacon, Clara L.** Prof. Emeritus, Goucher Coll., Baltimore, Md. 2316 N. Calvert St.
Bacon, H. M. Asst. Prof., Stanford Univ., Stanford University, Calif. Box 1144.
Baer, Reinhold. Asso. Prof., Univ. of Illinois, Urbana, Ill.
Bagby, L. C. Educational Dir., Aviation School, Hartung Aircraft Corp., Detroit, Mich. 7027 E. Warren.
Baidaff, B. I. Prof., Univ. of Buenos Aires, Buenos Aires, Argentina. Av. de Mayo 560.
Bailey, H. W. Dir., Personnel Bureau, and Asso. Prof., Univ. of Illinois, Urbana, Ill.
Bailey, R. P. Asst. Prof., Lafayette Coll., Easton, Pa. 628 Monroe St.
Baker, Frances E. Asso. Prof., Vassar Coll., Poughkeepsie, N.Y.

Baker, Dr. G. A. Instr. and Junior Statistician, Coll. of Agric., Univ. of California, Davis, Calif.

Bal, Y. K. Columbia Univ., New York, N.Y. 500 Riverside Dr.

Ball, N. H. Asst. Prof., U.S. Naval Acad., Annapolis, Md.

Ballantine, Dr. Constance R. (Mrs. J. P.). Asso., Univ. of Washington, Seattle, Wash. 1802 Ravenna Blvd.

Ballantine, J. P. Prof., Univ. of Washington, Seattle, Wash 1802 Ravenna Blvd.

Ballard, Dr. Ruth M. (Mrs. F. K.). Wright Junior Coll., Chicago, Ill. 636 Wellington Ave.

Ballou, D. H. Asst. Prof., Middlebury Coll., Middlebury, Vt. 18 S. Pleasant St.

Bamforth, F. R. Asso. Prof., Ohio State Univ., Columbus, Ohio.

Bankier, Dr. J. D. Instr., Johns Hopkins Univ., Baltimore, Md.

Banks, G. B. Prof., Niagara Univ., Niagara University, N.Y.

Baños, Alfredo. Dir., Inst. of Physics, and Prof., Univ. of Mexico, Tacuba 5, Mexico, D.F., Mexico.

§Barankin, E. W. Asst., Univ. of California, Berkeley, Calif. 2540 LeConte Ave

Barber, Dr. S. F. Instr., Coll. of the City of New York, 139th St. and Convent Ave., New York, N.Y. *Instr., Defense Training Inst., 375 Pearl St., Brooklyn, N Y.*

Bardell, R. H. Asst. Prof. and Acting Asst. Dir., Univ. of Wisconsin at Milwaukee, Milwaukee, Wis.

Bareis, Grace M. Asst. Prof., Ohio State Univ, Columbus, Ohio. Canal Winchester, Ohio.

§Bargmann, Dr. Valentin. Asst., Inst. for Advanced Study, Princeton, N.J.

Barker, J. E. Asso. Mathematician, U.S. Naval Proving Ground, Dahlgren, Va

Barlaz, Joshua. Instr., Coll. of Engineering and Commerce, Univ. of Cincinnati, Cincinnati, Ohio.

Barnard, R. W. Asso. Prof., Univ. of Chicago, Chicago, Ill. Eckhart Hall.

Barnes, J. L. Prof., Tufts Coll., Medford, Mass. 41 Harold Ave., Princeton, N.J.

Barnett, I. A. Asso. Prof., Univ. of Cincinnati, Cincinnati, Ohio.

Barnett, Joseph. Asst. Prof, Oklahoma Agric. and Mech. Coll., Stillwater, Okla. 520 Maple Ave.

Baron, Louis. Instr., Washington Square Coll., New York Univ., New York, N.Y.

Barral-Souto, José. Prof., Univ. of Buenos Aires, Buenos Aires, Argentina. Córdoba 1459.

Barron, J. J. Asso. Prof., St. Joseph Coll., West Hartford, Conn.

Barrow, D. F. Prof., Univ. of Georgia, Athens, Ga. 260 Cherokee Ave.

Barta, V. P. Research Physicist, Massachusetts Inst. of Tech, Cambridge, Mass. 247 N. 11th Ave., Corvallis, Ore.

Bartels, Dr. R. C. F. Instr., Univ. of Michigan, Ann Arbor, Mich. *Engineer, Structures Section, Bureau of Aeronautics, U. S. Navy Dept, Washington, D.C.*

Barten, H. J. Johns Hopkins Univ, Baltimore, Md. 1518 Moreland Ave.

Bartky, Walter. Asso. Dean of Physical Sciences and Prof., Univ. of Chicago, Chicago, Ill.

Bartleson, E. L. Asst. Mathematician, Actuarial Dept. General, Prudential Ins. Co. of America, Newark, N.J.

Basch, Dr. A. E. Coll. of the City of New York, New York, N.Y 503 W. 121st St.

Basoco, M. A. Prof., Univ. of Nebraska, Lincoln, Neb.

§Bassford, H. R. Actuary, Metropolitan Life Ins Co., 1 Madison Ave., New York, N.Y.

Basye, Dr. R. E. 5529 Brooklyn Ave., Kansas City, Mo. *Lt. (j g.), U S.N.R.*

Batchelder, P. M. Asso. Prof., Univ. of Texas, Austin, Tex. 910 W. 22d St.

Bateman, Harry. Prof., California Inst. of Tech., Pasadena, Calif.

Bateman, P. T. 421 Lawnton Road, Willow Grove, Pa.

Bates, O. K. Prof., St. Lawrence Univ., Canton, N.Y. 44 E. Main St.

†Battig, Leon. Instr., Univ. of Wisconsin at Milwaukee, Milwaukee, Wis. 2535 Elizabeth St., Sheboygan, Wis.

Bauer, P. S. Vice Pres., North Shore News Co., 50 Exchange St., Lynn, Mass.

Baumgart, J. K. 919 Adams St., Maywood, Ill. *Instr., U.S.A., Air Corps Tech. School, Madison, Wis*

Beal, F. W. Prof, Univ. of Pennsylvania, Philadelphia, Pa College Hall.

Beale, F. S. Asst. Prof., Lehigh Univ., Bethlehem, Pa. 316 W. Packer Ave.
Beamish, Sister M. Amator. 5601 Kingsessing Ave., Philadelphia, Pa.
Bearman, J. E. 1811 Queen Ave. N., Minneapolis, Minn. *U.S.A., Coast Artillery Replacement Training Center, Fort Eustis, Va. Care of H. Nisselson, 1630 R St. N.W., Washington, D.C.*
Beatley, Ralph. Asso. Prof., Graduate School of Education, Harvard Univ., Cambridge, Mass.
Beatty, Samuel. Dean, Faculty of Arts, Univ. of Toronto, Toronto, Ont., Canada.
Beaty, Dr. **Marjorie H.** (Mrs. D. W.). 717 E. Main St., Vermillion, S.Dak.
Beaumont, Dr. **R. A.** Instr., Univ. of Washington, Seattle, Wash.
Beaver, R. A. Asst. Prof., New York State Coll. for Teachers, Albany, N.Y.
Beckenbach, E. F. Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.
Becker, F. A. Instr., St. Mary's Univ., San Antonio, Tex.
Becker, Dr. **Miriam F.** Care of A. Mazur, 65 Park Ter. E., New York, N.Y.
Beckwith, Ethelwynn R. (Mrs. W. E.). Prof., Milwaukee-Downer Coll., Milwaukee, Wis.
Beckwith, W. S. Asso. Prof., Univ. of Georgia, Athens, Ga.
Beegle, B. L. Dean and Prof., Seattle Pacific Coll., Seattle, Wash.
Beenken, May M. Prof., State Teachers Coll., Oshkosh, Wis.
Beesley, E. M. Asst. Prof., Univ. of Nevada, Reno, Nev.
Belcher, D. R. Asst. Chief Statistician, American Telephone and Telegraph Co., 195 Broadway, New York, N.Y.
Bell, Clifford. Asso. Prof., Univ. of California at Los Angeles, Los Angeles, Calif. 10514 Rochester Ave.
Bell, E. T. Prof., California Inst. of Tech., Pasadena, Calif.
§**Bell, J. C.** Asst., Univ. of Illinois, Urbana, Ill. 160 Math. Bldg.
Bell, Dr. **J. H.** Research Asst., National Research Council, Ottawa, Ont., Canada. 95 Barker St., London, Ont., Canada.
Bell, Lois E. Instr., Independence Junior Coll., Independence, Kan. Box 83.
Bell, P. O. Asst. Prof., Univ. of Kansas, Lawrence, Kan. 2201 New Hampshire St.
Bellamy, B. C. Civil Engineer. Lamont, Wyo.
§**Bellman, Richard.** No address.
Benac, Dr. **T. J.** Instr., U.S. Naval Acad., Annapolis, Md.
Bender, J. R. Asso. Prof., Univ. of Idaho, Moscow, Idaho.
†**Bennett, A. A.** Prof., Brown Univ., Providence, R.I.
Bennett, B. M. Columbia Univ., New York, N.Y. *Asst. Mathematician, U.S. Naval Proving Ground, Dahlgren, Va. 1410 M St. N.W., Washington, D.C.*
Bennett, Theodore. Asst. Prof., Marietta Coll., Marietta, Ohio.
Benscoter, S. U. Asst. Engineer, U.S. Engineer Office, Vicksburg, Miss.
§**Benson, Kathryn E.** Asst., Univ. of California, Berkeley, Calif.
Bentley, Dr. **A. F.** R.F.D. 2, Paoli, Ind.
§**Bentz, R. P.** Instr., Pennsylvania State Coll., State College, Pa.
Berg, Dr. **W. D.** Instr., State Univ. of Iowa, Iowa City, Iowa. 211 Physics Bldg.
Bergman, Dr. **Stefan.** School of Mech., Brown Univ., Providence, R.I.
Bergmann, P. G. Asst. Prof., Physics Dept., Lehigh University, Bethlehem, Pa.
§**Bernard, R. R.** Univ. of Virginia, University Station, Charlottesville, Va.
Bernstein, B. A. Prof., Univ. of California, Berkeley, Calif.
Bernstein, Dr. **Dorothy L.** Instr. and Research Asso., Univ. of California, Berkeley, Calif.
Bernstein, Felix. Prof., New York Univ., 477 1st Ave., New York, N.Y.
Bernstorf, Ella E. Dean, State Teachers Coll., California, Pa.
Berry, A. C. Asso. Prof., Lawrence Coll., Appleton, Wis.
Berry, W. J. Prof., Polytech. Inst. of Brooklyn, Brooklyn, N.Y. 224 St. John's Pl.
Berwick, W. E. H. Prof. Emeritus, University Coll. of North Wales, Bangor, North Wales. Care of E. G. Phillips.
Besancon, R. M. Coll. of Pharmacy, Univ. of Illinois, Chicago, Ill. *2d Lt, A U.S.*
Beth, R. A. Asso. Prof, Michigan State Coll., East Lansing, Mich. *Consultant, National Defense Research Com.; Physicist, Com. on Passive Protection Against Bombing Palmer Physics Lab., Princeton Univ., Princeton, N.J.*

Betz, Dr. **E. E.** Instr., U.S. Naval Acad., Annapolis, Md. 1106 Maple St. *Lt. (j.g.), U.S.N.R.*
Betz, Herman. Asst. Prof., Univ. of Missouri, Columbia, Mo.
Betz, Dr. **William.** Mathematics Specialist, Public Schools of Rochester, Rochester, N.Y. 652 Melville St.
Beveridge, H. R. Prof., Monmouth Coll., Monmouth, Ill. 1043-E. Detroit Ave.
Bibb, S. F. Asso. Prof., Illinois Inst. of Tech., Chicago, Ill.
Biesele, Dr. **F. C.** Instr., Univ. of Utah, Salt Lake City, Utah.
Bilgery, Conrad. Prof., Regis Coll., Denver, Colo.
Biot, M. A. Asst. Prof., Columbia Univ., New York, N.Y. *Research Asso. in Aeronautics, California Inst. of Tech., Pasadena, Calif.*
Birchby, W. N. Asst. Prof., California Inst. of Tech., Pasadena, Calif.
Birchenough, Harry. Prof., New York State Coll. for Teachers, Albany, N.Y.
Bird, M. T. Asst. Prof., Utah State Agric. Coll., Logan, Utah.
Birkhoff, Garrett. Asso. Prof., Harvard Univ., Cambridge, Mass. 45 Fayerweather St.
Birkhoff, G. D. Prof., Harvard Univ., Cambridge, Mass. 987 Memorial Dr.
Birnbaum, Z. W. Asst. Prof., Univ. of Washington, Seattle, Wash.
Bisshopp, K. E. 1136 LaSalle St., Beloit, Wis.
§**Bissinger, B. H.** Instr., Cornell Univ., Ithaca, N.Y.
Black, A. H. Prof., Lebanon Valley Coll., Annville, Pa. 485 Maple St.
Black, Florence L. Asso. Prof., Univ. of Kansas, Lawrence, Kan.
Black, H. L. Prof., Westminster Coll., New Wilmington, Pa. 200 High St.
Blackall, Dr. **C. J.** North Baltimore, Ohio. *U.S.A., Camp Wallace, Tex.*
Blackwell, Dr. **David.** Instr., Southern Univ., Scotlandville, La.
Blake, Dr. **Archie.** Engineer, National Inventors Council, Washington, D.C. 2812 Cathedral Ave. N.W.
Blake, Dr. **E. M.** Drawer A, Pratt Station, Brooklyn, N.Y.
Blake, R. G. Hernando High School, Brooksville, Fla. 303 E. Jefferson St.
Blanc, Charles. Prof., Univ. of Lausanne, Lausanne, Switzerland. Ave. Jolimont 13.
Blanch, Dr. **Gertrude.** Asso. Mathematician, Mathematical Tables Project, National Bureau of Standards, U.S. Dept. of Commerce, 70 Columbus Ave., New York, N.Y. 4823 Clarendon Road, Brooklyn, N.Y.
Blanche, Dr. **E. E.** Instr., Michigan State Coll., East Lansing, Mich.
§**Blankinship, W. A.** Industrial Mathematician, U.S. Rubber Co., Detroit, Mich. 5026 Chalmers Ave.
Blaquier, Juan. Prof., Univ. of Buenos Aires, Buenos Aires, Argentina. Arroyo 1000.
Blau, J. H. Ordnance Lab., Navy Yard, U.S. Navy Dept., Washington, D.C.
Bleick, Dr. **W. E.** Instr., U. S. Naval Acad., Annapolis, Md. 114 Monticello Ave. *Lt., U.S.N.R., Bureau of Ordnance, U.S. Navy Dept., Washington, D.C.*
Blichfeldt, H. F. Prof. Emeritus, Stanford Univ., Stanford University, Calif. 520 W. Crescent Dr., Palo Alto, Calif.
†**Bliss, G. A.** Prof. Emeritus, Univ. of Chicago, Chicago, Ill.
Blumberg, Henry. Prof., Ohio State Univ., Columbus, Ohio. 76 E. Blake Ave.
Blumenthal, L. M. Asso. Prof., Univ. of Missouri, Columbia, Mo.
Boas, Mary L. (Mrs. R. P.). Instr., Duke Univ., Durham, N.C. 2708 Legion Ave.
Boas, Dr. **R. P.** 2708 Legion Ave., Durham, N.C. *Junior Ground School Instr., U.S.N. Pre-Flight School, Chapel Hill, N.C.*
Bobonis, Augusto. Asso. Prof., Univ. of Puerto Rico, Rio Piedras, Puerto Rico.
Bochner, Salomon. Asso. Prof., Princeton Univ., Princeton, N.J. Fine Hall.
Bode, Dr. **H. W.** Bell Telephone Labs., 463 West St., New York, N.Y.
Boeder, Dr. **Paul.** American Optical Co., Southbridge, Mass.
Boehm, Frank. Manager, Life Ins. Dept., 80 Maiden Lane, New York, N.Y.
Boeker, Mary L. D. (Mrs.). Instr., Brooklyn Coll., Brooklyn, N.Y. Collingwood Hotel, 45 W. 35th St., New York, N.Y.
Boggs, Dr. **Elizabeth M.** (Mrs. F. W.). Lecturer, Physics Dept., Univ. of Pittsburgh, Pittsburgh, Pa.
Bohnenblust, H. F. Asso. Prof., Princeton Univ., Princeton, N.J.

Bohr, H. A. Prof., Univ. of Copenhagen, Copenhagen, Denmark. Maglevaenget 9, Charlottenlund, Denmark.
Boldyreff, A. W. Asso. Prof., Univ. of Arizona, Tucson, Ariz. 2309 E. 5th St.
Bolza, Oskar. Prof., Univ. of Freiburg, Freiburg i. Br., Germany. Hotel Ochsen, Tubingen, Germany.
Bond, W. M. Instr., Polytech. Inst. of Brooklyn, Brooklyn, N.Y
Borgman, W. M. Asst. Prof., Wayne Univ., Detroit, Mich.
Borofsky, Samuel. Asst. Prof., Brooklyn Coll., Brooklyn, N.Y. 245 Lenox Road.
Boswell, J. M. Prof., Cumberland Coll., Williamsburg, Ky. *Lt., U S N R*
Botts, T. A. Fellow, Univ. of Virginia, University Station, Charlottesville, Va
Bouckaert, Dr. L. P. 112 Rue Christine, Ostend, Belgium.
Bourgin, D. G. Asst. Prof., Univ. of Illinois, Urbana, Ill.
§**Bourne, S. G.** Johns Hopkins Univ., Baltimore, Md.
Bowden, Joseph. Prof. Emeritus, Adelphi Coll., Garden City, N.Y. 21 Carleton Pl., Baldwin, N.Y.
Bowden, Muriel. 405 Park Ave, New York, N.Y
Bower, Julia W. Asso. Prof., Connecticut Coll., New London, Conn.
Bower, Dr. O. K. Instr, Univ. of Illinois, Urbana, Ill. 369 Math. Bldg.
§**Bowie, Clagett.** Johns Hopkins Univ., Baltimore, Md. Rowland Hall.
Bowie, H. E. Prof., American International Coll, Springfield, Mass. 14 Gunn Sq.
Boyce, M. G. Asst. Prof., Western Reserve Univ., Cleveland, Ohio.
Boyd, P. P. Dean, Coll. of Arts and Sciences, Univ of Kentucky, Lexington, Ky.
Boyer, C. B. Asst. Prof., Brooklyn Coll., Brooklyn, N.Y.
Bradley, A. D. Asst. Prof., Hunter Coll., Bedford Park Blvd. and Navy Ave, New York, N.Y. 66 Villard Ave., Hastings-on-Hudson, N Y.
§**Bradshaw, C. H.** Johns Hopkins Univ., Baltimore, Md.
Bradshaw, J. W. Prof., Univ. of Michigan, Ann Arbor, Mich. 1304 Cambridge Road.
Bradt, Paul. Physicist, National Bureau of Standards, U.S. Dept. of Commerce, Washington, D.C. 1847 Mintwood Pl. N.W.
Brady, Dr. C. P. Los Angeles City Coll, Los Angeles, Calif. *Lt., U.S N R, U. S. Naval Acad., Annapolis, Md Box 232 A, Fairfax Road, R F.D. 1.*
§**Brady, G. V.** Metropolitan Life Ins. Co., 180 Wellington St., Ottawa, Ont, Canada.
Brady, Mary L. Asst. Prof., Coll. of Mt. St. Vincent, New York, N.Y.
Brahana, H. R. Prof., Univ. of Illinois, Urbana, Ill. 707 S. McCullough.
Brainard, Millar. 17220 Aldersyde Dr., Shaker Heights, Cleveland, Ohio.
Bramble, C. C. Prof., Postgraduate School, U. S. Naval Acad., Annapolis, Md. 145 Monticello Ave.
Brand, Louis. Prof., Univ. of Cincinnati, Cincinnati, Ohio. *Coordinator of Research in Math. and Mech., Wright Field, Dayton, Ohio.*
Brandeberry, J. B. Prof., Univ. of Toledo, Toledo, Ohio.
Branson, H. R. Asst. Prof., Howard Univ., Washington, D.C. Chemistry Bldg.
Brant, Dr. Laura. 33 Capwell Ave., West Warwick, R.I.
Brasefield, S. E. Prof., Rutgers Univ, New Brunswick, N.J.
Bratton, W. A. Prof., Whitman Coll., Walla Walla, Wash.
§**Brauer, A. T.** Asst. Prof., Univ. of North Carolina, Chapel Hill, N.C. 408 North St.
Brauer, Richard. Asst. Prof., Univ. of Toronto, Toronto, Ont., Canada.
Bray, H. E. Prof., Rice Inst., Houston, Tex.
Breit, Gregory. Prof., Physics Dept., Univ. of Wisconsin, Madison, Wis. Sterling Hall.
Brenke, W. C. Prof., Univ. of Nebraska, Lincoln, Neb 1250 S. 21st St.
Brennan, J. F. 1769 Broadway, San Francisco, Calif. *Lt., U S N.R*
Brenner, Dr. J. L. 2431 N. Underwood St., East Falls Church, Arlington, Va
Brewer, B. W. Asst. Prof., Agric. and Mech. Coll. of Texas, College Station, Tex. 17 S. Bluff St., Janesville, Wis. *Lt. (j.g), U.S N R.*
Briant, R. C. Fellow, Mellon Inst., Pittsburgh, Pa. 403 Glasgow Road.
§**Bridgman, P. W.** Prof., Harvard Univ., Cambridge, Mass. Research Physics Lab.
Bright, S. K. Instr., Angleton Public Schools, Angleton, Tex.

Brink, R. W. Prof, Univ. of Minnesota, Minneapolis, Minn. 2243 Hoyt Ave., St. Paul, Minn.

Brinkmann, H. W. Asso. Prof., Swarthmore Coll., Swarthmore, Pa.

Bristow, Dr. Leonard. Instr., Univ. of Illinois, Urbana, Ill. 360 Math. Bldg.

Britton, J. R. Asst. Prof., Univ. of Colorado, Boulder, Colo.

Brixey, J. C. Asso. Prof., Univ. of Oklahoma, Norman, Okla. 927 S. Pickard.

Brodie, Helen C. Fellow, Coll. of the City of New York, New York, N.Y. 29 Washington Sq. W.

Brooks, F. L. Asst. Prof., Kent State Univ., Kent, Ohio. Box 224.

Brothers, W. H. Asso. Prof., Talladega Coll., Talladega, Ala.

Brown, Dr. A. A. F. Research Dept., L. Bamberger and Co., Newark, N.J. 11 Edgehill St., Princeton, N.J.

Brown, A. B. Asst. Prof., Queens Coll., Flushing, N.Y.

Brown, B. H. Prof., Dartmouth Coll., Hanover, N.H. 7 Ripley Road.

Brown, B. L. Asso. Prof., Amherst Coll., Amherst, Mass. 19 Hitchcock Road.

Brown, Dr. F. W. Asso. Physicist, Bureau of Mines, U.S. Dept. of Interior, Bruceton, Pa.

Brown, Dr. G. W. Research Asst., Princeton Univ., Princeton, N.J. 172 Nassau St.

Brown, Dr. H. K. Instr., Rensselaer Polytech. Inst., Troy, N.Y. 703 Grand St.

Brown, Myrtle C. Asso. Prof., North Texas State Teachers Coll., Denton, Tex. 5334 Teachers College Station.

Brown, O. E. Asso. Prof., Case School of Applied Science, Cleveland, Ohio. 1842 Taylor Road, East Cleveland, Ohio.

Brown, R. D. Engineer, Patent Dept., Philco Corp, Philadelphia, Pa. Blue Bell, Montgomery Co., Pa.

Brown, T. H., Prof., Graduate School of Business Administration, Harvard Univ., Soldiers Field, Boston, Mass.

Browne, E. T. Prof., Univ. of North Carolina, Chapel Hill, N.C. 730 E. Franklin St.

Bruce, R. E. Prof., Boston Univ., Boston, Mass.

Bruck, Dr. R. H. Instr., Univ. of Wisconsin, Madison, Wis. 1430 Chandler St.

Brunk, H. D. Fellow, Rice Inst., Houston, Tex.

Bruton, G. S. Asso. Prof., Univ. of the South, Sewanee, Tenn.

Bryan, N. R. Asso. Prof., Univ. of Maine, Orono, Me. 4 University Pl.

Bubb, C. R. Univ. of Oregon, Eugene, Ore.

Buchanan, Daniel. Dean, Faculty of Arts and Science, Univ. of British Columbia, Vancouver, B C., Canada.

Buchanan, H. E. Prof., Tulane Univ., New Orleans, La.

Buck, Thomas. Prof., Univ. of California, Berkeley, Calif. Faculty Club.

Buell, Dr. E. L. Instr., Northwestern Univ., Evanston, Ill.

Bula, Clotilde A. Asst. Prof., Univ. of Buenos Aires, Buenos Aires, Argentina; Research Worker, Univ. of Rosario, Rosario, Argentina. Calle Moreno 364, Rosario, Argentina.

Bull, Adelaide W. Research Representative, Encyclopoedia Britannica, Inc., 20 Wacker Dr, Chicago, Ill. 47 Montclair Ave., Montclair, N J.

Bullard, J. A. Prof., Coll. of Engineering, Univ. of Vermont and State Agric. Coll, Burlington, Vt. 110 Summit St.

Bullitt, W. M. Lawyer, 1711 Kentucky Home Life Bldg., Louisville, Ky.

Bulloch, G. E. Asst. Engineer, New Jersey Bell Telephone Co., 540 Broad St., Newark, N.J. 290 Livingston Ave., Lyndhurst, N.J.

Bullock, R. C. Asso. Prof., North Carolina State Coll., Raleigh, N.C. 5548 State College Station.

Bumer, C. T. Prof., Kenyon Coll., Gambier, Ohio.

Bunyan, L. H. Asst. Prof., Rutgers Univ., New Brunswick, N.J.

Burcham, Dr. P. B. Instr., Central Coll., Fayette, Mo. 510 N. Church St.

Bures, Dr. C. E. National Jewish Hospital, Denver, Colo.

Burgess, Louis. Attorney, 233 Broadway, New York, N.Y.

Burgess, Dr. R. W. Chief Economist, Western Electric Co, 195 Broadway, New York, N.Y.

Burington, R. S. Asso. Prof., Case School of Applied Science, Cleveland, Ohio. *Research Mathematician, Bureau of Ordnance, U. S. Navy Dept., Washington, D.C. 5200 Carlin Springs Road, Arlington Forest, Va.*
Burk, J. D. Asst. Prof., Univ. of Toronto, Toronto, Ont., Canada.
§**Burke, Barbara H.** (Mrs. T. F.). 581 San Antonio St., San Diego, Calif.
Burke, Sister Leonardo. Prof., Regis Coll., Weston, Mass.
Burkett, F. J. H. Asst. Prof., Union Coll., Schenectady, N.Y. 1030 Park Ave.
Burks, Dr. A. W. Instr., Moore School of Electrical Engineering, Univ. of Pennsylvania, Philadelphia, Pa.
Burnam, J. E. Prof., Hardin-Simmons Univ., Abilene, Tex. 1141 Grape St.
Burns, H. E. Asst. Prof., Indiana Univ. Extension, East Chicago, Ind. 3901 Indianapolis Blvd.
Burns, Sister M. Roberdette. Instr., Clarke Coll., Dubuque, Iowa.
Burr, I. W. Asst. Prof., Purdue Univ., Lafayette, Ind.
Burt, Mary M. (Mrs. L. N.). Retired. 800 24th Ave. N., St. Petersburg, Fla.
Burton, E. T. Research Engineer, Bell Telephone Labs, 463 West St., New York, N.Y.
§**Burton, L. P.** No address.
Burwell, Dr. W. R. 3311 Perkins Ave, Cleveland, Ohio.
Busemann, Dr. Herbert. Instr., Illinois Inst. of Tech., Chicago, Ill.
Bush, K. A. 7 Walling Blvd., Oneonta, N.Y.
Bush, L. E. Prof., Coll. of St. Thomas, St. Paul, Minn.
Bush, Vannevar. Pres., Carnegie Inst., Washington, D.C. 1530 P St. N.W. *Dir., Office of Scientific Research and Development.*
Bushey, Hobart. Asso. Prof., Hunter Coll., 695 Park Ave., New York, N.Y.
Bushey, Jewell H. (Mrs. Hobart). Asso. Prof., Hunter Coll., 695 Park Ave., New York, N.Y.
Bushkovitch, A. V. Asso. Prof. of Physics and Philosophy, Coll. of Charleston, Charleston, S.C.
Bussey, W. H. Asst. Dean, Coll. of Science, Literature and the Arts, and Prof., Univ. of Minnesota, Minneapolis, Minn. 106 Folwell Hall.
Butler, J. F. Univ. of Detroit, Detroit, Mich.
Butler, L. G. Instr., State Coll. of Washington, Pullman, Wash.
Butter, F. A. Asst. Prof., Univ. of Southern California, Los Angeles, Calif. Box 24.
§**Byerly, R. W.** 225 Broadway, New York, N.Y.
Byrne, W. E. Prof, Virginia Military Inst., Lexington, Va. Box 836. *Col, Virginia Militia.*

Cahn, Albert. Research Asst., Inst. of Meteorology, Univ. of Chicago, Chicago, Ill.
Cairns, S. S. Asst. Prof., Queens Coll., Flushing, N.Y. 42-14 149th Pl.
Cairns, W. D. Prof. Emeritus, Oberlin Coll., Oberlin, Ohio. 153 E. College St.
Calkin, J. W. Asst. Prof., Illinois Inst. of Tech., Chicago, Ill. *Bureau of Ordnance, U. S Navy Dept., Washington, D.C. 7813 Old Chester Road, Bethesda, Md.*
Calkins, Helen. Prof., Pennsylvania Coll. for Women, Pittsburgh, Pa.
Callahan, Ethel B. Asso. Prof., Shepherd State Teachers Coll., Shepherdstown, W.Va.
Callaway, Iris. Asso. Prof., Univ. of Georgia, Athens, Ga. 258 Hill St.
Cameron, E. A. Asso. Prof., Univ. of North Carolina, Chapel Hill, N.C. 121 Kenan St. *Ground School Instr., U.S.N. Pre-Flight School, Chapel Hill, N C.*
Cameron, R. H. Asst. Prof., Massachusetts Inst. of Tech., Cambridge, Mass. 17 Frost Road, Belmont, Mass.
†**Camp, B. H.** Prof., Wesleyan Univ., Middletown, Conn. 110 Mt. Vernon St.
Camp, C. C. Prof., Univ. of Nebraska, Lincoln, Neb.
Camp, E. J. Asso. Prof., Macalester Coll., St. Paul, Minn.
Campaigne, Dr. H. H. Instr., Univ. of Minnesota, Minneapolis, Minn. *Lt. (j.g.), U S N.R., U. S. Navy Dept., Washington, D.C. 232 N. Fillmore St., Arlington, Va.*
Campbell, A. D. Prof., Syracuse Univ., Syracuse, N.Y. 208 Westminster Ave.

Campbell, Dr. D. F. Consulting Actuary. 160 N. LaSalle St., Chicago, Ill.
Campbell, Dr. G. A. Retired. 129 Bellevue Ave., Upper Montclair, N.J.
Campbell, J. W. Prof., Univ. of Alberta, Edmonton, Alta., Canada.
Campbell, W. B. Asso. Prof. of Engineering, Pennsylvania Military Coll., Chester, Pa. 6848 N. 7th St., Philadelphia, Pa.
Candy, A. L. Prof. Emeritus, Univ. of Nebraska, Lincoln, Neb. Station A.
†**Canning, Joseph.** Roundhouse Foreman, Northern Pacific Railway Co., Garrison, Mont.
Cannon, Dr. E. W. Instr., Univ. of Delaware, Newark, Del. Cannon, Del.
Capecelatro, Dr. Achille. 100 E. Mosholu Pkwy. S., Bronx, New York, N.Y.
Caraher, Loula M. Asso. Prof., Arkansas State Teachers Coll., Conway, Ark. 342 Donaghey Ave.
Carey, C. E. Prof., State Teachers Coll., Clarion, Pa. 125 S. 5th Ave.
Carey, E. F. A. Asso. Prof., Montana State Univ., Missoula, Mont.
Caris, P. A. Asso. Prof., Univ. of Pennsylvania, Philadelphia, Pa. 717 Shadeland Ave., Drexel Hill, Pa.
Carlen, Mildred E. Registrar, Graduate School, and Instr., Brown Univ., Providence, R.I.
Carlitz, Leonard. Asso. Prof., Duke Univ., Durham, N.C. 4674 Duke Station.
§**Carlson, B. E.** Asst., Yale Univ., New Haven, Conn. 109 Davenport Ave.
Carlson, Elizabeth. Asst. Prof., Univ. of Minnesota, Minneapolis, Minn. 3024 14th Ave. S.
Carman, M. G. Prof., Murray State Teachers Coll., Murray, Ky.
Carmichael, R. D. Dean, Graduate School, Univ. of Illinois, Urbana, Ill. 207 W. Washington Blvd.
Carner, F. E. Stress Analyst, Airplane Div., Curtiss-Wright Corp., Buffalo, N.Y. 303 Humboldt Pkwy.
Carpenter, A. F. Prof., Univ. of Washington, Seattle, Wash.
Carpenter, R. V. A. Retired. 66 Park Ave., New York, N.Y.
Carroll, C. L. Instr., Georgia School of Tech., Atlanta, Ga.
Carroll, I. S. Asso. Prof., Syracuse Univ., Syracuse, N.Y. 511 Comstock Ave.
Carroll-Rusk, Evelyn (Mrs. W. S. Rusk). Dean and Prof., Wells Coll., Aurora, N.Y.
§**Carruth, Dr. P. W.** 2843 Southington Road, Shaker Heights, Cleveland, Ohio. *2d Lt., U.S.A., Fort Bliss, Tex.*
†**Carruth, W. M.** Prof., Hamilton Coll., Clinton, N.Y. College Hill.
Carslaw, H. S. Prof. Emeritus, Univ. of Sydney, Sydney, Australia. Burradoo, Australia.
Carson, Dr. A. B. Instr., Louisiana State Univ., University, La. Care of Mrs. Jessie Roth, 748 N. Laramie Ave., Chicago, Ill. *2d Lt., A.U.S., Miami Beach, Fla.*
Cartan, E. J. Prof., Univ. of Paris, Paris, France. 95 Blvd. Jourdan, Paris, 14, France.
Carter, E. R. Actuary, Washington National Ins. Co., Evanston, Ill.
Carter, H. C. Prof., Mary Washington Coll., Fredericksburg, Va. 1307 Littlepage St.
Carter, P. S. Research Engineer, Radio Corp. of America Labs., Rocky Point, N.Y.
§**Carter, W. C.** 5737 University Ave., Chicago, Ill. *Ensign, U.S.N.R.*
Carver, W. B. Prof., Cornell Univ., Ithaca, N.Y. White Hall.
Case, J. E. Prof., St. Louis Univ., St. Louis, Mo.
Cassidy, Dr. W. F. St. John's Univ., Brooklyn, N.Y.
Cassity, Dr. C. R. Instr., Univ. of Alabama, University, Ala. *Office of the Chief Signal Officer, U. S. War Dept., Washington, D.C. 1110 N. Kensington St., Arlington, Va.*
Castle, S. N. Engineer, Kalanianaole Highway, Honolulu, Hawaii.
Caton, W. B. Asso. Prof., Southwestern Coll., Winfield, Kan.
Caywood, T. E. Fellow, Harvard Univ., Cambridge, Mass. 1734 Cambridge St. *Research Asso., Cruft Lab., Harvard Univ.*
Cederberg, W. E. Prof., Augustana Coll., Rock Island, Ill. 2542 22½ Ave.
Cell, J. W. Asso. Prof., North Carolina State Coll., Raleigh, N.C.

de Cesare, E. A. Prof., Univ. of LaPlata, LaPlata, Argentina. Calle Rivadavia 1757, Buenos Aires, Argentina.
§**Chandrasekhar, Subrahmanyan.** Asso. Prof., Yerkes Observ., Univ. of Chicago, Williams Bay, Wis.
Chaney, J. G. Asst. Prof., Agric. and Mech. Coll. of Texas, College Station, Tex. *Lt. (j.g.), U.S.N.R.*
Chanler, Dr. **Josephine H.** Asso., Univ. of Illinois, Urbana, Ill. 364 Math. Bldg.
Chapman, D. G. Meteorologist, Meteorological Services of Canada 25 Arlington Ave., Westmount, Quebec, Canada.
Charnes, Abraham. Harvard Univ., Cambridge, Mass. 4 Arrow St.
Charney, J. G. Asst., Univ. of California at Los Angeles, Los Angeles, Calif.
Chatland, Harold. Asst. Prof., Montana State Univ., Missoula, Mont. 518 Evans Ave.
Cheney, W. F. Prof., Univ. of Connecticut, Storrs, Conn.
Cherry, T. M. Prof., Melbourne Univ., Melbourne, Australia.
Chesna, John. Physicist, Bausch and Lomb Optical Co., Rochester, N.Y. 29 Barons St.
Chevalley, Claude. Asst. Prof., Princeton Univ., Princeton, N.J. Fine Hall.
Chittenden, E. W. Prof., State Univ. of Iowa, Iowa City, Iowa. 221 Physics Bldg.
Christie, Dr. **D. E.** Instr., Bowdoin Coll., Brunswick, Me. Searles Physics Lab.
Church, Alonzo. Asso. Prof., Princeton Univ., Princeton, N.J. Fine Hall.
Church, Randolph. Asst. Prof., Postgraduate School, U.S. Naval Acad., Annapolis, Md. 316 N. Glen Ave. *Lt., U.S.N.R., Office of Chief of Naval Operations, U.S. Navy Dept., Washington, D.C.*
Churchill, R. V. Prof., Univ. of Michigan, Ann Arbor, Mich. 924 S. Forest Ave.
§**Civin**, Dr. **Paul.** Instr., Univ. of Michigan, Ann Arbor, Mich.
Clark, C. E. Asst. Prof., Purdue Univ., Lafayette, Ind.
§**Clark, C. L.** Univ. of Virginia, University Station, Charlottesville, Va. Monroe Apts., Brandon Ave.
Clarke, E. H. Prof., Hiram Coll., Hiram, Ohio.
Clarkson, Helen E. Prof., Jacksonville Coll., Jacksonville, Tex. 1806 Emmet St., Omaha, Neb.
Clarkson, J. A. Asst. Prof., Univ. of Pennsylvania, Philadelphia, Pa. College Hall.
Clarkson, J. M. Asso. Prof., North Carolina State Coll., Raleigh, N.C. State College Station.
Claytor, Dr. **W. W. S.** 1710 P St. N.W., Washington, D.C.
Clement, Dr. **Mary Dean.** Instr., Wells Coll., Aurora, N.Y.
§**Clement, P. A.** Asst., Univ. of California at Los Angeles, Los Angeles, Calif. 220 Diamond, Laguna Beach, Calif.
Clements, G. R. Prof., U.S. Naval Acad., Annapolis, Md. 7 Thompson St.
Clifford, A. H. Asst. Prof., Massachusetts Inst. of Tech., Cambridge, Mass. *Lt. (j.g.), U.S.N.R., 3130 Wisconsin Ave. N.W., Washington, D.C.*
§**Clifford, H. E.** Dean, Engineering School, Harvard Univ., Cambridge, Mass. Pierce Hall
Clippinger, Dr. **R. F.** Instr., Carnegie Inst. of Tech., Pittsburgh, Pa. 235 Lehigh St.
Coble, A. B. Prof., Univ. of Illinois, Urbana, Ill. 702 W. Washington Blvd.
Coburn, Dr. **Nathaniel.** Instr., Univ. of Texas, Austin, Tex. 5401 Aurora Dr.
Coddington, E. A. Junior Physicist, Ordnance Lab., Navy Yard, U.S. Navy Dept., Washington, D.C. 808 Portland St. S.E.
Coe, C. J. Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.
Coffin, L. M. Prof., Coe Coll., Cedar Rapids, Iowa. 1027 2d Ave.
Cohen, Abraham. Prof. Emeritus, Johns Hopkins Univ., Baltimore, Md.
§**Cohen**, Dr. **I. S.** Asst., Inst. for Advanced Study, Princeton, N.J.
Cohen, Leo. 1337 Tuckerman St. N.W., Washington, D.C.
Cohen, L. W. Prof., Univ. of Kentucky, Lexington, Ky.
Cohen, Teresa. Asso. Prof., Pennsylvania State Coll., State College, Pa. 315 S. Atherton St.
§**Cohn, Harvey.** Asst., New York Univ., New York, N.Y. 847 E. 172d St., Bronx
§**Cohn, R. M.** 905 West End Ave., New York, N.Y. *Inspector of Radio Equipment, U.S.A., Signal Corps. 5000 Wissahickon Ave., Philadelphia, Pa.*

Coker, E. C. Asst. Prof., Clemson Coll., Clemson, S.C. Box 284.

§Cole, K. S. Asso. Prof., Columbia Univ., New York, N.Y. Inst. for Advanced Study, Princeton, N.J.

Cole, Dr. Nancy. Instr., Sweet Briar Coll., Sweet Briar, Va.

Cole, Dr. R. H. Instr., Univ., of Western Ontario, London, Ont., Canada.

Coleman, A. J. Sec., Student Christian Movement, Univ. of Toronto, Toronto, Ont., Canada Hart House.

Coleman, Dr. J. B. 620 Bull St., Columbia, S.C.

Coleman, W. B. Instr., Agric. and Mech. Coll. of Texas, College Station, Tex. 253 Faculty Exch. *2d Lt, U.S.A., Miami Beach, Fla*

Coler, Dr. M. A. Tech. Dir., Engineering Products Div., Paragon Paint and Varnish Corp., Long Island City, N.Y. 328 W. 86th St., New York, N.Y.

Collier, Myrtie. Prof., Immaculate Heart Coll., Los Angeles, Calif. 1307 S. Beverly Glen Blvd.

Collingwood, Dr. E. F. Trinity Coll., Univ. of Cambridge, Cambridge, England. Lilburn Tower, Alnwick, Northumberland, England.

†Collins, O. C. Asst. Prof. of Astronomy, Univ. of Nebraska, Lincoln, Neb. The Observatory.

Colvin, B. H. Asst., Univ. of Wisconsin, Madison, Wis.

Comegys, Dr. Esther. Instr., Univ. of Maine, Orono, Me. 6 University Pl.

Comenetz, Dr. George. Tutor, St. John's Coll., Annapolis, Md.

§Comer, Louise M. Asst., Barnard Coll., Columbia Univ., New York, N.Y. Milbank Hall.

Comfort, Dr. E. G. H. Instr., Univ. of Arkansas, Fayetteville, Ark. *Capt, U.S.A., Kelly Field, Tex*

Comstock, C. E. Prof. Emeritus, Bradley Polytech. Inst., Peoria, Ill. 203 Fredonia Ave.

Conkwright, N. B. Asst. Prof., State Univ. of Iowa, Iowa City, Iowa. 209-B Physics Bldg.

Constable, Mary L. Philadelphia High School for Girls, Philadelphia, Pa. The Whittier, 140 N. 15th St.

Conwell, Dr. G. M. St. Paul's School, Concord, N.H. 307 Pleasant St.

Conwell, H. H. Dean and Prof., Beloit Coll., Beloit, Wis. 1621 Emerson St.

Cook, A. J. Asso. Prof., Univ. of Alberta, Edmonton, Alta., Canada.

Cook, R. H. Asst. Prof., South Dakota State School of Mines, Rapid City, S.Dak.

Cook, Sister Rose M. Prof., Loretto Heights Coll, Loretto, Colo.

Cooley, H. R. Asst. Prof., Washington Square Coll., New York Univ., New York, N.Y.

†Coolidge, J. L. Prof. Emeritus, Harvard Univ, Cambridge, Mass. 27 Fayerweather St.

§Coolidge, T. J. Vice Pres., The First National Bank of Boston, 67 Milk St, Boston, Mass.

Coombs, W. C. Radio Div, Bureau of Ships, U S. Navy Dept., Washington, D C. 1833 Ingelside Ter. N.W

Cooper, Clara M. St. Vincent's Coll., Shreveport, La.

Cooper, Dr. Elizabeth M. Hunter Coll. High School, New York, N.Y. 201 E. 71st St.

Cope, T. F. Asso. Prof, Queens Coll., Flushing, N.Y. 33-69 167th St.

Copeland, A. H. Asso. Prof., Univ. of Michigan, Ann Arbor, Mich. 616 Oswego St.

†Copeland, Lennie P. Prof., Wellesley Coll, Wellesley, Mass.

Copenhagen, Helen M. Univ. of Washington, Seattle, Wash.

Copson, E. T. Prof., University Coll, St. Andrews Univ., Dundee, Scotland.

Coral, Max. Asst. Prof., Wayne Univ, Detroit, Mich.

Cordrey, W. A. Prof., Southeastern Louisiana Coll., Hammond, La. 3930 Clermont Dr., New Orleans, La.

Corliss, J. J. Prof., De Paul Univ., Chicago, Ill.

Corona, Sister Maria. Dean, Coll. of Mt. St Joseph, Mt. St. Joseph, Ohio.

Corral-Alémán, J. I. Dir., Mountains and Mines of the Republic of Cuba, Havana, Cuba. Calzada esquina 13, Vedado.

Cosby, Dr. Byron. Instr., U. S. Naval Acad., Annapolis, Md.

Cothran, Dr. **J. C.** Chemistry Dept, State Teachers Coll., Duluth, Minn. 512 N. 19th Ave. E.
Courant, Richard. Prof., Washington Square Coll., New York Univ., New York, N.Y. 142 Calton Road, New Rochelle, N.Y.
Court, N. A. Prof., Univ. of Oklahoma, Norman, Okla. 425 W. Eufaula St.
Cowan, G. P. St. John's Univ., Brooklyn, N.Y.
Cowgill, A. P. Asst. Prof., Syracuse Univ., Syracuse, N.Y. 519 Allen St. *Maj., U.S.A., retired*
Cowles, W. H. H. Acting Dir, School of Science and Tech., Pratt Inst., Brooklyn, N.Y. 132 Joralemon St.
Cox, Dr. **L. C.** English Hotel, Indianapolis, Ind.
Cox, Mary J. Univ. of Virginia, University Station, Charlottesville, Va. 21 University Circle.
Coxeter, H. S. M. Asst. Prof., Univ. of Toronto, Toronto, Ont., Canada. 24 Strathearn, Blvd.
Craig, A. T. Asso. Prof., State Univ. of Iowa, Iowa City, Iowa. 119 Physics Bldg.
Craig, C. C. Prof., Univ. of Michigan, Ann Arbor, Mich. 3020 Angell Hall.
Craig, H. V. Prof., Univ. of Texas, Austin, Tex. 1816 W. 36th St.
Cramblet, W. H. Pres., Bethany Coll., Bethany, W.Va.
Cramer, G. F. Asst. Prof., Tulane Univ., New Orleans, La. 1333 Lowerline St. *Lt., U.S.N.R. 3026 Porter St. N.W., Washington, D.C.*
Cramer, R. H. Aerodynamics Section, Curtiss-Wright Corp., Buffalo, N.Y. 500 Grand Ave., Rochester, N.Y.
Cramlet, C. M. Asso. Prof., Univ. of Washington, Seattle, Wash.
§**Crane, C. H.** Pres., St. Joseph Lead Co., 250 Park Ave., New York, N.Y.
Crathorne, A. R. Prof., Univ. of Illinois, Urbana, Ill. 802 Pennsylvania Ave.
Craw, A. R. Instr., De Paul Univ., Chicago, Ill. 5601 Drexel Ave. *Ensign, U.S.N.R.*
Crawford, Jane E. Editorial Asst., American Mathematical Society, 531 W. 116th St., New York, N.Y.
§**Crawford, W. S. H.** Asst., Univ. of Minnesota, Minneapolis, Minn. 103 Folwell Hall.
Creane, Sister James S. Instr., Sacred Heart High School, Indianapolis, Ind. 10 E. Palmer St.
Cromwell, J. W. 1815 13th St., Washington, D.C.
Cronvich, Dr. **L. L.** Asst Stress Analyst, McDonnell Aircraft Corp., St. Louis, Mo. 7023 Stanford Ave.
Crosby, D. R. Box 543, Banff, Alta., Canada. *Flying Officer, R.C.A.F., No. 9 Air Observors School, St. Johns, Quebec, Canada*
Crout, P. D. Asso. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.
Crow, Dr. **E. L.** *Bureau of Ordnance, U.S. Navy Dept., Washington, D.C. 119 5th St., N.E.*
Culver, M. M. Prof., Univ. of Pittsburgh, Pittsburgh, Pa.
Cummings, H. K. Chief, Automotive Power Plants Section, National Bureau of Standards, U. S. Dept. of Commerce, Washington, D.C. 4213 Ellicott St. N.W.
§**Cunningham,** Dr. **A. B.** Instr., DuBois Undergraduate Center, Pennsylvania State College, Dubois, Pa.
§**Curran, Mary T.** Johns Hopkins Univ., Baltimore, Md. 5 E. Preston St.
Currier, A. E. Asst. Prof, U. S. Naval Acad., Annapolis, Md.
Curry, H. B. Prof., Pennsylvania State Coll., State College, Pa. *Asso. Mathematician, Frankford Arsenal, Philadelphia, Pa. 6709 N. 6th St.*
Curtis, H. B. Prof., Lake Forest Coll., Lake Forest, Ill. 11 College Campus.
Curtiss, D. R. Prof., Northwestern Univ, Evanston, Ill.
Curtiss, J. H. Asst. Prof., Cornell Univ., Ithaca, N.Y. White Hall.
Cutler, E. H. Asst. Prof., Lehigh Univ., Bethlehem, Pa.

Dadourian, H. M. Prof., Trinity Coll., Hartford, Conn. 125 Vernon St.
Dalaker, H. H. Prof. Emeritus, Univ. of Minnesota, Minneapolis, Minn. 523 Walnut St. S.E.
†**Dalal, R. D.** Care of Swiss Bank Corp., 11c Regent St., London, S.W. 1, England.
Daly, Dr. **J. F.** Instr., Catholic Univ. of America, Washington, D.C.

Danielson, J. O. 2319 Maryland Ave., Superior, Wis.
Dantzig, G. B. Asso. Statistician, U. S. War Dept., Washington, D.C. 4720 7th Road N., Arlington, Va.
Dantzig, Tobias. Prof., Univ. of Maryland, College Park, Md. University Lane.
Darkow, Marguerite D. Asso. Prof., Hunter Coll., 695 Park Ave., New York, N.Y. 16 E. 82nd St.
Daugherty, R. D. Asst. Prof., Kansas State Coll. of Agric. and Applied Science, Manhattan, Kan. Box 64.
Daum, Dr. **J. A.** Instr., Agric. and Mech. Coll. of Texas, College Station, Tex. 3804 Ames Ave., Omaha, Neb. *2d Lt., U.S.A., Camp Bowie, Tex.*
Daus, P. H. Asso. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.
Davids, Dr. **Norman.** Junior Physicist, U. S. Engineer Div., Cincinnati Testing Lab., Mariemont, Ohio.
Davidson, Anthony. Prof., Wilberforce Univ., Wilberforce, Ohio. 316 E. Church St., Xenia, Ohio.
Davis, D. R. Prof., New Jersey State Teachers Coll., Upper Montclair, N.J.
Davis, H. A. Asso. Prof., West Virginia Univ., Morgantown, W.Va. 307 Duquesne Ave.
†**Davis, H. N.** Pres., Stevens Inst. of Tech., Hoboken, N.J.
Davis, James Edgar. Asso. Prof., Central Y.M.C.A. Coll., 19 S. LaSalle St., Chicago, Ill.
Davis, James Elmer. Prof., Drexel Inst. of Tech., Philadelphia, Pa. *Colonel, U.S.A.*
§**Davis, M. E.** Asso. Actuary, Metropolitan Life Ins. Co., 1 Madison Ave., New York, N.Y.
Davis, W. M. Asst. Prof., Cornell Coll., Mt. Vernon, Iowa. 603 N. 6th St. W.
Davison, Charlotte I. Prof., Wilson Coll., Chambersburg, Pa.
Dawkins, W. S. 3310 Carlisle Ave., Baltimore, Md.
§**Day, A. S.** Asst., Yale Univ., New Haven, Conn. 2723 Yale Station.
Day, Dr. **M. M.** Instr., Univ. of Illinois, Urbana, Ill. 312 E. Healey St., Champaign, Ill.
Dean, Alice C. Fellow and Acting Lib., Rice Inst., Houston, Tex.
Dean, J. P. *Radio Engineer, U.S.A. Signal Corps, Hq. Bldg. 100, Presidio of San Francisco, Calif. 1635 Gough St.*
Dearborn, D. C. Asso. Prof. and Registrar, Catawba Coll., Salisbury, N.C.
Dearman, C. C. Instr., East Central Junior Coll., Decatur, Miss. *Ground School Instr., U. S. Naval Air Station, Pensacola, Fla. 7 S. H St.*
Decherd, Mary E. Asst. Prof., Univ. of Texas, Austin, Tex. 2313 Nueces St.
DeCicco, Dr. **John.** Instr., Illinois Inst. of Tech., Chicago, Ill. 6340 Blackstone Ave.
Deck, L. J. Prof., Muhlenberg Coll., Allentown, Pa. 232 N. 15th St.
Decker, F. F. Prof., Syracuse Univ., Syracuse, N.Y. 312 Marshall St.
DeCleene, Rev. L. A. V. Prof., St. Norbert's Abbey, West De Pere, Wis.
DeCou, E. E. Prof. Emeritus, Univ. of Oregon, Eugene, Ore. 929 Hilyard St.
†**Dederick,** Dr. **L. S.** Prin. Mathematician, Ballistic Research Lab., Aberdeen Proving Ground, Md.
DeFoe, O. K. Prof., St. Louis Coll. of Pharmacy, St. Louis, Mo. 4588 Parkview Pl.
Dehn, Dr. **M. W.** Visiting Lecturer, Illinois Inst. of Tech., Chicago, Ill.
Deimel, R. F. Prof., Stevens Inst. of Tech., Hoboken, N.J.
†**DeLong, I. M.** Prof. Emeritus, Univ. of Colorado, Boulder, Colo. 2009 13th St.
DeLury, Dr. **D. B.** Lecturer, Univ. of Toronto, Toronto, Ont., Canada.
Demers, M. R. *1st Lt., U.S.A., Air Forces, Gravelly Point, Washington, D.C.*
Denbow, Carl. Asst. Prof., Ohio Univ., Athens, Ohio. Valley View Dr.
Dennis, J. J. Prof., Clark Univ., Atlanta, Ga. 1315 Marcy St.
Denton, W. W. Dean, Great Lakes Coll., 4211 3d Ave., Detroit, Mich.
DePrima, C. R. Asst., New York Univ., New York, N.Y. 307 6th Ave., Paterson, N.J.
Derby, G. T. Maj., U.S.A., Corps of Engineers, New Orleans, La. 1403 Calhoun St.
Derry, Douglas. Asst. Prof., Univ. of Saskatchewan, Saskatoon, Sask., Canada. 619 Temperance St.

§**Derthick, P. W.** Asst., Univ. of Kentucky, Lexington, Ky. *Signal Corps, U.S. War Dept., Washington, D.C. 5 Columbia Ave., Takoma Park, D.C.*
Diamond, A. H. Prof., Oklahoma Agric. and Mech. Coll., Stillwater, Okla. *2d Lt., U.S.A., Air Forces, Monroe Tower Hotel, Miami Beach, Fla.*
Dickson, L. E. Prof. Emeritus, Univ. of Chicago, Chicago, Ill.
Dieckmann, H. C. Asst. Prof., Occidental Coll., Los Angeles, Calif.
Dietzold, R. L. Research Engineer, Bell Telephone Labs., 463 West St., New York, N.Y. 34 W. 11th St.
Dilworth, Dr. R. P. Instr., Yale Univ., New Haven, Conn. North Sheffield Hall.
Dimick, C. E. Prof., U.S. Coast Guard Acad., New London, Conn. *Com., U.S. C.G.*
Dimsdale, Dr. Bernard. Instr., Univ. of Idaho, Moscow, Idaho. Hastings, Neb.
Dines, L. L. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa.
Disert, Margaret C. Dean, Wilson Coll., Chambersburg, Pa. *Lt., U.S.N.R., Widener Bldg., Philadelphia, Pa.*
Dix, Dr. C. H. Chief Seismologist, United Geophysical Co. 628 Alameda St., Altadena, Calif.
Dixon, A. L. Prof., Magdalen Coll., Univ. of Oxford, Oxford, England.
Dixon, W. J. Instr., Univ. of Oklahoma, Norman, Okla.
Dobbie, Dr. J. M. Instr., Northwestern Univ., Evanston, Ill.
DoBell, H. A. Prof., New York State Coll. for Teachers, Albany, N.Y.
Dodd, E. L. Prof., Univ. of Texas, Austin, Tex. 3012 West Ave.
Doob, J. L. Asst. Prof., Univ. of Illinois, Urbana, Ill. 4431 Greenwich Pkwy N.W., Washington, D.C.
Doole, H. P. Asst. Prof., Univ. of Nebraska, Lincoln, Neb.
Dorroh, J. L. Asst. Prof., Louisiana State Univ., University, La.
Dorwart, H. L. Asst. Prof., Washington and Jefferson Coll., Washington, Pa. Gabby Heights, R.F.D. 6.
Dostal, B. F. Asst. Prof., Univ. of Florida, Gainesville, Fla. Peabody Hall.
Dotterer, R. H. Prof., Philosophy Dept., Pennsylvania State Coll., State College, Pa. 825 W. Foster Ave.
Douglas, Jesse. Prof., Columbia Univ., New York, N.Y.
Dowker, Dr. C. H. Instr., Johns Hopkins Univ., Baltimore, Md.
Downey, J. F. Consulting Engineer, American Associated Consultants Inc., 250 Park Ave., New York, N.Y. 20 N. Broadway, White Plains, N.Y.
Downing, H. H. Prof., Univ. of Kentucky, Lexington, Ky. 138 State St.
Downs, Dr. T. L. Instr., Trinity Coll., Hartford, Conn. *Lt. (j.g.) U.S.N.R., U.S. Naval Acad., Annapolis, Md.*
§**Doyle, Martina E.** Connecticut General Life Ins. Co., Hartford, Conn. 316 White St.
Doyle, Dr. T. C. Instr., Stanford Univ., Stanford University, Calif.
Doyle, W. C. Asst. Prof., Rockhurst Coll., 5225 Troost Ave., Kansas City, Mo.
Dresch, Dr. F. W. Instr., Univ. of California, Berkeley, Calif. *Lt. (j.g.), U.S.N.R., U.S. Naval Proving Ground, Dahlgren, Va.*
Dresden, Arnold. Prof., Swarthmore Coll., Swarthmore, Pa. 606 Elm Ave.
Dressel, F. G. Asst. Prof., Duke Univ., Durham, N.C. 309 Francis St.
Dressler, R. F. Univ. of Pennsylvania, Philadelphia, Pa. 115 S. 42d St. *Electrical Engineer, Radio Corp. of America, Camden, N.J.*
Dribin, Dr. D. M. *Junior Research Analyst, Office of Chief Signal Officer, U.S. War Dept., Washington, D.C.*
Drum, M. L. Prof. of Surveying, Bucknell Univ., Lewisburg, Pa. 55 S. Water St.
§**Dubisch, Roy.** Instr., Montana State Univ., Missoula, Mont.
Duffin, Dr. R. J. Physicist, Terrestrial Magnetism Dept., 5241 Broad Branch Road N.W., Washington, D.C.
Duncan, Dr. D. C. Instr., Los Angeles City Coll., Los Angeles, Calif.
Dunford, Margaret J. Southeastern High School, Detroit, Mich. 2526 Glendale Ave.
Dunford, Nelson. Asst. Prof., Yale Univ., New Haven, Conn. 215 Norton St.
Dunkel, Otto. Asso. Prof. Emeritus, Washington Univ., St. Louis, Mo.
Durand, Janet C. Prof., Beaver Coll., Jenkintown, Pa. 144 Walnut St.
†**Durand, W. F.** Prof. Emeritus, Stanford Univ., Stanford University, Calif. 540 Aero Lab.

Durell, Dr. Fletcher. Belleplain, N.J.
Duren, W. L. Asso. Prof., Tulane Univ., New Orleans, La.
Durfee, Walter H. Dean, Hobart Coll., Geneva, N.Y. Coxe Hall.
§**Durfee, William H.** Instr., Cornell Univ., Ithaca, N.Y.
Dushnik, Ben. Asst. Prof., Univ. of Michigan, Ann Arbor, Mich. 205 W. Engineering Bldg.
Duthie, Dr. W. D. Instr., Univ. of Michigan, Ann Arbor, Mich. *Lt., U S N.R., Postgraduate School, U.S. Naval Acad., Annapolis, Md.*
§**Dutka, Jacques.** 56 W. 180th St., Bronx, New York, N.Y.
Dwyer, P. S. Asst. Prof., Univ. of Michigan, Ann Arbor, Mich. 2509 James St.
Dye, L. A. Asst. Prof., The Citadel, Charleston, S.C.
Dyer-Bennet, Dr. John. Instr., Vanderbilt Univ., Nashville, Tenn. *U.S.A., Camp Blanding, Fla.*
Dysart, Robert. Certified Public Accountant, 50 State St., Boston, Mass.

Eachus, Dr. J. J. Instr., Purdue Univ., Lafayette, Ind. *Lt. (j.g.), U.S N.*
Eagle, E. L. 1320 Huffman Ave., Dayton, Ohio. *Supervisor of Training, U.S.A., Air Corps, Patterson Field, Air Depot, Fairfield, Ohio.*
Earl, J. M. Prof., Municipal Univ. of Omaha, Omaha, Neb. 528 S. 53d St.
Eaton, Dr. J. E. Instr., Queens Coll., Flushing, N.Y. 43-55 Kissena Blvd.
Eaves, E. D. Asso. Prof., Univ. of Tennessee, Knoxville, Tenn. University Station.
Eberhart, Paul. Asso. Prof., Washburn Municipal Univ. of Topeka, Topeka, Kan.
Eberlein, W. F. Propeller Research Analyst, U. S. Navy Dept., Washington, D.C. 2223 H St. N.W.
Eberman, J. W. 141-72 85th Road, Jamaica, N.Y.
Echols, R. L. Asst. Prof., Postgraduate School, U. S. Naval Acad., Annapolis, Md.
Eckweiler, H. J. Instr., New York Univ., New York, N.Y. 9303 49th Ave., Elmhurst, N.Y.
§**Eddy, R. P.** Rockton, Ill. *U.S.A., Signal Corps, Burr School, 1621 Wabansia, Chicago, Ill.*
Edington, W. E. Prof., DePauw Univ., Greencastle, Ind. E. Franklin St.
Edison, T. M. Pres., Calibron Products Inc., 51 Lakeside Ave., West Orange, N.J.
Efird, Laura C. Hugh Morson High School, Raleigh, N.C. 506 Cutler St.
Eide, Margaret C. (Mrs. R. B.). Asso. Prof., State Teachers Coll., River Falls, Wis. 308 S. 2d St.
Eiesland, J. A. Prof. Emeritus, West Virginia Univ., Morgantown, W.Va. 316 Demain Ave.
Ellenberg, Samuel. Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.
Eisenhart, Churchill. Asst. Prof., Univ. of Wisconsin, Madison, Wis. 1662 Monroe St.
†**Eisenhart, L. P.** Dean, Princeton Univ., Princeton, N.J.
Elder, Alfonso. Dean, Liberal Arts, North Carolina Coll. for Negroes, Durham, N.C.
Elder, C. C. Hydraulic Engineer, Metropolitan Water District of Southern California, 306 W. 3d St., Los Angeles, Calif.
Ellingson, H. E. Luther Coll., Decorah, Iowa.
Elliott, E. R., Asst. Prof., Hampden-Sydney Coll., Hampden-Sydney, Va. 906 Buford St., Corpus Christi, Tex.
Elliott, W. W. Prof., Duke Univ., Durham, N.C. 4721 Duke Station.
Ellis, Dr. G. A. Lawyer, 120 Broadway, New York, N.Y.
Ellis, J. R. Booker T. Washington High School, Tulsa, Okla. 1110 E. Pine St.
§**Ellis, Wade.** 921 Woodlawn, Ann Arbor, Mich.
Elston, J. S. Asst. Actuary, Travelers Ins. Co., Hartford, Conn.
Elveback, Mary L. Hunter Coll., 695 Park Ave., New York, N.Y.
§**Embree, R. S.** Asst., Univ. of Wisconsin, Madison, Wis.
Emch, Arnold. Prof. Emeritus, Univ. of Illinois, Urbana, Ill. 1002 S. Orchard St.
Engstrom, H. T. Asso. Prof., Yale Univ., New Haven, Conn. *Lt. Com., U.S N.R., U.S. Navy Dept., Washington, D.C. 406 N. Thomas St., Arlington, Va.*
Epstein, Dr. Benjamin. Physicist, Ballistic Lab., Frankford Arsenal, Philadelphia, Pa. 4721 Leiper St.
§**Epstein, Dr. Marion G.** (Mrs. Jess). 132 W. Maple Ave., Moorestown, N.J.

Erdős, Dr. Paul. Univ. of Pennsylvania, Philadelphia, Pa. Bennett Hall.
Erickson, Dr. R. W. Instr., Hibbing Junior Coll., Hibbing, Minn.
Erickson, W. S. Instr., St. Olaf Coll., Northfield, Minn.
Erkiletian, D. H. Instr., School of Mines and Metallurgy, Univ. of Missouri, Rolla, Mo.
†**Escott, E. B.** Actuary. 1019 S. East Ave., Oak Park, Ill.
§**Escott, F. H.** R.F.D. 1, Bridgeport, Conn.
Esty, T. C. Prof. Emeritus, Amherst Coll., Amherst, Mass. First National Bank
Ettlinger, H. J. Prof., Univ. of Texas, Austin, Tex. 3110 Harris Park Ave.
†**Evans, G. C.** Prof., Univ. of California, Berkeley, Calif. Wheeler Hall.
Evans, G. W. 16 Beverly Road, Swampscott, Mass.
Evans, H. B. Prof. Emeritus, Univ. of Pennsylvania, Philadelphia, Pa. 88 Merbrook Lane, Merion Station, Pa.
Evans, H. P. Prof., Univ. of Wisconsin, Madison, Wis. North Hall.
Evans, J. O. Consulting Aeronautical Engineer, 558 14th St. S.E., Washington, D.C.
Everett, Dr. C. J. Instr., Univ. of Wisconsin, Madison, Wis. 2554 Kendall Ave.
Everett, H. S. Prof., Univ. of Chicago, Chicago, Ill.
§**Ewing, Frances E.** Asst., Univ. of Illinois, Urbana, Ill. 710 Nevada St.
Ewing, G. M. Asst. Prof., Univ. of Missouri, Columbia, Mo. 213 Engineering Bldg.
Ewing, (W.) M. Asso. Prof. of Geophysics, Lehigh Univ., Bethlehem, Pa *Asso. in Submarine Geology, Woods Hole Oceanographic Inst., Woods Hole, Mass*

Falconer, Dr. B. L. 315 Ward St., Marlin, Tex.
Falvey, Frances E. Ward-Belmont School, Nashville, Tenn.
Fan, Dr. S. C. Care of L. C. Chia, Lane 483-56 Yu Yuan Road, Shanghai, China.
§**Farley, B. G.** Radiation Lab., Massachusetts Inst. of Tech., Cambridge, Mass.
Farnau, E. F. Prof, Chemistry Dept., Univ. of Cincinnati, Cincinnati, Ohio.
Farrell, O. J. Asso. Prof., Union Coll., Schenectady, N.Y. Stone Ridge Road, R.F D. 1.
Faulkner, Donald. Prof., John B. Stetson Univ., De Land, Fla. 335 W. Wisconsin Ave.
Fay, E. A. Central Y.M.C.A., Rochester, N.Y.
Fazel, Dr. C. S. Chief of Research, Nitrogen Div., The Solvay Process Co., Hopewell, Va.
Feather, E. H. Capt., U.S.A., retired. Prof., Valley Forge Military Junior Coll, Wayne, Pa. 526 W. Beechtree Lane.
Fejér, Lipót. Prof., Univ. of Budapest, Budapest, Hungary. Mathematical Seminary VIII, Muzeum-Körut 6-8.
Feld, Dr. J. M. Instr., Queens Coll., Flushing, N.Y. Care of Miss Helen Kogut, 751 St. Marks Ave, Brooklyn, N.Y. *U.S.A.*
Feller, W. K. Asso. Prof., Brown Univ., Providence, R.I.
§**Fenichel, S. A.** 301 S. Illinois St., Belleville, Ill.
Fenn, L. H. Prof., Polytech. Inst. of Brooklyn, Brooklyn, N.Y.
§**Ferguson, W. A.** Roanoke, Mo. *Lt., U.S.A., Engineer Replacement Training Center, Camp Claiborne, La.*
Ferguson, W. E. Instr., Univ. of Missouri, Columbia, Mo 511 S. 6th St.
Fernbach, Sidney. Junior Physicist, Ballistic Lab., Frankford Arsenal, Philadelphia, Pa. 1011 N. 6th St.
†**Ferry, F. C.** Pres. Emeritus, Hamilton Coll., Clinton, N.Y. 324 Hart St., New Britain, Conn.
§**Fialkow, Dr. A. D.** Lecturer, School of Pharmacy, Columbia Univ., New York, N.Y. 1755 Weeks Ave., Bronx. *Instr., Defense Training Inst., 375 Pearl St., Brooklyn, N.Y.*
Ficken, F. A. Prof., Univ. of Tennessee, Knoxville, Tenn.
Field, Peter. Prof., Univ. of Michigan, Ann Arbor, Mich. 904 Olivia Ave.
Finan, E. J. Asso. Prof., Catholic Univ. of America, Washington, D.C. 604 Girard St. N.E.
§**Finch, J. V.** 1226 Sweet Briar Road, Shorewood Hills, Madison, Wis.

Findlay, William. Prof., McMaster Univ., Hamilton, Ont., Canada. 131 Chedoke Ave.
§**Fine, N. J.** Instr., Purdue Univ., Lafayette, Ind.
§**Finkel, B. F.** Prof. Emeritus, Drury Coll., Springfield, Mo. N. Clay Ave.
Fischer, C. H. Asst. Prof., Univ. of Michigan, Ann Arbor, Mich. 3016 Angell Hall.
Fisher, Dr. F. G. Instr., Univ. of Santa Clara, Santa Clara, Calif.
Fisher, H. A. Prof., North Carolina State Coll., Raleigh, N.C. State College Station.
†**Fiske, T. S.** Prof. Emeritus, Columbia Univ., New York, N.Y. 2 Oak Crescent, Poughkeepsie, N.Y.
†**Fite, Dr. W. B.** Retired. Columbia Univ., New York, N.Y.
Fithian, J. H. Prof., Newark Coll. of Engineering, Newark, N.J. 157 Halsted St., East Orange, N.J.
Fitterer, J. C. Prof., Colorado School of Mines, Golden, Colo. 1620 Maple St.
Fitzpatrick, J. D. Instr., Creighton Univ., Omaha, Neb. 2102 W. Wisconsin Ave., Milwaukee, Wis.
Flanders, D. A. Asso. Prof., New York Univ., University Heights, New York, N.Y. Bedford Road, Chappaqua, N.Y.
Fleiger, A. G. Research Engineer, Detroit Edison Co., Detroit, Mich. 13193 Wisconsin St.
Flexner, W. W. Asso. Prof., Cornell Univ., Ithaca, N.Y. 109 Iroquois Road.
Flood, Dr. M. M. Research Asso., Princeton Univ., Princeton, N.J. 20 Nassau St. *Official Investigator, National Defense Research Com*
Focke, T. M. Dean, Case School of Applied Science, Cleveland, Ohio. 10900 Euclid Ave.
Folley, K. W. Asso. Prof., Wayne Univ., Detroit, Mich.
§**Folk, C. H.** Asst. Actuary, Western and Southern Life Ins. Co., 4th and Broadway, Cincinnati, Ohio.
Foraker, F. A. Prof., Univ. of Pittsburgh, Pittsburgh, Pa. 1313 Macon Ave.
Ford, L. R. Prof., Illinois Inst. of Tech , Chicago, Ill.
†**Ford, Dr. W. B.** Retired. Hayts Corners, N.Y.
Forder, H. G. Prof., University Coll., Auckland, New Zealand.
Forrest, J. H. Prof , Langston Univ., Langston, Okla. Box 185.
Forsyth, C. H. Prof., Dartmouth Coll., Hanover, N.H.
§**Forsythe, Alexandra I.** (Mrs. G. E.). Research Dept., Douglas Aircraft Co., El Segunda, Calif. 555 Kelton Ave., West Los Angeles, Calif.
Forsythe, Dr. G. E. 1819 Hill St., Ann Arbor, Mich. *U S A., Air Corps.*
Fort, Tomlinson. Dean, Graduate School, Lehigh Univ., Bethlehem, Pa.
Foster, A. L. Asst. Prof., Univ. of California, Berkeley, Calif.
Foster, M. C. Prof., Wesleyan Univ., Middletown, Conn. 30 Gordon Pl.
Foster, R. M. Bell Telephone Labs., 463 West St., New York, N.Y. 122 E. Dudley Ave., Westfield, N.J.
Fowler, F. H. Engineer, Propeller Div., Curtiss-Wright Corp., Caldwell, N.J.
Fowler, Sister M. Charlotte. Nazareth Coll , Louisville, Ky.
Fox, A. H. Asst. Prof., Union Coll., Schenectady, N.Y. 1101 Millington Road.
Fox, Mother Ethelyn. Prof., Barat Coll. of the Sacred Heart, Lake Forest, Ill.
Fox, Dr. R. H. Asso , Univ. of Illinois, Urbana, Ill. 507 S. McCullough St.
Foyle, Grace A. (Mrs. W. R.). 21-45 77th St., Jackson Heights, N.Y.
Frame, J. S. Asso. Prof., Allegheny Coll., Meadville, Pa. 351 Woodland Pl
Frankel, E. T. Asso. Statistician, New York State Dept. of Social Welfare, Albany, N.Y. 491 State St.
§**Frankel, S. P.** Univ. of California, Berkeley, Calif.
Franklin, Constance W. (Mrs. Philip). 312 Pleasant St., Belmont, Mass.
Franklin, Philip. Prof., Massachusetts Inst. of Tech., Cambridge, Mass. 312 Pleasant St., Belmont, Mass.
Fraser, Peter. Prof., Univ. of Bristol, Bristol, England.
§**Fraser, Dr. W. C. G.** Asst Gauge Examiner, Gauge Testing Lab , National Research Council, Ottawa, Ont., Canada.
Fréchet, Maurice. Prof., Inst. Henri Poincaré, Univ. of Paris, Paris, France.
Frecheville, George. Agric. Economics Research Inst., Univ. of Oxford, Oxford, England. 56 Holywell St.

French, A. T. Instr., State Teachers Coll., Winona, Minn.
Frey, E. U. 2962 Decatur Ave., Bronx, New York, N.Y.
Fried, Dr. **Hans.** Research Asst., Sproul Observ., Swarthmore Coll., Swarthmore, Pa. 361 Lancaster Ave., Haverford, Pa.
Friedman, Dr. **Bernard.** Instr., Wilson Junior Coll., Chicago, Ill. 5454 Cornell Ave.
§**Friedman, Morris.** Instr., Wayne Univ., Detroit, Mich. 3764 Clements Ave.
Friedrichs, K. O. Asso. Prof., New York Univ., New York, N.Y. 39 Glenorchy Road, New Rochelle, N.Y.
Frink, Dr. **Aline H.** (Mrs. Orrin). Instr., Pennsylvania State Coll., State College, Pa. 706 Sunset Road.
Frink, Orrin. Prof., Pennsylvania State Coll., State College, Pa. 706 Sunset Road.
Fry, Dr. **T. C.** Mathematical Research Dir., Bell Telephone Labs., 463 West St., New York, N.Y.
Fry, W. J. Physicist, Sound Div., Research Lab., U. S. Navy Dept., Washington, D.C.
§**Fubini,** Dr. **Guido.** Inst. for Advanced Study, Princeton, N.J.; Lecturer, New York Univ, New York, N.Y. 88 Central Park W., New York, N.Y.
Fudge, Dr. **Helen G.** Holmes Junior High School, Philadelphia, Pa. Rosemont, Pa.
Fuller, Dr. **D. L.** Physical Chemist, Shell Development Co., Emeryville, Calif.
§**Fullerton, R. E.** Asst., Yale Univ., New Haven, Conn. 2727 Yale Station.
Fulton, D. G. Asst. Prof., Ohio Northern Univ., Ada, Ohio. 805 S. Main St.

Gaba, M. G. Prof., Univ. of Nebraska, Lincoln, Neb. 2755 Rathbone Road.
Gage, W. H. Asso. Prof., Univ. of British Columbia, Vancouver, B.C., Canada.
Gaines, R. E. Prof., Univ. of Richmond, University of Richmond, Va.
Galbraith, A. S. Asst. Prof., Colby Coll, Waterville, Me. 47 Redington St.
Gale, A. S. Dean Emeritus and Prof., Univ. of Rochester, Rochester, N.Y. River Campus.
Garabedian, C. A. Prof., Wheaton Coll. Norton, Mass.
Garabedian, H. A. Asst. Actuary, John Hancock Mutual Life Ins. Co., 197 Clarendon St., Boston, Mass.
Garabedian, H. L. Asso. Prof., Northwestern Univ., Evanston, Ill.
García, Godofredo. Dean, Univ. of San Marcos, Lima, Peru. Apartado 1979, Casa San Miguel, Bolivar, 280.
§**García, Mariano.** Fellow, Univ. of Virginia, University Station, Charlottesville, Va. Box 1411
Gardner, R. S. 711 W. High St., Bryan, Ohio. *U.S.A., Air Forces, Wendover Field, Utah.*
§**Garfin,** Dr. **Louis.** Instr., U.S.A., Air Corps Tech. School, Scott Field, Ill. 525 E. Washington, Belleville, Ill.
Garis, C. F. F. Dean and Prof, Union Coll., Schenectady, N.Y.
Garrahan, Sister M. Immaculata. Instr., Coll Misericordia, Dallas, Pa.
Garrison, Dr. **G. N.** Instr., Coll. of the City of New York, 139th St. and Convent Ave., New York, N.Y.
Garvin, Sister M. Cleophas. Prof., Science Dept., Notre Dame Coll., South Euclid, Ohio.
Gaskell, Dr. **R. E.** Fellow, School of Mech., Brown Univ., Providence, R.I. 61 Barnes St.
Gaspar, F. L. Dir., Inst. of Applied Math., and Prof., Univ. of Rosario, Rosario, Argentina. Calle 25 de Diciembre 1433.
§**Gassensmith,** Rev. **F. M.** Asso. Prof, Univ. of Notre Dame, Notre Dame, Ind.
Gatewood, B. E. Asst. Prof., Louisiana Polytech. Inst., Ruston, La. *Stress Engineer, McDonnell Aircraft Corp., St. Louis, Mo. 9038 McNulty Dr., Overland, St Louis, Mo.*
Gay, H. J. Prof, Worcester Polytech. Inst., Worcester, Mass. 7 Belvidere Ave.
Gehman, H. M. Prof., Univ. of Buffalo, Buffalo, N.Y. 163 Winspear Ave.
§**Geiringer,** Dr. **Hilda P.** Visiting Lecturer, Bryn Mawr Coll., Bryn Mawr, Pa.
Gelbart, Dr. **Abe.** Instr., North Carolina State Coll., Raleigh, N.C. School of Mech., Brown Univ., Providence, R.I.

Gentry, F. C. Asso. Prof., Louisiana Polytech. Inst., Ruston, La. 2644 Dwight Way, Berkeley, Calif. *Asso. Instr, U.S.N. Pre-Flight School, St. Mary's College, Calif.*

George, Dr. T. S. Instr., Univ. of Florida, Gainesville, Fla. *Lt. (j.g.), U.S.N.R. U.S.S. Sangamon, Care of Postmaster, New York, N.Y.*

Gere, Dr. B. H. Arnold, Md. *Lt (j.g.), U.S.N.R., Postgraduate School, U. S. Naval Acad., Annapolis, Md*

Gergen, J. J. Prof., Duke Univ., Durham, N.C. 4771 Duke Station.

Gerhardt, Henry. Merchant. 8 N Royal St., Mobile, Ala.

Gerst, Rev. F. J. Prof., Loyola Univ., Chicago, Ill.

Getchell, B. C. Asst. Prof., Butler Univ, Indianapolis, Ind.

Ghent, K. S. Asst. Prof, Univ. of Oregon, Eugene, Ore.

Gibbens, Gladys. Asst. Prof., Univ. of Minnesota, Minneapolis, Minn. 121 Folwell Hall.

Gibson, J. L. Prof. Emeritus, Univ. of Utah, Salt Lake City, Utah. 1337 Harrison Ave.

§**Gibson, R. W.** Asst., Univ. of Illinois, Urbana, Ill.

Giddings, H. A. Asso. Prof., Univ. of New Hampshire, Durham, N.H.

Giese, Dr. J. H. Instr., Purdue Univ., Lafayette, Ind

Gilbarg, Dr. David. Asst. Physicist, Ordnance Lab., Navy Yard, U. S. Navy Dept., Washington, D.C. 2701 14th St. N.W.

Gilbert, Dr. P. W. Instr., Texas Tech. Coll., Lubbock, Tex.

Gilchrist, Lachlan. Asso. Prof., Physics Dept., Univ. of Toronto, Toronto, Ont., Canada.

Gill, B. P. Prof, Coll. of the City of New York, 139th St. and Convent Ave., New York, N.Y. 493 Warwick Ave., West Englewood, N.J.

Gillam, Dr. B. E. Instr., Univ. of Missouri, Columbia, Mo. 208 Engineering Bldg.

Gillespie, William. Prof. Emeritus, Princeton Univ., Princeton, N.J. Fine Hall.

Gillis, Dr. Joseph. Lecturer, Queen's Univ., Belfast, Northern Ireland.

§**Gillman, Leonard.** Fellow, Columbia Univ., New York, N.Y. 549 Riverside Dr.

Gilman, R. E. Asso. Prof., Brown Univ., Providence, R.I.

§**Gingerich, H. F.** Asst., Univ. of Illinois, Urbana, Ill. 160 Math. Bldg.

Gingery, W. G. Prin., George Washington High School, Indianapolis, Ind. 2215 W. Washington St.

§**Ginsberg, N. M.** Fellow, Univ. of Illinois, Urbana, Ill. 306 E. Healey St., Champaign, Ill.

Ginsburg, Jekuthiel. Prof., Yeshiva Coll., New York, N.Y. 610 W. 139th St.

Girshick, M. A. Senior Agric. Statistician, Div. of Statistical and Historical Research, Bureau of Agric. Economics, U. S. Dept. of Agric., Washington, D.C.

Givens, Dr. J. W. Instr., Northwestern Univ, Evanston, Ill.

Glass, T. F. Instr., Marion Inst, Marion, Ala.

Glazier, Harriet E. Asst. Prof. Emeritus, Univ. of California at Los Angeles, Los Angeles, Calif. 1307 Lucile Ave.

§**Gleason, A. M.** 21 Howe Pl., Bronxville, N.Y. *Ensign, U.S.N.R., 2017 Eye St., Washington, D.C. Arts Club.*

Gleason, R. E. Prof., Temple Univ., Philadelphia, Pa. 2022 N. Park Ave.

Glenn, W. H. Instr., Pasadena Junior Coll., Pasadena, Calif. 1425 Beech St., South Pasadena, Calif.

Gleyzal, André. Prof., St. Michael's Coll., Winooski Park, Vt.

Glover, R. E. Senior Engineer, Bureau of Reclamation, U. S. Dept. of Interior, Denver, Colo.

Godfrey, E. L. Instr., Fenn Coll., Cleveland, Ohio.

Gödel, Dr. Kurt. Inst. for Advanced Study, Princeton, N.J.

Goff, J. A. Dean, Towne Scientific School, Univ. of Pennsylvania, Philadelphia, Pa 109 Engineering Bldg.

Goheen, Dr. H. E. *Ensign, U S N.R, Midshipmen's School, Northwestern Univ., Chicago, Ill.*

Gokhale, V. D. Prof., Univ. of the Philippines, Manila, Philippine Islands.

Gold, J. S. Asso. Prof., Bucknell Univ., Lewisburg, Pa. 306 S. 3d St.

Goldberg, Michael. Engineer, Bureau of Ordnance, U S. Navy Dept., Washington, D.C. 5823 Potomac Ave. N.W.

Goldstine, Dr. H. H. Instr., Univ. of Michigan, Ann Arbor, Mich. *1st Lt, U.S.A, Air Forces.*
Goldsworthy, E. C. Asst. Prof., Univ. of California, Berkeley, Calif. 201 Admin. Bldg.
Golomb, Dr. Michael. Instr., Purdue Univ., Lafayette, Ind.
Gonzalez, J. M. Asst. Prof., Spring Hill Coll., Spring Hill, Ala.
González-Dominguez, Dr. Alberto. Univ. of Buenos Aires, Buenos Aires, Argentina. Perú 222.
González-Rodriguez, M. O. Asst. Prof., Univ. of Havana, Havana, Cuba. Faculty of Sciences.
Good, R. A. Asst., Univ. of Wisconsin, Madison, Wis. North Hall.
Goodier, J. N. Prof., Sibley School of Mechanical Engineering, Cornell Univ., Ithaca, N.Y. 109 Cayuga Heights Rd.
Gordon, W. O. Asst. Prof., Pennsylvania State Coll., State College, Pa.
Gore, G. D. Prof., Central Y.M.C.A. Coll., 19 S. LaSalle St., Chicago, Ill. 1707 Burlington Ave., Downers Grove, Ill.
§**Gorn, Dr. Saul.** 2390 Tiebout Ave., Bronx, New York, N.Y. *U.S.A., Air Forces, Tampa, Fla.*
Gottschalk, W. H. Asst., Univ. of Virginia, University Station, Charlottesville, Va. 1229 Wertland St.
Gough, Sister M. de Lellis. Prof., Incarnate Word Coll., San Antonio, Tex.
†**Gould, Alice B.** Care of Samuel Vaughan, 35 Congress St., Boston, Mass.
Gould, S. H. Asst. Prof., Victoria Coll., Univ of Toronto, Toronto, Ont., Canada.
Gourin, Dr. Eli. Instr., Univ. Extension, Columbia Univ., New York, N.Y. 33 E. 208th St., Bronx.
Gouwens, Cornelius. Asso. Prof., Iowa State Coll., Ames, Iowa.
Graber, M. E. Prof., Morningside Coll., Sioux City, Iowa.
Grable, E. S. Instr., Univ. of Richmond, University of Richmond, Va.
Graesser, R. F. Prof., Univ. of Arizona, Tucson, Ariz. 1648 E. 5th St.
Graham, P. H. Asso. Dean and Prof., Washington Square Coll., New York Univ., New York, N.Y.
†**Grant, Alice A.** Ontario Coll. of Education, Toronto, Ont., Canada. 215 College St.
Grant, Dr. Anna M. C. 185 N. Village Ave., Rockville Centre, N.Y.
Grant, H. S. Asst. Prof., Rutgers Univ., New Brunswick, N.J.
§**Grau, A. A.** Fellow, Univ. of Michigan, Ann Arbor, Mich. 713 North Ave. N.E., Grand Rapids, Mich.
Graustein, Dr. Mary C. (Mrs. W. C.). Radcliffe Coll., Cambridge, Mass. 32 Shepard St.
Graves, C. H. Asst. Prof., Pennsylvania State Coll., State College, Pa. *Asso Educational Statistician, Federal Security Agency, U.S. Office of Education, Washington, D.C. 6201 Broad St., Brookmont, Md*
Graves, G. H. Asso. Prof., Purdue Univ., Lafayette, Ind. 227 S. Grant, West Lafayette, Ind.
Graves, L. M. Prof., Univ. of Chicago, Chicago, Ill. Eckhart Hall.
Graves, W. L. Asst. Prof., Drury Coll, Springfield, Mo.
Gray, Dr. Marion C. Tech. Research, Bell Telephone Labs., 463 West St., New York, N.Y.
Greeley, J. B. Instr., Polytech. Inst. of Brooklyn, Brooklyn, N.Y.
Green, Sister Agnes A. Instr., Immaculate Heart Coll., Los Angeles, Calif. 5515 Franklin Ave.
Green, Dr. F. W. Vice Pres., St. Louis Southwestern Railway Lines, St. Louis, Mo. 6203 Washington Ave.
Green, J. W. Asst. Prof., Univ. of Rochester, Rochester, N.Y. River Campus.
Green, Dr. L. J. Instr., Case School of Applied Science, Cleveland, Ohio.
§**Greenberg, H. J.** Fellow, School of Mech., Brown Univ., Providence, R.I.
§**Greenspan, Bernard.** Instr., Brooklyn Coll., Brooklyn, N.Y. 2438 E. 26th St.
Greenwald, Lewis. Junior Physicist, U.S.A., Signal Corps, Radar Lab., Rumson, N.J. Box 199 C, R.F.D. 2, Neptune, N.J.
Greenwood, Dr. R. E. Instr., Univ. of Texas, Austin, Tex. *U.S. Navy Dept., Washington, D.C. 1374 Harvard St. N.W.*

Greitzer, S. L. 1535 Undercliff Ave., Bronx, New York, N.Y.
Grennan, Dr. Elizabeth B. (Mrs. John). 719 S. 7th St., Ann Arbor, Mich.
Greville, Dr. T. N. E. Actuarial Mathematician, Bureau of the Census, U. S. Dept. of Commerce, Washington, D.C. 1714 34th St. N.W.
Griffin, F. L. Prof., Reed Coll., Portland, Ore.
Griffin, Harriet. Asst. Prof., Brooklyn Coll., Brooklyn, N.Y. 1575 Brooklyn Ave.
Griffiths, Lois W. Asso. Prof., Northwestern Univ., Evanston, Ill.
†**Groat, B. F.** Consulting Engineer. 6733 Emlen St., Philadelphia, Pa.
Gross, Dr. G. L. Asst. Prof., Agric. and Mech. Coll. of Texas, College Station, Tex. 13 Faculty Exch.
Grossman, E. A. Actuarial Dept., U.S. Life Ins. Co., 101 5th Ave., New York, N.Y. *U S A, Aberdeen Proving Ground, Md.*
§**Groth, A. O.** Asst. Actuary, Equitable Life Ins. Co. of Iowa, Des Moines, Iowa.
Grove, C. C. Asst. Prof., Coll. of the City of New York, 17 Lexington Ave., New York, N.Y. 143 Milburn Ave., Baldwin, N.Y.
Grove, V. G. Prof., Michigan State Coll., East Lansing, Mich. 438 Rosewood Ave.
§**Guenther, Dr. P. E.** Instr., Case School of Applied Science, Cleveland, Ohio.
Guggenbühl, Laura. Asst. Prof., Hunter Coll., Bedford Park Blvd. and Navy Ave., New York, N.Y.
Guinand, Dr. A. P. Care of J. F. Turnbull, Bramshott Farm, Orillia, Ont., Canada. *Flight Lt., R.C.A.F., Chatham, N B., Canada.*
Gumbel, E. J. Visiting Prof., New School for Social Research, 66 W. 12th St., New York, N.Y. 3820 Waldo Ave.
†**Gummere, H. V.** Dir., Strawbridge Memorial Observ., Haverford Coll., Haverford, Pa.
Gunder, D. F. Asso. Prof., Colorado State Coll. of Agric. and Mech. Arts, Fort Collins, Colo.
§**Gunderson, N. G.** Instr., Cornell Univ., Ithaca, N.Y. White Hall.
Gurney, Dr. Margaret. Asso. Economist, Bureau of the Budget, Washington, D.C. 6102 Lombard St., Cheverly, Hyattsville, Md.
Gutzman, Dr. W. W. Instr., Hunter Coll., 695 Park Ave., New York, N.Y.

Hackney, Lilian. Retired. 89 Kingwood St., Morgantown, W.Va.
§**Hadamard, Jacques.** Prof. Emeritus, Coll. of France and Polytech. School, Paris, France; Visiting Prof., Columbia Univ., New York, N.Y. 523 W. 112th St., New York, N.Y.
Hadley, Laurence. Prof., Purdue Univ., Lafayette, Ind.
Hagen, Beatrice L. Asst. Prof., Pennsylvania State Coll., State College, Pa.
Haight, N. L. No address.
§**Hailperin, Theodore.** Instr., Cornell Univ., Ithaca, N.Y. White Hall.
Halbert, K. W. Statistician, American Telephone and Telegraph Co., 195 Broadway, New York, N.Y.
§**Halfar, Edwin.** Asst., State Univ. of Iowa, Iowa City, Iowa. 316A Physics Bldg.
Hall, Dr. D. W. Instr., Brown Univ., Providence, R.I.
Hall, F. C. Instr., Evening Session, Hunter Coll., New York, N.Y. 430 W. 119th St.
Hall, Marshall. Asst. Prof., Yale Univ., New Haven, Conn. 1805 Silliman Coll.
Hall, Dr. N. A. Asst. Admin. Engineer, Vought-Sikorsky Aircraft, Stratford, Conn.
Hall, Philip. Lecturer, King's Coll., Univ. of Cambridge, Cambridge, England.
§**Hallam, Sarah M.** Univ. of California, Berkeley, Calif.
Hallett, Dr. W. N. Instr., Maryland State Teachers Coll., Frostburg, Md. 31 Park Ave.
Halmos, Dr. P. R. Asso., Univ. of Illinois, Urbana, Ill. 909 W. California St.
Halperin, Israel. Asst. Prof., Queen's Univ., Kingston, Ont., Canada.
Haltiner, G. J. *Ensign, U.S.N.R., Inst. of Meteorology, Univ. of Chicago, Chicago, Ill.*
Hamilton, H. J. Asst Prof., Pomona Coll., Claremont, Calif. 451 W. 7th.
Hamilton, J. A. Actuarial Div., Metropolitan Life Ins. Co., 1 Madison Ave., New York, N.Y.
Hamilton, O. H. Asso. Prof., Oklahoma Agric. and Mech. Coll., Stillwater, Okla. 503 Pine St. *Lt., U.S.N.R.*

Hammer, Dr. Hans-Karl. Research Lab., Texas Co., Beacon, N.Y. *Instr. in German, U S. Military Acad., West Point, N.Y.*
Hammer, Dr. P. C. Instr., Oregon State Coll., Corvallis, Ore.
§**Hamming, Dr. R. W.** Instr., Univ. of Illinois, Urbana, Ill. 359 Math. Bldg.
Hammond, E. S. Dir. of Admissions and Prof., Bowdoin Coll., Brunswick, Me. 9 Thompson St.
Hancock, Harris. Prof. Emeritus, Univ. of Cincinnati, Cincinnati, Ohio. University Station, Charlottesville, Va.
Hansman, Dr. Margaret M. Instr., Colorado Coll., Colorado Springs, Colo. 2038 Armstrong Ave.
Hanson, E. H. Prof., North Texas State Teachers Coll., Denton, Tex. 619 W. Oak St.
Hanson, W. T. Asst. Prin., Tech. High School, Atlanta, Ga. R.F D 1, Smyrna, Ga.
Hardy, G. H. Prof., Trinity Coll., Univ. of Cambridge, Cambridge, England.
†**Hardy, Dr. J. G.** Retired. Box 32, Williamstown, Mass.
Harkin, D. C. 3000 Crest Ave., Cheverly, Hyattsville, Md.
Harrington, C. H. Instr., Glendale Junior Coll., Glendale, Calif. 1330 Hillside Dr.
Harrington, Dr. W. J. Instr., Pennsylvania State Coll., State College, Pa.
Harris, V. C. 2517 Greenwood, Wilmette, Ill. *Ensign, U S.N R., Great Lakes, Ill.*
§**Harrison, Gerald.** Fellow, California Inst. of Tech., Pasadena, Calif.
Harrison, M. G. Math. Consultant, Aerodynamics Group, Republic Aviation Corp., Farmingdale, N.Y. 94 Bayview, Port Washington, N Y.
Harrold, O. G. Asst. Prof., Louisiana State Univ., University, La
Harshbarger, Frances. Asso. Prof, Kent State Univ., Kent, Ohio.
Hart, W. L. Prof., Univ. of Minnesota, Minneapolis, Minn. 1955 E. River Ter.
Hart, W. W. Retired. 139 Tudor Pl, Kenilworth, Ill.
Harter, G. A. Prof. Emeritus, Univ. of Delaware, Newark, Del. 4577 Boston Post Road, Pelham, Manor, N.Y.
Hartman, Dr. Philip. Instr, Queens Coll., Flushing, N.Y. *1st Lt, A.U.S., Field Artillery School, Fort Sill, Okla.*
Hartnell, George. Retired Wyoming, N.Y.
Hartung, M. L. Asst. Prof., Univ. of Chicago, Chicago, Ill. Graduate Education Bldg.
Hartzler, H. H. Asso. Prof., Goshen Coll., Goshen, Ind.
Harvey, G. G. Asst. Prof, Physics Dept., Massachusetts Inst. of Tech., Cambridge, Mass.
Harvey, W. R. Tech. Staff, Bell Telephone Labs, 463 West St., New York, N.Y.
Haskins, C. N. Prof., Dartmouth Coll., Hanover, N.H. Hanover Road, Lebanon, N.H.
Haskins, E. E. Asso. Prof., Northeastern Univ., Boston, Mass. 213 Jackson St., Newton Centre, Mass.
Hassler, J. O. Prof., Univ. of Oklahoma, Norman, Okla. 425 Lahoma Ave.
Hastings, Cecil. Shelton Hotel, Lexington Ave. and 49th St, New York, N.Y.
Hatcher, T. W. Prof., Virginia Polytech Inst., Blacksburg, Va.
§**Hattan, Corinne.** Asst., Univ. of Illinois, Urbana, Ill. 153 Math. Bldg.
Hausman, B. A. Prof., Univ. of Detroit, Detroit, Mich
Haviland, E. K. Prof., Lincoln Univ., Lincoln University, Pa. Box 382, Oxford, Pa.
Hawkes, H. E. Dean, Columbia Coll., Columbia Univ, New York, N.Y.
Hawkins, Ernest. Asso. Prof., U. S. Naval Acad., Annapolis, Md 4 Steele Ave
Hay, G. E. Asst. Prof, Univ. of Michigan, Ann Arbor, Mich.
Hayden, Camilla. Prof., St. Mary-of-the-Woods Coll., St. Mary-of-the-Woods, Ind.
Hayes, Dr. C. A. U.S.A. Engineers, 74 New Montgomery St., San Francisco, Calif. 1417 Dolores St.
Hayes, J. J. Instr., Univ. of Utah, Salt Lake City, Utah.
Haynes, Dr. Nola A. (Mrs. E. S.). 1408 Rosemary Lane, Columbia, Mo.
§**Hayward, Nathan.** Engineer, 12 S. 12th St., Philadelphia, Pa.
Hazard, C. T. Prof., Purdue Univ., Lafayette, Ind.
Hazard, Dr. Katherine E. Instr., Wellesley Coll., Wellesley, Mass.
Hazeltine, (L.) A. Prof., Physics Dept., Stevens Inst. of Tech., Hoboken, N.J.

†**Hazlett, Olive C.** Asso. Prof., Univ. of Illinois, Urbana, Ill. 574 Station A, Champaign, Ill.
H'Doubler, Dr. F. T. R.F.D. 7, Springfield, Mo.
Heaslet, M. A. Asso. Prof., San Jose State Coll., San Jose, Calif.
Hebbert, Dr. C. M. Tech. Staff, Bell Telephone Labs., 463 West St., New York, N.Y.
Hedberg, E. A. Asso. Prof., Baylor Univ., Waco, Tex. *Asso. Instr., U.S.N. Pre-Flight School, Athens, Ga. 180 Milledge Circle.*
Hedge, L. B. Asst. Prof., The Citadel, Charleston, S.C. *Capt., U.S.A., Ordnance Dept., Kansas Ordnance Plant, Parsons, Kan.*
Hedlund, G. A. Prof., Univ. of Virginia, University Station, Charlottesville, Va. 1834 Fendall Ave.
Hedrick, Dr. E. R. Retired. Univ. of California at Los Angeles, Los Angeles, Calif.
Hefner, R. A. Prof., Georgia School of Tech., Atlanta, Ga. Faculty Exch.
Heins, A. E. Asst. Prof., Purdue Univ., Lafayette, Ind. *Radiation Lab., Massachusetts Inst. of Tech., Cambridge, Mass.*
§**Heins, M. H.** Asst. Prof., Illinois Inst. of Tech., Chicago, Ill.
§**Heller, G. S.** Radiation Lab., Massachusetts Inst. of Tech., Cambridge, Mass.
Hellinger, Dr. E. D. Lecturer, Northwestern Univ., Evanston, Ill. 2215 Maple Ave.
Helly, Dr. Edward. Instr., Monmouth Junior Coll., Long Branch, N.J. 369 Broadway.
Helmer, Dr. Olaf. Instr., Coll. of the City of New York, New York, N.Y. 496 Hudson St.
§**Helton, F. F.** Asst., Univ. of Illinois, Urbana, Ill. 153 Math. Bldg.
§**Hemmingsen, Erik.** Asst. Instr., Univ. of Pennsylvania, Philadelphia, Pa. 125 S. 37th St. *Junior Physicist, Navy Yard, U. S. Navy Dept., Philadelphia, Pa.*
Hendrickson, Dr. Morris. Instr., Bowling Green State Univ., Bowling Green, Ohio.
Hennel, Cora B. Prof., Indiana Univ., Bloomington, Ind. 410 S. Park Ave.
§**Herman, J. R.** Asso. Actuary, Metropolitan Life Ins. Co., 1 Madison Ave., New York, N.Y.
Herpel, Coleman. Asst. Prof., Undergraduate Center, Pennsylvania State Coll., Hazleton, Pa.
Herr, Gertrude A. Asso. Prof., Iowa State Coll., Ames, Iowa. 2404½ Knapp St.
Herriot, Dr. J. G. Instr., Stanford Univ., Stanford University, Calif.
Herschfeld, Aaron. Junior Actuarial Mathematician, Office of the Actuary, U. S. Social Security Board, Washington, D.C.
Herzberger, Dr. Max. Research Lab., Eastman Kodak Co., Rochester, N.Y.
Herzog, Dr. Fritz. Instr., Cornell Univ., Ithaca, N.Y. White Hall.
Hess, G. W. Asso. Prof., Howard Coll., Birmingham, Ala. 8009 4th Ave. S.
Hestenes, Dr. A. D. Instr., Carnegie Inst. of Tech., Pittsburgh, Pa.
Hestenes, M. R. Asst. Prof., Univ. of Chicago, Chicago, Ill. Eckhart Hall.
Hett, J. H. Asst. Prof., Manhattan Coll., New York, N.Y. 3140 Netherland Ave., Bronx.
§**Hewitt, Dr. Edwin.** Instr., Harvard Univ., Cambridge, Mass. Kirkland House.
Heyda, Dr. J. F. Instr., Michigan State Coll., East Lansing, Mich.
Hickman, J. S. Instr., Rochester Junior Coll., Rochester, Minn.
Hicks, H. C. Asst. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa. 844 Country Club Dr. 16. *Aeronautical Engineer, Waco Aircraft Co., Troy, Ohio*
Hickson, A. O. Asst. Prof., Duke Univ., Durham, N.C.
Higdon, Archie. Asst. Prof., Theoretical and Applied Mech. Dept., Iowa State Coll., Ames, Iowa. *Maj., U.S.A., Air Corps, Jefferson Barracks, Mo.*
Highberg, I. E. Asst. Prof., Whitman Coll., Walla Walla, Wash.
Hightower, Ruby U. Prof., Shorter Coll., Rome, Ga.
§**Hildebrand, Dr. F. B.** Instr., Massachusetts Inst. of Tech., Cambridge, Mass.
Hildebrandt, E. H. C. Asst. Prof., New Jersey State Teachers Coll., Upper Montclair, N.J.
Hildebrandt, T. H. Prof., Univ. of Michigan, Ann Arbor, Mich. 1930 Cambridge Road.
Hildner, R. C. Asso. Prof., Mt. Union Coll., Alliance, Ohio.

Hill, C. G. 28 Scott St., Chicago, Ill.
Hill, E. L. Asso. Prof., Physics Dept., Univ. of Minnesota, Minneapolis, Minn.
Hill, J. D. Asst. Prof., Michigan State Coll., East Lansing, Mich.
Hill, L. S. Asso. Prof., Hunter Coll., 695 Park Ave., New York, N.Y.
Hill, Sister M. Laetitia. Prof., Our Lady of the Lake Coll., San Antonio, Tex.
Hille, Einar. Prof., Yale Univ., New Haven, Conn. 125 Armory St., Hamden, Conn.
§**Hillman, Abraham.** Asst. Mathematician, Mathematical Tables Project, National Bureau of Standards, U. S. Dept. of Commerce, 70 Columbus Ave., New York, N.Y. 3021 Ave. I, Brooklyn, N.Y.
Hinds, Frances C. (Mrs.). Asst. Prof., George Pepperdine Coll., 1121 W. 79th St., Los Angeles, Calif.
Hinrichsen, J. J. L. Asso. Prof., Iowa State Coll., Ames, Iowa.
Hobbs, A. W. Dean, Coll. of Arts and Sciences, Univ. of North Carolina, Chapel Hill, N.C.
Hochschild, Dr. G. P. 730 Ft. Washington Ave., New York, N.Y. *U.S.A., Air Forces, Tampa, Fla.*
Hodell, L. R. Research Supervisor, Carter Oil Co., Box 801, Tulsa, Okla.
Hodge, F. H. Asst. Prof. Emeritus, Purdue Univ., Lafayette, Ind. 820 N. Main St., West Lafayette, Ind.
Hoel, P. G. Asst. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.
Hoersch, Dr. V. A. Asso., Univ. of Illinois, Urbana, Ill. 909 S. 1st St., Champaign, Ill.
Hoffmann, Banesh. Asst. Prof., Queens Coll., Flushing, N.Y. 149-30 Hollywood Ave.
Hofmann, Dr. Lulu. Box 302, Grass Valley, Calif.
§**Hohaus, R. A.** Asst. Actuary, Metropolitan Life Ins. Co., 1 Madison Ave., New York, N.Y.
Hohn, Dr. F. E. Instr., Univ. of Arizona, Tucson, Ariz.
Holgate, T. F. Prof. Emeritus, Northwestern Univ., Evanston, Ill. 617 Library Pl.
Holl, D. L. Prof., Iowa State Coll., Ames, Iowa. 2323 Donald St.
Holland, Sister M. Charlotte. Prof. and Registrar, St. Francis Xavier Coll. for Women, 4900 Cottage Grove Ave., Chicago, Ill.
Hollcroft, Mary P. (Mrs. T. R.). Aurora, N.Y.
Hollcroft, T. R. Prof., Wells Coll., Aurora, N.Y.
Holley, Dr. J. L. 1318 L St. N.W., Washington, D.C.
Holly, Melita A. Research Mathematician, Ordnance Lab., Navy Yard, U.S. Navy Dept., Washington, D.C. 2139 R St. N.W.
Holmes, C. T. Asso. Prof., Bowdoin Coll., Brunswick, Me.
Hooper, Dorothy S. (Mrs. W. J.). Asst. Prof., Principia Coll., Elsah, Ill.
Hopf, Eberhard. Prof., Univ. of Leipzig, Leipzig, Germany. Talstrasse 35.
Hopkins, L. A. Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.
Hopkins, Dr. Margarete W. (Mrs.). 196 Clinton Ave., Brooklyn, N.Y.
Hopper, F. F. Dir., New York Public Lib., 476 5th Ave., New York, N.Y.
Hopper, Grace M. (Mrs. V. F.), Asst. Prof., Vassar Coll., Poughkeepsie, N.Y.
Hopwood, Muriel A. Kauai High School, Lihue, Kauai, Hawaii.
§**Horn, Alfred.** Asst., Univ. of California, Berkeley, Calif.
Hornberger, H. J. Actuary, Great Northern Life Ins. Co., 110 S. Dearborn St, Chicago, Ill
Horsfall, I. O. Dir., Extension Div., and Prof., Univ. of Utah, Salt Lake City, Utah.
Horst, Dr. A. P. Supervisor, Selection Research, Procter and Gamble Co., Cincinnati, Ohio.
Horton-Porter, Goldie. See Porter, Goldie Horton.
Horvay, Dr. Gabriel, Instr., Univ. of Cincinnati, Cincinnati, Ohio. *Asst. Physicist, David Taylor Model Basin, U.S. Navy Dept., Carderock, Md. 13 McKay Circle, Bethesda, Md.*
Hosford, H. M. Dean, Coll. of Arts and Sciences, Univ. of Arkansas, Fayetteville, Ark. 722 W. Maple.
Hostetter, I. M. Asst. Prof, Oregon State Coll., Corvallis, Ore.
Hotelling, Harold. Prof, Columbia Univ., New York, N.Y. Mountain Lakes, N.J.
Hotelling, Susanne E. (Mrs Harold). Mountain Lakes, N. J.

Householder, Dr. A. S. Research Asso. in Biophysics, Univ. of Chicago, Chicago, Ill. 644 Euclid Ave., Glen Ellyn, Ill.

Houston, W. V. Prof., Physics Dept., California Inst. of Tech., Pasadena, Calif. *Div. of National Defense Research, Columbia Univ., New York, N.Y. 60 Gramercy Park.*

Hove, Ethel M. Asso. Prof., Nebraska State Teachers Coll., Wayne, Neb. 416 Main St.

Howard, Awghtum S. (Mrs.). Fellow, Univ. of Kentucky, Lexington, Ky. 121 McVey Hall.

Howland, L. A. Prof., Wesleyan Univ., Middletown, Conn.

Hoy, E. A. Asst. Prof., Univ. of Hawaii, Honolulu, Hawaii. 509 W. 121st St., New York, N.Y.

Hoy, Ruth Y. (Mrs. E. A.). Instr., Univ. of Hawaii, Honolulu, Hawaii. 509 W. 121st St., New York, N.Y.

Hoyt, R. S. Tech. Staff, Bell Telephone Labs., 463 West St., New York, N.Y.

Hsü, Dr. H. Y. Lecturer, Yenching Univ., Peking, China.

Hubbs, H. N. Prof., Hobart Coll., Geneva, N.Y.

Hubert, W. G. Asso. Prof., Coll. of the City of New York, 139th St. and Convent Ave., New York, N.Y.

Hudson, Dr. C. M. Junior Physicist, Ballistic Lab., Frankford Arsenal, Philadelphia, Pa. 1052 Granite St.

Huff, W. N. Instr., The Hill School, Pottstown, Pa.

Hughes, H. K. Asso. Prof., Purdue Univ., Lafayette, Ind.

Hull, Evelyn M. Office Manager, American Mathematical Society, 531 W. 116th St., New York, N.Y.

Hull, Ralph. Prof., Univ. of British Columbia, Vancouver, B.C., Canada.

Hume, Alfred. Chancellor Emeritus and Prof., Univ. of Mississippi, University, Miss.

Humm, Dr. D. G. Personnel Specialist, 416 W. 8th St., Los Angeles, Calif.

Hummel, P. M. Asst. Prof., Univ. of Alabama, University, Ala. Box 1251.

Humphreys, Mabel G. Asst. Prof., Newcomb Coll., Tulane Univ., New Orleans, La.

Humphreys, W. J. Prof. Emeritus, George Washington Univ., Washington, D.C. Cosmos Club.

Hunt, G. H. Asst. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.

Hunt, Mildred. Prof., Illinois Wesleyan Univ., Bloomington, Ill.

Hunter, Louise S. (Mrs.). Asst. Prof., Virginia State Coll., Ettrick, Va.

§**Hunter, Dr. R. G.** Actuary, Equitable Life Ins. Co. of Iowa, Des Moines, Iowa.

†**Huntington, E. V.** Prof. Emeritus, Harvard Univ., Cambridge, Mass. 48 Highland St.

Huntley, H. B. Mathematician, Research and Development Dept., Cox and Stevens Aircraft Corp., Box 30, Mineola, N.Y.

Hurd, C. C. Asst. Prof., Michigan State Coll., East Lansing, Mich. *Lt. (j.g.), U.S.C.G.R., U.S. Coast Guard Acad., New London, Conn.*

Hurewicz, Witold. Asst. Prof., Univ. of North Carolina, Chapel Hill, N.C. Box 127.

Hurry, J. A. Prof., Physics Dept., San Antonio Junior Coll., San Antonio, Tex.

Hurst, J. W. Asso. Prof., Montana State Coll., Bozeman, Mont. 522 S. 6th.

Hurst, Mary J. 1105 N. Broad St., Galesburg, Ill.

Hurt, J. T. Asst. Prof., Agric. and Mech. Coll. of Texas, College Station, Tex.

†**Hurwitz, W. A.** Prof., Cornell Univ., Ithaca, N.Y. White Hall.

Huston, R. E. Asst. Prof., Rensselaer Polytech. Inst., Troy, N.Y. 4 Broadview Ter., S. Lake Ave.

Hutcherson, W. R. Prof., Berea Coll., Berea, Ky.

Hutchings, W. L. Asso. Prof., Rollins Coll., Winter Park, Fla. 357 Knowles Ave.

Hutchinson, C. A. Prof., Univ. of Colorado, Boulder, Colo.

Hutchinson, J. D. Asso. Prof., Univ. of Houston, Houston, Tex. 2904 Milam St.

Hutchinson, Dr. L. C. Fellow, School of Mech., Brown Univ., Providence, R.I.

Hyers, Dr. D. H. Fellow, Mech. Engineering Dept., California Inst. of Tech., Pasadena, Calif. 101 N. Hill Ave.

Ingersoll, B. M. *1st Lt., A.U.S., Instr., U.S. Military Acad., West Point, N.Y.*
Ingham, A. E. Lecturer, King's Coll., Univ. of Cambridge, Cambridge, England.
Ingraham, M. H. Dean, Coll. of Letters and Science, and Prof., Univ. of Wisconsin, Madison, Wis. South Hall.
Ingram, W. H. 150 Claremont Ave., New York, N.Y.
Irwin, Frank. Asso. Prof. Emeritus, Univ. of California, Berkeley, Calif. 2921 Regent St.
§**Irwin, P. C.** Asso. Actuary, Equitable Life Ins. Co. of Iowa, Des Moines, Iowa.
Isaacs, Dr. R. P. Analytic Engineer, Hamilton Standard Propellers, United Aircraft Corp., East Hartford, Conn.
Iyengar, K. S. K. Prof., Central Coll., Bangalore, India.

Jablon, Seymour. Asst., Columbia Univ., New York, N.Y. 200 W. 108th St.
†**Jackson, Dunham.** Prof., Univ. of Minnesota, Minneapolis, Minn. 119 Folwell Hall.
Jackson, J. B. Prof., Univ. of South Carolina, Columbia, S.C. 227 S. Waccamaw Ave.
Jackson, Dr. L. L. Retired. 4335 University Station, Tucson, Ariz.
Jackson, O. B. Seismologist, Shell Oil Co., Tulsa, Okla. Box 638, Graham, Tex.
Jackson, Rosa L. Prof., Alabama Coll., Montevallo, Ala.
Jackson, S. B. Asst. Prof., Univ. of Maryland, College Park, Md.
Jacobsen, R. S. Instr., Luther Coll., Decorah, Iowa. 602 North St.
Jacobson, Nathan. Asso. Prof., Univ. of North Carolina, Chapel Hill, N.C. Box 264. *Instr., U.S.N., Pre-Flight School, Chapel Hill, N.C.*
Jacobus, Dr. D. S. Stevens Inst. of Tech., Hoboken, N.J. 93 Harrison Ave., Montclair, N.J.
Jaeger, C. G. Prof., Pomona Coll., Claremont, Calif.
Jaeger, Dr. J. C. Lecturer, Univ. of Tasmania, Hobart, Australia.
Jaffe, Dr. Haym. Penn Treaty Junior High School, Philadelphia, Pa. 3800 Chestnut St.
James, Glenn. Asso. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.
§**James, R. C.** Fellow, California Inst. of Tech., Pasadena, Calif. Box 216, Glendora, Calif.
James, R. D. Prof., Univ. of Saskatchewan, Saskatoon, Sask., Canada.
Janes, W. C. Asst. Prof., Kansas State Coll. of Agric., and Applied Science, Manhattan, Kan.
Jaramillo, Dr. T. J. Actuary, Bureau of Treasury, Manila, Philippine Islands. Box 1045.
Jarnagin, M. P. Asst., Duke Univ., Durham, N.C. 630 Milledge Circle, Athens, Ga.
Jeffery, R. L. Prof., Acadia Univ., Wolfville, N.S., Canada.
Jenkins, E. D. Asso. Prof., Eastern Kentucky State Teachers Coll., Richmond, Ky. 1405 Howard Ave., Utica, N.Y.
Jennings, Dr. S. A. Lecturer, Univ. of British Columbia, Vancouver, B.C., Canada.
Jennings, Walter. Instr., Agric. and Mech. Coll. of Texas, College Station, Tex. *Lt. (j.g.), U.S.N.R., U.S. Naval Proving Ground, Dahlgren, Va.*
Jensen, C. M. Prof., Kansas Wesleyan Univ., Salina, Kan.
§**Jerbert, A. H.** Fellow, Univ. of Washington, Seattle, Wash. Philosophy Hall.
Jerbert, A. R. Asso. Prof., Univ. of Washington, Seattle, Wash. 5033 Pullman Ave.
Jessen, Børge. Prof., Royal Tech. Coll., Copenhagen, Denmark. Maltegaardsvej 11, Gentofte, Denmark.
†**Joffe, S. A.** Retired. 515 W. 110th St., New York, N.Y.
Johanson, R. N. Asst. Prof., Bradley Polytech. Inst., Peoria, Ill.
John, Fritz. Asso. Prof., Univ. of Kentucky, Lexington, Ky. 569 Webster Ave., New Rochelle, N.Y.
Johns, A. E. Asso. Prof., McMaster Univ., Hamilton, Ont., Canada.
Johnson, Evan. Asst. Prof., Pennsylvania State Coll., State College, Pa. Box 321.
Johnson, Marie M. Asst. Prof., Oberlin Coll., Oberlin, Ohio. 257 W. College St.
§**Johnson, P. B.** Asst., Univ. of Illinois, Urbana, Ill. *Civilian Public Service Camp, Glendora, Calif. Box 65.*

Johnson, R. A. Prof., Brooklyn Coll., Brooklyn, N.Y. 3421 80th St., Jackson Heights, N.Y.
Johnson, Dr. R. E. Instr., Univ. of Wisconsin, Madison, Wis.
Johnson, Roberta F. Asst. Prof., Wilson Coll., Chambersburg, Pa.
Johnston, F. E. Prof., George Washington Univ., Washington, D.C.
Jones, A. W. Instr., Yale Univ., New Haven, Conn. Johnson Rd., Woodbridge, Conn.
Jones, B. W. Asso. Prof., Cornell Univ., Ithaca, N.Y. White Hall.
Jones, C. D. 13445 Rosemary, Detroit, Mich.
Jones, F. B. Asst. Prof., Univ. of Texas, Austin, Tex. *Research Asso., Underwater Sound Lab., Harvard Univ., Cambridge, Mass. 77 Martin St.*
Jones, Dr. G. M. Architect and Civil Engineer. 437 E. Oakwood Blvd., Chicago, Ill.
Jones, Harris. Prof., U.S. Military Acad., West Point, N.Y. *Col., U.S.A.*
Jones, P. C. Bureau of Publication, Bell Telephone Labs., 463 West St., New York, N.Y.
Jones, R. W. Asst. Prof., Univ. of Delaware, Newark, Del. 121 Townsend Road.
Jordan, Dr. H. A. Catholic Univ. of America, Washington, D.C. 2512 Que St. N.W.
Jordan, H. E. Asso. Prof., Univ. of Kansas, Lawrence, Kan. 1600 Kentucky St.
Joseph, J. A. Instr., Civil Engineering Dept., New Mexico Coll. of Agric. and Mech. Arts, State College, N.Mex.
Judd, T. E. R.S.C., Auditing Div., Lafayette Bldg., Washington, D.C.
Justice, H. K. Prof., Coll. of Engineering and Commerce, Univ. of Cincinnati, Cincinnati, Ohio. 927 Kreis Lane.

Kac, Dr. Mark. Instr., Cornell Univ., Ithaca, N.Y.
Kagno, Dr. I. N. 3326 Bainbridge Ave., Bronx, New York, N.Y.
§**Kakutani, Shizuo.** Asst. Prof., Osaka Imperial Univ., Osaka, Japan.
Kales, Dr. M. L. Univ. of Maine, Orono, Me.
Kalinowski, W. C. St. John's Univ., Collegeville, Minn.
§**Kalisch, Dr. G. K.** Instr., Univ. of Kansas, Lawrence, Kan.
§**Kaplan, Samuel.** Fellow, Univ. of Michigan, Ann Arbor, Mich. 1104 Catherine.
Kaplan, Dr. Wilfred. Instr., Univ. of Michigan, Ann Arbor, Mich. 1135 Lincoln Ave.
Kaplansky, Dr. Irving. Instr., Harvard Univ., Cambridge, Mass. 17 Sumner Road.
†**Karapetoff, Vladimir.** Prof. Emeritus, Cornell Univ., Ithaca, N.Y. 107 Romaine Pl., Leonia, N.J.
§**Karlin, Rabbi Meyer.** 1890 E. 5th St., Brooklyn, N.Y.
Karnes, H. T. Asst. Prof., Louisiana State Univ., University, La.
Karpinski, L. C. Prof., Univ. of Michigan, Ann Arbor, Mich. 1315 Cambridge Road.
§**Karush, Dr. William.** Mathematician Geophysical Lab., Carnegie Inst., Washington, D.C. 4844 Western Ave. N.W.
†**Kasner, Edward.** Prof., Columbia Univ., New York, N.Y. 430 W. 116th St.
Kato, Chosaburo. Asso. Prof., Denison Univ., Granville, Ohio.
Katz, William. Attorney, 70 Pine St., New York, N.Y. 1574 Crotona Park E., Bronx.
Kazarinoff, D. K. Univ. of Michigan, Ann Arbor, Mich. 1515 Cambridge Road.
Kearney, Dora E. Asst. Prof., Iowa State Teachers Coll., Cedar Falls, Iowa.
Keeler, L. F. Prof., North Central Coll., Naperville, Ill.
Keffer, Ralph. Actuary, Aetna Life Ins. Co., Hartford, Conn.
Kelley, J. L. Asst. Prof., Univ. of Notre Dame, Notre Dame, Ind. *Asso. Mathematician, Ballistic Research Lab., Aberdeen Proving Ground, Md.*
Kelley, T. L. Prof., Education Dept., Harvard Univ., Cambridge, Mass. 90 Follen Road, Lexington, Mass.
Kells, L. M. Asso. Prof., U. S. Naval Acad., Annapolis, Md. 23 Thompson St.
§**Kemble, E. C.** Prof., Physics Dept., Harvard Univ., Cambridge, Mass.
Kemmer, P. H. *Col., U.S.A., Air Corps, Wright Field, Dayton, Ohio*
Kempner, A. J. Prof., Univ. of Colorado, Boulder, Colo.
Kendall, Claribel. Asso. Prof., Univ. of Colorado, Boulder, Colo. 1305 Euclid Ave.

Kenna, Rev. J. H. Asst. Prof., Univ. of Notre Dame, Notre Dame, Ind. Moreau Seminary.
Kennedy, E. C. Asso. Prof., Texas Coll. of Arts and Industries, Kingsville, Tex.
Kennedy, Dr. E. S. Instr., Univ. of Alabama, University, Ala. *1st Lt., U.S.A., Military Intelligence Div., General Staff, U. S. War Dept., Washington, D.C.*
Kenney, J. F. Asst. Prof., Univ. of Wisconsin at Milwaukee, Milwaukee, Wis. North Hall, Univ. of Wisconsin, Madison, Wis.
Kennison, L. S. Asst. Prof., Brooklyn Coll., Brooklyn, N.Y. *Lt. U.S.N.R., 9 Twin Oak Dr., Hyattsville, Md.*
Kent, J. R. F. Instr., Univ. of Arkansas, Fayetteville, Ark. 90 Bagot St., Kingston, Ont., Canada. *Sub.-Lt., R.C.N.V.R.*
Kernaghan, Mother Marie. Prof., Maryville Coll., St. Louis, Mo.
Kershner, Dr. R. B. Asso., Johns Hopkins Univ., Baltimore, Md. 3321 Winterbourne Road.
Kersten, H. J. Asst. Prof., Univ. of Cincinnati, Cincinnati, Ohio.
Ketchum, Dr. Gertrude S. (Mrs. P. W.). 803 W. Illinois St., Urbana, Ill.
Ketchum, P. W. Asst. Prof., Univ. of Illinois, Urbana, Ill
†Keyser, C. J. Prof. Emeritus, Columbia Univ., New York, N.Y.
Keyser, Sarah Y. (Mrs. C. J.). Asst. Headmistress, Nightingale-Bamford School, New York, N.Y. 50 Morningside Dr.
Kibbey, Dr. D. E. Instr., Michigan State Coll., East Lansing, Mich.
Kiefer, Dr. Lois. Instr., Univ. of Illinois, Urbana, Ill. 153 Math. Bldg.
§Kieval, H. S. Instr., North Carolina State Coll., Raleigh, N.C.
Kimball, Dr. B. F. Senior Statistician, New York State Public Service Com., New York, N.Y. 33 Bogart Ave., Port Washington, N.Y,
Kimball, S. H. Asso. Prof, Univ. of Maine, Orono, Me. 25 Forest Ave.
Kimball, W. S. Prof., Robert Coll., Istanbul, Turkey.
King, F. G. No address.
Kingston, H. R. Prof., Univ. of Western Ontario, London, Ont., Canada.
Kingston, Dr. J. M. Asso., Univ. of Washington, Seattle, Wash.
Kinney, J. M. Prof., Wilson Junior Coll., Chicago, Ill. 8058 Bennett Ave.
Kinsman, A. D. 80 County Road, Ipswich, Mass.
Kiokemeister, Dr. Fred. Instr., Cornell Univ., Ithaca, N.Y.
Kirby, A. R. Lecturer, Fordham Univ., New York, N.Y. 2921 Briggs Ave., Bronx.
Kirby, Mary D. (Mrs. A. R.). Instr., Manhattanville Coll. of the Sacred Heart, New York, N.Y. 2921 Briggs Ave., Bronx.
Kirchen, C. J. *U.S. Inspector, Zenith Machine Co., Duluth, Minn. 1101 E. 3d St.*
Kiss, S. A. Chemical Engineer. 12 E. 88th St., New York, N.Y.
Kitchens, J. W. Naval Observ, U. S. Navy Dept., Washington, D.C.
Klauber, Dr. L. M. Vice Pres. and General Manager, San Diego Gas and Electric Co., San Diego, Calif. 233 W. Juniper St.
Kleene, S. C. Asso. Prof., Amherst Coll, Amherst, Mass. *Lt. (j g), U.S.N.R.*
Kleinschmidt, R. B. Instr., Coll. of Engineering, Rutgers Univ., New Brunswick, N.J.
Kline, J. R. Prof., Univ. of Pennsylvania, Philadelphia, Pa. 529 Riverview Road, Swarthmore, Pa.
Kline, Dr. Morris. Instr., New York Univ., New York, N.Y. *Physicist, U.S.A., Signal Corps, Fort Monmouth, N.J. Care of Wainwright, Little Silver, N.J.*
Klipple, E. C. Asso. Prof., Agric. and Mech. Coll. of Texas, College Station, Tex. 107 Faculty Exch.
Knebelman, M. S. Prof., State Coll. of Washington, Pullman, Wash. 2008 Indiana St.
Knedler, P. A. Prof., State Teachers Coll., Kutztown, Pa. East Texas, Pa.
Knipp, J. C. Asst. Prof., Univ. of Pittsburgh, Pittsburgh, Pa. 450 Cathedral of Learning.
§Knobelauch, E. A. Instr., Drew Univ., Madison, N.J.
Knowler, L. A. Asst. Prof., State Univ. of Iowa, Iowa City, Iowa. 212 Physics Bldg.
Knox, R. H. Asst. Prof., Virginia Military Inst., Lexington, Va. 313 Letcher Ave.
Koch, R. F. *2d Lt., U.S.A., Signal Corps, Radar School, Camp Murphy, Fla. 436 43d St., West Palm Beach, Fla.*

Koehler, Dr. **Fulton.** Actuarial Dept., Northwestern National Life Ins. Co., Minneapolis, Minn. R.F.D. 2, Hopkins, Minn.

Koehler, T. L. Asst. Prof., Muhlenberg Coll., Allentown, Pa. 625 N. 24th St.

§**Kolchin,** Dr. **E. R.** U.S.A., Air Forces, Office of Ordnance Officer, Tampa, Fla. 218 Hyde Park Ave.

Kolsrud, E. R. Univ. of Illinois, Urbana, Ill. 201 S. Gregory St.

§**Komm, Horace.** Engineering Dept., Curtiss-Wright Corp., Buffalo, N.Y. 399 Jefferson Ave.

Koopman, B. O. Asso. Prof., Columbia Univ., New York, N.Y.

Korgen, R. L. Instr., Bowdoin Coll., Brunswick, Me. 38 College St.

Kormes, Joan P. (Mrs. Mark). Instr., Brooklyn Coll., Brooklyn, N.Y. 119 E. 84th St., New York, N.Y.

Kormes, Dr. **Mark.** Asst. Dir., New York State Ins. Fund, 625 Madison Ave., New York, N.Y. 119 E. 84th St.

Korn, Arthur. Prof. of Electrical Engineering, Stevens Inst. of Tech., Hoboken, N.J. 523 River St.

Korzybski, Alfred. Dir., Inst. of General Semantics, 1234 E. 56th St., Chicago, Ill.

Kossack, C. F. Asst. Prof., Univ. of Oregon, Eugene, Ore.

Kovarik, A. F. Prof., Physics Dept., Yale Univ., New Haven, Conn.

Krabill, Dr. **D. M.** Instr., Potomac State School, West Virginia Univ., Keyser, W.Va. *Ensign, U.S.N.R.*

Kraft, Ona. Collinwood High School, Cleveland, Ohio. 11328 Euclid Ave.

Krall, H. L. Asst. Prof., Pennsylvania State Coll., State College, Pa. 1420 N. Nichols St., Arlington, Va.

Kramer-Lassar, Dr. **Edna** (Mrs. I. B. Lassar). Thomas Jefferson High School, Brooklyn, N.Y. 500 St. John's Pl.

Krathwohl, W. C. Dir., Dept. of Educational Tests and Measurements, and Prof., Illinois Inst. of Tech., Chicago, Ill.

Kratz, P. H. 1623 W. Westmoreland, Philadelphia, Pa.

Krieger, Cecilia. Asst. Prof., Univ. of Toronto, Toronto, Ont., Canada. 382 Roxton Road.

Kryloff, Dr. **Nicolas.** St. Novo-Mosto-Vaya N. 17/A, Ufa, U.S.S.R.

Kubis, J. F. Asst. Prof., Fordham Univ., New York, N.Y.

Kubota, Tadahiko. Prof., Tôhoku Imperial Univ., Sendai, Japan. Coll. of Science.

Kuhn, H. W. Prof., Ohio State Univ., Columbus, Ohio. 1179 Fairview Ave.

Kuniyeda, Motoji. Prof., Tokyo Univ. of Literature and Science, Tokyo, Japan.

†**Kuschke, C. G. P.** Prof., Univ. of Puerto Rico, Rio Piedras, Puerto Rico. 723 W. 3d St., Los Angeles, Calif.

Kwerel, S. M. Statistician and Economist, National Resources Planning Board, Washington, D.C.

Lackay, Sister M. Pachomia. St. Teresa's Coll., Kansas City, Mo.

Laden, Dr. **H. N.** Instr., Univ. of Wisconsin, Madison, Wis. *Ensign, U.S.N.R.*

Laderman, Jack. Asso. Statistician, Chemical Warfare Service, U.S. War Dept., 292 Madison Ave., New York, N.Y. 3224 Bronx Blvd., Bronx.

Ladner, A. C. Asst. Prof., Denison Univ., Granville, Ohio. Box 253.

Ladue, Dr. **Mary E.** Instr., Barnard Coll., Columbia Univ., New York, N.Y.

LaFon, Gladys Q. (Mrs. J. E.). 609 S. Flood, Norman, Okla.

Lambert, W. D. Chief, Section of Gravity and Astronomy, and Prin. Mathematician, Coast and Geodetic Survey, U. S. Dept. of Commerce, Washington, D.C.

Lampland, Dr. **C. O.** Astronomer, Lowell Observ., Flagstaff, Ariz. Box 1640.

Lamson, K. W. Asso. Prof., Lehigh Univ., Bethlehem, Pa.

Lancaster, O. E. Asst. Prof., Univ. of Maryland, College Park, Md.

Lanckton, A. L. Box 163, Athens, Greece.

Lanczos, Cornelius. Prof., Purdue Univ., Lafayette, Ind.

Landau, H. G. Coal Research Lab., Carnegie Inst. of Tech., Pittsburgh, Pa. *Asso. Statistician, Office of Quartermaster General, U. S. War Dept., Washington, D C 2408 20th St N E.*

Landers, Dr. **A. W.** Instr., Brooklyn Coll., Brooklyn, N.Y. 108-48 67th Dr., Forest Hills, N.Y.

Landers, Dr. Mary K. (Mrs. A. W.). Instr., Hunter Coll., Bedford Park Blvd. and Navy Ave., New York, N.Y. 108-48 67th Dr., Forest Hills, N.Y.
Landin, Joseph. Asst., Univ. of Notre Dame, Notre Dame, Ind.
Landry, A. E. Prof., Catholic Univ. of America, Washington, D.C. 3624 13th St. N.E., Brookland, D.C.
§**Landshoff, Rolf.** Prof., Coll. of St. Thomas, St. Paul, Minn.
Lane, E. P. Prof., Univ. of Chicago, Chicago, Ill.
Lane, R. E. Statistician, Flitcraft, Inc., 613-615 S. Maple Ave., Oak Park, Ill.
Lang, Dr. G. B. Instr., Univ. of Florida, Gainesville, Fla.
Langer, R. E. Prof., Univ. of Wisconsin, Madison, Wis. 1802 Jefferson St.
Langhaar, Dr. H. L. Structures Engineer, Consolidated Aircraft Corp, Chula Vista, Calif., R.F D. 1, Box 31-B-1.
LaPaz, Lincoln. Prof. Ohio State Univ., Columbus, Ohio.
§**Lapsley, Janie C.** Asst., Univ. of Illinois, Urbana, Ill. 153 Math. Bldg.
Larew, Gillie A. Prof., Randolph-Macon Woman's Coll, Lynchburg, Va
Larguier, E. H. Instr, Loyola Univ., New Orleans, La 4425 S. Johnson St
Larkin, Sister Francis X. Prof., St. Joseph's Coll. for Women, 245 Clinton Ave., Brooklyn, N.Y.
Larrivee, Dr. J. A. Asso. Astronomer, Nautical Almanac Office, U.S. Naval Observ., Washington, D.C.
Larsen, Thorstein. Asso., School of Engineering, Johns Hopkins Univ., Baltimore, Md.
Larus, J. R. Vice Pres. and Actuary, Phoenix Mutual Life Ins. Co., Hartford, Conn.
LaSalle, Dr. J. P. Instr., Massachusetts Inst. of Tech., Cambridge, Mass. 383 Harvard St.
Lasley, J. W. Prof., Univ. of North Carolina, Chapel Hill, N.C. 523 E. Rosemary Lane.
Lass, Harry. Asst., Univ. of California at Los Angeles, Los Angeles, Calif 2805½ Fairmount St.
Latimer, C. G. Prof., Univ. of Kentucky, Lexington, Ky.
Latshaw, V. V. Asst. Prof., Lehigh Univ, Bethlehem, Pa. 1228 W. North St.
Lawrence, V. S. Asso. Prof, Cornell Univ., Ithaca, N.Y. *Maj, U S A, U. S. Military Acad., West Point, N Y.*
Lawton, W. S. Asst. Prof., Temple Univ., Philadelphia, Pa. 7500 New 2d St., Melrose Park, Pa.
Leach, C. E. R.F.D. 2, Marion, Ohio.
§**Learson, R. J.** Actuary, Western and Southern Life Ins. Co., 4th and Broadway, Cincinnati, Ohio.
Leavens, D. H. Research Asso, Cowles Com., Univ. of Chicago, Chicago, Ill.
Leavitt, W. G. *Ground School Instr, U. S Naval Air Station, Corpus Christi, Tex. 1117 2d St*
LeCaine, Dr. Jeanne. Instr., Smith Coll., Northampton, Mass. *Junior Professional, National Research Council, Ottawa, Ont., Canada. 107 Dalhousie St.*
Lee, (T.) H. Asst., Univ. of Wisconsin, Madison, Wis. North Hall
Leech, J. S. Yale Univ., New Haven, Conn. 2737 Yale Station.
Lefschetz, Solomon. Prof., Princeton Univ., Princeton, N.J.
Lehmann, C. H. Instr., School of Engineering, Cooper Union, New York, N Y. 144-17 29th Ave., Flushing, N.Y. *Defense Training Inst., 375 Pearl St., Brooklyn, N Y.*
Lehmer, D. H. Asso Prof., Univ. of California, Berkeley, Calif. 942 Hilldale Ave.
Lehner, Dr. Joseph. Instr., Cornell Univ., Ithaca, N.Y. White Hall.
Lehr, (A.) Marguerite (M.). Asso. Prof., Bryn Mawr Coll., Bryn Mawr, Pa. Cartref.
Leibler, Dr. R. A. Instr., Purdue Univ., Lafayette, Ind 560 Aldine, Chicago, Ill. *Ensign, U S N.R*
Leifer, H. R. Admin. Asst., U. S. Dept. of Labor, Pittsburgh, Pa. 5500 Hays St. *U S A., Ordnance Replacement Training Center, Aberdeen Proving Ground, Md.*
Leighton, Dr. Walter, Lecturer, Rice Inst, Houston, Tex.
Leimanis, Eugene. Asso. Prof., Univ. of Latvia, Riga, Latvia. Jumaras ielá 11, dz. 10.

LeLeiko, Max. Statistician, 547 W. 142d St., New York, N.Y.
§**Lemmer, J. G.** Asst. Prof. of Philosophy, West Baden Coll., Loyola Univ., West Baden Springs, Ind.
Lengyel, Dr. **B. A.** Physics Dept., Coll. of the City of New York, 139th St. and Convent Ave., New York, N.Y. 69 W. 90th St.
†**Lennes, N. J.** Prof., Montana State Univ., Missoula, Mont.
Lenzen, V. F. Prof., Physics Dept., Univ. of California, Berkeley, Calif.
Leonard, H. B. Prof., Univ. of Arizona, Tucson, Ariz. 4024 University Station.
Lester, Caroline A. Asst. Prof., New York State Coll. for Teachers, Albany, N.Y.
LeStourgeon, Flora E. Asso. Prof., Univ. of Kentucky, Lexington, Ky. 630 Maxwelton Ct.
Leuschner, A. O. Dir. Emeritus, Student's Observ., and Prof. of Astronomy, Univ. of California, Berkeley, Calif. 1816 Scenic Ave.
Levi, Dr. **Beppo.** Dir., Mathematical Inst., Univ. of Rosario, Rosario, Argentina. San Lorenzo 2133.
§**Levi**, Dr. **Howard.** Instr., U.S.N. Pre-Flight School, Iowa City, Iowa.
§**Levin**, Dr. **J. H.** 12515 Broadstreet Blvd., Detroit, Mich. *2d Lt., U.S.A., Aberdeen Proving Ground, Md.*
Levin, Dr. **Madeline.** Instr., Hunter Coll., 695 Park Ave., New York, N.Y.
Levine, Jack. Asso. Prof., North Carolina State Coll., Raleigh, N.C. 5548 State College Station.
Levinson, Dr. **H. C.** Treas., L. Bamberger and Co., Newark, N.J.
Levinson, Norman. Asst. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.
Levit, Dr. **R. J.** *Ensign, U.S.N.R., Cruft Lab., Harvard Univ., Cambridge, Mass.*
Levitzki, Dr. **Jakob.** Asst., Hebrew Univ., Jerusalem, Palestine. Math. Inst.
Levy, Harry. Asst. Prof., Univ. of Illinois, Urbana, Ill.
Levy, Sophia H. Asso. Prof., Univ. of California, Berkeley, Calif. 453 Wheeler Hall.
§**Lewis, Anne L.** Fellow, Univ. of Chicago, Chicago, Ill.
Lewis, D. C. Asso. Prof., Univ. of New Hampshire, Durham, N.H. Box 53.
Lewis, F. A. Prof., Univ. of Alabama, University, Ala. Box 1444.
Lewis, Florence P. Prof., Goucher Coll., Baltimore, Md.
§**Lewis, J. V.** Asst., Univ. of California, Berkeley, Calif. 24 Wheeler Hall.
Lewis, Dr. **P. E.** Instr., Oklahoma Agric. and Mech. Coll., Stillwater, Okla.
Lewy, Hans. Asso. Prof., Univ. of California, Berkeley, Calif. 417 Wheeler Hall.
Lieber, H. G. Asst. Prof., Long Island Univ., Brooklyn, N.Y. 258 Clinton Ave.
Lieber, Dr. **Lillian R.** (Mrs. H. G.). Dir., Galois Inst. of Math., Long Island Univ., Brooklyn, N.Y. 258 Clinton Ave.
§**Light**, Dr. **F. W.** Johns Hopkins Univ., Baltimore, Md. 2410 St. Paul St.
§**Lin, C. C.** Asst., California Inst. of Tech., Pasadena, Calif. Guggenheim Lab.
Lindquist, Dr. **C. B.** Instr., U. S. Naval Acad., Annapolis, Md. *Lt. (j.g.), U.S.N.*
Lindsay, R. B. Prof., Physics Dept., Brown Univ., Providence, R.I.
Lindstrum, Dr. **A. O.** Instr., Univ. of Wisconsin, Madison, Wis. *U.S.A., Air Forces, Tech. Training Detachment, Univ. of Chicago, Chicago, Ill.*
Linehan, P. H. Prof., Coll. of the City of New York, New York, N.Y. 346 Convent Ave.
Linfield, B. Z. Asso. Prof., Univ. of Virginia, University Station, Charlottesville, Va. Hilltop Road.
Ling, D. P. 377 Hudson St., New York, N.Y.
Ling, G. H. Dean Emeritus and Prof. Emeritus, Univ. of Saskatchewan, Saskatoon, Sask., Canada. 10 Wanless Crescent, Toronto, Ont., Canada.
Linscheid, H. W. Asst. Prof., Bluffton Coll., Bluffton, Ohio.
§**Lintvedt, D. R.** 806 N. Spring Ave., Sioux Falls, S.Dak.
Litterick, Dr. **W. S.** Dir. of Studies, The Peddie School, Hightstown, N.J.
Litzinger, Marie. Prof., Mt. Holyoke Coll., South Hadley, Mass. Mosier St.
Livens, G. H. Prof., University Coll., Cardiff, Wales.
§**Livesay, Naomi M.** Asst., Univ. of Illinois, Urbana, Ill. 153 Math. Bldg.
Livingood, J. N. B. Instr., Gettysburg Coll., Gettysburg, Pa. 152 Spruce St., Birdsboro, Pa.
Locke, L. L. 950 St. John's Pl., Brooklyn, N.Y.
§**Lockhart, B. J.** Asst., Univ. of Illinois, Urbana, Ill. 153 Math. Bldg.
Loewner, Charles. Asst. Prof., Univ. of Louisville, Louisville, Ky. 125 W. Lee St.

Logsdon, Mayme I. (Mrs.). Asso. Prof., Univ. of Chicago, Chicago, Ill.
Long, T. R. Asst. Prof., Univ. of Rochester, Rochester, N.Y. 75 Cypress St.
Longenecker, J. V. S. Farmers and Bankers Life Ins. Co., Wichita, Kan.
Longley, W. R. Prof., Yale Univ., New Haven Conn. 305 Lawrence St.
Lonseth, Dr. A. T. Instr., Iowa State Coll., Ames, Iowa. *Bureau of Ordnance, U. S. Navy Dept., Washington, D.C. 3027 Newark St.*
Loomer, A. P. 7762 S. Foord St., Stellarton, N.S., Canada.
Loomis, Dr. L. H. Instr., Harvard Univ., Cambridge, Mass. 43 Museum St.
Lorch, E. R. Asst. Prof., Barnard Coll., Columbia Univ., New York, N.Y. 435 W. 123d St.
Lorch, Dr. Lee. Asst. Mathematician, National Advisory Com. for Aeronautics, Langley Field, Hampton, Va. Box 293.
Lorell, Jack. 10 Liberty St., Aberdeen, Md.
§**Lorenz, E. N.** Fellow, Harvard Univ., Cambridge, Mass. 56 W. Hill Dr., West Hartford, Conn.
Lorenzen, Robert. 2156 Lexington Ave., New York, N.Y. *Radio Engineer, Camp Murphy, Florida.*
Loring, S. J. Research Engineer, Vought-Sikorsky Aircraft, Stratford, Conn. 63 Camden St., Devon, Conn.
de Losada y Puga, Christóbal. Prof., Catholic Univ. of Peru, Lima, Peru. Apartado 2708.
Lotka, Dr. A. J. Asst. Statistician, Metropolitan Life Ins. Co., 1 Madison Ave., New York, N.Y. Beattie Park, Red Bank, N.J.
§**Loud, W. S.** Massachusetts Inst. of Tech., Cambridge, Mass. 171 Commercial St., Weymouth, Mass.
Love, C. E. Prof., Univ. of Michigan, Ann Arbor, Mich. 1915 Scottwood.
†**Lovett, E. O.** Pres. Emeritus, Rice Inst., Houston, Tex.
Lowan, A. N. Asso. Prof., Yeshiva Coll., New York, N Y.; Senior Mathematician, Mathematical Tables Project, National Bureau of Standards, U. S. Dept. of Commerce, 70 Columbus Ave., New York, N.Y. 312 Schenectady Ave., Brooklyn, N.Y.
Lowenstein, L. L. Asso. Prof., Alfred Univ., Alfred, N.Y.
Lubben, R. G. Asst. Prof., Univ. of Texas, Austin, Tex. W.H. 16, University Station.
Lubin, C. I. Asso. Prof., Univ. of Cincinnati, Cincinnati, Ohio. 3612 Washington Ave.
§**Luckey, Dr. R. R. R.** Instr., Houghton Coll., Houghton, N.Y.
§**Lufkin, Dr. H. M.** Instr., Univ. of Pennsylvania, Philadelphia, Pa. Bennett Hall.
Lukacs, Dr. Eugene. 5510 Pimlico Road, Baltimore, Md.
Luke, Y. L. Care of Alper, 1839 N. Whipple, Chicago, Ill.
Luneburg, Dr. R. K. Research Dept., Spencer Lens Co , Buffalo, N.Y.
Lunn, A. C. Prof. Emeritus, Univ. of Chicago, Chicago, Ill. 5211 Kenwood Ave.
Luther, C. F. Dean, Coll. of Liberal Arts, and Prof., Willamette Univ., Salem, Ore.
§**Lyman, Theodore.** Prof. Emeritus, Harvard Univ., Cambridge, Mass. Research Physics Lab.
Lynch, Rev. J. J. Prof., Fordham Univ., New York, N.Y.

§**McCankie, R. C.** Equitable Life Ins Co. of Iowa, Des Moines, Iowa.
McCarthy, E. D. Asst. Prof., Coll. of Engineering, Univ. of Detroit, Detroit, Mich.
McCarthy, J. J. Asst. Prof., St. John's Univ., Brooklyn, N.Y. 211 Lafayette Ave.
McCarty, A. L. Prof., San Francisco Junior Coll., San Francisco, Calif. 1798 Grove St.
McCauley, C. E. Woodstock Coll., Woodstock, Md.
McClay, Dr. D. T. Instr., Univ. of Vermont and State Agric. Coll., Burlington, Vt. 37 Colonial Sq.
†**McClenon, R. B.** Prof., Grinnell Coll., Grinnell, Iowa. 1105 Park.
McConnell, A. J. Prof. of Philosophy, Univ. of Dublin, Dublin, Ireland. 36 Trinity Coll.
McConnell, H. J. Cartographic Engineer. 3328 Madison Ave., San Diego, Calif.
McCormick, J. T. Prof., Holy Cross Coll., Worcester, Mass.

McCoy, Dorothy. Prof., Belhaven Coll., Jackson, Miss.
McCoy, N. H. Prof., Smith Coll., Northampton, Mass. 53 Ridgewood Ter.
McCrea, W. H. Prof., Queen's Univ., Belfast, Northern Ireland. *Flying Officer, R.A.F.V.R.*
§**McCuskey, S. W.** Asso. Prof., Case School of Applied Science, Cleveland, Ohio.
McDaniel, R. R. Asso. Prof., Virginia State Coll., Ettrick, Va.
McDaniel, W. C. Asst. Prof., Southern Illinois State Normal Univ., Carbondale, Ill.
McDermott, Eugene. Pres., Geophysical Service, Inc., 1311 Republic Bank, Dallas, Tex. *Consultant, National Defense Research Com.*
McDonough, Dr. D. L. 1205 Belfield Ave., Drexel Hill, Pa.
McDougle, Edith A. Instr., Women's Coll., Univ. of Delaware, Newark, Del.
McEwen, Dr. G. F. Scripps Inst., Univ. of California, La Jolla, Calif. Box 109.
McEwen, W. H. Prof., Mt. Allison Univ., Sackville, N.B., Canada.
McEwen, W. R. Instr., Univ. of Minnesota, Minneapolis, Minn.
McFadden, Dr. Leonard. Instr., Virginia Polytech. Inst., Blacksburg, Va. Box 65.
McFarlan, L. H. Asso. Prof., Univ. of Washington, Seattle, Wash. 158 Philosophy Hall.
McFarland, Dora. Asso. Prof., Univ. of Oklahoma, Norman, Okla. Faculty Exch.
McGaughey, Dr. A. W. Instr., U. S. Naval Acad., Annapolis, Md.
McGrath, Rev. P. H. Prof., St. Peter's Coll., 144 Grand St., Jersey City, N.J.
McKee, Dr. Ruth S. 3607 Brisban St., Harrisburg, Pa.
McKelvey, J. V. Prof., Iowa State Coll., Ames, Iowa. 2117 Graeber St.
McKelvey, Martha M. (Mrs. J. V.). Instr., Iowa State Coll., Ames, Iowa. 2117 Graeber St.
McKenney, Rev. J. L. Prof., Providence Coll., Providence, R.I.
McKinsey, Dr. J. C. C. Instr., Washington Square Coll., New York Univ., New York, N.Y. Guggenheim Fellow. Inst. for Advanced Study, Princeton, N.J.
McLaughlin, K. F. Junior Instr., U.S.A., Air Corps Tech. School, Scott Field, Ill. 307 E. Washington St., Belleville, Ill.
McMahon, F. A. Brooklyn Coll., Brooklyn, N.Y. 40-18 Forley St., Elmhurst, N.Y.
McMaster, A. S. Asst. Prof., Engineering Math. Dept., Univ. of Colorado, Boulder, Colo.
McMillan, Dr. Audrey W. (Mrs. B.). Instr., Vassar Coll., Poughkeepsie, N.Y. Asst., Inst. for Advanced Study, Princeton, N.J.
McMillan, Dr. Brockway. Instr., Princeton Univ., Princeton, N.J.
McMullen, C. B. Prof., State Teachers Coll., Hyannis, Mass.
McNeil, Sister M. Gertrude. Prof., Seton Hill Coll., Greensburg, Pa.
McShane, E. J. Prof., Univ. of Virginia, University Station, Charlottesville, Va. 209 Maury Ave.
MacColl, Dr. L. A. Research Mathematician, Bell Telephone Labs., 463 West St., New York, N.Y.
MacCullough, R. H. Prof., Defiance Coll., Defiance, Ohio. Sisson Hall.
MacDonald, J. K. L. Asst. Prof., Cooper Union, New York, N.Y.
MacDougal, H. B. Prof., South Dakota State Coll., Brookings, S.Dak.
MacDuffee, C. C. Prof., Hunter Coll., 695 Park Ave., New York, N.Y.
§**Mackey, Dr. G. W.** Instr., Illinois Inst. of Tech., Chicago, Ill.
Mackie, E. L. Prof., Univ. of North Carolina, Chapel Hill, N.C. 702 Gimghoul Road.
§**MacKinnon, W. D.** Equitable Life Ins. Co. of Iowa, Des Moines, Iowa.
MacLane, G. R. Radiation Lab., Massachusetts Inst. of Tech., Cambridge, Mass. 7 Avon St.
MacLane, Saunders. Asso. Prof., Harvard Univ., Cambridge, Mass. 7 Avon St.
MacLean, N. B. Prof., McGill Univ., Montreal, Quebec, Canada.
MacNeille, H. M. Prof., Kenyon Coll., Gambier, Ohio.
MacNeish, H. F. Prof., Brooklyn Coll., Brooklyn, N.Y. 48 Taunton Road, Scarsdale, N.Y.
Macphail, M. S. Asst. Prof., Acadia Univ., Wolfville, N.S., Canada.
MacQueen, M. L. Prof., Southwestern Coll., Memphis, Tenn.
Maddaus, Ingo. Asst. Prof., Univ. of Oregon, Eugene, Ore.
Madden, Sister Teresa M. Prof., Coll. of Our Lady of the Elms, Chicopee, Mass.

Madow, Dr. W. G. Asso. Statistician Bureau of the Census, U. S. Dept. of Commerce, Washington, D. C. 1445 Ogden St. N.W.
§**Maharam, Dr. Dorothy.** Inst. for Advanced Study, Princeton, N.J.
Makarov, A. G. Instr., Univ. of Delaware, Newark, Del.
Maker, P. T. Asst. Prof., Duke Univ., Durham, N.C. 2803 Nation Ave.
Malin, Dr. Henry. Instr., Coll. of the City of New York, 139th St. and Convent Ave., New York, N.Y.
Mallory, V. S. Prof., New Jersey State Teachers Coll., Upper Montclair, N.J.
Maloney, Sister M. Michael. Instr., D'Youville Coll., 320 Porter Ave., Buffalo, N.Y.
Maloney, Sister Thomas M. Asso. Prof., Trinity Coll., Washington, D.C.
Malti, M. G. Asst. Prof., Cornell Univ., Ithaca, N.Y. Franklin Hall.
Manchester, R. E. Dean, Kent State Univ., Kent, Ohio.
Mancill, J. D. Asso. Prof., Univ. of Alabama, University, Ala.
Mandelbrojt, Szolem. Prof., Coll. of France, Paris, France; Visiting Prof., Rice Inst., Houston, Tex.
Mann, Dr. H. B. Research Asso., Economics Dept., Columbia Univ., New York, N.Y. 189 8th Ave., Brooklyn, N.Y.
§**Mann, W. R.** Asst., Univ. of California, Berkeley, Calif. *American Friends Service Com., Paso de Ovejas, Veracruz, Mexico.*
†**Manning, H. P.** Asso. Prof. Emeritus, Brown Univ., Providence, R.I. 148 Governor St.
Manning, Dr. Rhoda. Instr., Oregon State Coll., Corvallis, Ore. 321 N. 23d St.
Manning, W. A. Prof., Stanford Univ., Stanford University, Calif. 649 Alvarado Row.
Manring, W. E. Sales Engineer, American Cyanamid and Chemical Corp., 30 Rockefeller Plaza, New York, N.Y.
Mansfield, Ralph. Prof., Chicago Teachers Coll., Chicago, Ill.
Many, Anna E. Prof., Newcomb Coll., Tulane Univ., New Orleans, La.
§**Mapes, G. S.** Asst., Univ. of Illinois, Urbana, Ill. 160 Math. Bldg.
March, H. W. Prof., Univ. of Wisconsin; Head Mathematician, U. S. Forest Products Lab., Madison, Wis. 1825 Summit Ave.
Marcou, R. J. Prof., Boston Coll., Chestnut Hill, Mass.
Marden, Morris. Asso. Prof., Univ. of Wisconsin at Milwaukee, Milwaukee, Wis. 403 E. Carlisle Ave.
Maria, Dr. A. J. Instr., Brooklyn Coll., Brooklyn, N.Y.
Maria, Dr. Deborah H. (Mrs. A. J.). Brooklyn Coll., Brooklyn, N.Y.
Marm, Anna. Prof., Bethany Coll., Lindsborg, Kan. 741 N. 2d St.
Marquis, R. H. Prof., Ohio Univ., Athens, Ohio. Box 216.
Marriott, R. W. Prof., Swarthmore Coll., Swarthmore, Pa. 213 Lafayette Ave.
Marshall, William. Prof. Emeritus, Purdue Univ., Lafayette, Ind. R.F.D. 9.
Marston, Dr. A. E. Box 630, Monterey, Calif. *Lt. (j.g.), U.S.N.R.*
§**Martin, Dr. F. L.** Meteorologist, Northwest Airlines, Felts Field, Spokane, Wash.
Martin, M. H. Asso. Prof., Univ. of Maryland, College Park, Md.
Martin, Margaret P. Instr. in Biostatistics, DeLamar Inst. of Public Health, Columbia Univ., 600 W. 168th St., New York, N.Y.
Martin, W. T. Asst. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.
Martyn, W. J. Otago Boys' High School, Dunedin, New Zealand.
§**Marx, Imanuel.** New York Univ., New York, N.Y. 53 Washington Sq. S.
†**Mason, Dr. Max.** Chairman, Observ. Council, California Inst. of Tech., Pasadena, Calif.
Mason, W. E. Asso. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.
Mason, Dr. W. P. Research Engineer, Bell Telephone Labs., 463 West St., New York, N.Y.
§**Mathews, C. W.** Asst., Univ. of Illinois, Urbana, Ill. Marissa, Ill. *Ensign, U.S.N.R.*
Mathewson, L. C. Prof., Dartmouth Coll., Hanover, N.H.
Matz, Joshua. Instr., Yeshiva Coll., New York, N.Y.
May, Dr. A. E. Instr., Univ. of Wisconsin at Kenosha, Kenosha, Wis. 416 68th St.
Mayer, Dr. Joanna I. 217 S. Moniteau, Sedalia, Mo.

Mayer, Dr. Walther. Asso., Inst. for Advanced Study, Princeton, N.J.
Mayor, J. R. Prof., Southern Illinois Normal Univ., Carbondale, Ill.
Meacham, E. D. Dean, Coll. of Arts and Sciences, and Prof. Univ. of Oklahoma, Norman, Okla. Faculty Exch.
Mead, Sallie P. (Mrs. C. E.). Tech. Staff, Bell Telephone Labs., 463 West St., New York, N.Y.
Mears, Florence M. Asso. Prof., George Washington Univ., Washington, D.C.
Meder, A. E. Prof., New Jersey Coll. for Women, Rutgers Univ., New Brunswick, N.J.
Mehlenbacher, L. E. Prof., Arizona State Teachers Coll., Flagstaff, Ariz.
Melcher, Dr. George. Asst. Supt. of Schools, Kansas City, Mo. 201 W. 51st St.
Melchor, Alejandro. Prof. of Engineering, Philippine Military Acad, Baguio, Philippine Islands. *Lt. Col., U.S.A, Engineer Board, Fort Belvoir, Va.*
Mendel, Dr. C. W. Asso., Univ. of Illinois, Urbana, Ill. 369 Math. Bldg.
§**Mendelsohn, Dr. N. S.** Supervisor, Munitions Gauge Lab., National Research Council, Ottawa, Ont., Canada.
Menger, Karl. Prof., Univ. of Notre Dame, Notre Dame, Ind. 1421 E. LaSalle St., South Bend, Ind.
Merrick, Rev. J. P. Prof., Baghdad Coll, Baghdad, Iraq.
Merrill, A. S. Prof., Montana State Univ., Missoula, Mont. *Consultant, Civil Aeronautics Admin*
†**Merrill, Helen A.** Prof. Emeritus, Wellesley Coll., Wellesley, Mass. 6 Waban St.
Merriman, G. M. Asso. Prof., Univ. of Cincinnati, Cincinnati, Ohio
Mersman, Dr. W. A. Instr., Coll. of Agric., Univ. of California, Davis; Calif. *Radiation Lab, Massachusetts Inst. of Tech, Cambridge, Mass. 395 Broadway.*
§**Meserve, B. E.** Asst., Duke Univ., Durham, N.C. 1625 Forest Ave., Portland, Me. *Civilian Public Service Camp, Buck Creek, Marion, N C.*
†**Metzler, Dr. W. H.** Retired. 172 High St., Lockport, N.Y.
Mewborn, A. B. Asst. Prof., Univ. of Arizona, Tucson, Ariz. 4425 University Station.
Meyer, Herman. Asst. Prof., Univ. of Miami, Coral Gables, Fla.
Meyer, H. A. Prof., Hanover Coll., Hanover, Ind.
Meyer, H. L. Instr., Coll. of Wooster, Wooster, Ohio.
Michal, A. D. Prof., California Inst. of Tech., Pasadena, Calif.
Michel, R. J. Prof., Southeast Missouri State Teachers Coll., Cape Girardeau, Mo 225 N. Ellis St.
Mickelson, Dr. E. L. Fox Construction Co., Tucson, Ariz. 607 Bayard St., Silver City, N.Mex.
Midgley, Dr. Thomas. General Motors Chemical Co., Dayton, Ohio. R.F.D. 2, Worthington, Ohio.
Mijares, J. M. Prof., Mapua Inst. of Tech., Manila, Philippine Islands.
Miles, E. J. Asso. Prof., Yale Univ., New Haven, Conn. 87 Marvel Road.
Miles, E. P. 837 7th St. W., Birmingham, Ala. *Ensign, U.S N R, Supply Dept., U. S. Naval Station, New Orleans, La.*
Miles, Dr. H. J. Asso., Univ. of Illinois, Urbana, Ill. 253 Math. Bldg.
Milgram, A. N. Asst. Prof., Univ. of Notre Dame, Notre Dame, Ind. 1006 E. Ewing St., South Bend, Ind.
Miller, Dr. A. L. Treas., C. N. Miller Co., Boston, Mass. 25 Clinton Road, Brookline, Mass.
Miller, C. E. Asst. Prof., Univ of New Brunswick, Fredericton, N.B., Canada.
Miller, Dr. D. D. Instr, Ohio Univ., Athens, Ohio. Faculty Club. *Lt. (j g), U.S N R.*
Miller, Dr. D. S. Instr., Yale Univ., New Haven, Conn.
Miller, E. A. 113 N. Tioga St., Ithaca, N.Y.
Miller, F. A. Instr., Wentworth Military Acad., Lexington, Mo *Capt., U.S A*
Miller, F. H. Asst. Prof, Cooper Union, New York, N Y.
†**Miller, G. A.** Prof., Univ. of Illinois, Urbana, Ill. 1203 W. Illinois St.
Miller, Harlan C. Asso. Prof., North Texas State Teachers Coll., Denton, Tex.
Miller, H. L. Asso. Prof., Univ. of Cincinnati, Cincinnati, Ohio.
Miller, Dr. J. A. Wallingford, Pa.

Miller, J. L. 542 Georgia Ave., Brooklyn, N.Y. *Junior Inspector, Office of Inspectors of Naval Materiel, U. S. Navy Dept. 30 Church St., New York, N.Y.*
Miller, Norman. Prof., Queen's Univ., Kingston, Ont., Canada.
Miller, O. M. Mathematical Geography Dept., American Geographical Society, Broadway and 156th St., New York, N.Y.
Miller, R. A. *Lt., A.U.S., Pre-Flight School, Maxwell Field, Ala.*
Miller, W. L. Asst. Prof., Bucknell Univ., Lewisburg, Pa. 220 S. 3d St.
§**Mills, M. M.** Research Asst. in Aeronautics, California Inst. of Tech., Pasadena, Calif. 405 Northcliffe Road.
§**Millsaps, K. T.** California Inst. of Tech., Pasadena, Calif.
Milne, W. E. Prof., Oregon State Coll., Corvallis, Ore.
Milne, W. P. Prof., Univ. of Leeds, Leeds, England. 16 Monkbridge Road, Headingly.
Mintzer, H. L. Fellow, Brown Univ., Providence, R.I.
Mira, J. A. Prof. Manhattanville Coll. of the Sacred Heart, New York, N.Y. 61 E. 86th St.
Mirick, G. R. Prin., High School, Lincoln School, 425 W. 123d St., New York, N.Y.
Miser, H. J. Instr., Illinois Inst., of Tech., Chicago, Ill.
Miser, W. L. Prof., Vanderbilt Univ., Nashville, Tenn. 1702 Cedar Lane.
von Mises, Dr. R. E. Lecturer, Harvard Univ., Cambridge, Mass.
Mitchell, A. K. Asst. Prof., Trinity Coll., Hartford, Conn.
Mitchell, H. B. Chairman of the Board, Griscom-Russell Co., 285 Madison Ave., New York, N.Y. Chapel Farm, Riverdale-on-Hudson, N.Y.
Mitchell, H. H. Prof., Univ. of Pennsylvania, Philadelphia, Pa. College Hall.
§**Mitchell, Dr. Josephine M.** Instr., Hollins Coll., Hollins College, Va.
Mitchell, Dr. W. L. Instr., Michigan State Coll., East Lansing, Mich. 317 Evergreen Ave.
§**Mittleman, Don.** 300 Fort Washington Ave., New York, N.Y.
Mode, E. B. Prof., Boston Univ., Boston, Mass. Longmeadow Road, Wellesley, Mass.
Modesitt, Dr. Virginia. Wright Junior Coll., Chicago, Ill.
Molina, E. C. Switching Theory Engineer, Bell Telephone Labs., 463 West St., New York, N.Y.
Monsky, Morris. Asst. Actuary, Mutual Life Ins. Co. of N.Y., 32 Nassau St., New York, N.Y.
Montague, Harriet F. Asst. Prof., Univ. of Buffalo, Buffalo, N.Y.
Montgomery, A. G. Asst., Univ. of Minnesota, Minneapolis, Minn. 204 Main Engineering Bldg.
Montgomery, Deane. Prof., Smith Coll., Northampton, Mass.
Montgomery, Dr. J. C. Instr., Univ. of Connecticut, Storrs, Conn.
Montgomery, P. V. Vice Pres. and Actuary, Southland Life Ins. Co., Dallas, Tex.
Montroll, Dr. E. W. Instr., Princeton Univ., Princeton, N.J. Palmer Physics Lab.
Moody, W. A. Prof. Emeritus, Bowdoin Coll., Brunswick, Me. 60 Federal St.
Moon, Parry. Asso. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.
§**Moore, B. C.** Asst. Prof., Agric. and Mech. Coll. of Texas, College Station, Tex. Box 424, Laguna Beach, Calif. *Lt., U.S.N.R.*
Moore, C. N. Dir. of Graduate Studies, and Prof., Univ. of Cincinnati, Cincinnati, Ohio. 219 Woolper Ave.
Moore, Dr. Lillian. Far Rockaway High School, Far Rockaway, N.Y.; General Asst., Central Evening High School, Brooklyn, N.Y. 885 N. 28th St., Philadelphia, Pa.
Moore, L. T. Asso. Prof., Brooklyn Coll., Brooklyn, N.Y. 205 Hicks St.
Moore, M. G. Prof., Tri-State Coll., Angola, Ind. 301 E. Gilmore.
†**Moore, R. L.** Prof., Univ. of Texas, Austin, Tex.
Moore, T. W. Asst. Prof., U. S. Naval Acad., Annapolis, Md.
Moore, W. L. Asso. Prof., Univ. of Louisville, Louisville, Ky. Box 163, R.F.D. 1, Coral Ridge, Ky.
Moots, E. E. Prof., Cornell Coll., Mt. Vernon, Iowa.
Morales, J. D. Prof., Univ. of Puerto Rico, Mayaguez, Puerto Rico. Box 882.
Moran, Dr. C. W. Lane Tech. School, Chicago, Ill. 2950 Jarlath St.
Mordell, L. J. Prof., Univ. of Manchester, Manchester, England.

Morehead, J. C. Prof. Emeritus, Carnegie Inst. of Tech., Pittsburgh, Pa. 4815 Bayard St.

Morenus, Eugenie M. Prof., Sweet Briar Coll., Sweet Briar, Va.

Morgan, Dr. F. M. Dir., Clark School, Hanover, N.H.

Morgan, W. B. 6728 Woodland Ave., Philadelphia, Pa. *Civilian Public Service Camp, Fresno, Ohio.*

Morkovin, Dr. Vladimir. Fellow, School of Mech., Brown Univ., Providence, R.I.

Morley, R. K. Prof., Worcester Polytech. Inst., Worcester, Mass.

Morrey, C. B. Asso. Prof., Univ. of California, Berkeley, Calif. *Mathematician, Ballistic Research Lab., Aberdeen Proving Ground, Md.*

Morrill, Dr. W. K. Asso., Johns Hopkins Univ., Baltimore, Md. 534 Stevenson Lane, Towson, Md.

Morris, F. R. Prof., Fresno State Coll., Fresno, Calif.

Morris, Max. Asso. Prof., Case School of Applied Science, Cleveland, Ohio.

Morris, Richard. Prof., Rutgers Univ., New Brunswick, N.J. 12 Johnson St.

Morris, Dr. W. L. Mathematician, Research Dept., Phillips Petroleum Co., Bartlesville, Okla. 519 Keeler St.

Morrison, Nathan. Senior Statistician, Div. of Research and Statistics, U. S. Employment Service, 342 Madison Ave., New York, N.Y. 420 Ave. F, Brooklyn, N.Y.

Morrow, D. C. Asso. Prof., Wayne Univ., Detroit, Mich.

§**Morrow, Dorothy J.** Fellow, Bryn Mawr Coll., Bryn Mawr, Pa. 4332 8th N.E., Seattle, Wash.

Morse, A. P. Asst. Prof., Univ. of California, Berkeley, Calif. 18 Oak Vale.

Morse, D. S. Prof., Union Coll., Schenectady, N.Y.

Morse, Marston. Prof., Inst. for Advanced Study, Princeton, N.J.

Mosesson, Dr. Z. I. Prudential Ins. Co. of America, Newark, N.J. *U.S.A., Box 192, Ft. Monroe, Va.*

Moskovitz, David. Asst. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa.

Mossman, Thirza A. Asst. Prof., Kansas State Coll. of Agric. and Applied Science, Manhattan, Kan.

Moulton, E. J. Prof., Northwestern Univ., Evanston, Ill. 1114 Colfax St.

Moulton, Dr. F. R. Permanent Sec., American Asso. for the Advancement of Science, Smithsonian Inst. Bldg., Washington, D.C.

Moursund, A. F. Asso. Prof., Univ. of Oregon, Eugene, Ore.

Mowbray, A. H. Prof., Univ. of California, Berkeley, Calif. 119 South Hall.

Muhly, Dr. H. T. Instr., U. S. Naval Acad., Annapolis, Md. 5 School St.

Muhly, Isabel F. (Mrs. H. T.). 5 School St., Annapolis, Md.

Mullemeister, Hermance. Asst. Prof., Univ. of Washington, Seattle, Wash. 1900 E. 68th St.

Mullings, M. E. Prof., Abilene Christian Coll., Abilene, Tex.

Mullins, G. W. Prof., Barnard Coll., Columbia Univ., New York, N.Y.

Mumford, C. G. Asso. Prof., North Carolina State Coll., Raleigh, N.C. State College Station.

Munro, G. C. Asso. Prof., Kansas State Coll. of Agric. and Applied Science, Manhattan, Kan.

§**Munro, W. D.** Asst., Univ. of Minnesota, Minneapolis, Minn. 123 Folwell Hall.

Munroe, Florence L. Retired. 51 Henshaw Ave., Northampton, Mass.

§**Munroe, M. E.** Fellow, Brown Univ., Providence, R.I.

Munshower, C. W. Asso. Prof., Colgate Univ., Hamilton, N.Y.

Murdoch, D. C. Asst. Prof., Univ. of Saskatchewan, Saskatoon, Sask., Canada.

Murnaghan, F. D. Prof., Johns Hopkins Univ., Baltimore, Md. 6202 Sycamore Road.

Murphy, R. D. Vice Pres. and Actuary, Equitable Life Assurance Society of the U.S., 120 Broadway, New York, N.Y. 28 Godfrey Road, Upper Montclair, N.J.

Murray, Dr. F. H. 5412 Kimbark Ave., Chicago, Ill.

Murray, F. J. Asst. Prof., Columbia Univ., New York, N.Y. 2 Arden St.

Murray, W. R. Asso. Prof., Franklin and Marshall Coll., Lancaster, Pa.

Murtaugh, T. H. Asst. Registrar, School of Education, Fordham Univ., New York, N.Y. 597 North Ave., New Rochelle, N.Y.

Musselman, J. R. Prof., Western Reserve Univ., Cleveland, Ohio.
Myers, F. G. Instr., Univ. of Virginia, University Station, Charlottesville, Va. 1410 Park Pl.
Myers, S. B. Asst. Prof., Univ. of Michigan, Ann Arbor, Mich 3020 Angell Hall.
Myers, W. H. Asst. Prof., San Jose State Coll., San Jose, Calif 1344 Heddings St. *2d Lt., U.S.A., Ordnance Presidio, San Francisco, Calif*

Nagel, Ernest. Asso. Prof., Philosophy Dept., Columbia Univ., New York, N.Y.
Nagle, J. L. Head Engineer, Caribbean Div., U.S.A., Corp of Engineers, 711 Park Ave., Elizabeth, N.J.
Nahikian, Dr. H. M. Instr., North Carolina State Coll., Raleigh, N.C. 5548 State College Station.
§**Naim, Yael.** Johns Hopkins Univ., Baltimore, Md.
Nakayama, Tadasi. Asst. Prof., Osaka Imperial Univ., Osaka, Japan.
Nardelli, M. J. Asst. Prof., Fordham Univ., New York, N.Y.
Nash, Dr. J. P. Instr., Univ. of Notre Dame, Notre Dame, Ind. *Research Asso., Radiation Lab., Massachusetts Inst. of Tech., Cambridge, Mass. 541 Commonwealth Ave., Boston, Mass.*
Neelley, J. H. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa. 300 Broadmoor Ave., Mt. Lebanon, Pa.
Neikirk, L. I. Asst. Prof., Univ. of Washington, Seattle, Wash. 4723 21st Ave. N.E.
Nelson, A. L. Prof., Wayne Univ., Detroit, Mich.
Nelson, C. A. Asso. Prof., New Jersey Coll. for Women, Rutgers Univ., New Brunswick, N.J.
Nelson, J. B. Instr., Evening Div., Univ. of Southern California, Los Angeles, Calif. 6345 Ben Ave., North Hollywood, Calif.
§**Nelson, M. L.** Asst., Univ. of California, Berkeley, Calif.
Nelson, N. D. Asst. Univ. of Wisconsin, Madison, Wis.
Nelson, Sara L. Prof., Georgia State Coll. for Women, Milledgeville, Ga. 512 N Columbia St.
Nesbitt, C. J. Asst. Prof., Univ. of Michigan, Ann Arbor, Mich. 3016 Angell Hall.
Ness, Marie M. Research Asso., Univ. of Minnesota, Minneapolis, Minn. 2530 Dupont Ave. S.
Netzorg, Dr. D. L. Box 2714, Manila, Philippine Islands.
Neugebauer, Otto. Prof., Brown Univ., Providence, R.I.
Neuhaus, Dr. Albert. Instr., Univ. of Nebraska, Lincoln, Neb. 1718 N St.
Neuman, Sidney. 717 W. 177th St., New York, N.Y.
von Neumann, John. Prof., Inst., for Advanced Study, Princeton, N.J. *Bureau of Ordnance, U. S Navy Dept, Washington, D C.*
Newell, Dr. H. E. Instr., Univ. of Maryland, College Park, Md.
Newman, J. R. 6019 N. 19th Road, Arlington, Va.
Newman, Dr. Philip. Coll. of the City of New York, New York, N.Y.
Newsom, C. V. Prof., Univ of New Mexico, Albuquerque, N.Mex. 105 S. 14th St.
Newson, Dr. Mary W. (Mrs. H. B.). Retired. Eureka Coll, Eureka, Ill.
Newton, Abba V. Prof., Hartwick Coll., Oneonta, N Y. 22 Maple St.
Newton, G. A. Prof., Trinity Univ., San Antonio, Tex.
Neyman, Jerzy. Dir., Statistical Lab., and Prof., Univ. of California, Berkeley, Calif. *National Defense Research Com.*
Nichols, G. D. Asso. Prof., Univ. of Arkansas, Fayetteville, Ark.
Nielsen, Dr. K. L. Instr., Louisiana State Univ, University, La.
Nightwine, Sister M. Cyprian. Asst. Prof., Xavier Univ., New Orleans, La.
Nilson, E. N. Asst Prof., Mt. Holyoke Coll., South Hadley, Mass. 31 Woodbridge St.
Niven, Dr. Ivan. Instr., Purdue Univ., Lafayette, Ind.
Noble, Dr. Andrewa R. Instr., San Francisco Junior Coll., San Francisco, Calif. 1495 7th Ave.
Noble, C. A. Prof. Emeritus, Univ. of California, Berkeley, Calif. 2224 Piedmont Ave.
Noble, C. E. Northeast Missouri State Teachers Coll., Kirksville, Mo. 407 S. Main St.
Nodelman, H. M. Instr., Radio Corp of America Inst., Inc., 75 Varick St., New York, N.Y.

Noether, G. E. *U S A, Antiaircraft Artillery School, Camp Davis, N.C*
Nordhaus, Dr. **E. A.** Instr., Univ. of Wisconsin at Milwaukee, Milwaukee, Wis.
Nordling, Dr. **C. G. A.** Instr., Univ. of Connecticut, Storrs, Conn.
Norlie, O. M. Prof., St. Olaf Coll., Northfield, Minn.
Nörlund, N. E. Prof., Univ. of Copenhagen, Copenhagen, Denmark. Stockholmsgade 33.
Norman, P. B. Instr., Univ. of Wisconsin, Madison, Wis.
§**Norris, M. J.** Fellow, Harvard Univ., Cambridge, Mass 3544 Portland Ave. S., Minneapolis, Minn.
Norris, Dr. **Nilan.** Instr., Economics Dept., Hunter Coll., 695 Park Ave., New York, N.Y.
Norris, R. E. Prof., State Teachers Coll., Milwaukee, Wis.
§**Northham, J. I.** Instr., U.S.A., Signal Corps, Chicago, Ill. 828 W. Kalamazoo Ave., Kalamazoo, Mich.
Northcott, J. A. Asso. Prof., Columbia Univ., New York, N.Y.
Northrop, Dr. **E. P.** Hotchkiss School, Lakeville, Conn.
Norton, K. A. Senior Radio Engineer, Federal Communications Com, Washington, D.C. 4623 Kenmore Dr. N W.
Nyswander, J. A. Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.

Oakley, C. O. Prof., Haverford Coll., Haverford, Pa.
Oberbeck, T. E. 7230 Glades Ave, Richmond Heights, Mo. *Ensign, U S N.R.*
Oberg, Dr. **E. N.** Instr., State Univ. of Iowa, Iowa City, Iowa.
Oboukhoff, N. M. Prof., Oklahoma Agric. and Mech. Coll., Stillwater, Okla. 402 Jefferson St.
O'Brien, Sister Catherine M. Prof., Coll. of Mt. St. Vincent, New York, N.Y.
O'Callahan, Rev. J. T. Holy Cross Coll., Worcester, Mass.
O'Connor, Rev. R. E. Prof., Loyola Coll., Montreal, Quebec, Canada.
Odle, Dr. **J. W.** Research Mathematician, U. S. Rubber Co., Detroit, Mich. 20506 Picadilly.
O'Donnell, G. A. Dean, Graduate School, Boston Coll., Chestnut Hill, Mass.
Ogden, E. B. Prof., Union Coll, Lincoln, Neb. 4626 Bancroft Ave.
Ogg, F. C. Prof., Bowling Green State Univ, Bowling Green, Ohio.
Oglesby, E. J. Prof., Univ. of Virginia, University Station, Charlottesville, Va. Box 1032.
Oldenburger, Rufus. Prof., Illinois Inst. of Tech. Chicago, Ill. *Math Consultant, Woodward Governor Co. and Askania Regulator Co.*
Olds, C. D. Instr., Purdue Univ., Lafayette, Ind. 208 W. Stadium, West Lafayette, Ind.
Olds, E. G. Asso. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa. 953 La Clair Ave., Regent Sq.
Ollivier, Arthur. Asst. Prof, Mississippi State Coll., State College, Miss. Box 405.
Ollmann, L. F. Asso. Prof., Hofstra Coll., Hempstead, N.Y.
Olmstead, Dr. **P. S.** Lecturer, Stevens Inst. of Tech., Hoboken, N.J.; Tech. Staff, Bell Telephone Labs., 463 West St., New York, N.Y. Box 72, Essex Fells, N J.
Olmsted, Dr. **J. M. H.** Instr., Univ. of Minnesota, Minneapolis, Minn. 111 Arthur Ave. S.E.
Olshen, Dr. **A. C.** Actuary and Chief Examiner, Oregon Ins. Dept., State Office Bldg., Salem, Ore.
Olson, Emma J. Asst. Prof., Kent State Univ., Kent, Ohio. 537 S. Lincoln St.
Olson, F. C. W. Physicist, Research Dept, American Can Co., Maywood, Ill.
Olson, H. L. Asst. Prof., Southwestern Univ., Georgetown, Tex. 1017 College St.
O'Neill, Sister Anna C. Dean, Coll. of St. Elizabeth, Convent Station, N.J.
Opatowski, Dr. **Isaac.** Armour Research Foundation, Illinois Inst. of Tech, Chicago, Ill.
Oppenheim, Alexander. Prof., Raffles Coll., Singapore, Straits Settlements.
Ore, Oystein. Prof., Yale Univ., New Haven, Conn. 326 Hall of Graduate Studies.
Oseen, C. W. Prof., Nobel Inst., Stockholm, 50, Sweden.
Osgood, W. F. Prof. Emeritus, Harvard Univ., Cambridge, Mass. 10 Dorset Road, Belmont, Mass.

Ott, E. R. Asso. Prof., Univ. of Buffalo, Buffalo, N.Y. 20 Nicholson Ave.
Ott, W. P. Prof., Univ. of Alabama, University, Ala.
Overman, J. R. Dean, Coll. of Liberal Arts, and Prof., Bowling Green State Univ., Bowling Green, Ohio.
Owens, F. W. Prof., Pennsylvania State Coll., State College, Pa.
§**Owens, Dr. Glynn.** Massachusetts Inst. of Tech., Cambridge, Mass.
Owens, Dr. Helen B. (Mrs. F. W.). 462 E. Foster Ave., State College, Pa.
Oxtoby, J. C. Asst. Prof., Bryn Mawr Coll., Bryn Mawr, Pa.

Page, Leigh. Prof., Physics Dept., Yale Univ., New Haven, Conn.
Paige, L. J. 686 W. Irving Park Rd., Chicago, Ill. *Ensign, U S N R.*
Pall, Gordon. Asst. Prof., McGill Univ., Montreal, Quebec, Canada. Engineering Bldg.
Palm, W. G. No address. *1st Lt , U.S.A.*
Palmer, T. P. Instr., Rose Polytech. Inst., Terre Haute, Ind. 592 Lincoln Ave., St. Paul, Minn. *2d Lt., U.S.A.*
Palmquist, Dr. K. L. Instr., Southern Methodist Univ., Dallas, Tex. *Asso. Ground School Instr., U. S. Naval Air Station, Cabaniss Field, Corpus Christi, Tex. Box 3031.*
§**Panofsky, Dr. W. K. H.** Asso. Physicist, National Defense Research Com., California Inst. of Tech., Pasadena, Calif.
†**Paranjpye, Sir R. P.** Retired. Purushattam Ashram, Poona, 4, India.
Park, Eugene. No address.
Park, H. V. Asst. Prof., North Carolina State Coll., Raleigh, N.C. 5548 State College Station.
Park, R. S. Prof., Eastern Kentucky State Teachers Coll., Richmond, Ky. 213 Burnam Ct.
Parke, N. G. Aeronautical Engineer, Bureau of Aeronautics, U.S. Navy Dept., Washington, D.C. Malvern Ave., Ruxton, Md.
Parker, S. T. Instr., Univ. of Louisville, Louisville, Ky. 2626 Pindell Ave.
Parker, W. V. Prof., Louisiana State Univ., University, La.
Parsons, C. H. 415 W. 115th St., New York, N.Y.
Pate, Dr. R. S. Instr., Univ. of Kansas, Lawrence, Kan.
Patten, W. E. Asst. Hydraulic Engineer, U.S. Dept. of Agric. 1306 Irish St., South Boston, Va.
Patterson, B. C. Prof., Hamilton Coll., Clinton, N.Y.
Patterson, G. W. Asst. Physicist, National Hydraulic Lab., National Bureau of Standards, U S. Dept. of Commerce, Washington, D C.
Patterson, Dr. J. O. Retired. 16 W. Lynnwood Ave., Glenside, Pa.
Patterson, W. A. Asst. Prof., Fenn Coll., Cleveland, Ohio. 16601 Raymond St., Maple Heights, Ohio.
Paxson, Dr. E. W. Instr., Wayne Univ., Detroit, Mich. *Mathematician, Jam Handy Organization, Detroit, Mich.*
Paxton, E. K. Asso. Prof., Washington and Lee Univ , Lexington, Va. Box 754.
Paydon, J. F. Instr., Northwestern Univ., Evanston, Ill.
Payne, C. K. Asst. Prof., Washington Square Coll., New York Univ., New York, N.Y. Butler, N.J.
§**Peabody, Mary K.** Yale Univ., New Haven, Conn. 246 Park St.
Peach, M. O. Instr. of Mech., Carnegie Inst. of Tech., Pittsburgh, Pa. 51 Maytide St.
Pease, E. M. J. Asst. Prof., Rhode Island State Coll., Kingston, R.I.
Peck, E. R. Research, National Defense Research Com , Physics Dept., Northwestern Univ., Evanston, Ill. 1233 Crain St.
§**Peck, L. G.** Johns Hopkins Univ., Baltimore, Md.
Peebles, G. H. Asst. Prof., Univ. of Minnesota, Minneapolis, Minn. 207 Main Engineering Bldg.
Pegram, Annie M. Prof., Greensboro Coll., Greensboro, N.C.
Pehrson, E. W. Prof , Univ. of Utah, Salt Lake City, Utah.
Pell-Wheeler, Anna. See Wheeler, Anna Pell.
Pence, Sallie E. Asst. Prof., Univ. of Kentucky, Lexington, Ky. 635 Maxwelton Ct.

Penney, W. F. U. S. Navy Dept., Washington, D.C. 621 19th St. N.W.

Pepper, Dr. Echo D. Counselor, Personnel Bureau, and Asso., Univ. of Illinois, Urbana, Ill. 356 Math. Bldg.

Pepper, P. M. Asst. Prof., Univ. of Notre Dame, Notre Dame, Ind.

Perez, Francisco. Asso. Prof., Univ. of the Philippines, Manila, Philippine Islands.

Perkins, F. W. Prof., Dartmouth Coll., Hanover, N.H. 8 Prospect St.

Perkins, H. A. Asso. Prof., Hampton Inst., Hampton, Va.

Perlin, Dr. I. E. Instr., Illinois Inst. of Tech., Chicago, Ill.

§**Perlis, Dr. Sam.** Lockheed Aircraft Corp., Burbank, Calif. 140 N. Jackson St., Glendale, Calif.

Peters, Dr. A. S. Instr., New York Univ., University Heights, New York, N.Y.

Peters, Dr. J. W. Instr., Univ. of Illinois, Urbana, Ill. 360 Math. Bldg.

Peters, Ruth M. Prof., Lake Erie Coll., Painesville, Ohio.

Peterson, O. J. Prof., Kansas State Teachers Coll., Emporia, Kan.

Peterson, T. S. Asst. Prof., Univ. of Oregon, Eugene, Ore. *Physicist, Degaussing Section, 13th Naval District, Seattle, Wash. 2408 N. 45th St.*

Petrie, G. W. *Lt. (j.g.), U.S.N.R., Midshipmen's School, New York, N.Y. 419 W. 119th St.*

Pettis, Dr. B. J. Box 673, Spartanburg, S.C. *2d Lt., U.S.A., Army Post Office, San Francisco, Calif.*

Pettis, C. S. Prof. of Physics, Wofford Coll., Spartanburg, S.C. Box 673.

Pettit, H. P. Prof., Marquette Univ., Milwaukee, Wis. Waterford, Wis.

Pfeiffer, G. A. Asso. Prof., Columbia Univ., New York, N.Y. 1110 Anderson Ave., Palisade, N.J.

Phalen, H. R. Asso. Prof., Coll. of William and Mary, Williamsburg, Va. Box 493.

Phelps, Dr. C. R. *Ensign, U.S.N.R., U. S. Naval Acad., Annapolis, Md.*

Philip, Maximilian. Prof., Coll. of the City of New York, 17 Lexington Ave., New York, N.Y.

Phillips, H. B. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.

Phillips, Dr. R. S. Instr., Harvard Univ., Cambridge, Mass. 20 Highland Ave.

Phipps, C. G. Asso. Prof., Univ. of Florida, Gainesville, Fla. 2132 University Station.

Pierce, J. A. Prof., Atlanta Univ., Atlanta, Ga.

Pierce, T. A. Prof., Univ. of Nebraska, Lincoln, Neb. 1811 Pepper Ave.

Pietenpol, H. W. Dean and Prof., Central Coll., Pella, Iowa.

§**Pinney, Dr. Edmund.** Research Asso., Radiation Lab., Massachusetts Inst. of Tech., Cambridge, Mass.

Pipes, Dr. L. A. Instr. in Electrical Engineering, Harvard Univ., Cambridge, Mass.

Piranian, George. Fellow, Rice Inst., Houston, Tex.

Pitcher, (A.) E. Asst. Prof., Lehigh Univ., Bethlehem, Pa. 422 W. Broad St. *2d Lt., A.U.S., Aberdeen Proving Ground, Md.*

Pixley, Emily C. (Mrs. H. H.). Prof., St. Francis Xavier Coll. for Women, 4900 Cottage Grove Ave., Chicago, Ill. 20194 Briarcliff Ave., Detroit, Mich.

Pixley, H. H. Asso. Prof., Wayne Univ., Detroit, Mich.

Plancherel, Michel. Prof., Federal Inst. of Tech., Zurich, Switzerland. Zeltweg 66.

Plant, L. C. Prof., Michigan State Coll., East Lansing, Mich.

Podmele, Theresa L. East High School, Buffalo, N.Y. 356 Lisbon Ave.

Polachek, Harry. Ballistic Research Lab., Aberdeen Proving Ground, Md. 25 Swan St., Aberdeen, Md.

Polansky, F. J. Actuarial Dept., Northwestern National Life Ins. Co., Minneapolis, Minn. 767 Portland Ave., St. Paul, Minn.

Polanyi, M. Helen (Mrs.). Bennington Coll., Bennington, Vt.

Pollard, Dr. Harry. Instr., Harvard Univ., Cambridge, Mass. 58 Waldeck St., Dorchester, Mass.

Pollard, H. S. Asso. Prof., Miami Univ., Oxford, Ohio. 350 Patterson Ave.

Polley, J. C. Prof., Wabash Coll., Crawfordsville, Ind.

Polya, George. Asso. Prof., Stanford Univ., Stanford University, Calif.

§**Pond, J. A.** Asst., Lehigh Univ., Bethlehem, Pa.

Poor, C. L. Prof., Columbia Univ., New York, N.Y. 35 E. 69th St.

Poor, V. C. Asso. Prof., Univ. of Michigan, Ann Arbor, Mich. 930 Packard St.
Poritsky, Dr. Hillel. General Electric Co., Schenectady, N.Y. 1012 Wendell Ave.
Porter, E. A. Sec. and Actuary, Manhattan Life Ins. Co., New York, N Y. 17 Glen Ridge Pkwy., Bloomfield, N.J.
Porter, Goldie Horton (Mrs. M. B.). Adj. Prof., Univ. of Texas, Austin, Tex. 2402 Windsor Road.
Porter, J. W. Asso. Prof., Bluefield State Teachers Coll., Bluefield, W.Va. *Asst Ground School Instr., U S.N , Chicago Vocational School, Chicago, Ill 528 E. 34th St.*
Porter, M. B. Prof., Univ. of Texas, Austin, Tex. 2402 Windsor Road
Porter, Ruth E. Albany Senior High School, Albany, Ore. Ellsworth Apts
Post, E. L. Asst. Prof., Coll. of the City of New York, New York, N.Y. 610 W. 173d St.
Potron, M. M. J. Prof., Catholic Inst., Paris, France. 42 rue de Grenelle, Paris, 7, France.
Pound, V. E. Prof., Univ. of Buffalo, Buffalo, N.Y. Hayes Hall.
Pounder, I. R. Prof., Univ. of Toronto, Toronto, Ont., Canada.
Povey, Kenneth. Lib., Queen's Univ., Belfast, Northern Ireland.
Powell, J. E. Asso. Prof., Michigan State Coll., East Lansing, Mich.
†Powers, R. E. Box 511, Denver, Colo.
§Prager, Willy. Prof. of Applied Mech., Brown Univ , Providence, R.I.
Pratt, E. A. Civil Engineer, Barber, N.J. R.F.D. 1, Scotch Plains, N.J.
Preiser, T. H. Financial Advisor. 10 Haim Cohen St., Petah Tikvah, Palestine.
Prenowitz, Walter. Asst. Prof., Brooklyn Coll., Brooklyn, N.Y. 2610 Glenwood Road.
Preston, Bernard. Certified Public Accountant, 103 Park Ave., New York, N.Y.
Pretz, P. S. Prof., St. Benedict's Coll., Atchison, Kan.
Price, G. B. Asso. Prof , Univ. of Kansas, Lawrence, Kan. 209 Frank Strong Hall.
Price, Dr. H. V. Instr., University High School and State Univ. of Iowa, Iowa City, Iowa.
Price, Irene. Prof., State Teachers Coll., Oshkosh, Wis.
§Price, J. F. Asst., Univ. of Illinois, Urbana, Ill. 303 2d St. S., Forest Grove, Ore. *U S A , Aberdeen Proving Ground, Md*
Protter, M. H. Asst , Brown Univ., Providence, R.I. 121 Power St.
Puckett, W. T. Asst. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.
Pulliam, F. M. Prof., Lambuth Coll., Jackson, Tenn 1420 N Main St., Decatur, Ill. *U S M C.R.*
Purcell, E. J. Asso. Prof., Univ. of Arizona, Tucson, Ariz. Box 738, R.F D. 5
§Putnam, Dr. A. L. Instr , Yale Univ., New Haven, Conn 781 Yale Station.
§Putnam, George. 50 State St., Boston, Mass.
Putnam, R. G. Asso. Prof., Washington Square Coll., New York Univ., New York, N.Y. 115 Riverview Ave., Tarrytown, N.Y.
Putnam, T. M. Dean, Univ. of California, Berkeley, Calif.

Quade, E. S. Asst. Prof., Univ of Florida, Gainesville, Fla.
Querry, J. W. Prof., Sam Houston State Teachers Coll., Huntsville, Tex. 1313 21st St.
Quilliam, V. V. Cost Accountant, Firestone Tire and Rubber Co , 2525 E. Firestone Blvd., Los Angeles, Calif. 3771 May St , Venice, Calif.
Quinn, Dr. Grace S. (Mrs. R. B.). U. S. Navy Dept., Washington, D.C. 3221 Wheeler Road S E.

Rademacher, Hans. Prof., Univ. of Pennsylvania, Philadelphia, Pa. Box 247, Swarthmore, Pa.
Rado, Tibor. Prof., Ohio State Univ., Columbus, Ohio. 92 Walhalla Road.
Rafferty, T. C. Consulting Actuary, 233 International Bldg , St Louis, Mo.
Rahn, Edris P. Hayward Union High School, Hayward, Calif. 1456 Glen Dr., San Leandro, Calif.
Rainich, G. Y. Prof., Univ. of Michigan, Ann Arbor, Mich. 602 Oswego St.

Rainville, E. D. Asst. Prof., Univ. of Michigan, Ann Arbor, Mich. 1459 Rosewood, R.F.D. 4.

Rambo, Susan M. Prof., Smith Coll., Northampton, Mass. 12 Barrett Pl.

Ramler, O. J. Prof., Catholic Univ. of America, Washington, D.C. 12 Girard St. N.E.

Ramsey, N. F. Asst. Prof., Physics Dept., Columbia Univ., New York, N.Y. *Research Asso., Radiation Lab., Massachusetts Inst. of Tech., Cambridge, Mass. 4 Farrar St.*

Randels, W. C. Asso. Prof., Univ. of Oklahoma, Norman, Okla. Faculty Exch.

Randolph, J. F. Asst. Prof., Cornell Univ., Ithaca, N.Y. White Hall.

Rankin, J. M. Prof., Coll. of Idaho, Caldwell, Idaho. 1810 Ash St.

Rao, K. N. Lib., Annamalai Univ., Annamalainagar, India.

Rasmusen, Dr. Ruth B. Instr., Wilson Junior Coll., Chicago, Ill. 6105 Woodlawn Ave.

Rasor, S. E. Prof., Ohio State Univ., Columbus, Ohio. 1594 Neill Ave.

Rauch, L. L. Fellow, Princeton Univ., Princeton, N.J.

Rauch, Dr. L. M. Seton Hall Coll., South Orange, N.J.

Rauch, S. E. Santa Barbara State Coll., Santa Barbara, Calif.

Raudenbush, H. W. Asst. Prof., Queens Coll., Flushing, N.Y. 84-41 169th St., Jamaica, N.Y.

Rawhouser, Robert. Fort Hill High School, Cumberland, Md. 409 Louisiana Ave.

Rawlins, C. H. Prof., Postgraduate School, U.S. Naval Acad., Annapolis, Md. 13 Franklin St.

Raybould, Ethel H. Lecturer, Univ. of Queensland, Brisbane, Australia.

Rayl, Adrienne S. Asst. Prof., Birmingham Center, Univ. of Alabama, Birmingham, Ala. 1214 N. 32d St.

Raymond, J. H. Instr., St. Martin's Coll., Lacey, Wash.

Raynor, G. E. Asso. Prof., Lehigh Univ., Bethlehem, Pa. 530 10th Ave.

Reade, Dr. Maxwell. Instr., Purdue Univ., Lafayette, Ind.

Reagen, L. M. Asst. Prof., Polytech. Inst. of Brooklyn, Brooklyn, N.Y.

Reaves, S. W. Dean Emeritus and Prof., Univ. of Oklahoma, Norman, Okla.

ReBarker, Herbert. Prof., East Carolina Teachers Coll., Greenville, N.C.

Rechard, O. H. Prof., Univ. of Wyoming, Laramie, Wyo.

†Reddick, H. W. Prof., Cooper Union, New York, N.Y. *Dir., Defense Training Inst., 375 Pearl St., Brooklyn, N.Y.*

Reed, F. W. Prof., Ohio Univ., Athens, Ohio. 61 Columbia Ave.

Reed, L. J. Prof., School of Hygiene, Johns Hopkins Univ., Baltimore, Md. 615 N. Wolfe St.

Rees, C. J. Prof., Univ. of Delaware, Newark, Del.

Rees, Mina. Asso. Prof., Hunter Coll., Bedford Park Blvd. and Navy Ave., New York, N.Y. 140 E. 40th St.

Rees, P. K. Asst. Prof., Southern Methodist Univ., Dallas, Tex.

Regan, Francis. Asso. Prof., St. Louis Univ., St. Louis, Mo.

Rehberg, C. F. Instr. in Electrical Engineering, New York Univ., New York, N.Y. 25-28 84th St., Jackson Heights, N.Y.

Reichelderfer, Dr. P. V. Instr., Univ. of Chicago, Chicago, Ill.

Reid, W. T. Asso. Prof., Univ. of Chicago, Chicago, Ill. Eckhart Hall.

Reingold, Dr. Haim. Instr., Our Lady of Cincinnati Coll., Cincinnati, Ohio. *Illinois Inst. of Tech., Chicago, Ill. 146 N. Parkside Ave.*

Reinsch, B. P. Prof., Florida Southern Coll., Lakeland, Fla. 191 Lake Morton Dr.

Reissner, Eric. Asst. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.

Remick, B. L. Prof., Kansas State Coll. of Agric. and Applied Science, Manhattan, Kan. 613 Houston St.

Rempfer, Dr. R. W. Instr., Rensselaer Polytech. Inst., Troy, N.Y. Carnegie Hall.

Reschovsky, Dr. Helene. Baldwin School, Bryn Mawr, Pa.

Reynolds, C. N. Prof., West Virginia Univ., Morgantown, W.Va. 217 McLane Ave.

Reynolds, Dr. F. G. Retired. 75 Central Park W., New York, N.Y.

Rhodes, E. E. 300 Broadway, Newark, N.J.

Riblet, H. J. Asst. Prof., Hofstra Coll., Hempstead, N.Y. *Research Asso., Radiation Lab., Massachusetts Inst. of Tech., Cambridge, Mass. 26 Concord Ave.*

Rice, Harris. Prof., Worcester Polytech. Inst., Worcester, Mass.

Rice, J. N. Asso. Prof., Catholic Univ. of America, Washington, D.C. 3326 13th St. N.E.
Rich, G. R. Chief Design Engineer, Tennessee Valley Authority, 305 Union Bldg., Knoxville, Tenn.
Richardson, A. V. Prof., Bishop's Coll., Lennoxville, Quebec, Canada.
Richardson, D. P. Asso. Prof., Univ. of Arkansas, Fayetteville, Ark. Cleveland Ave. *Capt., U.S.A.*
Richardson, Dr. Moses. Instr., Brooklyn Coll., Brooklyn, N.Y.
†**Richardson, R. G. D.** Dean, Graduate School, Brown Univ., Providence, R.I. 44 University Ave.
Richert, D. H. Prof., Bethel Coll., North Newton, Kan.
Richmond, C. A. Retired. Tyngsboro, Mass. *Instr., U.S.A., Signal Corps, Harvard Univ., Cambridge, Mass.*
Richmond, D. E. Prof., Williams Coll., Williamstown, Mass. Mt. Pleasant.
†**Richmond, Dr. H. W.** Fellow, King's Coll., Univ. of Cambridge, Cambridge, England.
Rickart, Dr. C. E. Instr., Harvard Univ., Cambridge, Mass. 5 Crawford St.
Ricker, Dr. N. (H.) Senior Geophysicist, Geophysical Research Labs., Carter Oil Co., Tulsa, Okla. 1718 E. 30th St.
Rickey, F. A. Asso. Prof., Louisiana State Univ., University, La.
Rider, P. R. Prof., Washington Univ., St. Louis, Mo.
Riess, J. K. Asst., Physics Dept., Brown Univ., Providence, R.I.
Riesz, Frederick. Prof., Univ. of Szeged, Szeged, Hungary. 2 Baross-u.
Rietz, H. L. Retired. State Univ. of Iowa, Iowa City, Iowa. Physics Bldg.
Rigby, Dr. F. D. Instr., Texas Tech. Coll., Lubbock, Tex.
Rinehart, R. F. Asst. Prof., Case School of Applied Science, Cleveland, Ohio.
Rines, David. Patent Attorney, 739 First National Bank Bldg., 1 Federal St., Boston, Mass.
Riordan, John. Tech. Staff, Bell Telephone Labs., 463 West St., New York, N.Y.
Ripandelli, J. S. 35 Summer St., Forest Hills, N.Y. *U.S.A., Camp Croft, S.C.*
Ritt, J. F. Prof., Columbia Univ., New York, N.Y.
§**Ritter, E. K.** Asst. Instr., Univ. of Pennsylvania, Philadelphia, Pa. *Lt. (j.g.), U.S.N.R., Postgraduate School, U.S. Naval Acad., Annapolis, Md.*
Ritter, Dr. L. F. Instr., Coll. of Engineering, New York Univ., University Heights, New York, N.Y.
Robbins, Dr. H. E. Instr., New York Univ., University Heights, New York, N.Y.
Robbins, Dr. R. B. Vice Pres. and Sec., Teachers Ins. and Annuity Asso., 522 5th Ave., New York, N.Y.
Roberts, J. H. Asso. Prof., Duke Univ., Durham, N.C. *Lt., U.S.N.R.*
Roberts, W. C. Instr., Glendale Junior Coll., Glendale, Calif. 403 E. Fairview Ave.
Robertson, H. P. Prof., Princeton Univ., Princeton, N.J. 175 Prospect Ave.
Robertson, M. S. Asst. Prof., Rutgers Univ., New Brunswick, N.J.
Robertson, W. M. 41 Narbrook Park, Narberth, Pa.
Robinson, Dr. C. V. Instr., Univ. of Notre Dame, Notre Dame, Ind.
Robinson, G. de B. Asst. Prof., Univ. of Toronto, Toronto, Ont., Canada. *Research Officer, National Research Council, Ottawa, Ont., Canada. 151 Metcalfe St.*
Robinson H. A. Prof., Agnes Scott Coll., Decatur, Ga. *Maj., U. S. Military Acad., West Point, N.Y.*
§**Robinson, Julia B.** (Mrs. R. M.). 2633 Durant Ave., Berkeley, Calif.
§**Robinson, L. B.** 131 E. North Ave., Baltimore, Md.
Robinson, L. V. Asst. Prof., Univ. of Mississippi, University, Miss.
Robinson, P. G. Asso. Prof., Iowa State Coll., Ames, Iowa.
Robinson, Robin. Prof., Dartmouth Coll., Hanover, N.H. 16 Allen St.
Robinson, R. M. Asst. Prof., Univ. of California, Berkeley, Calif.
Robinson, Dr. S. L. Instr., Coll. of the City of New York, 139th St. and Convent Ave., New York, N.Y. 4138 Westmoreland St., Little Neck, N.Y.
Robinson, Dr. V. N. Instr., Louisiana State Univ., University, La. 27 15th St. N.E., Atlanta, Ga. *Lt. (j.g.), U.S.N.R., Amphibian Force, Norfolk, Va.*
Robinson, W. J. Asst. Prof., Washington Coll., Chestertown, Md.
Rochford, Sister M. de Pazzi. Briar Cliff Coll., Sioux City, Iowa.

Rock, Dr. **D. H.** Instr., Univ. of Nebraska, Lincoln, Neb. Fellow, School of Mech., Brown Univ., Providence, R.I.

Rodabaugh, L. D. Asst. Prof., Butler Univ., Indianapolis, Ind.

§**Rodd, T. E.** Asst., Univ. of Wisconsin, Madison, Wis.

Rodgers, T. G. Dean Emeritus, New Mexico State Normal Univ., Las Vegas, N.Mex. 1018 4th St.

†**Roe**, Dr. **Josephine R.** (Mrs. E. D.). Retired. 123 W. Ostrander Ave., Syracuse, N.Y.

Roessler, E. B. Asst. Prof. and Asst. Statistician, Coll. of Agric., Univ. of California, Davis, Calif.

Roever, W. H. Prof., Washington Univ., St. Louis, Mo.

Rogers, F. W. Asst. Prof., Univ. of Cincinnati, Cincinnati, Ohio.

Roos, Dr. **C. F.** 500 5th Ave., New York, N.Y.

Roosevelt, G. E. Roosevelt and Son, 30 Pine St., New York, N.Y.

Root, R. E. Prof., U. S. Naval Acad., Annapolis, Md. 7 Franklin St.

Rorer, Dr. **J. T.** Lecturer, Temple Univ., Philadelphia, Pa. 333 N. 34th St.

Rosenbach, J. B. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa. 2550 Beechwood Blvd.

Rosenbaum, Dr. **Benjamin.** Milford School, Milford, Conn. 1251 Boulevard, New Haven, Conn.

Rosenbaum, Ira. *U.S.A., Branch Immaterial Replacement Training Center, Fort McClellan, Ala.*

Rosenbaum, Dr. **Louise Johnson** (Mrs. R. A.). Instr., Reed Coll. Portland, Ore.

Rosenbaum, R. A. Instr., Reed Coll., Portland, Ore.

Rosenblatt, Alfred. Prof., Univ. of San Marcos, Lima, Peru. Calle Atahualpa 192, Miraflores, Peru.

Rosenbloom, J. H. Junior Physicist, Fire Control Lab., U. S. Gov't Service, Philadelphia, Pa. 2913 Tulip St.

§**Rosenbloom, P. C.** Asst., Stanford Univ., Stanford University, Calif

Rosenthal, Arthur. Asst. Prof., Univ. of New Mexico, Albuquerque, N.Mex. 314 S. Maple St.

Rosenthal, Dr. **Jenny E.** Physicist, U.S.A. Signal Corps, Radar Lab., Fort Monmouth, N.J.

Ross, A. E. Asst. Prof., St. Louis Univ., St. Louis, Mo. 1342 McCutcheon, Richmond Heights, St. Louis, Mo.

Rosser, J. B. Asso. Prof., Cornell Univ., Ithaca, N.Y. White Hall.

Rosskopf, Dr. **M. F.** John Burroughs School, 755 S. Price Road, Clayton, Mo.

Roth, S. G. Instr., Washington Square Coll., New York Univ., New York, N.Y.

Roth, W. E. Asso. Prof., Univ. of Wisconsin at Milwaukee, Milwaukee, Wis.

Rothe, E. H. Prof., William Penn Coll., Oskaloosa, Iowa.

Rothrock, D. A. Prof. Emeritus, Indiana Univ., Bloomington, Ind. 1000 Atwater Ave.

Rowe, C. H. Prof., Univ. of Dublin, Dublin, Ireland. 38 Trinity Coll.

Rowland, S. A. Prof., Ohio Wesleyan Univ., Delaware, Ohio. 49 Oak Hill Ave. *Lt. Col., U.S.A., New York Port of Embarkation, Brooklyn, N.Y.*

Royall, N. N. Asso. Prof., Winthrop Coll., Rock Hill, S.C. *Lt. (jg.), U.S.N.R., Communications Office, Navy Yard, U.S. Navy Dept., Charleston, S.C.*

Ruger, Dr. **H. A.** Retired. 526 N. Main St., Wellington, Ohio.

Ruley, B. T. Fellow, School of Mech. and Instr., Engineering Dept., Brown Univ., Providence, R. I.

Rulon, P. J. Asso. Prof., Education Dept., Harvard Univ., Cambridge, Mass. 13 Kirkland St.

Runge, Lulu L. Asst. Prof., Univ. of Nebraska, Lincoln, Neb.

Running, T. R. Prof. Emeritus, Univ. of Michigan, Ann Arbor, Mich. 1019 Michigan Ave.

Rupp, C. A. Prof., Pennsylvania State Coll., State College, Pa.

Rusk, W. J. Retired, 1415 Park St., Grinnell, Iowa.

Russell, Helen G. Asst. Prof., Wellesley Coll., Wellesley, Mass.

Rust, Dr. **W. M.** Head of Geophysics Research, Humble Oil and Refining Co., Houston, Tex.

Rutt, N. E. Prof., Louisiana State Univ., University, La.

§**Ryan, R. A.** Treas., Western and Southern Life Ins. Co., 4th and Broadway, Cincinnati, Ohio. Box 869.

§**Saastad, Arthur.** Asst., Northwestern Univ., Evanston, Ill.
Sabin, Mary S. 1333 E. 10th Ave., Denver, Colo.
Sadowsky, M. A. Asst. Prof., Illinois Inst. of Tech., Chicago, Ill.
†**Safford, F. H.** Prof. Emeritus, Univ. of Pennsylvania, Philadelphia, Pa. 4527 Osage Ave.
Saibel, E. A. Asso Prof. of Mech., Carnegie Inst. of Tech, Pittsburgh, Pa. *Engineer, Waco Aircraft Co, Troy, Ohio.*
Salem, Dr. Raphael. Lecturer, Massachusetts Inst. of Tech., Cambridge, Mass
§**Samuelson, Dr. Hans.** Univ. of Wyoming, Laramie, Wyo.
Samuelson, P. A. Asst. Prof., Economics Dept., Massachusetts Inst. of Tech., Cambridge, Mass.
Sánchez-Díaz, Rafael. Prof., Coll of Agric. and Mech. Arts, Univ. of Puerto Rico, Mayaguez, Puerto Rico. Box 231.
Sanders, S. T. Retired. Box 1322, Baton Rouge, La.
Sanders, S. T., Jr. Prof., Southwestern Louisiana Inst., Lafayette, La. 310 General Mouton St.
Sandidge, W. L. Texas Military Coll., Terrell, Tex. *U S A, Air Forces, S.A T S, Decatur, Ala.*
Sandt, J. E. Asst. Prof., Marietta Coll., Marietta, Ohio. *U.S.A.*
Sanger, R. G. Asst. Prof., Univ. of Chicago, Chicago, Ill. Eckhart Hall.
Sard, Arthur. Asst. Prof., Queens Coll., Flushing, N.Y. 146-19 Beech Ave.
Sarratt, C. M. Prof., Vanderbilt Univ, Nashville, Tenn.
Sasuly, Max. U. S. Social Security Board, Washington, D C 3815 Upton St. N.W.
Satterthwaite, Dr. F. E. Group Statistical Div., Aetna Life Ins. Co., Hartford, Conn.
Saunders, T. J. Prof., Univ. of San Francisco, San Francisco, Calif.
Savage, Dr. L. J. Instr., Cornell Univ., Ithaca, N.Y.
Scammon, R. E. Prof., Univ. of Minnesota, Minneapolis, Minn. 172 Bedford St. S.E.
Scarborough, J. B. Prof., U S. Naval Acad., Annapolis, Md. Ferry Farms.
§**Schaaf, S. A.** Univ. of California, Berkeley, Calif. 24 Wheeler Hall.
Schaefer, M. B. Scientist, International Pacific Salmon Fisheries Com., Univ. of Washington, Seattle, Wash *Ensign, U S.N R., D C O, 12th Naval District, San Francisco, Calif*
Schaeffer, A. C. Asst. Prof, Stanford Univ., Stanford University, Calif.
§**Schafer, Dr. R. D.** *Ensign, U.S N R., U.S S. Essex Care of Postmaster, New York, N Y.*
Schart, Dr. W. J. Asst. Physicist, Aircraft Radio Lab, Wright Field, Dayton, Ohio. 425 Grand Ave.
§**Schatten, Robert.** *U S A, Station Hospital, Fort Benning, Ga*
Scheffé, Dr. Henry. Instr., Princeton Univ., Princeton, N.J. Fine Hall.
Scheier, Rev. M. A. Prof, St. Bonaventure Coll., St Bonaventure, N.Y.
Schelkunoff, Dr. S. A. Consultant in Electromagnetic Theory, Bell Telephone Labs., 463 West St., New York, N.Y.
Scherberg, Dr. M. G. Aerodynamicist, Vought-Sikorsky Aircraft, Stratford, Conn. 87 Locust St., Bridgeport, Conn.
Scherk, Dr. Peter. Asst., Indiana Univ., Bloomington, Ind
§**Schilansky, Dr. Ivor.** Univ. of Michigan, Ann Arbor, Mich.
Schilling, Dr. O. F. G. Instr., Univ. of Chicago, Chicago, Ill. Eckhart Hall.
Schmidt, O. H. Instr., Brown Univ., Providence, R.I.
†**Schmiedel, Oscar.** San Blas Ave., R.F.D. 1, Atascadero, Calif.
Schneckenburger, Dr. Edith R. Instr., Michigan State Normal Coll., Ypsilanti, Mich 210 N. Washington St.
Schneider, Sister M. Christopher. 45 Candler Ave., Highland Park, Mich.
Schoenberg, I. J. Asst Prof, Univ. of Pennsylvania, Philadelphia, Pa. 19 S. Wycombe Ave., Lansdowne, Pa.
Schraut, K. C. Asst. Prof., Univ. of Dayton, Dayton, Ohio.
§**Schuettler, Alberta.** Instr., Wellesley Coll, Wellesley, Mass.
Schwartz, Dr. Abraham. Instr., Pennsylvania State Coll., State College, Pa.
Schwartz, Dr. H. M. Instr, Univ of Illinois, Urbana, Ill 367 Math. Bldg.
Schweigert, Dr. G. E. Instr., Univ. of Missouri, Columbia, Mo.

Schweitzer, Dr. A. R. 452 W. Oakdale Ave., Chicago, Ill.
Schweitzer, E. O. Electrical Engineer, 1241 Waukegan Road, Northbrook, Ill.
Scott, Dr. W. M. Federal Bureau of Investigation, U. S. Dept. of Justice, Washington, D.C.
Scott, Dr. W. T. Instr., Northwestern Univ., Evanston, Ill. *2d Lt., A.U.S., Ordnance Replacement Training Center, Aberdeen Proving Ground, Md.*
Sealander, C. E. Asst. Prof., Univ. of South Dakota, Vermillion, S.Dak.
Sedgewick, C. H. W. Asso. Prof., Univ. of Connecticut, Storrs, Conn
Sedgewick, Dr. Rose W. (Mrs. C. H. W.). Storrs, Conn.
Seebeck, Dr. C. L. Instr., Randolph Field, Tex. Box 303, New Braunfels, Tex
Seekins, C. W. *Ensign, U.S.N.R., U. S. Naval Acad., Annapolis, Md 114 Severn Ave., West Annapolis, Md*
§Seely, Dr. Caroline E. Willseyville, Tioga Co, N.Y
Segal, Dr. I. E. Research Asst., Princeton Univ., Princeton, N.J.
Seidel, Wladimir. Asso. Prof., Univ. of Rochester, Rochester, N.Y. River Campus.
§Seidenberg, Abraham. Johns Hopkins Univ., Baltimore, Md.
Seiler, Brother Louis de LaSalle. Dean of Studies and Prof., St. Mary's Coll, Winona, Minn.
Sellew, G. T. Prof. Emeritus, Knox Coll., Galesburg, Ill. 140 W. 8th St, Claremont, Calif.
Semple, J. G. Prof, King's Coll., Univ. of London, London, W. C. 2, England. The University, Bristol, 8, England.
Senior, Dr. J. K. Research Asso., Chemistry Dept., Univ. of Chicago, Chicago, Ill. 5612 Kenwood Ave.
Serbin, Dr. Hyman. Chief Aerodynamicist, Fairchild Aircraft Co., Hagerstown, Md.
Seth, B. R. Prof, Hindu Coll., Delhi, India.
Sewell, W. E. Asst. Prof., Georgia School of Tech., Atlanta, Ga Care of T. W. Sewell, Newnan, Ga.
†Shah, S. M. Lecturer, Muslim Univ., Aligarh, India. Jail Road.
Shamos, M. H. 45 Christopher St., New York, N.Y.
§Shand, William. Research Asst., California Inst. of Tech, Pasadena, Calif. 551 S. Hill Ave.
§Shanks, E. B. Asst., Univ. of Illinois, Urbana, Ill. 160 Math. Bldg.
Shanks, Dr. M. E. Instr., Univ. of Missouri, Columbia, Mo. 208 Engineering Bldg.
Shannon, Dr. C. E. Bell Telephone Labs, 463 West St, New York, N Y.
§Shapley, Harlow. Prof., Harvard Univ, Cambridge, Mass. Harvard Observ.
§Sharpe, F. R. Prof. Emeritus, Cornell Univ., Ithaca, N.Y. 57 Central Ave., Ocean City, N.J.
Sharpe, Dr. J. A. Chief Physicist, Stanolind Oil and Gas Co., Tulsa, Okla. Box 591 *Special Advisor, National Defense Research Com*
Shaub, H. C. Prof., Washington and Jefferson Coll., Washington, Pa.
Shaw, A. N. Prof, McGill Univ., Montreal, Quebec, Canada. Physics Bldg.
Shaw, W. F. 1733 Ohio Ave., Mercedes, Tex.
§Sheedy, J. W. Asst. Prof., Michigan State Coll., East Lansing, Mich
Sheffer, H. M. Prof., Philosophy Dept., Harvard Univ., Cambridge, Mass. Emerson Hall.
Sheffer, I. M. Prof, Pennsylvania State Coll., State College, Pa. 212 Sparks Bldg.
Sheldon, E. W. Prof., Univ. of Alberta, Edmonton, Alta., Canada.
§Shepard, E. I. Asso. Prof., Williams Coll., Williamstown, Mass.
Shephard, Dr. R. W. Instr., Univ. of California, Berkeley, Calif.
Sheridan, L. W. Prof., Coll. of Mt. St. Vincent, New York, N.Y.
§Sherman, Bernard. Johns Hopkins Univ., Baltimore, Md.
Sherman, Dr. Jacob. Test Lab., Navy Yard, U. S. Navy Dept., Philadelphia, Pa. 5822 Pentridge St.
Sherman, Dr. Seymour. Research Lab., Curtiss-Wright Corp., Buffalo, N.Y
Sherwood, G. E. F. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.

Shewhart, Dr. W. A. Research Statistician, Bell Telephone Labs., 463 West St., New York, N.Y. 158 Lake Dr., Mountain Lakes, N.J.
Shiffman, Dr. Max. Instr., Coll. of the City of New York, 139th St. and Convent Ave., New York, N.Y. 110 E. 177th St., Bronx.
Shirk, J. A. G. Prof., Kansas State Teachers Coll., Pittsburg, Kan.
Shohat, J. A. Prof., Univ. of Pennsylvania, Philadelphia, Pa. 600 S. Eagle Road, Manoa, Pa.
Shonka, E. A. Instr., St. Procopius Coll., Lisle, Ill.
Shook, C. A. Asso. Prof., Lehigh Univ., Bethlehem, Pa. 1122 W. Broad St.
Shook, Dr. R. C. International Statistical Bureau, Inc., 70 5th Ave., New York, N.Y.
Short, R. L. Beaumont High School, St. Louis, Mo. 1234 Broadway, Quincy, Ill.
Shreve, Dr. D. R. Stress Analyst, McDonnell Aircraft Corp., St. Louis, Mo. 7108 Dale Ave., Richmond Heights, St. Louis, Mo.
Shugert, S. P. Prof., Univ. of Pennsylvania, Philadelphia, Pa. College Hall.
Shulits, Samuel. *1st Lt., U.S.A., Corps of Engineers, Engineer Office, Charleston, S.C. Box 516*
Shumway, R. R. Asst. Dean and Prof., Univ. of Minnesota, Minneapolis, Minn.
Shuster, C. N. Prof., New Jersey State Teachers Coll., Trenton, N.J.
Siceloff, L. P. Prof., Columbia Univ., New York, N.Y.
Sigley, D. T. Asso. Prof., Kansas State Coll. of Agric. and Applied Science, Manhattan, Kan. 1710 Fairchild.
Siller, Harry. Asst. Statistician, National Resources Planning Board, Washington, D.C. 42 Independence Ave. S.W.
Silverman, L. L. Prof., Dartmouth Coll., Hanover, N.H.
Simmons, D. H. Fellow, Univ. of California at Los Angeles, Los Angeles, Calif. 260 Aster St., Laguna Beach, Calif. *Instr , U.S.A., Air Corps, Santa Ana, Calif.*
Simmons, H. A. Asso. Prof., Northwestern Univ., Evanston, Ill.
Simon, L. G. Statistician, 225 W. 34th St., New York, N.Y.
Simon, W. G. Vice Pres., Dean and Prof., Western Reserve Univ., Cleveland, Ohio.
Simond, Ruth G. Asso. Prof., Hampton Inst., Hampton, Va. 123 Orchard St., Franklin, N.H. *U. S Navy Dept.*
Simons, Dr. Leo G. Retired. 875 West End Ave., New York, N.Y.
§**Simons, W. H.** Asst., Univ. of California, Berkeley, Calif.
Simpson, Harold. Prof., Bedford Coll., Univ. of London, London, N.W. 1, England.
§**Simpson, R. C.** Asst., Univ. of Wisconsin, Madison, Wis. North Hall.
Simpson, T. M. Prof., Univ. of Florida, Gainesville, Fla.
Sinclair, Mary E. Prof., Oberlin Coll., Oberlin, Ohio. 260 Oak St.
Singer, Dr. James. Instr., Brooklyn Coll., Brooklyn, N Y. 3054 Bedford Ave.
Singer, Dr. P. M. Instr., Extension Div., Univ. of California, Berkeley, Calif. 1288 Campus Dr.
Singleton, H. E. 141 Anacostia Road S.E., Washington, D.C.
Singleton, R. R. Research Asso., Princeton Univ., Princeton, N.J. 20 Nassau St.
Sinkov, Dr. Abraham. Cryptanalyst, Signal Corps, Signal Office, Munitions Bldg., Washington, D.C. *Maj., U.S A.*
†**Sisam, C. H.** Prof., Colorado Coll., Colorado Springs, Colo. 816 N. Weber St.
Slepian, Dr. Joseph. Asso. Dir., Research Labs., Westinghouse Electric and Mfg. Co., East Pittsburgh, Pa.
†**Slichter, C. S.** Dean Emeritus and Prof. Emeritus, Univ. of Wisconsin, Madison, Wis. 636 N. Frances St.
Slobin, H. L. Dean, Univ. of New Hampshire, Durham, N.H.
Slotnick, Dr. M. M. Humble Oil and Refining Co., Houston, Tex. *Supervisor, Ordnance Lab., Navy Yard, U. S. Navy Dept, Washington, D.C. 12 Ridge Road S.E.*
Small, L. L. Prof., Lehigh Univ., Bethlehem, Pa.
Smiley, C. H. Asso. Prof. of Astronomy, Brown Univ., Providence, R.I.
Smiley, Dr. Dorothy M. (Mrs. M. F.). 204 E. Market St., Bethlehem, Pa.
Smiley, M. F. Asst. Prof., Lehigh Univ., Bethlehem, Pa. 204 E. Market St.

Smith, A. H. Asst. Prof., Purdue Univ., Lafayette, Ind.
Smith, A. J. Prof., Susquehanna Coll., Selinsgrove, Pa.
Smith, Dr. **Burke.** Transmission Engineer, Illinois Bell Telephone Co., 212 W. Washington St., Chicago, Ill. 5729 Kimbark Ave.
Smith, Dr. **C. B.** Instr., Montana State Coll., Bozeman, Mont.
†**Smith, Clara E.** Prof. Emeritus, Wellesley Coll., Wellesley, Mass. 14 Waban St.
Smith, C. W. Acting Pres., State Teachers Coll., Superior, Wis.
†**Smith, D. E.** Prof. Emeritus, Teachers Coll., Columbia Univ., New York, N.Y. 501 W. 120th St.
Smith, D. M. Prof., Georgia School of Tech., Atlanta, Ga.
Smith, E. R. Prof., Iowa State Coll., Ames, Iowa.
Smith, E. S. Prof., Univ. of Cincinnati, Cincinnati, Ohio. *Maj., U.S.A., Engineer Depot Columbus, Ohio.*
Smith, F. C. Prof., Coll. of St. Francis, Joliet, Ill.
Smith, Gertrude. Asso. Prof., Vassar Coll., Poughkeepsie, N.Y.
Smith, G. W. Prof., Univ. of Kansas, Lawrence, Kan. 1730 Illinois St.
Smith, Helen F. Instr., Iowa State Coll., Ames, Iowa. 113 Welch Ave.
Smith, H. L. Prof., Louisiana State Univ., University, La. Nicholson Hall.
Smith, H. W. Asso. Prof., Oklahoma Agric. and Mech. Coll., Stillwater, Okla.
Smith, I. W. Prof. Emeritus, North Dakota State Coll., Fargo, N.Dak. Actuary, Ancient Order of United Workmen of North Dakota, 203 10th St. N., Fargo, N.Dak.
Smith, J. C. Instr., Cornell Univ., Ithaca, N.Y.
Smith, Rev. J. P. Church of Gesu, 18th and Thompson Sts., Philadelphia, Pa.
Smith, P. A. Asso. Prof., Columbia Univ., New York, N.Y.
Smith, R. E. Prof., Atlantic Christian Coll., Wilson, N.C. 110 N. Kincaid Ave.
Smith, R. G. Prof., Kansas State Teachers Coll., Pittsburg, Kan.
Smith, S. S. Asst. Prof., Univ. of Utah, Salt Lake City, Utah.
Smith, T. L. Asst. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa.
Smith, W. F. Prof., West Virginia Inst. of Tech., Montgomery, W.Va. Box 788.
Smith, W. H. 320 T St. N.W., Washington, D.C.
Smith, W. M. Prof., Lafayette Coll., Easton, Pa.
Smith, W. N. Asst., Univ. of Wisconsin, Madison, Wis.
Smithies, Dr. **Frank.** Fellow, St. John's Coll., Univ. of Cambridge, Cambridge, England. *Experimental Officer, Ministry of Supply, London, England.*
Snapper, Dr. **Ernst.** Instr., Princeton Univ., Princeton, N.J. 82 Linden Lane.
Snook, T. E. Architect, 299 Broadway, New York, N.Y.
Snow, Dr. **Chester.** Senior Physicist, National Bureau of Standards, U.S. Dept. of Commerce, Washington, D.C. 110 Radio Bldg.
§**Snowdon, S. C.** Fellow, Physics Dept., California Inst. of Tech., Pasadena, Calif.
Snyder, A. D. Asso. Prof., Union Coll., Schenectady, N.Y. 1592 Union St.
§**Snyder, Virgil.** Prof. Emeritus, Cornell Univ., Ithaca, N.Y. 214 University Ave. Visiting Prof., Brown Univ, Providence, R.I.
Snyder, Dr. **W. S.** Instr., Illinois Inst. of Tech., Chicago, Ill. 5649 S. Drexel Ave.
Sobczyk, Dr. **Andrew.** Instr., Oregon State Coll., Corvallis, Ore. *Radiation Lab., Massachusetts, Inst. of Tech., Cambridge, Mass.*
§**Sohl, H. K.** Fellow, School of Mech., Brown Univ, Providence, R.I.
Sohn, Harry. No address.
Sohon, Rev. F. W. Dir., Seismological Observ., and Prof., Georgetown Univ, Washington, D.C.
Sokolnikoff, Dr. **Elizabeth S.** (Mrs I. S.). Lecturer, Univ. of Wisconsin, Madison, Wis. 1111 Oak Way, Shorewood.
Sokolnikoff, I. S. Prof., Univ. of Wisconsin, Madison, Wis. North Hall.
Solomon, Herbert. Statistician, Quartermaster Depot, Jersey City, N.J. 801 E. Tremont Ave., Bronx, New York, N.Y.
Solt, M. R. Asst. Prof., Univ. of New Hampshire, Durham, N.H. Box 283.
Sorgenfrey, Dr. **R. H.** Instr., Case School of Applied Science, Cleveland, Ohio.
Southard, Dr. **T. H.** Instr., Wayne Univ., Detroit, Mich. *Statistician, War Production Board, Washington, D.C. 1504 Seminary Road, Silver Spring, Md.*

Spear, Joseph. Prof., Northeastern Univ., Boston, Mass. 360 Huntington Ave.
Specht, Dr. R. D. Instr., Univ. of Florida, Gainesville, Fla.
Speeker, G. G. Asso. Prof., Michigan State Coll., East Lansing, Mich. Box 861.
Speer, Dr. Mary T. (Mrs.). Instr., Univ. of Pittsburgh, Pittsburgh, Pa.
Spencer, D. C. Asso. Prof., Stanford Univ., Stanford University, Calif.
Spencer, Domina E. Asst. Prof, American Univ., Washington, D.C. 106 Hurst Hall.
Spencer, Dr. Vivian E. Senior Mineral Economist, U.S. Dept. of Commerce, Washington, D C. Room 5122, Commerce Bldg.
Sperry, Pauline. Asso. Prof., Univ. of California, Berkeley, Calif. 28 Wheeler Hall.
Spicer, C. A. Prof., Western Maryland Coll., Westminster, Md. 17 Ridge Road.
Spinks, M. J. 425 W Main St., Wilmington, Ohio.
Spitzbart, Dr. Abraham. Instr., Univ. of Minnesota, Minneapolis, Minn. 126 Folwell Hall
Spooner, C. C. Prof., Northern Michigan Coll of Education, Marquette, Mich. 117 E. Ridge St.
Sprague, A. H. Prof., Amherst Coll., Amherst, Mass. 227 S. Pleasant St.
Springer, C. E. Prof., Univ. of Oklahoma, Norman, Okla.
§Springer, M. D. Asst., Univ. of Illinois, Urbana, Ill. 160 Math. Bldg.
Stabler, E. R. Asst. Prof, Hofstra Coll., Hempstead, N.Y
§Staderman, O. P. Asst., Univ. of Illinois, Urbana, Ill. 160 Math. Bldg.
Stafford, Anna A. Asst. Prof., Univ. of Utah, Salt Lake City, Utah.
Stanwick, C. A. Electrical Engineer. 131 Rynda Road, South Orange, N.J.
Starcher, G. W. Prof, Ohio Univ., Athens, Ohio. R.F.D. 3.
Stark, Marion E. Asso Prof, Wellesley Coll., Wellesley, Mass. 6 Waban St.
Stark, R. H. Instr., Northwestern Univ., Evanston, Ill.
Starke, E. P. Asso Prof., Rutgers Univ., New Brunswick, N.J.
Starr, Dr. D. W. Instr., Southern Methodist Univ., Dallas, Tex.
Stauffer, J. R. K. Kingston, R.I.
§Stechschulte, V. C. Prof., Xavier Univ., Cincinnati, Ohio.
Stecker, Ormelle (Mrs.). Asst. Prof., Pennsylvania State Coll., State College, Pa. 306 Miles St.
Steed, D. V. Asso. Prof., Univ. of Southern California, Los Angeles, Calif.
Steen, F. H. Asst. Prof., Georgia School of Tech., Atlanta, Ga. 532 Peeples St. S.W.
Steenrod, N. E. Asst. Prof, Univ. of Chicago, Chicago, Ill.
Steinhaus, Dr. H. W. Chief, Research Div., Group Dept., Equitable Life Assurance Society of the U.S, 393 7th Ave., New York, N.Y. Box 386, Scarsdale, N.Y.
Stelson, H. E. Prof., Kent State Univ., Kent, Ohio.
Stephan, F. F. Prof. of Sociology and Statistics, Cornell Univ., Ithaca, N.Y. *Chief, Statistical Service, War Manpower Com, Washington, D C. 4924 Little Falls Road, Arlington, Va*
Stephany, E. O. 92 Tyler St., Rochester, N Y.
Stephens, R. C. Prof, Knox Coll, Galesburg, Ill.
Stephens, R. P. Dean, Graduate School, Univ of Georgia, Athens, Ga.
§Sternberg, Dr. Wolfgang. 17 Francis Ave., Cambridge, Mass.
†Stetson, J. M. Prof, Coll. of William and Mary, Williamsburg, Va.
Stevenson, Guy. Prof, Univ. of Louisville, Louisville, Ky.
Stewart, Dr. B. M. Instr., Michigan State Coll., East Lansing, Mich. 705 Grove St.
§Stewart, F. M. Fellow, Harvard Univ., Cambridge, Mass. 31 Conant Hall.
§Stewart, J. C. Asst, Univ. of Illinois, Urbana, Ill
Stewart, S. W. Junior Radio Engineer, U S.A, Signal Corps, Radar Lab, Fort Monmouth, N.J. 503 W. Oliver St., Owosso, Mich
Stibitz, Dr. G. R. Mathematician, Bell Telephone Labs., 463 West St., New York, N.Y. 748 Green St., Boonton, N.J. *Prin Tech Aide, National Defense Research Com*
Stoker, J. J. Asso. Prof., New York Univ., New York, N.Y. 66 Wayne Ave., White Plains, N.Y.

Stokes, C. N. Prof., Temple Univ., Philadelphia, Pa. 6416 N. Camac St.
Stokes, Dr. Ellen C. Instr., New York State Coll. for Teachers, Albany, N.Y.
Stokes, Ruth W. Prof., Winthrop Coll., Rock Hill, S.C. Box 113
Stoll, R. R. Instr., Williams Coll., Williamstown, Mass.
§Stone, Dr. A. H. Inst. for Advanced Study, Princeton, N.J. Fuld Hall.
Stone, M. H. Prof., Harvard Univ., Cambridge, Mass. 30 Hillside Ave.
Stone, R. B. Asso. Prof. and Registrar, Purdue Univ., Lafayette, Ind. 615 Russell St., West Lafayette, Ind.
Stopher, Dr. E. C. Brockport State Normal School, Brockport, N.Y.
Story, Helen F. Instr., St. Petersburg Junior Coll., St. Petersburg, Fla.
Stouffer, E. B. Dean, Univ. of Kansas, Lawrence, Kan. 1019 Maine St.
§Strange, W. J. Asst., Yale Univ., New Haven, Conn. 323 Harkness.
Stratton, W. T. Prof., Kansas State Coll. of Agric. and Applied Science, Manhattan, Kan.
§Straus, E. G. Fellow, Columbia Univ., New York, N.Y. 98 Thayer St.
Strayhorn, Elizabeth C. Asst. Prof., Western Kentucky State Teachers Coll., Bowling Green, Ky. *W A.A.C., Fort Des Moines, Iowa*
Street, Dr. R. E. Asst. Physicist, National Advisory Com. for Aeronautics, Langley Field, Hampton, Va. 146 Hanover Ave.
Strobel, Dr. C. F. Instr., North Carolina State Coll., Raleigh, N.C.
Strodt, Ruth C. (Mrs. W. C.). 45 Tiemann Pl., New York, N.Y.
Strodt, Dr. W. C. Instr., Columbia Univ., New York, N Y. 45 Tiemann Pl.
Struik, D. J. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.
Struik, Dr. Saly R. R. (Mrs. D. J.). Fellow, Radcliffe Coll., Cambridge, Mass. 179 Oakley Road, Belmont, Mass.
§Stump, Margaret E. Asst., Indiana Univ., Bloomington, Ind. 127 Blue Ridge Road, Indianapolis, Ind.
Sturley, E. A. Berkshire School, Sheffield, Mass.
Sturm, Dr. R. G. Research Engineer Physicist, Aluminum Co. of America, New Kensington, Pa. Box 772.
Suffa, Mary C. Prof., Elmira Coll., Elmira, N.Y.
Sugar, Dr. Alvin. 2450 Blaine, Detroit, Mich. *2d Lt, U.S A, Air Corps*
Sullivan, C. T. Prof., McGill Univ., Montreal, Quebec, Canada. Engineering Bldg.
Sullivan, Sister Helen. Prof., Mt. St. Scholastica Coll., Atchison, Kan.
Sullivan, Dr. Mildred M. Instr., Queens Coll., Flushing, N Y
Sunde, E. D. Engineer, Bell Telephone Labs., 463 West St., New York, N.Y.
Supnick, Fred. 4037 New Hampshire Ave. N.W., Washington, D.C
Sutherland, Ethel. Asso., Teachers Coll., Columbia Univ., New York, N.Y.
Sutton, C. S. Asst. Prof., The Citadel, Charleston, S.C.
Sutton, Dr. Flora D. 308 Southway St., Baltimore, Md.
Sutton, F. X. Instr., Suffolk Univ., Boston, Mass
Swain, Dr. R. L. Instr., Univ. of Wisconsin, Madison, Wis.
Swann, Dr. W. F. G. Dir., Bartol Research Foundation, Franklin Inst., Swarthmore, Pa.
Swanson, L. W. Asst. Prof., Coe Coll., Cedar Rapids, Iowa.
†Swift, Elijah. Dean and Prof., Univ. of Vermont and State Agric. Coll., Burlington, Vt. 415 S. Willard St.
Swift, J. D. *1st Lt., U S.A., Camp Bowie, Tex.*
Swinford, Dr. L. H. Instr., Univ. of California, Berkeley, Calif. 1619 Oxford St
Swingle, P. M. Asso. Prof., New Mexico Coll. of Agric. and Mech. Arts, State College, N.Mex. Box 277
Swope, W. G. Instr., Yale Univ., New Haven, Conn. 32 N. Sheffield Hall.
Synge, J. L. Prof., Univ. of Toronto, Toronto, Ont., Canada.
Szász, Dr. Otto. Research Lecturer, Univ. of Cincinnati, Cincinnati, Ohio.
§Szatrowski, Dr. Zenon. Instr., Economics Dept., Northwestern Univ., Evanston, Ill. 1762½ Hinman Ave
Szegö, Gabor. Prof., Stanford Univ., Stanford University, Calif.

Takasu, Tsurusaburo. Prof., Tôhoku Imperial Univ., Sendai, Japan. Math. Inst., Coll. of Science.

Tamarkin, J. D. Prof., Brown Univ., Providence, R.I.
Tansola, J. J. Chief, Pension Div., Dept. of Finance, New York, N.Y. 2041 Watson Ave., Bronx.
Tappan, A. Helen. Dean and Prof., Western Coll., Oxford, Ohio.
Tarski, Dr. Alfred. Lecturer, Univ. of California, Berkeley, Calif.
Taub, A. H. Asst. Prof., Univ. of Washington, Seattle, Wash. Palmer Physical Lab., Princeton Univ., Princeton, N.J.
Taussky, Dr. Olga (Mrs. John Todd). Lecturer, Westfield Coll., Univ. of London, Oxford, England. St. Peter's Hall.
Taylor, A. E. Asst. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.
Taylor, B. P. Instr., Extension Div., Univ. of California at Los Angeles, Los Angeles, Calif. 11326 Gladwin St., West Los Angeles, Calif. *Senior Mathematician, Douglas Aircraft Co., El Segundo, Calif.*
Taylor, Eugene. Prof., Univ. of Idaho, Moscow, Idaho.
Taylor, J. H. Prof., George Washington Univ., Washington, D.C.
Taylor, J. S. Prof., Univ. of Pittsburgh, Pittsburgh, Pa.
Taylor, Mildred E. Prof., Mary Baldwin Coll., Staunton, Va.
Taylor, Dr. W. C. Instr., Univ. of Cincinnati, Cincinnati, Ohio.
Tepping, Dr. B. J. Asso. Statistician, Bureau of the Census, U. S. Dept. of Commerce, Washington, D.C.
Terracini, Alejandro. Prof., Univ. of Tucumán, Tucumán, Argentina. Calle Salta 417.
Terry, M. E. 14 Elm St., Springfield, Vt. *R.C.A.*
Thaxton, H. M. Prof., Agric. and Tech. Coll. of North Carolina, Greensboro, N.C. Box 1074.
Theobald, John. Prof., Loras Coll. of Dubuque, Dubuque, Iowa.
Thielman, H. P. Prof., Coll. of St. Thomas, St. Paul, Minn.
Thomas, C. F. Prof., Case School of Applied Science, Cleveland, Ohio.
Thomas, Dr. G. B. Instr., Massachusetts Inst. of Tech., Cambridge, Mass.
Thomas, J. M. Prof., Duke Univ., Durham, N.C. 4785 Duke Station.
Thomas, Sister M. Placide. Pres., Mt. St. Agnes Junior Coll., Mt. Washington, Baltimore, Md.
Thomas, T. Y. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.
Thompson, E. L. Prof., Texas Tech. Coll., Lubbock, Tex.
Thompson, J. E. Prof., Pratt Inst., Brooklyn, N.Y. 183 Steuben St.
Thompson, Dr. J. M. Western Div., Agric. Adjustment Admin., U.S. Dept. of Agric., Washington, D.C.
Thompson, J. S. Vice Pres., Mutual Benefit Life Ins. Co., 300 Broadway, Newark, N.J.
Thompson, Dr. W. R. 883 Warren St., Albany, N.Y.
Thomson, J. F. Asst. Prof., Tulane Univ., New Orleans, La.
Thorne, Dr. C. J. Instr., Univ. of Michigan, Ann Arbor, Mich.
Thorne, H. H. Lecturer, Univ. of Sydney, Sydney, Australia.
Thornton, Dr. Marian W. (Mrs.). Statistician, Minnesota Dept. of Health, Minneapolis, Minn. 852 Warren St., Albany, N.Y.
Thrall, R. M. Asst. Prof., Univ. of Michigan, Ann Arbor, Mich. 319 W. Engineering Bldg.
Thron, W. J. Fellow, Rice Inst., Houston, Tex.
Thuener, Sister M. Domitilla. Prof., Villa Madonna Coll., Covington, Ky. Villa Madonna Convent, Box 33, R.F.D. 2.
Thullen, Dr. Peter. Asst. Dir., Actuarial Dept., Inst. Nacional de Previsión, Quito, Ecuador. Apartado 636.
Thurman, Dr. G. R. Instr., Univ. of Missouri, Columbia, Mo. 208 Engineering Bldg.
Thurston, H. S. Asso. Prof., Univ. of Alabama, University, Ala.
Tilley, Arthur. Asso. Prof., Washington Square Coll., New York Univ., New York, N.Y.
Timoshenko, S. P. Prof., Stanford Univ., Stanford University, Calif. 536 W. Crescent Dr., Palo Alto, Calif.
Tintner, Gerhard. Asso. Prof., Economics Dept., Iowa State Coll., Ames, Iowa.

Titt, E. W. Asso. Prof., Univ. of Texas, Austin, Tex. 2206 Newfield Lane.
Tolar, Blanche H. (Mrs. M. B.). Asst. Prof., Fenn Coll., Cleveland, Ohio. 3124 E. Overlook Road, Cleveland Heights, Ohio.
Tolle, L. F. Prof., Teachers Coll., St. John's Univ., Brooklyn, N.Y. 240-23 141st Ave., Rosedale, N.Y.
Tolman, R. C. Dean, Graduate School, California Inst. of Tech., Pasadena, Calif.
Tolonen, E. O. Cost Auditor, U.S.A., Air Forces, Detroit, Mich. 15764 Kentucky Ave.
Tompkins, Dr. C. B. Instr., Princeton Univ., Princeton, N.J. Care of R. D. Tompkins, Mountain Home, Tenn. *Lt. (j.g.), U.S.N.R.*
Tonelli, Leonida. Prof., Univ. of Pisa, Pisa, Italy.
Toops, H. A. Prof., Ohio State Univ., Columbus, Ohio. 1430 Cambridge Blvd.
Topp, C. W. Instr., Fenn Coll., Cleveland, Ohio. 1946 Woodward Ave., Cleveland Heights, Ohio.
Tordella, L. W. Asst. Prof., Loyola Univ., Chicago, Ill. 7431 Kenwood Ave. *Lt. (j.g.), U.S.N.R., U. S. Navy Dept., Washington, D.C.*
§**Tornheim, Dr. Leonard.** 4246 West End Ave., Chicago, Ill. *U.S.A., Fort Sill, Okla.*
Torrance, C. C. Asst. Prof., Case School of Applied Science, Cleveland, Ohio.
Torrance, Dr. Esther M. (Mrs. C. C.). 3607 Randolph Road, Cleveland Heights, Ohio.
Torrey, Marian M. Prof., Goucher Coll., Baltimore, Md.
Townsend, E. J. Prof. Emeritus, Univ. of Illinois, Urbana, Ill. 510 E. John St., Champaign, Ill.
Tracey, J. I. Asso. Prof., Yale Univ., New Haven, Conn 84 McKinley Ave.
Trachtenberg, Sidney. *2d Lt, U.S A., Air Forces, Kelly Field, Tex*
Transue, Dr. W. R. Asst., Inst. for Advanced Study, Princeton, N.J. 9 Patton Ave.
Trapani, B. J. 1406 Farr St., Scranton, Pa. *Capt., U.S.A., Camp Livingston, La.*
Tremblay, Althéod. Prof., Laval Univ., Quebec, Quebec, Canada. 367 St. Cyrille St.
Tricomi, Francesco. Prof., Univ. of Turin, Turin, Italy. Via Po 15.
Trimble, Dr. H. C. Instr., Iowa State Teachers Coll., Cedar Falls, Iowa.
Tripp, M. O. Prof. Emeritus, Wittenberg Coll., Springfield, Ohio. 218 W. Cecil St.
§**Tripp, Dr. R. H.** Grumman Aircraft Engineering Corp., Bethpage, N.Y.
Trjitzinsky, W. J. Prof., Univ. of Illinois, Urbana, Ill.
Trott, G. R. Prof., Blue Mountain Coll., Blue Mountain, Miss.
§**Truesdell, C. A.** Instr., Princeton Univ., Princeton, N.J.
Trump, P. L. Asst. Prof., University High School, Univ. of Wisconsin, Madison, Wis.
Tryon, G. M. V. 105 W. Shiawassee Ave., Fenton, Mich.
Tuan, H. F. Princeton Univ., Princeton, N.J.
Tucker, A. W. Asso. Prof., Princeton Univ., Princeton, N.J. Fine Hall.
Tucker, C. B. Asst. Prof., Kansas State Teachers Coll., Emporia, Kan.
Tuckerman, Bryant. Cosmos Club, Washington, D.C.
Tukan, K. H. Prof. and Vice Prin., Najah National Coll., Nablus, Palestine. Box 31.
Tukey, J. W. Asst. Prof., Princeton Univ., Princeton, N.J. Fine Hall.
Tuller, Dr. Annita. Instr., Hunter Coll., Bedford Park Blvd. and Navy Ave., New York, N.Y. 139-62 Pershing Crescent, Jamaica, N.Y.
Tulloch, L. H. Instr., San Antonio Junior Coll., San Antonio, Tex.
§**Turner, Dr. Alice E.** Instr., Connecticut Coll., New London, Conn.
Turner, Bird M. Prof., West Virginia Univ., Morgantown, W.Va. 354 Spruce St.
Turner, Dr. Mary H. (Mrs. M. J.). 150 Washington Ter., Bridgeport, Conn.
Turquette, A. R. Instr., Cornell Univ., Ithaca, N.Y. 1 Sage Ave.
Turrittin, H. L. Asst. Prof., Univ. of Minnesota, Minneapolis, Minn. 4046 Beard Ave. S.

Uhler, H. S. Prof. Emeritus and Lecturer, Yale Univ., New Haven, Conn. 217 Prospect St.
Ulam, S. M. Asst. Prof., Univ. of Wisconsin, Madison, Wis.

Ulrich, Dr. F. E. Instr., Rice Inst., Houston, Tex.
Umberger, E. H. Instr., Pennsylvania State Coll., State College, Pa. 511 E. Foster Ave.
Underhill, A. L. Asso. Prof., Univ. of Minnesota, Minneapolis, Minn. 126 Folwell Hall.
Upton, C. B. Prof. Emeritus, Teachers Coll., Columbia Univ., New York, N.Y.
Urban, F. W. Prof., Central Missouri State Teachers Coll., Warrensburg, Mo. 418 N. Maguire St.
Urner, Dr. S. E. Instr., Los Angeles City Coll., Los Angeles, Calif. 821 N. Mariposa Ave.
Uspensky, J. V. Prof., Stanford Univ., Stanford University, Calif. 511 Hanover St., Palo Alto, Calif.

Vaage, E. F. Telephone Engineer, Bell Telephone Labs., 463 West St., New York, N.Y.
Valentine, F. A. Asst. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.
Vallarta, M. S. Prof., Physics Dept., Massachusetts Inst. of Tech., Cambridge, Mass.
Van Buskirk, H. C. Prof., California Inst. of Tech., Pasadena, Calif. 390 S. Holliston Ave.
Vance, E. P. Asst. Prof., Univ. of Nevada, Reno, Nev.
Vanderslice, J. L. Asst. Prof., Univ. of Maryland, College Park, Md.
Vandiver, H. S. Prof., Univ. of Texas, Austin, Tex.
Van Engen, Henry. Asst. Prof., Iowa State Teachers Coll., Cedar Falls, Iowa.
†**Van Orstrand, Dr. C. E.** Retired. 3906 Morrison St., Chevy Chase Branch, Washington, D.C.
Van Schaack, G. B. Asst. Prof., Michigan State Coll., East Lansing, Mich.
Van Vleck, E. B. Prof. Emeritus, Univ. of Wisconsin, Madison, Wis. 519 N. Pinckney St.
Van Vleck, J. H. Prof., Harvard Univ., Cambridge, Mass. Research Physics Lab.
Varineau, Dr. V. J. *Ground School Instr., U S. Naval Air Station, Pensacola, Fla. 700 E. Mallory St.*
Vass, Dr. J. L Instr., Univ. of Wisconsin at Milwaukee, Milwaukee, Wis.
Vassallo, Brother Godfrey. Prof., Univ. of Portland, Portland, Ore.
Vandreuil, Sister M. Felice. Prof., Mt. Mary Coll., Milwaukee, Wis.
Vaughan, Dr. H. E. Asso., Univ. of Illinois, Urbana, Ill. 907 S. Vine St.
†**Veblen, Oswald.** Prof., Inst. for Advanced Study, Princeton, N.J. 58 Battle Road.
Vest, M. L. Instr., West Virginia Univ., Morgantown, W.Va.
Vezeau, W. A. Instr., Univ. of Detroit, Detroit, Mich.
Vickery, Dr. C. W. Asso. Statistician, Public Roads Admin., Washington, D.C.
†**Vijayaraghavan, Dr. Thirukkannapuram.** Reader, Univ. of Dacca, Dacca, India.
§**Vinograde, Dr. Bernard.** Instr., Univ. of Wisconsin, Madison, Wis.
Virata, E. T. Prof., Univ. of the Philippines, Manila, Philippine Islands.
Vitale, R. L. Junior Electrical Engineer, Board of Transportation of the City of New York, New York, N.Y. *2d Lt., A.U.S., Signal Corps, Radar School, Camp Murphy, Fla.*
Votaw, D. F. Princeton Univ., Princeton, N.J. 73 Graduate Coll.

Waddell, Mary E. G. Univ. of Toronto, Toronto, Ont., Canada. 72 Madison Ave.
Wade, Dr. L. I. National Research Fellow. Inst. for Advanced Study, Princeton, N.J.
Wade, T. L. Asst. Prof., Univ. of Alabama, University, Ala. 2400 Glendale Gardens, Tuscaloosa, Ala.
Wagner, C. C. Prof., Pennsylvania State Coll., State College, Pa. 122 S. Atherton St.
Wagner, Dr. R. W. *Lt. (j.g.), U.S.N.R., U. S. Naval Acad., Annapolis, Md.*
Wagner, T. C. G. Instr., Univ. of Maryland, College Park, Md. 3728 Northampton St. N.W., Washington, D.C.
Wahlin, G. E. Prof., Univ. of Missouri, Columbia, Mo. 1401 Anthony St.
Wakerling, R. K. Asst. Prof., Fresno State Coll., Fresno, Calif.

Wald, Abraham. Asst. Prof., Columbia Univ., New York, N.Y.
Walder, O. E. Asso. Prof., South Dakota State Coll., Brookings, S.Dak. State College Station.
Walker, B. M. Prof. Emeritus, Mississippi State Coll., State College, Miss.; Pres., West Point Oil Mill, Starkville, Miss.
§**Walker, Dr. G. L.** Instr., Univ. of Delaware, Newark, Del.
Walker, Rev. G. W. 47 Grove St., Perry, N.Y.
Walker, R. J. Asst. Prof., Cornell Univ., Ithaca, N.Y. White Hall.
Wall, H. S. Asso. Prof., Northwestern Univ., Evanston, Ill.
Wallace, Dr. A. D. Instr., Univ. of Pennsylvania, Philadelphia, Pa.
Wallace, Dr. P. R. Instr., Massachusetts Inst. of Tech. Cambridge, Mass. 17A Forest St.
Wallman, Dr. Henry. Radiation Lab., Massachusetts Inst. of Tech., Cambridge, Mass. 5 Crawford St.
Walmsley, Charles. Asso. Prof. Dalhousie Univ., Halifax, N.S., Canada. 6 Studley Ave.
Walsh, J. L. Prof., Harvard Univ., Cambridge, Mass. 474 Widener Lib. *Lt. Com., U.S.N.R.*
Walter, R. M. Asst. Prof., New Jersey Coll. for Women, Rutgers Univ., New Brunswick, N.J.
Walton, Dr. L. F. Instr., San Diego State Coll., San Diego, Calif.
Walton, Sister M. Bertrand. Marywood Coll., Scranton, Pa.
Walton, T. O. Prof., Kalamazoo Coll., Kalamazoo, Mich.
Wannemacher, Joseph. 63 Valley Way, West Orange, N.J. *U.S.A., Quartermaster Replacement Training Center, Camp Lee, Va*
Ward, J. A. Prof., Delta State Teachers Coll., Cleveland, Miss. *Asso. Ground School Instr., U. S. Naval Air Station, Corpus Christi, Tex. 1317 Logan St.*
Ward, L. E. Asso. Prof., State Univ. of Iowa, Iowa City, Iowa. 209 Physics Bldg.
Ward, Morgan. Prof., California Inst. of Tech., Pasadena, Calif.
Wardwell, J. F. Asst. Prof., Colgate Univ., Hamilton, N.Y.
Warnock, W. G. Asst. Prof., Univ. of Alabama, University, Ala.
Warschawski, S. E. Asst. Prof., Washington Univ., St. Louis, Mo.
Washburne, A. C. Actuary Emeritus, Berkshire Life Ins. Co., Pittsfield, Mass.
§**Wasow, Dr. W. R.** 166 E. 91st St., New York, N.Y.
Watkeys, C. W. Prof., Univ. of Rochester, Rochester, N.Y. River Campus.
Watson, Marjorie. Asst. Prof., Huntingdon Coll., Montgomery, Ala. Box 6.
Wayne, Alan. Instr., Rhodes School, New York, N.Y.; Junior High School 184, New York, N.Y. 141-24 78th Road, Flushing, N.Y.
Wear, L. E. Asso. Prof., California Inst. of Tech., Pasadena, Calif. 2247 Lambert Dr.
Weatherburn, C. E. Prof., Univ. of Western Australia, Perth, Australia.
Weaver, Dr. Warren. Dir., Div. of Natural Sciences, Rockefeller Foundation, 49 W. 49th St., New York, N.Y. *Chairman, Section D-2, National Defense Research Com.*
Webb, D. L. Asst. Prof., Georgia School of Tech., Atlanta, Ga.
Webber, G. C. Asst. Prof., Univ. of Delaware, Newark, Del.
Webber, W. J. Asso. Prof., Univ. of Toronto, Toronto, Ont., Canada.
Weber, Ernst. Prof., Electrical Engineering Dept., Polytech. Inst. of Brooklyn, Brooklyn, N.Y. *Official Investigator, Office of Scientific Research and Development.*
Webster, M. S. Asst. Prof., Purdue Univ., Lafayette, Ind.
†**Wedderburn, J. H. M.** Prof., Princeton Univ., Princeton, N.J. Fine Hall.
Wegner, K. W. Prof., Coll. of St. Catherine, St. Paul, Minn.
Wehausen, Dr. J. V. Instr., Univ. of Missouri, Columbia, Mo.
Weida, F. M. Prof., Statistics Dept., George Washington Univ., Washington, D.C.
§**Weil, André.** Asst. Prof., Lehigh Univ., Bethlehem, Pa.
Weinberg, I. T. Instr., McMichael Intermediate School, Detroit, Mich. 2611 Gladstone Ave.
Weinstein, Dr. Alexander. Lecturer, Univ. of Toronto, Toronto, Ont., Canada.
Weisner, Louis. Asso. Prof., Hunter Coll., 695 Park Ave., New York, N.Y. 117 W. 197th St., Bronx.

Weiss, Marie J. Prof., Newcomb Coll., Tulane Univ., New Orleans, La.
Welch, Brother B. A. Asst. Prof., Manhattan Coll., New York, N.Y.
Welch, F. P. Asso. Prof., Mississippi State Coll., State College, Miss. 1216 18th Ave., Meridian, Miss. *1st Lt., U.S.A., Chemical Warfare Service, Huntsville Arsenal, Huntsville, Ala.*
Welker, Dr. E. L. Instr., Univ. of Illinois, Urbana, Ill. 1207 W. Daniel St., Champaign, Ill.
Wells, Dr. C. P. Instr., Michigan State Coll., East Lansing, Mich.
Wells, Mary E. Prof., Vassar Coll., Poughkeepsie, N.Y.
Wells, N. W. Instr., Springfield Junior Coll., Springfield, Ill.
Wells, V. H. Asso. Prof., Williams Coll., Williamstown, Mass. 3 Chapin Ct.
Welmers, E. T. Asst. Prof., Michigan State Coll., East Lansing, Mich.
Wente, Irene L. Asst. Prof., South Dakota State Coll., Brookings, S.Dak.
Wernick, Dr. William. McKee Vocational High School, St. George, Staten Island, N.Y. 2515 Davidson Ave., Bronx, New York, N.Y.
Wertheimer, Albert. Engineer, Bureau of Ordnance, U. S. Navy Dept., Washington, D.C.
Wescott, M. E. Asst. Prof., Northwestern Univ., Evanston, Ill. 1936 Greenwood Ave., Wilmette, Ill.
Westbrook, H. S. *U.S.A., Army Post Office, San Francisco, Calif.*
§**Westergaard, H. M.** Dean, Graduate School of Engineering, Harvard Univ., Cambridge, Mass.. Pierce Hall.
†**Western, Dr. A. E.** Windwhistle, Grayshott, Hindhead Surrey, England.
Western, D. W. Instr., Brown Univ., Providence, R.I.
Westfall, W. D. A. Prof., Univ. of Missouri, Columbia, Mo. 11 S. Glenwood Ave.
Westlund, Dr. Jacob. Retired. 301 W. Oak St., West Lafayette, Ind.
de Wet, J. S. Prof., Univ. of Pretoria, Pretoria, South Africa.
§**Wetzel, Marion D.** Fellow, Northwestern Univ., Evanston, Ill.
Wetzel, R. A. Retired. 218 Tecumseh Ave., Mt. Vernon, N.Y.
Weyl, Dr. F. J. Instr., Indiana Univ., Bloomington, Ind.
Weyl, Hermann. Prof., Inst. for Advanced Study, Princeton, N.J.
Whaples, Dr. George. Instr., Johns Hopkins Univ., Baltimore, Md.
Wheeler, A. H. North High School, Worcester, Mass. 59 Davidson Road.
Wheeler, Anna Pell (Mrs. A. L.). Prof., Bryn Mawr Coll., Bryn Mawr, Pa.
Wheeler, C. H. Prof. and Treas., Univ. of Richmond, University of Richmond, Va.
§**Wheeler, D. B.** Instr. in Physics, Occidental Coll., Los Angeles, Calif.
Wheeler, J. J. Asst. Prof., Univ. of Kansas, Lawrence, Kan.
Whelan, A. Marie. Asst. Prof., Hunter Coll., 695 Park Ave., New York, N.Y. 210 E. 68th St.
White, E. F. Asst. Prof., Univ. of Cincinnati, Cincinnati, Ohio.
White, H. S. Prof. Emeritus, Vassar Coll., Poughkeepsie, N.Y. Box 41
White, Dr. Marion B. Retired. 1949 Woodlyn Road, Pasadena, Calif.
§**White, M. E.** 2 Judson St., Hartford, Conn.
White, Dr. Paul. Structural Engineer, North American Aviation, Inglewood, Calif. 3626 W. Adams St., Los Angeles, Calif.
White, R. L. 2122 W. Washington Blvd., Los Angeles, Calif. *Ensign, U.S.N.R.*
†**Whited, Dr. Willis.** Retired. 1407 Market St., Harrisburg, Pa.
Whitehead, Dr. G. W. Instr., Purdue Univ., Lafayette, Ind.
Whiteman, Dr. A. L. Instr., Harvard Univ., Cambridge, Mass. *Ensign, U.S.N.R., 5529 Osage Ave., Philadelphia, Pa.*
Whitford, D. E. Asst. Prof., Polytech Inst. of Brooklyn, Brooklyn, N.Y.
Whitman, E. A. Asso. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa.
Whitman, Dr. P. M. Instr., Univ. of Pennsylvania, Philadelphia, Pa.
Whitmore, Dr. W. F. Asst. Physicist, Ordnance Lab., Navy Yard, U. S. Navy Dept., Washington, D.C.
Whitney, A. W. Consulting Dir., National Conservation Bureau, 60 John St., New York, N.Y.
Whitney, D. R. 551 Dunbar Ter., Lincolnshire, Crete, Ill. *Ensign, U.S.N.R.*
Whitney, Hassler. Asso. Prof., Harvard Univ., Cambridge, Mass. Sudbury Road, Weston, Mass.

†**Whittaker, E. T.** Prof., Univ. of Edinburgh, Edinburgh, Scotland. 48 George Sq., Edinburgh, 8, Scotland.

†**Whittemore, J. K.** Asso. Prof., Yale Univ., New Haven, Conn. 45 Lincoln St.

Whyburn, G. T. Prof., Univ. of Virginia, University Station, Charlottesville, Va. Colonnade Club.

Whyburn, W. M. Prof., Univ. of California at Los Angeles, Los Angeles, Calif. 715 Malcolm Ave.

†**Widder, D. V.** Prof, Harvard Univ., Cambridge, Mass. 572 Widener Lib.

Widder, Dr. **Vera A.** (Mrs. D. V.). 43 Linnaean St., Cambridge, Mass.

Widmark, L. E. Chief Engineer, Star Electric Motor Co., Newark, N.J. 6 Pen Bryn Road, West Orange, N.J.

§**Wiegmann, N. A.** Asst., Univ. of Wisconsin, Madison, Wis. North Hall.

Wiener, Norbert. Prof, Massachusetts Inst. of Tech., Cambridge, Mass.

Wiesenberg, Albert. Columbia Univ., New York, N.Y. 80 Van Cortlandt Park S., Bronx.

Wiggin, Evelyn P. Prof., Randolph-Macon Woman's Coll., Lynchburg, Va. Lecturer, Wellesley Coll., Wellesley, Mass. 26 Horton House.

Wigner, E. P. Prof., Princeton Univ., Princeton, N.J. Fine Hall.

Wilcox, L. R. Asst. Prof., Illinois Inst. of Tech., Chicago, Ill.

Wilczewski, Rev. **Joseph.** Prof., Marquette Univ., Milwaukee, Wis.

Wilder, C. E. Prof., Dartmouth Coll., Hanover, N.H. 5 Dana Road.

Wilder, R. L. Prof., Univ. of Michigan, Ann Arbor, Mich. Angell Hall.

Wiley, F. B. Prof, Denison Univ., Granville, Ohio.

§**Wilkins,** Dr. **J. E.** Inst. for Advanced Study, Princeton, N.J.

Wilks, S. S. Asso. Prof., Princeton Univ., Princeton, N.J.

Williams, A. R. Asst. Prof., Univ. of California, Berkeley, Calif. 455 Wheeler Hall.

Williams, C. W. Prof., Armstrong Junior Coll., Savannah, Ga.

Williams, Dr. **Emily C.** (Mrs. W. H.). 7 E. 63d St., New York, N.Y.

Williams, F. G. Prof. and Registrar, Pennsylvania Military Coll., Chester, Pa.

Williams, H. B. Prof. Emeritus, Columbia Univ., 630 W. 168th St., New York, N.Y.

Williams, J. D. Box 587, R.F.D. 10, Phoenix, Ariz. *Official Investigator, National Defense Research Com.*

Williams, J. E. Dean, Virginia Polytech. Inst., Blacksburg, Va.

Williams, K. P. Prof., Indiana Univ., Bloomington, Ind. 523 E. 3d St.

Williams, L. B. Instr., Hamilton Coll., Clinton, N.Y.

Williams, Dr. **Martha H.** (Mrs. J. W.). Instr., Univ. of Maryland, College Park, Md. Prince George Apts., Hyattsville, Md.

Williams, W. J. Prof., Baylor Univ., Waco, Tex.

Williams, W. L. G. Prof., McGill Univ., Montreal, Quebec, Canada. Engineering Bldg.

Williamson, C. O. Prof., Coll. of Wooster, Wooster, Ohio 1141 Beall Ave.

Williamson, John. Asso. Prof., Queens Coll., Flushing, N.Y.

Wilson, A. H. Prof. Emeritus, Haverford Coll., Haverford, Pa.

Wilson, E. B. Prof., School of Public Health, Harvard Univ., Boston, Mass. 55 Van Dyke St.

Wilson, Dr. **Elizabeth W.** 1 Waterhouse St., Cambridge, Mass.

Wilson, H. A. Prof., Physics Dept., Rice Inst., Houston, Tex. *Official Investigator, Office of Scientific Research and Development.*

Wilson, L. T. Prof., U. S. Naval Acad., Annapolis, Md. 20 Thompson St.

§**Wilson, N. R.** Prof., Univ. of Manitoba, Winnipeg, Man., Canada.

Wilson, R. L. Asst. Prof. of Military Science and Tactics, Alabama Polytech. Inst., Auburn, Ala. *1st Lt., U.S.A.*

Wilson, Dr. **Wilfrid.** Asso., Univ. of Illinois, Urbana, Ill. 362 Math. Bldg.

Wilson, W. A. Prof., Yale Univ., New Haven, Conn. 1960 Chapel St.

Wilson, W. H. Asso. Dean, Coll. of Arts and Sciences, Univ. of Florida, Gainesville, Fla. 2227 University Station.

†**Wilton, J. R.** Prof, Univ of Adelaide, Adelaide, Australia.

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Winton, L. S. Asso. Prof., North Carolina State Coll., Raleigh, N.C. 5548 State College Station.
Wirth, H. P. Asst. Prof., Coll. of the City of New York, 139th St. and Convent Ave., New York, N.Y.
Wise, Dr. **W. H.** Bell Telephone Labs., 463 West St., New York, N.Y.
Witmer, E. E. Asst. Prof., Univ. of Pennsylvania, Philadelphia, Pa. Randall Morgan Lab.
Wolf, Dr. **František.** Instr., Univ. of California, Berkeley, Calif.
Wolf, Louise A. Asst. Prof., Univ. of Wisconsin at Milwaukee, Milwaukee, Wis. 3700 S. 116th St., West Allis, Wis.
Wolfe, Herbert. Brooklyn Coll., Brooklyn, N.Y. 314 Pulaski St.
Wolfe, H. E. Asso. Prof., Indiana Univ., Bloomington, Ind. 812 S. Fess Ave.
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Wolfenden, H. H. Consulting Actuary and Statistician. 182 Rosedale Heights Dr., Toronto, Ont., Canada.
Wolfowitz, Dr. **Jacob.** Research Asso., Div. of War Research, Columbia Univ., 401 W. 118th St., New York, N.Y.
Wolkow, Ruth. Barret Junior High School, Louisville, Ky. 2114 Bonnycastle Ave.
Wong, B. C. Asst. Prof., Univ. of California, Berkeley, Calif. 1933 Grant St.
§**Wong,** Dr. **Y. K.** Research Asst., Univ. of Chicago, Chicago, Ill.
Wood, Frederick. Dean, Coll. of Arts and Sciences, Univ. of Nevada, Reno, Nev.
Wood, F. E. Asso. Prof., Northwestern Univ., Evanston, Ill.
Wood, Dr. **H. A.** Applied Mathematician, Vought-Sikorsky Aircraft, Stratford, Conn. 1843 Elm St.
Woodard, D. W. Prof., Howard Univ., Washington, D.C. 127 W St. N.W.
Woodbridge, Dr. **Margaret Young** (Mrs. D. E.). Instr., Brooklyn Coll., Brooklyn, N.Y. 169 Columbia Heights.
Woodbury, M. A. 248 University St., Salt Lake City, Utah. *2d Lt., U.S.A., Air Corps, Colorado Springs, Colo.*
Woods, F. S. Prof. Emeritus, Massachusetts Inst. of Tech., Cambridge, Mass. 123 Sumner St., Newton Centre, Mass.
Woods, Roscoe. Asso. Prof., State Univ. of Iowa, Iowa City, Iowa. 210B Physics Bldg.
Woolson, J. R. Asst. Seismologist, United Geophysical Co., 1255 E. Green St., Pasadena, Calif.
Worth, C. R. Asst. Prof., Rutgers Univ., New Brunswick, N.J. *Lt., U.S.N.R., U. S. Naval Acad., Annapolis, Md.*
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Wundheiler, Dr. **Alexander.** Instr., Physics Dept., Coll. of the City of New York, 139th St. and Convent Ave., New York, N.Y. 452 W. 144th St.
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Currier
Echols
Hawkins
Kells
Lindquist
McGaughey
Moore, T. W.
Muhly, H. T.
Phelps
Rawlins
Root
Scarborough
Seekins
Wagner, R. W.
Wilson, L. T.
Arnold
Gere
Baltimore
Dawkins
Lukacs
Robinson, L. B.
Sutton, F. D.
Goucher Coll.
Bacon, C. L.
Lewis, F. P.
Torrey
Johns Hopkins Univ.
Bankier
Barten
Bourne
Bowie, C.
Bradshaw, C. H.
Cohen, A.
Curran
Dowker
Kershner
Larsen
Light
Morrill
Murnaghan
Nairn
Peck, L. G.
Reed, L. J.
Seidenberg
Sherman, B.
Whaples
Wintner
Zariski
Mt. St Agnes Junior Coll.
Thomas, M. P.
Chestertown
Robinson, W. J.
College Park
Univ. of Maryland
Dantzig, T.
Jackson, S. B.
Lancaster
Martin, M. H.
Newell
Vanderslice
Wagner, T. C. G.
Williams, M. H.
Cumberland
Rawhauser
Frostburg
Hallett
Hagerstown
Serbin
Hyattsville
Harkin
Westminster
Spicer
Woodstock
McCauley

MASSACHUSETTS

Amherst
Amherst Coll.
Brown, B. L.
Esty
Kleene
Sprague
Belmont
Franklin, C. W.
Boston
Coolidge, T. J.
Dysart
Garabedian, H. A.
Gould, A. B.
Miller, A. L.
Putnam, G.
Rines
Boston Univ.
Bruce
Mode
Harvard Univ.
Brown, T. H.
Wilson, E. B.
Northeastern Univ
Haskins, E. E.
Spear
Suffolk Univ.
Sutton, F. X.
Cambridge
Sternberg
Widder, V. A.
Wilson, E. W.
Harvard Univ.
Beatley
Birkhoff, G.
Birkhoff, G. D.
Bridgman
Caywood
Charnes
Clifford, H. E.
Coolidge, J L.
Hewitt
Huntington
Kaplansky
Kelley, T. L.
Kemble
Levit
Loomis
Lorenz
Lyman
MacLane, S.
von Mises
Norris, M. J.
Osgood
Phillips, R. S.
Pipes
Pollard, H.
Rickart
Rulon
Shapley
Sheffer, H. M.
Stewart, F. M.
Stone, M. H.
Van Vleck, J. H.
Walsh
Westergaard
Whiteman
Whitney, H.
Widder, D. V.
Massachusetts Inst. of Tech.
Arnold, K. J.
Barta
Cameron, R. H.
Clifford, A. H.
Crout

Farley
Franklin, P.
Harvey, G. G.
Heller
Hildebrand
LaSalle
Levinson, N.
Loud
MacLane, G. R.
Martin, W. T.
Moon
Owens, G.
Phillips, H. B.
Pinney
Reissner
Salem
Samuelson
Struik, D. J.
Thomas, G. B.
Vallarta
Wallace, P. R.
Wallman
Wiener
Woods, F. S.
Yagi
Zeldin
Radcliffe Coll.
Adams, R. B.
Graustein
Struik, S. R. R.
Chestnut Hill
Marcou
O'Donnell
Chicopee
Madden
Hyannis
McMullen
Ipswich
Kinsman
Lynn
Bauer
Medford
Barnes
Northampton
Munroe, F. L.
Smith Coll.
LeCaine
McCoy, N. H.
Montgomery, D.
Rambo
Norton
Garabedian, C. A.
Pittsfield
Washburne
Sheffield
Sturley
Southbridge
Boeder
South Hadley
Mt. Holyoke Coll.
Litzinger
Nilson
Zygmund
Springfield
Bowie, H. E.
Swampscott
Evans, G. W.
Tyngsboro
Richmond, C. A.
Wellesley
Wellesley Coll.
Copeland, L. P.
Hazard, K. E
Merrill, H. A.
Russell
Schuettler
Smith, C. E.
Stark, M. E.
Young, M. M.
Weston
Burke, L.
Williamstown
Williams Coll.
Agard
Hardy, J. G.
Richmond, D. E.
Shepard
Stoll
Wells, V. H.
Wray
Worcester
Wheeler, A. H.
Holy Cross Coll.
McCormick
O'Callahan
Worcester Polytech. Inst.
Gay
Morley
Rice, H.

MICHIGAN

Ann Arbor
Ellis, W.
Forsythe, G. E.
Grennan
Univ. of Michigan
Bartels
Beckenbach
Bradshaw, J. W.
Churchill
Civin
Coe
Copeland, A. H.
Craig, C. C.
Dushnik
Duthie
Dwyer
Eilenberg
Field
Fischer
Goldstine
Grau
Hay
Hildebrandt, T. H.
Hopkins, L. A.
Kaplan, S.
Kaplan, W.
Karpinski
Kazarinoff
Love
Myers, S. B.
Nesbitt
Nyswander
Poor, V. C.
Rainich
Rainville
Running
Schilansky
Thorne, C. J.
Thrall
Wilder, R. L.
Detroit
Bagby
Blankinship
Dunford, M. J.
Fleiger
Jones, C. D.
Levin, J. H.
Odle
Sugar
Tolonen
Weinberg
Great Lakes Coll.
Denton
Univ. of Detroit
Butler, J. F.
Hausmann
McCarthy, E. D.
Vezeau
Wayne Univ.
Borgman
Coral
Folley
Friedman, M.
Morrow, D. C.
Nelson, A. L.
Paxson
Pixley, H. H.
Southard
East Lansing
Michigan State Coll.
Beth
Blanche
Grove, V. G.
Heyda
Hill, J. D.
Hurd
Kibbey
Mitchell, W. L.
Plant
Powell
Sheedy
Speeker
Stewart, B. M.
Van Schaack
Wells, C. P.
Welmers
Zimmer
Fenton
Tryon
Highland Park
Schneider
Kalamazoo
Walton, T. O.
Marquette
Spooner
Owosso
Stewart, S. W.
Ypsilanti
Schneckenburger

MINNESOTA

Collegeville
Kalinowski
Duluth
Cothran
Kirchen
Hibbing
Erickson, R. W.
Minneapolis
Bearman
Koehler, F.
Polansky
Thornton
Univ. of Minnesota
Amundson
Brink
Bussey
Campaigne
Carlson, E.
Crawford, W S. H.
Dalaker
Gibbens
Hart, W. L.
Hill, E. L.
Jackson, D.
McEwen, W. R.
Montgomery, A. G
Munro, W. D.
Ness

Olmsted
Peebles
Scammon
Shumway
Spitzbart
Turrittin
Underhill

Moorhead
Anderson, M. R.
Northfield
Erickson, W. S.
Norlie
Rochester
Hickman

St. Paul
Coll. of St. Catherine
Wegner
Coll. of St. Thomas
Bush, L. E.

Landshoff
Thielman
Macalester Coll
Camp, E. J.
Winona
French
Seiler

MISSISSIPPI

Blue Mountain
Trott
Cleveland
Ward, J. A
Decatur
Dearman

Jackson
McCoy, D.
State College
Mississippi State Coll.
Ollivier

Walker, B. M.
Welch, F. P.
University
Hume
Robinson, L. V.

Vicksburg
Benscoter

MISSOURI

Cape Girardeau
Michel
Clayton
Rosskopf
Columbia
Haynes
Univ. of Missouri
Betz, H.
Blumenthal
Ewing, G. M.
Ferguson, W. E.
Gillam
Schweigert
Shanks, M. E.
Thurman
Wahlin
Wehausen

Westfall
Fayette
Burcham
Kansas City
Basye
Melcher
Rockhurst Coll.
Doyle, W. C.
St. Teresa's Coll.
Lackay
Kirksville
Noble, C. E.
Lexington
Miller, F. A.
Richmond Heights
Oberbeck

Roanoke
Ferguson, W. A.
Rolla
Erkiletian
St. Louis
Cronvich
Green, F. W.
Rafferty
Short
Shreve
Maryville Coll.
Kernaghan
St. Louis Coll of Pharmacy
DeFoe
St. Louis Univ.
Case

Regan
Ross
Washington Univ
Avann
Dunkel
Rider
Roever
Warschawski
Sedalia
Mayer, J. I.
Springfield
H'Doubler
Drury Coll.
Finkel
Graves, W. L.
Warrensburg
Urban

MONTANA

Bozeman
Hurst, J. W.
Smith, C. B.

Garrison
Canning

Missoula
Montana State Univ.
Carey, E. F. A.

Chatland
Dubisch
Lennes
Merrill, A. S.

NEBRASKA

Lincoln
Arnold, H. A.
Union Coll.
Ogden
Univ of Nebraska
Basoco

Brenke
Camp, C. C.
Candy
Collins
Doole
Gaba

Neuhaus
Pierce, T. A.
Rock
Runge
Omaha
Earl

Fitzpatrick
Wayne
Hove

NEVADA

Reno
Univ. of Nevada
Beesley

Vance
Wood, F.

NEW HAMPSHIRE

Concord
Conwell, G. M.
Durham
Univ. of New Hampshire
Giddings

Lewis, D. C.
Slobin
Solt
Hanover
Morgan, F. M.

Dartmouth Coll.
Brown, B. H.
Forsyth
Haskins, C. N.
Mathewson

Perkins, F. W.
Robinson, R.
Silverman
Wilder, C. E.

NEW JERSEY

Barber
Pratt
Belleplain
Durrell
Caldwell
Fowler, F. H.
Convent Station
O'Neill
Elizabeth
Nagle
Fort Monmouth
Rosenthal, J. E.
Hightstown
Litterick
Hoboken
Stevens Inst. of Tech.
Davis, H. N.
Deimel
Hazeltine
Jacobus
Korn
Olmstead
Jersey City
McGrath
Solomon
Long Branch
Helly
Madison
Knobelauch
Moorestown
Epstein, M. G.
Mountain Lakes
Hotelling, S. E.
Newark
Bartleson
Brown, A. A. F.
Bulloch
Levinson, H. C.
Mosesson
Rhodes
Thompson, J. S.
Widmark
Newark Coll of Engineering
Fithian
New Brunswick
Rutgers Univ.
Brasefield
Bunyan
Grant, H. S.
Kleinschmidt
Meder
Morris, R.
Nelson, C. A.
Robertson, M. S.
Starke
Walter
Worth
Princeton
Inst. for Advanced Study
Alexander
Bargmann
Cohen, I. S.
Fubini
Gödel
Maharam
Mayer, W.
Morse, M.
von Neumann
Stone, A. H.
Transue
Veblen
Wade, L. I.
Weyl, H.
Wilkins
Princeton Univ.
Adams, E. P.
Ambrose
Anderson, T. W.
Bochner
Bohnenblust
Brown, G. W.
Chevalley
Church, A.
Eisenhart, L. P.
Flood
Gillespie
Lefschetz
McMillan
Montroll
Rauch, L. L.
Robertson, H. P.
Scheffé
Segal
Singleton, R. R.
Snapper
Tompkins
Truesdell
Tuan
Tucker, A. W.
Tukey
Votaw
Wedderburn
Wigner
Wilks
Rumson
Greenwald
South Orange
Rauch, L. M.
Stanwick
Trenton
Shuster
Upper Montclair
Campbell, G. A.
New Jersey State Teachers Coll.
Davis, D. R.
Hildebrandt, E. H. C.
Mallory
West Orange
Edison
Wannemacher

NEW MEXICO

Albuquerque
Newsom
Rosenthal, A.
Las Vegas
Rodgers
Roswell
Alden
State College
Joseph
Swingle

NEW YORK

Albany
Frankel, E. T.
Thompson, W. R.
New York State Coll. for Teachers
Beaver
Birchenough
DoBell
Lester
Stokes, E. C.
Alfred
Lowenstein
Aurora
Hollcroft, M. P.
Wells Coll.
Carroll-Rusk
Clement, M. D.
Hollcroft, T. R.
Beacon
Hammer, H.-K.
Bethpage
Tripp, R. H.
Brockport
Stopher
Bronxville
Gleason, A. M.
Brooklyn
Annenberg
Blake, E. M.
Hopkins, M. W.
Karlin
Kramer-Lassar
Locke
Miller, J. L.
Wynne
Brooklyn Coll
Boeker
Borofsky
Boyer
Greenspan
Griffin, H.
Johnson, R. A.
Kennison
Kormes, J. P.
Landers, A. W.
McMahon
MacNeish
Maria, A. J.
Maria, D. H.
Moore, L. T.
Prenowitz
Richardson, M.
Singer, J.
Wolfe, H.
Wolfe, J.
Woodbridge
Long Island Univ.
Lieber, H. G.
Lieber, L. R.
Polytech. Inst. of Brooklyn
Berry, W. J.
Bond
Fenn
Greeley
Reagen
Weber
Whitford
Pratt Inst.
Cowles
Thompson, J. E.
St. John's Univ.
Cassidy
Cowan
McCarthy, J. J.
Tolle
St. Joseph's Coll. for Women
Larkin

Buffalo
Carner
Cramer, R. H.
Komm
Luneburg
Podmele
Sherman, S.
D'Youville Coll.
Maloney, M. M.
Univ. of Buffalo
Gehman
Montague
Ott, E. R.
Pound
Canton
Bates
Clinton
Hamilton Coll.
Carruth, W. M.
Ferry
Patterson, B. C.
Williams, L. B.
Elmira
Suffa
Farmingdale
Harrison, M. G.
Far Rockaway
Moore, L.
Flushing
Queens Coll.
Archibald, R. G.
Brown, A. B.
Cairns, S. S.
Cope
Eaton
Feld
Hartman
Hoffmann
Raudenbush
Sard
Sullivan, M. M.
Williamson, J.
Zippin
Forest Hills
Ripandelli
Garden City
Bowden, J.
Geneva
Durfee, Walter H.
Hubbs
Hamilton
Munshower
Wardwell
Hayts Corners
Ford, W. B.
Hempstead
Hofstra Coll.
Ollmann
Riblet
Stabler
Houghton
Luckey
Ithaca
Miller, E. A.
Cornell Univ.
Agnew
Bissinger
Carver
Curtiss, J. H.
Durfee, William H.
Flexner
Goodier
Gunderson
Hailperin
Herzog
Hurwitz
Jones, B. W.
Kac
Karapetoff
Kiokemeister
Lawrence
Lehner
Malti
Randolph
Rosser
Savage
Sharpe, F. R.
Smith, J. C.
Snyder, V.
Stephan
Turquette
Walker, R. J.
Jackson Heights
Foyle
Jamaica
Eberman
Keuka Park
Archibald, H. F.
Lockport
Metzler
Long Island City
Coler
Mineola
Huntley
Mt. Vernon
Wetzel, R. A.
New York City
Abramowitz
Becker, M. F.
Blanch
Boehm
Bowden, M.
Burgess, L.
Burgess, R. W.
Byerly
Capecelatro
Carpenter, R. V. A.
Cohn, R. M.
Crane
Downey
Dutka
Ellis, G. A.
Frey
Gorn
Greitzer
Grossman
Hastings
Hillman
Hachschild
Hopper, F. F.
Ingram
Joffe
Kagno
Katz
Keyser, S. Y.
Kimball, B. F.
Kiss
Kormes, M.
Laderman
LeLeiko
Ling, D. P.
Lorenzen
Manring
Miller, O. M.
Mirick
Mitchell, H. B.
Mittleman
Morrison
Neuman
Nodelman
Parsons
Petrie
Preston
Reynolds, F. G.
Roos
Roosevelt
Schafer
Shamos
Shook, R. C.
Simon, L. G.
Simons, L. G.
Snook
Strodt, R. C.
Tanzola
Vitale
Wasow
Wayne
Weaver
Whitney, A. W.
Williams, E. C.
York
American Mathematical Society
Crawford, J. E.
Hull, E. M.
American Telephone and Telegraph Co.
Belcher
Halbert
Bell Telephone Labs.
Bode
Burton, E. T.
Dietzold
Foster, R. M.
Fry, T. C.
Gray
Harvey, W. R.
Hebbert
Hoyt
Jones, P. C.
MacColl
Mason, W. P.
Mead
Molina
Riordan
Schelkunoff
Shannon
Shewhart
Stibitz
Sunde
Vaage
Wise
Zobel
Coll. of Mt St. Vincent
Brady, M. L.
O'Brien
Sheridan
Coll. of the City of New York
Barber
Basch
Brodie
Garrison
Gill
Grove, C. C.
Helmer
Hubert
Lengyel
Linehan
Malin
Newman, P.
Philip
Post
Robinson, S. L.
Shiffman
Wirth
Wright, H. N.
Wundheiler
Columbia Univ.
Arrow
Bal
Bennett, B. M.
Biot
Cole, K. S.
Comer
Douglas
Fialkow
Fiske
Fite
Gillman
Gourin
Hadamard
Hawkes
Hotelling, H.
Jablon

Kasner
Keyser, C. J.
Koopman
Ladue
Lorch, E. R.
Mann, H. B.
Martin, M. P.
Mullins
Murray, F. J.
Nagel
Northcott
Pfeiffer
Poor, C. L.
Ramsey
Ritt
Siceloff
Smith, D. E.
Smith, P. A.
Straus
Strodt, W. C.
Sutherland
Upton
Wald
Wiesenberg
Williams, H. B.
Wolfowitz
Cooper Union
Lehmann
MacDonald
Miller, F. H.
Reddick
Equitable Life Assurance Society of the U. S.
Murphy
Steinhaus
Fordham Univ.
Kirby, A. R.
Kubis
Lynch
Murtaugh
Nardelli
Hunter Coll.
Anderson, R. L.
Aroian
Bradley
Bushey, H.
Bushey, J. H.
Cooper, E. M
Darkow
Elveback
Guggenbuhl
Gutzman
Hall, F. C.
Hill, L. S.
Landers, M. K.
Levin, M.
MacDuffee
Norris, N.
Rees, M.
Tuller
Weisner
Whelan
Manhattan Coll.
Hett
Welch, B. A.
Manhattan Life Ins. Co.
Porter, E. A.
Manhattanville Coll. of the Sacred Heart
Kirby, M. D.
Mira
Metropolitan Life Ins. Co.
Alfieri
Bassford
Davis, M. E.
Hamilton, J. A.
Herman
Hohaus
Lotka
Mutual Life Ins. Co. of N.Y.
Monsky
New School for Social Research
Gumbel
New York Univ.
Assadourian
Baron
Bernstein, F.
Cohn, H.
Cooley
Courant
DePrima
Eckweiler
Flanders
Friedrichs
Graham
Kline, M.
McKinsey
Marx
Payne
Peters, A. S.
Putnam, R. G.
Rehberg
Ritter, I. F.
Robbins, H. E.
Roth, S. G.
Stoker
Tilley
Yanosik
Teachers Ins. and Annuity Asso.
Robbins, R. B.
Yeshiva Coll.
Ginsburg
Lowan
Matz
Niagara University
Banks
Oneonta
Bush, K. A.
Newton, A. V.
Perry
Walker, G. W.
Poughkeepsie
Vassar Coll.
Baker, F. E.
Hopper, G. M.
McMillan, A. W.
Smith, G.
Wells, M. E.
White, H. S.
Rochester
Betz, W.
Chesna
Fay
Herzberger
Stephany
Univ. of Rochester
Atkins
Gale
Green, J. W.
Long
Seidel
Watkeys
Rockville Centre
Grant, A. M. C.
Rocky Point
Carter, P. S.
St. Albans
Adler, C. F
St. Bonaventure
Scheier
Schenectady
Alger
Poritsky
Union Coll.
Burkett
Farrell
Fox, A. H.
Garis
Morse, D. S.
Snyder, A. D.
Staten Island
Wernick
Syracuse
Roe
Syracuse Univ.
Campbell, A. D.
Carroll, L. S.
Cowgill
Decker
Troy
Rensselaer Polytech Inst.
Allen, E. B.
Ames, D. B.
Brown, H. K.
Huston
Rempfer
West Point
Ingersoll
Jones, H.
Willseyville
Seely
Wyoming
Hartnell

NORTH CAROLINA

Camp Davis
Noether
Chapel Hill
Univ. of North Carolina
Brauer, A. T.
Browne
Cameron, E. A.
Hobbs
Hurewicz
Jacobson
Lasley
Mackie
Durham
Boas, R. P.
Duke Univ.
Boas, M. L.
Carlitz
Dressel
Elliott, W. W.
Gergen
Hickson
Jarnagin
Maker
Meserve
Roberts, J. H.
Thomas, J. M.
North Carolina Coll. for Negroes
Elder, A.
Greensboro
Pegram
Thaxton
Greenville
ReBarker
Raleigh
Efird
North Carolina State Coll.
Bullock
Cell
Clarkson, J. M.
Fisher, H. A.
Gelbart

Kieval
Levine
Mumford
Nahikian
Park, H. V.
Strobel
Winton
Salisbury
Dearborn
Wilson
Smith, R. E.

NORTH DAKOTA

Fargo
Smith, I. W.

OHIO

Ada
Fulton
Alliance
Hildner
Athens
Ohio Univ.
Denbow
Marquis
Miller, D. D.
Reed, F. W.
Starcher
Bluffton
Linscheid
Bowling Green
Bowling Green State Univ.
Hendrickson
Ogg
Overman
Bryan
Gardner
Cincinnati
Folz
Horst
Learson
Ryan
Our Lady of Cincinnati Coll.
Reingold
Univ. of Cincinnati
Barlaz
Barnett, I. A.
Brand
Farnau
Hancock
Horvay
Justice
Kersten
Lubin
Merriman
Miller, H. L.
Moore, C. N.
Rogers
Smith, E. S.
Szász
Taylor, W. C.
White, E. F.
Yowell
Xavier Univ.
Stechschulte
Cleveland
Brainard
Burwell
Carruth, P. W.
Kraft
Case School of Applied Science
Brown, O. E.
Burington
Focke
Green, L. J.
Guenther
McCuskey
Morris, M.
Rinehart
Sorgenfrey
Thomas, C. F.
Torrance, C. C.
Fenn Coll.
Godfrey
Patterson, W. A.
Tolar
Topp
Western Reserve Univ.
Boyce
Musselman
Simon, W. G.
Cleveland Heights
Torrance, E. M.
Columbus
Ohio State Univ
Albert, G. E.
Bamforth
Bareis
Blumberg
Kuhn
LaPaz
Rado
Rasor
Toops
Wylie
Dayton
Eagle
Kemmer
Midgley
Schart
Univ. of Dayton
Schraut
Defiance
MacCullough
Delaware
Rowland
Gambier
Bumer
MacNeille
Granville
Denison Univ
Kato
Ladner
Wiley
Hiram
Clarke
Kent
Kent State Univ.
Brooks
Harshbarger
Manchester
Olson, E. J.
Stelson
Mariemont
Davids
Marion
Leach
Marietta
Bennett, T.
Sandt
Mt. St. Joseph
Corona
North Baltimore
Blackall
Oberlin
Oberlin Coll.
Cairns, W. D.
Johnson, M. M.
Sinclair
Yeaton
Oxford
Miami Univ.
Anderson, W. E.
Pollard, H. S.
Western Coll.
Tappan
Painesville
Peters, R. M.
South Euclid
Garvin
Springfield
Tripp, M. O.
Toledo
Brandeberry
Wellington
Ruger
Wilberforce
Davidson
Wilmington
Spinks
Wooster
Coll. of Wooster
Meyer, H. L.
Williamson, C. O.
Yanney
Yellow Springs
Astrachan

OKLAHOMA

Bartlesville
Morris, W. L.
Langston
Forrest
Norman
LaFon
Univ. of Oklahoma
Brixey
Court
Dixon, W. J.
Hassler
McFarland
Meacham
Randels
Reaves
Springer, C. E.
Stillwater
Oklahoma Agric. and Mech. Coll.
Barnett, J.
Diamond
Hamilton, O. H.
Lewis, P. E.
Oboukhoff
Smith, H. W.
Tulsa
Ellis, J. R.
Hodell
Jackson, O. B.
Ricker
Sharpe, J. A.

OREGON

Albany
Porter, R. E.
Corvallis
Oregon State Coll.
Hammer, P. C.
Hostetter
Manning, R.
Milne, W. E.
Sobczyk
Eugene
Univ of Oregon
Bubb
DeCou
Ghent
Kossack
Maddaus
Moursund
Peterson, T. S.
Portland
Reed Coll
Griffin, F. L.
Rosenbaum, L. J.
Rosenbaum, R. A.
Univ. of Portland
Vassallo
Salem
Luther
Olshen

PENNSYLVANIA

Allentown
Deck
Koehler, T. L.
Annville
Black, A. H.
Bethlehem
Smiley, D. M.
Lehigh Univ.
Beale
Bergmann
Cutler
Ewing, (W.) M.
Fort
Lamson
Latshaw
Pitcher, (A.) E.
Pond
Raynor
Shook, C. A.
Smail
Smiley, M. F.
Weil
Bruceton
Brown, F. W.
Bryn Mawr
Reschovsky
Bryn Mawr Coll.
Geiringer
Lehr
Morrow, D. J.
Oxtoby
Wheeler, A. P.
California
Bernstorf
Carlisle
Ayres, F.
Chambersburg
Wilson Coll.
Davison
Disert
Johnson, R. F.
Chester
Campbell, W. B.
Williams, F. G.
Clarion
Carey, C. E.
Dallas
Garrahan
Drexel Hill
McDonough
Dubois
Cunningham
Easton
Bailey, R. P.
Smith, W. M.
East Pittsburgh
Slepian
Gettysburg
Arms
Livingood
Glenside
Patterson, J. O.
Greensburg
McNeil
Harrisburg
McKee
Whited
Haverford
Haverford Coll
Allendoerfer
Gummere
Oakley
Wilson, A. H.
Hazleton
Herpel
Immaculata
Anastasia Maria
Jenkintown
Durand, J. C.
Kutztown
Knedler
Lancaster
Murray, W. R.
Latrobe
Yahner
Lewisburg
Bucknell Univ.
Drum
Gold
Miller, W. I.
Lincoln University
Haviland
Meadville
Frame
Narberth
Robertson, W. M.
New Kensington
Sturm
New Wilmington
Black, H. L.
Philadelphia
Beamish
Brown, R. D.
Constable
Epstein, B.
Fernbach
Fudge
Groat
Hayward
Hudson
Jaffe
Kratz
Morgan, W. B.
Rosenbloom, J. H.
Sherman, J.
Smith, J. P.
Drexel Inst. of Tech.
Davis, J. Elmer
Temple Univ.
Gleason, R. E.
Lawton
Rorer
Stokes, C. N.
Univ. of Pennsylvania
Beal
Burks
Caris
Clarkson, J. A.
Dressler
Erdös
Evans, H. B.
Goff
Hemmingsen
Kline, J. R.
Lufkin
Mitchell, H. H.
Rademacher
Ritter, E. K.
Safford
Schoenberg
Shohat
Shugert
Wallace, A. D.
Whitman, P. M.
Witmer
Pittsburgh
Briant
Leifer
Carnegie Inst. of Tech.
Clippinger
Dines
Hestenes, A. D.
Hicks
Landau
Morehead
Moskovitz
Nelley
Olds, E. G.
Peach
Rosenbach
Saibel
Smith, T. L.
Whitman, E. A.
Pennsylvania Coll. for Women
Calkins
Univ. of Pittsburgh
Boggs
Culver
Foraker
Knipp
Speer
Taylor, J. S.
Pottstown
Huff
Scranton
Trapani
Walton, M. B.
Selinsgrove
Smith, A. J.
State College
Owens, H. B.
Pennsylvania State Coll.
Bentz
Cohen, T.
Curry
Dotterer
Frink, A. H.
Frink, O.
Gordon
Graves, C. H.
Hagen
Harrington, W. J.
Johnson, E.
Krall
Owens, F. W.
Rupp
Schwartz, A.
Sheffer, I. M.
Stecker
Umberger
Wagner, C. C.

Swarthmore
Swann
Swarthmore Coll.
Brinkmann
Dresden
Fried
Marriott
Wallingford
Miller, J. A.
Washington
Dorwart
Shaub
Wayne
Feather
Willow Grove
Bateman, P. T.

PHILIPPINE ISLANDS

Baguio
Melchor
Manila
Jaramillo
Netzorg
Mapua Inst. of Tech
Mijares
Univ. of the Philippines
Gokhale
Perez
Virata

PUERTO RICO

Mayaquez
Morales
Sánchez-Díaz
Rio Piedras
Bobonis
Kuschke

RHODE ISLAND

Kingston
Pease
Stauffer
Providence
Brown Univ
Adams, C. R.
Archibald, R. C.
Bennett, A. A.
Bergman
Carlen
Feller
Gaskell
Gilman
Greenberg
Hall, D. W.
Hutchinson, L. C.
Lindsay
Manning, H. P.
Mintzer
Morkovin
Munroe, M. E.
Neugebauer
Prager
Protter
Richardson, R. G. D.
Riess
Ruley
Schmidt
Smiley, C. H
Sohl
Tamarkin
Western, D. W.
Providence Coll.
McKenney
West Warwick
Brant

SOUTH CAROLINA

Charleston
Shulits
Coll. of Charleston
Bushkovitch
The Citadel
Dye
Hedge
Sutton, C. S.
Clemson
Coker
Columbia
Coleman, J. B.
Jackson, J. B.
Rock Hill
Royall
Stokes, R. W
Spartanburg
Pettis, B. J.
Wofford Coll.
Ader
Pettis, C. S.

SOUTH DAKOTA

Brookings
South Dakota State Coll.
MacDougal
Walder
Wente
Rapid City
Cook, R. H.
Sioux Falls
Lintvedt
Vermillion
Beaty
Sealander

TENNESSEE

Jackson
Pulliam
Knoxville
Rich
Univ. of Tennessee
Eaves
Ficken
Memphis
MacQueen
Nashville
Vanderbilt Univ.
Dyer-Bennet
Miser, W. L
Sarratt
Ward-Belmont School
Falvey
Sewanee
Bruton

TEXAS

Abilene
Burnam
Mullings
Angleton
Bright
Austin
Univ. of Texas
Batchelder
Coburn
Craig, H. V.
Decherd
Dodd
Ettlinger
Greenwood
Jones, F. B.
Lubben
Moore, R. L
Porter, G. H.
Porter, M. B
Titt
Vandiver
Camp Bowie
Swift, J. D.
College Station
Agric. and Mech. Coll. of Texas
Brewer
Chaney
Coleman, W. B.
Daum
Gross
Hurt
Jennings, W.
Klipple
Moore, B. C.

Commerce
Wright, C. B.
Corpus Christi
Leavitt
Dallas
McDermott
Montgomery, P. V.
Southern Methodist Univ.
Palmquist
Rees, P. K.
Starr
Denton
North Texas State Teachers Coll.
Brown, M. C.
Hanson, E. H.
Miller, H. C.
Georgetown
Olson, H. L.

Houston
Rust
Slotnick
Rice Inst.
Bray
Brunk
Dean, A. C.
Leighton
Lovett
Mandelbrojt
Piranian
Thron
Ulrich
Wilson, H. A.
Univ. of Houston
Hutchinson, J. D.
Huntsville
Querry
Jacksonville
Clarkson, H. E.

Kelly Field
Trachtenberg
Kingsville
Kennedy, E. C.
Lubbock
Texas Tech. Coll.
Gilbert
Rigby
Thompson, E. L.
Marlin
Falconer
Mercedes
Shaw, W. F.
Randolph Field
Seebeck
San Antonio
Incarnate Word Coll.
Gough

Our Lady of the Lake Coll.
Hill, M. L.
St. Mary's Univ.
Becker, F. A.
San Antonio Junior Coll.
Hurry
Tulloch
Trinity Univ.
Newton, G. A.
Terrell
Sandidge
Waco
Hedberg
Williams, W. J.

UTAH

Logan
Bird
Salt Lake City
Woodbury

Univ. of Utah
Biesele
Gibson, J. L.

Hayes, J. J.
Horsfall
Pehrson

Smith, S. S.
Stafford

VERMONT

Bennington
Polanyi
Burlington
Univ. of Vermont
Bullard

McClay
Swift, E.
Middlebury
Ballou

Springfield
Terry

Winooski Park
Gleyzal

VIRGINIA

Arlington
Brenner
Newman, J. R.
Blacksburg
Virginia Polytech. Inst.
Hatcher
McFadden
Williams, J. E.
Charlottesville
Univ. of Virginia
Bernard
Botts
Clark, C. L.
Cox, M. J.
Garcia, M
Gottschalk
Hedlund

Linfield
McShane
Myers, F. G.
Oglesby
Whyburn, G. T.
Dahlgren
Barker
Ettrick
Hunter, L. S.
McDaniel, R. R.
Fredericksburg
Carter, H C.
Hampden-Sydney
Elliott, E. R.
Hampton
Lorch, L.
Street

Hampton Inst.
Perkins, H. A.
Simond
Hollins College
Allen, H. W.
Mitchell, J. M.
Hopewell
Fazel
Lexington
Virginia Military Inst.
Byrne
Knox
Washington and Lee Univ.
Paxton
Lynchburg
Larew

Wiggin
South Boston
Patten
Staunton
Taylor, M. E.
Sweet Briar
Cole, N.
Morenus
University of Richmond
Univ of Richmond
Gaines
Grable
Wheeler, C. H.
Williamsburg
Phalen
Stetson

WASHINGTON

Lacey
Raymond
Pullman
Butler, L. G.
Knebelman

Seattle
Seattle Pacific Coll.
Beegle

Univ. of Washington
Ballantine, C. R.
Ballantine, J. P
Beaumont

Birnbaum
Carpenter, A. F.
Copenhagen
Cramlet
Jerbert, A. H.

Jerbert, A. R.
Kingston, J. M.
McFarlan
Mullemeister
Neikirk
Schaefer
Taub
Winger
Zuckerman
Spokane
Martin, F. L.
Walla Walla
Bratton
Highberg
Woodinville
Anthony

WEST VIRGINIA

Bethany
Cramblet
Bluefield
Porter, J. W.
Keyser
Krabill
Montgomery
Smith, W. F.
Morgantown
Hackney
West Virginia Univ.
Davis, H. A.
Eiesland
Reynolds, C. N.
Turner, B. M.
Vest
Shepherdstown
Callahan

WISCONSIN

Appleton
Berry, A. C.
Beloit
Bisshopp
Conwell, H. H.
Kenosha
May
Madison
Finch
Univ. of Wisconsin
Adler, F. T.
Allen, F. E.
Breit
Bruck
Colvin
Eisenhart, C.
Embree
Evans, H. P.
Everett, C. J.
Good
Ingraham
Johnson, R. E.
Laden
Langer
Lee
Lindstrum
March
Nelson, N. D.
Norman
Rodd
Simpson, R. C.
Slichter
Smith, W. N.
Sokolnikoff, E. S.
Sokolnikoff, I. S.
Swain
Trump
Ulam
VanVleck, E. B.
Vinograde
Wiegmann
Milwaukee
Marquette Univ.
Pettit
Wilczewski
Milwaukee-Downer Coll.
Beckwith, E. R.
Mt. Mary Coll.
Vaudreuil
State Teachers Coll.
Norris, R. E.
Univ. of Wisconsin at Milwaukee
Bardell
Battig
Kenney
Marden
Nordhaus
Roth, W. E.
Vass
Wolf, L. A.
Oshkosh
Beenken
Price, I.
River Falls
Eide
Superior
Danielson
Smith, C. W.
West De Pere
DeCleene
Williams Bay
Chandrasekhar

WYOMING

Lamont
Bellamy
Laramie
Rechard
Samelson

CANADA

ALBERTA
Banff
Crosby
Edmonton
Univ. of Alberta
Campbell, J. W.
Cook, A. J.
Sheldon
BRITISH COLUMBIA
Vancouver
Univ. of British Columbia
Buchanan, D.
Gage
Hull, R.
Jennings, S. A.
MANITOBA
Winnipeg
Wilson, N. R.
NEW BRUNSWICK
Fredericton
Miller, C. E.
Sackville
McEwen, W. H.
NOVA SCOTIA
Halifax
Adshead
Walmsley
Stellarton
Loomer
Wolfville
Jeffery
Macphail
ONTARIO
Hamilton
Findlay
Johns
Kingston
Halperin
Miller, N.
London
Cole, R. H.
Kingston, H. R.
Orillia
Guinand
Ottawa
Bell, J. H.
Brady, G. V.
Fraser, W. C. G.
Mendelsohn
Toronto
Wolfenden
Ontario Coll of Education
Grant, A. A.
Univ of Toronto
Beatty
Brauer, R.
Burk
Coleman, A. J.
Coxeter
DeLury
Gilchrist
Gould, S. H.
Krieger
Pounder
Robinson, G. de B.
Synge
Waddell
Webber, W. J.
Weinstein
QUEBEC
Lennoxville
Richardson, A. V.
Montreal
Loyola Coll.
O'Connor
McGill Univ.
MacLean
Pall
Shaw, A. N.
Sullivan, C. T.
Williams, W. L. G.
Quebec
Tremblay
Westmount
Chapman
SASKATCHEWAN
Saskatoon
Univ. of Saskatchewan
Derry
James, R. D.
Ling, G. H.
Murdoch

AFRICA

SOUTH AFRICA
Pretoria
de Wet

ASIA

CHINA
Peking
Hsü
Shanghai
Fan
INDIA
Aligarh
Shah
Annamalainagar
Rao
Bangalore
Iyengar
Dacca
Vijayaraghavan
Delhi
Seth
Madras
Ananda-Rau
Poona
Paranjpye
IRAQ
Baghdad
Merrick
JAPAN
Osaka
Kakutani
Nakayama
Sendai
Kubota
Takasu
Tokyo
Kuniyeda
Yoshida
PALESTINE
Jerusalem
Levitski
Nablus
Tukan
Petah Tikvah
Preiser
STRAITS SETTLEMENTS
Singapore
Oppenheim
TURKEY
Istanbul
Kimball, W. S.

AUSTRALASIA

AUSTRALIA
Adelaide
Wilton
Brisbane
Raybould
Hobart
Jaeger, J. C.
Melbourne
Cherry
Perth
Weatherburn
Sydney
Carslaw
Thorne, H. H.
NEW ZEALAND
Auckland
Forder
Dunedin
Martyn

EUROPE

BELGIUM
Ostend
Bouckaert
DENMARK
Copenhagen
Royal Tech. Coll.
Jessen
Univ. of Copenhagen
Bohr
Nörlund
ENGLAND
Bristol
Fraser, P.
Cambridge
Univ. of Cambridge
Collingwood
Hall, P.
Hardy, G. H.
Ingham
Richmond, H. W.
Smithies
Hindhead Surrey
Western, A. E.
Leeds
Milne, W. P.
London
Dalal
Univ. of London
Semple
Simpson, H.
Manchester
Mordell
Oxford
Univ. of London
Taussky
Univ. of Oxford
Dixon, A. L.
Frecheville
FRANCE
Paris
Catholic Inst.
Potron
Univ. of Paris
Cartan
Fréchet
GERMANY
Freiburg i. Br.
Bolza
Leipzig
Hopf
GREECE
Athens
Lanckton
HUNGARY
Budapest
Fejér
Szeged
Riesz
IRELAND
Dublin
McConnell, A. J.
Rowe
ITALY
Pisa
Tonelli
Turin
Tricomi
LATVIA
Riga
Leimanis
NORTHERN IRELAND
Belfast
Queen's Univ.
Gillis
McCrea
Povey
SCOTLAND
Dundee
Copson
Edinburgh
Whittaker
SWEDEN
Stockholm
Oseen
SWITZERLAND
Lausanne
Blanc
Zurich
Plancherel
U.S.S.R.
Ufa
Kryloff
WALES
Bangor
Berwick
Cardiff
Livens

CENTRAL AND SOUTH AMERICA

ARGENTINA
Buenos Aires
Univ. of Buenos Aires
Baidaff
Barral-Souto
Blaquier
Bula
González-Domínguez
La Plata
de Cesare
Rosario
Gaspar
Levi, B.
Tucumán
Terracini
CUBA
Havana
Corral-Aléman
González-Rodríguez
ECUADOR
Quito
Thullen
MEXICO
Mexico
Baños
PERU
Lima
Catholic Univ. of Peru
de Losada y Puga
Univ. of San Marcos
García, G.
Rosenblatt

BY-LAWS OF THE AMERICAN MATHEMATICAL SOCIETY

ARTICLE I

OFFICERS

Section 1. There shall be a President, three Vice Presidents, a Secretary, four Associate Secretaries, a Treasurer, and a Librarian.

Section 2. It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

Section 3. The Librarian shall have charge of arrangements for the exchange of the Society's publications.

ARTICLE II

BOARD OF TRUSTEES

Section 1. There shall be a Board of Trustees consisting of five trustees elected by the Society, in accordance with Article V.

Section 2. The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

Section 3. The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporative offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

ARTICLE III

EDITORIAL COMMITTEES

Section 1. There shall be five Editorial Committees—one of four members for the *Bulletin*, one of three members for the *Transactions*, one of three members for the *Mathematical Reviews*, one of three members for the *Colloquium Publications*, and one consisting of three representatives of the Society on the Board of Editors of the *American Journal of Mathematics*.

ARTICLE IV

COUNCIL

Section 1. The Council shall consist of fifteen members at large, and the following *ex officio* members: the officers of the Society specified in Article I, the members of the Editorial Committees, ex-secretaries who have served as secretaries for ten years or more, and ex-presidents for a period of six years after the completion of their respective presidential terms.

Section 2. The Council shall formulate and administer the scientific policies of the Society and shall act in an advisory capacity to the Board of Trustees.

Section 3. All members of the Council shall be entitled to be present at any meeting of the Council and to take part in the discussion at that meeting.

Section 4. In the absence of the Secretary from any meeting of the Council, one of the Associate Secretaries present may be designated as Acting Secretary for the meeting, either by written authorization of the Secretary, or, failing that, by majority agreement among the Associate Secretaries present.

Section 5. The voting members at any meeting of the Council shall be the following: the members at large of the Council; the President; the Vice Presidents; the Secretary; the Acting Secretary for the meeting, if one is designated in the absence of the Secretary; the Treasurer; and a member of each Editorial Committee chosen annually by the committee, provided that if such voting member is absent at any meeting of the Council an alternate designated by him from among the membership of the committee shall have the power to vote.

Section 6. The method for settling matters before the Council at any meeting shall be by majority vote of the voting members present. If the result of such a vote is challenged at the time by any member of the Council, it shall be the duty of the Secretary—or of the Acting Secretary for the meeting—or of a member present who shall be designated by the Council for that purpose—to prepare a list of the voting members present, on the basis of which the presiding officer shall determine the true vote by a roll call.

Section 7. A quorum for the transaction of business at any meeting of the Council shall consist of five voting members as defined in Section 5 of this Article.

Section 8. Between meetings of the Council, business may be transacted by a mail vote. Voting members for this purpose shall be the members at large of the Council, the President, the Vice Presidents, the Secretary, the Treasurer, and the voting member of each Editorial Committee as designated in Section 5. An affirmative vote on any proposal by mail shall require at least a majority of the voting members, and at least three-quarters of the total number of votes transmitted to the Secretary before the time announced for the closing of the polls. If two or more voting members request postponement at the time of voting, action on the matter at issue shall be postponed until the next meeting of the Council, unless at the discretion of the Secretary the question is made the subject of a second vote by mail, in connection with which brief statements of reasons, for and against, are circulated.

ARTICLE V

ELECTIONS AND TERMS OF OFFICERS

Section 1. The term of office shall be two years in the case of the trustees, the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years in the case of the Librarian and the members of the Editorial Committees. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. In every case, however, the officials specified in Articles I, II, III, and IV shall continue to serve until their successors have been duly elected and qualified.

Section 2. The President and Vice Presidents shall not be eligible for immediate re-election to their respective offices. A member at large or an *ex officio* member of the Council shall not be eligible for immediate election (or re-election) as a member at large of the Council.

Section 3. Election of officers specified in Article I, trustees, members of Editorial Committees, and members at large of the Council shall be by ballot at the Annual Meeting. An official ballot shall be sent to each member by the Secretary at least one month before the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the name of the voter and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain one or more names proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A plurality of all votes cast, whether cast in person or by mail, shall be necessary for election. In case of failure to secure a plurality for any office, the members present at the Annual Meeting shall choose by ballot between the two having the highest number of votes.

Section 4. If the President of the Society die or resign before the expiration of his term of office, the Council, may, with the approval of the trustees, designate one of the Vice Presidents to serve as President until the next Annual Meeting, when the Society shall elect a President for a two-year term. Such vacancies as may occur at any time in the group consisting of the Vice Presidents, the Secretary, the Associate Secretaries, the Treasurer, the Librarian, and the members of the Editorial Committees may be filled by the Council with the approval of the trustees.

Section 5. If any trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

Section 6. If any member at large of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

ARTICLE VI

ELECTION OF MEMBERS

Section 1. Election of members shall be by vote of the Council.

Section 2. There shall be four classes of members, ordinary, contributing, institutional contributing, and sustaining.

Section 3. A firm, corporation, institution, association or individual interested in the support of mathematics, may be elected to sustaining membership. A sustaining member shall receive the *Bulletin*, the *Transactions*, and *Mathematical Reviews* (see Article IX), and shall have the privilege of nominating one or more persons for election by the Council to ordinary membership in the Society, the number to be determined by the Council. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 4. A firm, corporation, institution, or association interested in the support of mathematics may be elected to institutional contributing membership. An institutional contributing member shall receive the *Bulletin* or have the privilege, for each twenty-five dollars contributed, of nominating one person to ordinary membership in the Society. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 5. Except in the case of nominees of sustaining members and of institutional contributing members, application for admission to ordinary membership shall be made by the applicant, on a blank provided by the Secretary, and shall be approved by two members of the Society. Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council.

Section 6. An ordinary member may become a contributing member by paying the dues for such membership. (See Article VII, Section 4.)

Section 7. An ordinary member or a contributing member shall receive the *Bulletin*, except in the case of members whose dues are remitted under Article VII, Section 10.

ARTICLE VII

DUES

Section 1. Persons elected to ordinary membership in the Society by the Council, under the provisions of Article VI, Section 5, shall be admitted to membership upon the payment, within sixty days of the date of their election, of an initiation fee of five dollars.

Section 2. The annual dues of persons elected by the Council to ordinary membership under the provisions of Article VI, Section 5, shall be eight dollars, with the following exceptions: (1) during the first three years of membership, the annual dues shall be six dollars; a person shall be considered to have completed his first year of membership on January 1 following his election; (2) the amount of dues may be altered by reciprocity agreements with other societies; (3) the Council may make special rulings in exceptional cases, with the approval of the Board of Trustees. Each new member shall pay in proportion to the unexpired fraction of the year at the time of his election.

Section 3. Four dollars and fifty cents of the annual dues of those members who receive the *Bulletin* under the provisions of Article VI shall be allocated in payment therefor.

Section 4. The minimum dues for a contributing member shall be fifteen dollars per year. Members may, upon their own initiative, pay larger dues.

Section 5. The dues for an institutional contributing member for any year shall not be less than twenty-five dollars. Institutions may pay larger dues.

Section 6. The dues of a sustaining member for any year shall not be less than one hundred dollars. A sustaining member who contributes annually at least five hundred dollars shall be designated as a Patron of the Society.

Section 7. Persons elected to ordinary membership as nominees of sustaining members or institutional contributing members under the provisions of Article VI, Sections 3-4, shall not be required to pay an initiation fee. They shall not be required to pay dues, so long as it is agreed that they are designated by the sustaining member or institutional contributing member under these provisions. If a nominee of a sustaining member or an institutional contributing member later becomes a dues-paying member, he shall pay dues at the rate of six dollars a year for the remainder (if any) of his first three years of membership, and eight dollars a year thereafter.

Section 8. If the annual dues of any member remain unpaid beyond a reasonable time, the Board of Trustees shall remove his name from the list of members, after due notice.

Section 9. Any member not in arrears of dues may become a life member on the payment of a sum to be determined in accordance with actuarial principles. A life member shall have for life the status and privileges of an ordinary member, without further payment of dues. After October 25, 1941, no additional applications for life memberships will be accepted.

Section 10. After retirement from active service on account of age, any member who is not in arrears of dues and with membership extending over at least twenty years may—by giving proper notification to the Secretary—have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings, but not the *Bulletin*.

ARTICLE VIII

MEETINGS

Section 1. The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council. No matter of general business shall be considered at any meeting of the Society except the Annual Meeting, without the recommendation of the Council.

Section 2. The President may call a meeting of the Council whenever the affairs of the Society make it desirable; he shall call a meeting of the Council on written request of five of its members.

Section 3. The Board of Trustees shall have its regular Annual Meeting on the first day of the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days' notice of such meeting mailed to the last known post office address of each trustee. He shall call a meeting upon the receipt of a written request of two of the trustees. Meetings for the transaction of business may also be held by common consent of all the trustees.

Section 4. Papers intended for presentation at any meeting of the Society shall be passed upon in advance by a program committee appointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

ARTICLE IX

PUBLICATIONS

Section 1. The Society shall publish an official organ called the *Bulletin of the American Mathematical Society*. It shall publish a journal called the *Transactions of the American Mathematical Society*, the object of which shall be to make known important researches presented at meetings of the Society. It shall publish a periodical called *Mathematical Reviews* containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called *Colloquium Publications* which shall embody in book form new mathematical developments. It shall also cooperate in the conduct of the *American Journal of Mathematics*.

Section 2. The editorial management of the *Bulletin, Transactions, Mathematical Reviews*, and *Colloquium Publications*, and the participation of the Society in the management of the *American Journal of Mathematics* shall be in charge of the respective Editorial Committees as provided in Article III, Section 1.

ARTICLE X

AMENDMENTS

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

REPORT OF THE TREASURER FOR THE PERIOD DECEMBER 1, 1939 TO NOVEMBER 30, 1940[1]

BALANCE SHEET

November 30, 1940

ASSETS

CASH:			
Corn Exchange Bank Trust Company	$ 1,897.45		
The Pennsylvania Company for Insurance on Lives and Granting Annuities	57.73		
Bank for Savings	2,891.93		
Emigrant Industrial Savings Bank	622.38		
Providence Institution for Savings	3,553.85		
Union Dime Savings Bank	6,230.24		
Cash on Hand	30.00	$ 15,283.58	
INVESTMENTS[2]		173,943.07	
Total Assets			$189 226 65

RESERVES AND SURPLUS

RESERVES:			
Endowment Fund[3]	$60,110.68		
Bôcher Fund	1,728.96		
Brown Fund	1,054.47		
Cole Fund	2,653.54		
Moore Fund	3,584.35		
Reilly Fund	19,063.51		
Life Memberships	5,507.90		
Sinking Fund	290.80		
Investment Losses	1,926.26		
Profit on Sales of Securities	1,665.73		
International Congress	5,853.67		
Bulletin Reprinting and Index Fund	457.61		
Transactions Reprinting and Index Fund	103.63		
Transactions Life Subscription	114.09		
Colloquium	9,718 95		
Mathematical Reviews	68,426.11	$182,260.26	
SURPLUS AT NOVEMBER 30, 1940		6,966.39	
Total Reserves and Surplus			$189,226.65

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[1] For report of the Treasurer for 1941, see this Bulletin, vol. 48, no. 3, pp. 196-198.

[2] Value of investments based on market quotations on November 30, 1940, is $169,778.42

[3] Former value $70,506.07, depreciation $10,395.39.

STATEMENT OF GENERAL RECEIPTS AND GENERAL DISBURSEMENTS

GENERAL RECEIPTS:		
Dues from Ordinary Memberships	$14,380.96	
Dues from Contributing Memberships	1,114.92	
Dues from Institutional Memberships	6,705.00	
Initiation Fees	584.75	
Income from Investments	3,083.37	
Miscellaneous	695.95	
Total General Receipts		$26,564.95
GENERAL DISBURSEMENTS:		
Secretaries	$ 5,800.00	
Treasurer	650.00	
Officers' Traveling	749.54	
Library	629.82	
Furniture and Fixtures	31.00	
Gibbs Lecture	19.88	
Membership Committee	161.24	
Committee on War Preparedness	50.00	
American Journal Subvention	2,500.00	
Emergency	287.86	
Total General Disbursements		10,879.34
EXCESS OF GENERAL RECEIPTS OVER GENERAL DISBURSEMENTS		$15,685.61

SURPLUS AT DECEMBER 1, 1939		$ 9,405.61
Additions:		
Transferred from Sinking Fund and Reserve for Investment Losses to Offset Depreciation of Value of Investments	$ 5,016.29	
Adjustment—Life Membership Reserve	212.25	
Adjustment—Transactions Life Subscription Reserve	1.28	
Excess of General Receipts over General Disbursements (see above)	15,685.61	20,915.43
Total		$30,321.04
Deductions:		
Appropriation to:		
Endowment Fund Principal Account	$ 1,000.00	
Reserve for Investment Losses	936.21	
Bulletin Account	10,773.56	
Transactions Account	3,109.80	
Colloquium Account	1,250.00	
Mathematical Reviews Account	1,000.00	
Investment Account for Write-down of Securities to Market Value as of July 1939	5,285.08	23,354.65
SURPLUS AT NOVEMBER 30, 1940		$ 6,966.39

BULLETIN

Receipts:		
Subscription Sales	$ 1,512 47	
Reprint Sales	119.18	$ 1,631.65
Appropriation from General Receipts		10,773.56
Total		$12,405 21
Disbursements:		
Printing (2 numbers for 1939, 10 for 1940)	$ 6,761.39	
Reprints	777.34	
List of Members	656.62	
Salaries, Annuity and Pension	3,422.50	
Advertising	26.09	
Miscellaneous—Office Furniture, Supplies, etc.	761.27	12,405.21

TRANSACTIONS

Receipts:		
Subscription Sales	$ 4,336.74	
Back Volume Sales	862.20	
Reprint Sales	195.31	
Contributions	17.00	$ 5,411.25
Appropriation from General Receipts		3,109.80
Total		$ 8,521.05
Disbursements:		
Printing (1 number for 1939, 5 for 1940)	$ 5,497.40	
Reprints	760.28	
Salaries, Annuity and Pension	1,778 95	
Advertising	30.00	
Miscellaneous—Postage, Supplies, etc.	454.42	8,521.05

COLLOQUIUM

BALANCE AT DECEMBER 1, 1939		$ 8,075.38
Receipts:		
Sales of Colloquium Volumes	$ 2,985.49	
Sales of Colloquium Volume 16 (Bliss) for National Research Council	135.62	3,121.11
Appropriation from General Receipts		1,250.00
Total		$12,446.49
Disbursements:		
Printing (Volume 25)	$ 1,236.58	
Salaries, Annuity and Pension	1,083.00	

Miscellaneous—Shipping, Supplies, etc. (Including Cost of Shipping Volume 16 for National Research Council)	187.13	
Circulars and Advertising	100.20	
Payment to National Research Council for Volume 16 (Bliss)	120.63	2,727.54
BALANCE AT NOVEMBER 30, 1940		$ 9,718.95

Mathematical Reviews

BALANCE AT DECEMBER 1, 1939		$61,431.89
Receipts:		
Contributions (Including $1,000.00 for Microfilm Project)	$10,026.52	
Subscription Sales	7,849.48	
Income from Investments	2,469.99	
Petty Cash	5.00	20,350.99
Appropriation from General Receipts		1,000.00
Total		$82,782.88
Disbursements:		
Printing (11 numbers)	$ 4,915.58	
Salaries, Annuity and Pension	4,521.75	
Periodicals	1,080.58	
Advertising	2,501.40	
Traveling	13.10	
Office Furniture	211.59	
Miscellaneous	612.77	
Microfilm Project	500.00	14,356.77
BALANCE AT NOVEMBER 30, 1940		$68,426.11

International Congress

BALANCE AT DECEMBER 1, 1939		$ 5,716.62
Receipts:		
Sale of Equipment	$ 70.00	
Income from Investments	222.48	292.48
Total		$ 6,009.10
Disbursements:		
Refunds of Contributions	$ 100.00	
Refunds of Fees	55.00	
Miscellaneous	.43	155.43
BALANCE AT NOVEMBER 30, 1940		$ 5,853.67

SPECIAL FUNDS

	Total	Principal	Income
ENDOWMENT FUND			
Balance at December 1, 1939	$70,506 07	$70,506.07	
Additions:			
Income from Investments	2,329 86		$ 2,329.86
By Order of the Board of Trustees	1,000 00	1,000.00	
From Sinking Fund and Reserve for Investment Losses	5,758.37	5,758.37	
	$79,594.30	$77,264.44	$ 2,329.86
Deductions:			
Distributed to General	$ 2,329.86		$ 2,329.86
Write-down for Depreciation of Value of Investments	17,153.76	$17,153.76	
	$19,483.62	$17,153.76	$ 2,329 86
Balance at November 30, 1940	$60,110.68	$60,110.68	
BÔCHER FUND			
Balance at December 1, 1939	$2,171 62	$ 1,188 00	$ 983.62
Addition:			
Income from Investments	65.64		65.64
	$ 2,237.26	$ 1,188.00	$ 1,049 26
Deductions:			
Adjustment of Allocation of 1939 Income	$ 2.69		$ 2.69
Write-down for Depreciation of Value of Investments	505.61		505.61
	$ 508.30		$ 508.30
Balance at November 30, 1940	$ 1,728.96	$ 1,188.00	$ 540.96
BROWN FUND			
Balance at December 1, 1939	$ 1,012.52	$ 1,000.00	$ 12.52
Additions:			
Income from Investments	40.07		40.07
Adjustment of Allocation of 1939 Income	1.88		1 88
Balance at November 30, 1940	$ 1,054 47	$ 1,000.00	$ 54.47

COLE FUND

Balance at December 1, 1939	$ 3,466.50	$ 2,093.13	$ 1,373.37
Addition:			
Income from Investments	100.88		100.88
	$ 3,567.38	$ 2,093.13	$ 1,474.25
Deductions:			
Prize	$ 100.00		$ 100.00
Adjustment of Allocation of 1939 Income	4.30		4.30
Write-down for Depreciation of Value of Investments	809.54		809.54
	$ 913.84		$ 913.84
Balance at November 30, 1940	$ 2,653.54	$ 2,093.13	$ 560.41

MOORE FUND

Balance at December 1, 1939	$ 4,504.31	$ 2,100.62	$ 2,403.69
Addition:			
Income from Investments	136.12		136.12
	$ 4,640.43	$ 2,100.62	$ 2,539.81
Deductions:			
Adjustment of Allocation of 1939 Income	$ 5.10		$ 5.10
Write-down for Depreciation of Value of Investments	1,050.98		1,050.98
	$ 1,056.08		$ 1,056.08
Balance at November 30, 1940	$ 3,584.35	$ 2,100.62	$ 1,483.73

REILLY FUND

Balance at December 1, 1939	$18,494.11	$17,014.34	$ 1,479.77
Additions:			
Income from Investments	731.70		731.70
Adjustment of Allocation of 1939 Income	69.37		69.37
	$19,295.18	$17,014.34	$ 2,280.84
Deduction:			
Fees for Trustee (The Pennsylvania Co.)	231.67	231.67	
Balance at November 30, 1940	$19,063.51	$16,782.67	$ 2,280.84

BENNINGTON P. GILL, *Treasurer*

New York, N.Y.
November 30, 1940

PUBLICATIONS

JOURNALS

Bulletin of the American Mathematical Society.
Transactions of the American Mathematical Society.
Mathematical Reviews.

BOOKS

F. Klein, **The Evanston Colloquium Lectures on Mathematics**, 1894, x, 109 pp.; reprinted, 1911, xii, 109 pp.	$1.00
Mathematical Papers Read at the International Mathematical Congress, 1896, xvi, 411 pp.	4.70
Semicentennial Publications.	
1. R. C. Archibald, **A Semicentennial History of the American Mathematical Society, 1888–1938, with Biographies and Bibliographies of the Past Presidents**, 1938, xii, 261 pp., 30 plates.	3.00
2. **Semicentennial Addresses**, 1938, vi, 315 pp.	4 50

COLLOQUIUM SERIES

1. H. S. White, **Linear Systems of Curves on Algebraic Surfaces;** F. S. Woods, **Forms of Non-Euclidean Space;** E. B. Van Vleck, **Selected Topics in the Theory of Divergent Series and of Continued Fractions;** 1905, xii, 187 pp.	3 00
2.* E. H. Moore, **Introduction to a Form of General Analysis;** M. Mason, **Selected Topics in the Theory of Boundary Value Problems of Differential Equations;** E. J. Wilczynski, **Projective Differential Geometry;** 1910, x, 222 pp.	
3_1. G. A. Bliss, **Fundamental Existence Theorems**, 1913, x, 107 pp.; reprinted, 1934, iv, 107 pp.	2.00
3_2. E. Kasner, **Differential-Geometric Aspects of Dynamics**, 1913, iv, 117 pp.; reprinted, 1934, iv, 117 pp.	2.00
4.* L. E. Dickson, **On Invariants and the Theory of Numbers;** W. F. Osgood, **Topics in the Theory of Functions of Several Complex Variables;** 1914, xviii, 230 pp.	
5_1.* G. C. Evans, **Functionals and their Applications. Selected Topics, Including Integral Equations**, 1918, xii, 136 pp.	
5_2.* O. Veblen, **Analysis Situs**, 1922, viii, 150 pp.; second ed., 1931, x, 194 pp.	
6. G. C. Evans, **The Logarithmic Potential. Discontinuous Dirichlet and Neumann Problems**, 1927, viii, 150 pp.	2 00
7. E. T. Bell, **Algebraic Arithmetic**, 1927, iv, 180 pp.	2.70
8. L. P. Eisenhart, **Non-Riemannian Geometry**, 1927, viii, 184 pp.; reprinted, 1934, viii, 184 pp.	2.70
9. G. D. Birkhoff, **Dynamical Systems**, 1927, viii, 295 pp.	3.35
10. A. B. Coble, **Algebraic Geometry and Theta Functions**, 1929, viii, 282 pp.	3.35
11.* D. Jackson, **The Theory of Approximation**, 1930, viii, 178 pp.	
12.* S. Lefschetz, **Topology**, 1930, x, 410 pp.	
13. R. L. Moore, **Foundations of Point Set Theory**, 1932, viii, 486 pp.	6.00

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* Out of print.

14.	J. F. Ritt, **Differential Equations from the Algebraic Standpoint**, 1932, x, 172 pp.	2.70
15.*	M. H. Stone, **Linear Transformations in Hilbert Space and their Applications to Analysis**, 1932, viii, 622 pp.	
16.	G. A. Bliss, **Algebraic Functions**, 1933, x, 218 pp.	3.35
17.	J. H. M. Wedderburn, **Lectures on Matrices**, 1934, viii, 200 pp.	3.35
18.	M. Morse, **The Calculus of Variations in the Large**, 1934, x, 368 pp.	5.35
19.	R. E. A. C. Paley and N. Wiener, **Fourier Transforms in the Complex Domain**, 1934, viii, 184 pp. + portrait plate.	3.35
20.	J. L. Walsh, **Interpolation and Approximation by Rational Functions in the Complex Domain**, 1935, x, 382 pp.	6.00
21.	J. M. Thomas, **Differential Systems**, 1937, x, 118 pp.	2.00
22.	C. N. Moore, **Summable Series and Convergence Factors**, 1938, vi, 105 pp.	2.00
23.	G. Szegö, **Orthogonal Polynomials**, 1939, x, 401 pp.	6.00
24.	A. A. Albert, **Structure of Algebras**, 1939, xii, 210 pp	4.00
25.	G. Birkhoff, **Lattice Theory**, 1940, vi, 155 pp.	2.50
26.	N. Levinson, **Gap and Density Theorems**, 1940, viii, 246 pp	4.00
27.	S. Lefschetz, **Algebraic Topology**, 1942, vi, 389 pp.	6.00
28.	G. T. Whyburn, **Analytic Topology**, 1942, x, 278 pp.	4.75

SOME TOPICS IN THE ARITHMETIC OF POLYNOMIALS

L. CARLITZ

1. **Introduction.** Let $GF(p^n)$ denote a fixed Galois (finite) field, and x an indeterminate. The arithmetic of polynomials in x with coefficients in $GF(p^n)$ is in many ways similar to ordinary arithmetic, and was discussed in some detail by Dedekind.[1] As a matter of fact it appears that in many instances the arithmetic of polynomials is the simpler. Thus, for example, in the case of the analogues of the familiar arithmetic functions, in place of asymptotic formulas there are exact formulas for the polynomial domain. This is perhaps due to the possibility of grouping polynomials according to degree. Again it is familiar that in the problem of representing a rational integer as a sum of an even number of squares there is a considerable difference between the case $2t \leqq 8$ and $2t > 8$; in the former case the number of representations can be expressed in terms of divisor functions, while in the latter case this is in general impossible. For the polynomial case however the number of representations by an even number of squares can always be expressed in terms of divisor functions. Similar remarks apply to the case of an odd number of squares.

In the present paper we rather arbitrarily select three or four topics in the arithmetic of polynomials in a Galois field. In §2 we consider the simplest arithmetic functions. In §3 we discuss the problem of representing a given polynomial as a sum of squares. In §4 we define various special polynomials and functions that are rather intimately connected with the arithmetic of polynomials in $GF(p^n)$; application to power sums are given in §5. Finally in §6 we define analogues of the ordinary Bernoulli numbers; the principal result here is the Staudt-Clausen theorem. We remark that for the most part the extension of theorems from the coefficient field[2] $GF(p)$ to $GF(p^n)$ is quite trivial; however, in at least the last topic mentioned there appears to be some difference between the special and the general case.

It is evident from the above that we are ignoring such questions as construction and distribution of irreducible polynomials, the existence of irreducibles in an arithmetic progression, theorems of reciproc-

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An address delivered before the meeting of the Society in Hanover, N. H., on September 12, 1940, by invitation of the Program Committee under the title *Arithmetic of polynomials in a Galois field*; received by the editors January 22, 1942.

[1] Journal für die reine und angewandte Mathematik, vol. 54 (1857), pp. 1–26.

[2] The field $GF(p)$ may be defined as the set of residues (mod p).

ity and higher congruences generally, to mention a few obvious omissions. We give a few references below.[3]

2. **Arithmetic functions.**[4] It will be convenient to denote polynomials in x by capitals $A, B, \cdots, Z$; elements of $GF(p^n)$ and ordinary integers will be denoted by small letters. We write deg M for the degree of M and put $|M| = p^{nm}$, $m = \deg M$; if the coefficient of the highest power of x in M is 1 we call M *primary*. The letter P will be reserved for *irreducible* polynomials. Evidently the number of primary polynomials of fixed degree m is p^{nm}. Hence the ζ-function for our domain becomes

$$(2.1) \qquad \zeta(s) = \sum_M \frac{1}{|M|^s} = \sum_{m=0}^{\infty} \frac{p^{nm}}{p^{nms}} = \frac{1}{1 - p^{n(1-s)}}$$

for $\Re(s) > 1$; the first summation in (2.1) is taken over all primary polynomials. Now since the unique factorization theorem applies here we have

$$\begin{aligned} \sum_M \frac{1}{|M|^s} &= \prod_P \left\{1 + \frac{1}{P^s} + \frac{1}{P^{2s}} + \cdots \right\} \\ &= \prod_P \left\{1 - \frac{1}{|P|^{-s}}\right\}^{-1} = \prod_{k=1}^{\infty} \left\{1 - \frac{1}{p^{ks}}\right\}^{-f(k)}, \end{aligned}$$

where $f(k)$ denotes the number of (primary) irreducible polynomials of degree k. Comparison with (2.1) leads to the identity

$$(2.2) \qquad 1 - p^{n(1-s)} = \prod_{k=1}^{\infty} (1 - p^{-ks})^{f(k)},$$

by means of which it is easy to derive the familiar formula for $f(k)$. We remark that (2.2) can be considerably extended.

Now define the Möbius function $\mu(M)$ by means of

$$\mu(1) = 1, \qquad \mu(M) = 0 \quad \text{for} \quad P^2 \mid M,$$
$$\mu(M) = (-1)^r \quad \text{for} \quad M = P_1 \cdots P_r.$$

Then

$$\sum_M \frac{\mu(M)}{|M|^s} = \prod_P \left(1 - \frac{1}{|P|^s}\right) = \frac{1}{\zeta(s)} = 1 - p^{n(1-s)},$$

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[3] See for example L. E. Dickson, *Linear Groups*, 1901, pp. 3–54; H. Kornblum, Mathematische Zeitschrift, vol. 5 (1919), p. 107; E. Artin, Mathematische Zeitschrift, vol. 19 (1924), pp. 153–246; O. Ore, Transactions of this Society, vol. 35 (1933), pp. 559–584; L. Carlitz, American Journal of Mathematics, vol. 59 (1937), pp. 618–628.

[4] Compare American Journal of Mathematics, vol. 54 (1932), pp. 39–50.

and therefore

$$\sum_{\deg M=m} \mu(M) = \begin{cases} -p^n & \text{for } m = 1, \\ 0 & \text{for } m > 1. \end{cases}$$

Similarly the number of quadrat-frei (simple) polynomials of degree m is $p^{nm}-p^{n(m-1)}$.

In exactly the same way if $\delta(M)$ denotes the number of (primary) divisors of M and $\phi(M)$ denotes the Euler function (number of residues in a reduced residue system (mod M)) then we have the identities

$$\sum_M \frac{\delta(M)}{|M|^s} = \zeta^2(s) = (1 - p^{n(1-s)})^{-2},$$

$$\sum_M \frac{\phi(M)}{|M|^s} = \frac{\zeta(s-1)}{\zeta(s)} = \frac{1 - p^{n(1-s)}}{1 - p^{n(2-s)}},$$

by means of which it is easy to evaluate

$$\sum_{\deg M=m} \delta(M), \qquad \sum_{\deg M=m} \phi(M).$$

We remark that $\delta(M)$ and $\phi(M)$ may be generalized in the following way

$$\delta_k(M) = \sum_{A|M,\ \deg A=k} 1, \tag{2.3}$$

that is, the number of divisors of given degree k. Similarly $\phi_k(M)$ denotes the number of primary polynomials of degree k, each prime to M. These functions are useful in certain problems (see, for example, (3.8) below).

3. **Sums of squares.**[5] Assume p odd. Let t be an integer greater than 0; $\alpha_1, \cdots, \alpha_t, \beta_1, \cdots, \beta_t$ elements of $GF(p^n)$ such that

$$\gamma_i = \alpha_i + \beta_i \neq 0, \qquad i = 1, \cdots, t. \tag{3.1}$$

Then if $\gamma = \gamma_1 + \cdots + \gamma_t \neq 0$, and M is primary of even degree $2k$, we seek the number of solutions of

$$\gamma M = \alpha_1 X_1^2 + \beta_1 Y_1^2 + \cdots + \alpha_t X_t^2 + \beta_t Y_t^2 \tag{3.2}$$

in primary X_i, Y_i of degree k. We call the problem (A).

Next if $\gamma_1 + \cdots + \gamma_t = 0$, $\alpha \neq 0$, and M is primary of degree less then $2k$, we seek the number of solutions of

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[5] Compare Transactions of this Society, vol. 35 (1933), pp. 397–410; Duke Mathematical Journal, vol. 1 (1935), pp. 298–315.

$$\alpha M = \alpha_1 X_1^2 + \beta_1 Y_1^2 + \cdots + \alpha_t X_t^2 + \beta_t X_t^2. \tag{3.3}$$

We call this problem (B).

The solution of these problems is given in terms of certain "divisor" functions now to be defined. For brevity we limit ourselves to the case of problem (A). Put

$$\rho_t(M) = \left(1 - \frac{1}{p^{nt}}\right) \sum_{A|M}^{a>k} |A|^t + \sum_{A|M}^{a=k} |A|^t, \tag{3.4}$$

where $a = \deg A$, $|A| = p^{na}$, the first summation is over all primary A dividing M and of degree greater than k, the second is over all primary A dividing M and of degree equal to k;

$$\omega_t(M) = \left(1 + \frac{1}{p^{nt}}\right) \sum_{A|M}^{a>k} (-1)^a |A|^t + \sum_{A|M}^{a=k} (-1)^a |A|^t, \tag{3.5}$$

the notation having the same meaning as in (3.4). *Then the number of solutions of* (3.2) *is* $\rho_{t-1}(M)$ *or* $\omega_{t-1}(M)$ *according as*

$$(-1)^t \alpha_1 \cdots \alpha_t \beta_1 \cdots \beta_t$$

is or is not a square in $GF(p^n)$.

The proof of this theorem is by induction. The case $t=1$ is easily proved. For the rest it suffices to prove the following three formulas

$$\begin{aligned} \sum \rho_s(A)\rho_t(B) &= \rho_{s+t+1}(M), \\ \sum \rho_s(A)\omega_t(B) &= \omega_{s+t+1}(M), \\ \sum \omega_s(A)\omega_t(B) &= \rho_{s+t+1}(M), \end{aligned} \tag{3.6}$$

where the summation is over all primary A, B of degree $2k$ such that

$$(\alpha + \beta)M = \alpha A + \beta B, \tag{3.7}$$

and α, β are elements of $GF(p^n)$, $\alpha\beta(\alpha+\beta) \neq 0$. We sketch briefly a proof of the first of (3.6); by introducing certain additional notation we may prove all three formulas simultaneously.

It is convenient to make use of the divisor function $\delta_s(M)$ defined in (2.3). For this function we have the theorem[6]

$$\sum \delta_i(A)\delta_j(B) = (1 - p^{-n}) \sum_{s=i+1}^{2k} \delta_s(M) + \delta_j(M), \tag{3.8}$$

where $k \leqq j \leqq i \leqq 2k$, and the summation on the left is over all A,

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[6] Proceedings of the London Mathematical Society, (2), vol. 38 (1934), pp. 116–124, in particular, p. 122.

B satisfying (3.7). Then by (3.4)

$$\sum \rho_s(A)\rho_t(B) = \epsilon_s\epsilon_t \sum_{i,j=k+1}^{2k} p^{n(is+jt)}g(i, j) + p^{nk(s+t)}g(k, k)$$
$$+ \epsilon_s p^{nkt} \sum_{i=k+1}^{2k} p^{nis}g(i, k) + \epsilon_t p^{nks} \sum_{j=k+1}^{2k} p^{njt}g(k, j),$$

where $g(i, j)$ denotes the left member of (3.8) and $\epsilon_t = 1-p^{-nt}$. Now using (3.8) the proof of the formula in question follows without much difficulty.

We have assumed p odd. For $p=2$ the right member of (3.2) or (3.3) is a perfect square and therefore the problem is of no interest. However, in this case we may consider the number of solutions of

$$\gamma M = \sum_{i=1}^{t} (\alpha_i X_i^2 + \beta_i X_i Y_i + \gamma_i Y_i^2),$$

where now $\prod \beta_i \neq 0$. It may be shown that results similar to the above still hold. The final form of the result depends on whether the quadratic form $\alpha_i X^2 + \beta_i XY + \gamma_i Y^2$ is irreducible in $GF(2^n)$. As a special case of some interest we take

$$\gamma M = \beta_1 X_1 Y_1 + \cdots + \beta_t X_t Y_t, \tag{3.9}$$

where

$$\gamma = \beta_1 + \cdots + \beta_t \neq 0, \qquad \beta_i \neq 0.$$

Evidently the number of solutions of this problem in primary X_i, Y_i of degree k is

$$\sum \delta_k(M_1) \cdots \delta_k(M_t), \tag{3.10}$$

the summation extending over primary M_i of degree $2k$ such that $\gamma M = \beta_1 M_1 + \cdots + \beta_t M_t$. But by (3.4) and the definition of $\delta_s(M)$ it is clear that $\delta_k(M) = \rho_0(M)$, so that (3.9) becomes

$$\sum \rho_0(M_1) \cdots \rho_0(M_t) = \rho_{t-1}(M),$$

as follows from (3.6). Hence the number of solutions of (3.9) is $\rho_{t-1}(M)$. This result holds for all p.

The situation for an odd number of squares is quite different. Corresponding to (3.2) we consider

$$\epsilon M = \alpha_1 X_1^2 + \cdots + \alpha_{2t+1} X_{2t+1}^2, \tag{3.11}$$

where $\epsilon = \alpha_1 + \cdots + \alpha_{2t+1} \neq 0$, while to (3.3) corresponds

$$(3.12)\qquad \alpha M = \alpha_1 X_1^2 + \cdots + \alpha_{2t+1} X_{2t+1}^2,$$

where $\alpha_1 + \cdots + \alpha_{2t+1} = 0$, $\alpha \neq 0$ arbitrary.

Put $\theta = (-1)^t \epsilon \alpha_1 \cdots \alpha_{2t+1}$ or $(-1)^t \alpha\alpha_1 \cdots \alpha_{2t+1}$ according as (3.11) or (3.12) is being considered. For simplicity we limit ourselves to the case M quadrat-frei. Then the number of solutions of (3.11) or (3.12) is given by

$$(3.13)\qquad \begin{aligned} p^{nk(t-1)}\{\sigma_k + p^{nt}\sigma_{k-1} + (p^{2nt} - p^n)\sigma_{k-2} + \cdots \\ + (p^{nkt} - p^{n(kt-2t+1)})\sigma_0\} \end{aligned}$$

where

$$\sigma_j = \sigma_j(\theta M) = \sum_{\deg A = j} (\theta M/A),$$

summed over primary A of degree j, and $(\theta M/A)$ is the quadratic residue symbol. The proof of (3.13) depends on the previous results for (3.2) and (3.3).

We remark finally that the results of this section can be obtained by an entirely different method which applies to either an odd or even number of squares. This was suggested by Hardy's paper *On the representation of a number as the sum of any number of squares and in particular of five.*[7] However, we shall not take the space to discuss this method.

4. Special polynomials and functions.[8] Put

$$(4.1)\qquad \begin{aligned} F_k &= [k][k-1]^{p^n} \cdots [1]^{p^{n(k-1)}}, & F_0 &= 1, \\ L_k &= [k][k-1] \cdots [1], & L_0 &= 1, \end{aligned}$$

where

$$[k] = x^{p^{nk}} - x.$$

Then as is well known, $[k]$ is the product of irreducible polynomials of degree a divisor of k. As for F_k and L_k, it may be shown that F_k is the product of the (primary) polynomials of degree k, while L_k is the least common multiple of these polynomials.

Next consider

$$(4.2)\qquad \psi_m(t) = \prod_{\deg A < m} (t + A), \qquad \psi_0(t) = t,$$

where t is a second indeterminate and the product is over all $A = A(x)$ of degree less than m. It is not difficult to show that (4.2) implies

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[7] Transactions of this Society, vol. 21 (1920), pp. 255–284.

[8] Compare Duke Mathematical Journal, vol. 1 (1935), pp. 137–168.

$$(4.3)\qquad \psi_m(t) = \sum_{i=0}^{m} (-1)^{m-i} \begin{bmatrix} m \\ i \end{bmatrix} t^{p^{ni}},$$

where the coefficients are determined by

$$\begin{bmatrix} m \\ i \end{bmatrix} = \frac{F_m}{F_i L_{m-i}^{p^{ni}}}, \qquad \begin{bmatrix} m \\ 0 \end{bmatrix} = \frac{F_m}{L_m}, \qquad \begin{bmatrix} m \\ m \end{bmatrix} = 1,$$

and are integral, that is, polynomials in x. Of the properties of $\psi_m(t)$ that follow from (4.3) we mention

$$(4.4)\qquad \psi_m(xt) = x\psi_m(t) + [m]\psi_{m-1}^{p^n}(t),$$

$$(4.5)\qquad \psi_m(t) = \psi_{m-1}^{p^n}(t) - F_{m-1}^{p^n-1}\psi_{m-1}(t).$$

Also as a consequence of (4.2) we have

$$(4.6)\qquad \psi_m(M) = F_m$$

for M primary of degree m. Note that

$$\psi_m(t+u) = \psi_m(t) + \psi_m(u), \quad \psi_m(ct) = c\psi_m(t) \quad \text{for } c \text{ in } GF(p^n);$$

we therefore call $\psi_m(t)$ a "linear" polynomial.

The formula (4.4) suggests the operator Δ defined by

$$\Delta g(t) = g(xt) - xg(t);$$

Δ^k is defined recursively by

$$\Delta^{k+1}g(t) = \Delta^k g(xt) - x^{p^{nk}}\Delta^k g(t).$$

From these formulas it follows in particular that

$$(4.7)\qquad \Delta^k\psi_m(t) = [m]\cdots[m-k+1]^{p^{n(k-1)}}\psi_{m-k}^{p^{nk}}(t), \qquad m \geqq k.$$

Now let $g(t) = \sum \alpha_i t^{p^{ni}}$ be any linear polynomial in t. Clearly it may be expressed in the form $\sum \beta_i \psi_i(t)$. The coefficients are readily determined by means of (4.7). Replacing t by tu we get the expansion

$$(4.8)\qquad g(tu) = \sum_i (1/F_i)\psi_i(u)\Delta^i g(t).$$

Thus for $g(t) = \psi_m(t)$, (4.8) becomes

$$(4.9)\qquad \psi_m(tu) = \sum_{i=0}^{m} \begin{Bmatrix} m \\ i \end{Bmatrix} \psi_i(u)\psi_{m-i}^{p^{ni}}(t),$$

while for $g(t) = t^{p^{nm}}$ we get

$$t^{p^{nm}} = \sum_{i=0}^{m} \left\{ {m \atop i} \right\} \psi_i(t). \tag{4.10}$$

The coefficients in (4.9) and (4.10) are given by

$$\left\{ {m \atop i} \right\} = \frac{F_m}{F_i F_{m-i}^{p^{ni}}}, \qquad \left\{ {m \atop 0} \right\} = \left\{ {m \atop m} \right\} = 1,$$

and are integral.

The polynomial $\psi_m(t)$ suggests the construction of a function vanishing for all $A = A(x)$. We find that the function

$$\psi(t) = \sum_{i=0}^{\infty} (-1)^i t^{p^{ni}}/F_i \tag{4.11}$$

has the product expansion

$$\psi(t) = t \prod_M \left\{ 1 - \frac{t^{p^n-1}}{(M\xi)^{p^n-1}} \right\}, \tag{4.12}$$

the product extending over *all* primary M. Hence $\psi(A\xi) = 0$ for all A. As for ξ it may be defined by

$$\xi = \lim_{k=\infty} \frac{[1]^{p^{nk}/(p^n-1)}}{L_k}. \tag{4.13}$$

It has recently been proved[9] that ξ is transcendental relative to the field $GF(p^n, x)$, that is, the field of rational functions in x with coefficients in $GF(p^n)$.

Applying the operator Δ to $\psi(t)$ we find

$$\psi(xt) = x\psi(t) - \psi^{p^n}(t); \tag{4.14}$$

repeated use of this formula leads to the multiplication formula

$$\psi(Mt) = \sum_{i=0}^{m} (-1)^i \frac{\psi_i(M)}{F_i} \psi^{p^{ni}}(t), \tag{4.15}$$

where M is primary of degree m. This in turn suggests the introduction of the linear polynomial $\omega_M(u)$ defined by

$$\omega_M(\psi(t)) = \psi(Mt);$$

next put

$$W_M(u) = \prod_{AB=M} \left\{ \omega_A(u) \right\}^{\mu(B)},$$

with $\mu(B)$ as in §2. Then $W_M(u)$ may be thought of as an analogue of the cyclotomic polynomial; in particular it is irreducible.[10]

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[9] L. I. Wade, Duke Mathematical Journal, vol. 8 (1941), pp. 701–720.

[10] See Transactions of this Society, vol. 43 (1938), pp. 167–182.

Returning to (4.11) we find that the inverse of $\psi(t)$ is also of simple form, namely

$$(4.16)\qquad \lambda(t) = \sum_{i=0}^{\infty} (1/L_i)t^{p^{ni}}.$$

As for the convergence of (4.11) and (4.16) if we take

$$(4.17)\qquad t = c_k x^k + \cdots + c_0 + \frac{c_{-1}}{x} + \cdots ,$$

where the c_j are all in $GF(p^n)$, then $\psi(t)$ converges for all t while $\lambda(t)$ converges only for $k \leq 1$.

It follows from (4.11) and (4.12) that

$$(4.18)\qquad \psi(t + M\xi) = \psi(t) + \psi(M\xi) = \psi(t),$$

for arbitrary M; in other words $\psi(t)$ has the period ξ. This property can be generalized and "linear" functions with any number of periods can be constructed. For example, if we write (4.17) in the form

$$\psi(t + M_0\xi + M_1\theta\xi + \cdots + M_{n-1}\theta^{n-1}\xi) = \psi(t),$$

where θ defines $GF(p^n)$ and the M_i have coefficients in $GF(p)$, then we may regard $\psi(t)$ as an n-ply periodic function with respect to the smaller field. For the general case we put

$$(4.19)\qquad f(t) = \sum_{i=0}^{\infty} (-1)^i A_i t^{p^{ni}}/F_i;$$

in place of (4.14) we now have

$$(4.20)\qquad f(xt) = xf(t) + \sum_{j=1}^{k} (-1)^j \gamma_j f^{p^{nj}}(t),$$

which is characteristic. Using (4.20) we get a recursion for A_i,

$$A_i = \sum_{j=1}^{k} \frac{F_{i-1}^{p^n}}{F_{i-j}^{p^{nj}}} \gamma_j A_{i-j}^{p^{nj}}.$$

It is easily verified that (4.19) converges for all t defined by (4.17). The multiplication formula (4.15) now becomes

$$f(Mt) = \sum_{i=0}^{mk} (-1)^i \frac{f_i(M)}{F_i} f^{p^{ni}}(t),$$

thus defining a set of linear polynomials $f_i(u)$.

5. **Power sums.**[11] Going back to (4.2) it is not difficult to show that it implies

$$\sum_{\deg M=m} \frac{1}{t+M} = (-1)^m \frac{F_m}{L_m} \frac{1}{\psi_m(t)+F_m}. \tag{5.1}$$

In this identity put $t=0$ and we get

$$\sum_{\deg M=m} (1/M) = (-1)^m/L_m, \tag{5.2}$$

so that by the remark at the beginning of §4, the sum of the reciprocals of the polynomials of degree m is, except for sign, the reciprocal of the L. C. M. If we expand (5.1) in descending powers of t it is easily seen that

$$\sum_{\deg M=m} M^{p^{nm}-1} = (-1)^m F_m/L_m \tag{5.3}$$

while

$$\sum_{\deg M=m} M^k = 0 \qquad \text{for} \quad k < p^{nm} - 1. \tag{5.4}$$

(5.2) and (5.3) are special cases of[12]

$$\sum_{\deg M=m} M^{p^{nk}-1} = (-1)^m \frac{F_k}{L_m F_{k-m}^{p^{nm}}}, \qquad k \geq m, \tag{5.5}$$

and

$$\sum_{\deg M=m} M^{1-p^{nk}} = \frac{L_{k+m-1}}{L_{k-1} L_m^{p^{nk}}}. \tag{5.6}$$

We outline a new proof of these formulas.

Consider the sum

$$\sum_{\deg M=m} M(t)/M(x). \tag{5.7}$$

Since (5.7) is a polynomial in t of degree m, it may be determined by assigning $m+1$ values to t. We take

$$t = x^{p^{ni}}, \qquad i = 0, 1, \cdots, m.$$

Using the Lagrange interpolation formula together with (5.3) and (5.4) leads to the identity

$$\sum_M M(t)/M(x) = ((-1)^m/L_m)(t-x)(t-x^{p^n}) \cdots (t-x^{p^{n(m-1)}}). \tag{5.8}$$

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[11] Compare Duke Mathematical Journal, vol. 5 (1939), pp. 941–947.

[12] More general formulas are proved by H. L. Lee in a Duke University thesis.

In (5.8) take $t = x^{p^{nk}}$ and we get (5.5); on the other hand interchange x and t, put $t = x^{p^{nk}}$, and the result is (5.6).

We next construct a polynomial[13] $G_m(t)$ which is useful in evaluating more general power sums. Put

$$m = a_0 + a_1 p^n + a_2 p^{2n} + \cdots, \qquad 0 \leqq a_i < p^n;$$

then define

$$G_m(t) = \psi_0^{a_0}(t)\psi_1^{a_1}(t)\psi_2^{a_2}(t) \cdots, \qquad G_0(t) = 1, \tag{5.9}$$

so that $G_m(t)$ is of degree m in t, and the coefficients are integral in x. We also define

$$g_m = F_1^{a_1} F_2^{a_2} \cdots, \qquad g_0 = 1. \tag{5.10}$$

We shall not now go into the connection between $G_m(t)$ and power sums but instead quote an application of a different sort. A polynomial $f(t)$ may be called integral-valued if $f(A)$ is integral for all integral A. Then we have the theorem:—*the polynomial*

$$\sum_{i=0}^{k} A_i G_i(t)/g_i$$

is integral-valued if and only if the coefficients A_i *are integral, that is, polynomials in* x.

6. **Bernoulli numbers.**[14] As analogues of the Bernoulli numbers we define a set of rational functions B_m by means of

$$\frac{t}{\psi(t)} = \sum_{m=0}^{\infty} \frac{B_m}{g_m} t^m, \tag{6.1}$$

where $\psi(t)$ is defined by (4.11) and g_m is given by (5.10). Note that B_m is defined only for m a multiple of $p^n - 1$. Then in the first place we have the formula

$$\sum_A \frac{1}{A^m} = \frac{B_m}{g_m} \xi^m, \qquad p^n - 1 \mid m, \tag{6.2}$$

the summation extending over all primary polynomials, and ξ as in (4.13). Thus (6.2) is the analogue of a familiar formula; in particular it shows that $B_m \neq 0$. Unfortunately no formula connecting B_m with finite power sums is available and therefore the usual methods for deriving arithmetic properties of the ordinary Bernoulli numbers paparently cannot be applied. Instead we make use of certain ideas due to Hurwitz.[15]

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[13] See Duke Mathematical Journal, vol. 6 (1940), pp. 486–504.

[14] Compare Duke Mathematical Journal, vol. 3 (1937), pp. 503–517; vol. 7 (1940), pp. 62–67.

We call a series of the form

$$H(t) = \sum_{m=0}^{\infty} A_m t^m / g_m, \tag{6.3}$$

where the A_m are integral, a Hurwitz series, briefly an H-series. It follows that the sum and product of two H-series are again H-series; if $A_0 = 1$ then the reciprocal is an H-series. If $A_0 = 0$ we call (6.3) an H_1-series. For this case we have the result that H_1^k / g_k is an H-series. This result applied to

$$\psi^{p^{nk}-1}(t) = \sum_m A_m^{(k)} t^m / g_m \tag{6.4}$$

shows that $A_m^{(k)}$ is a multiple of F_k / L_k.

Now returning to the definition of B_m we have

$$\frac{t}{\psi(t)} = \frac{\lambda(\psi(t))}{\psi(t)} = \sum_{k=0}^{\infty} \frac{1}{L_k} \psi^{p^{nk}-1}$$

by (4.15) and therefore using (6.4) we get the formula

$$B_m = \sum_k \frac{1}{L_k} A_m^{(k)}. \tag{6.5}$$

From this result it follows that the denominator of B_m contains only simple factors. To improve this we require the following lemma: *Let P be irreducible of degree k. Then*

$$\psi^{p^{nk}-1}(t) \equiv \left\{ \sum_{i=0}^{\infty} (-1)^i \frac{t^{p^{nki}}}{F_{ki}} \right\}^{p^{nk}-1} \pmod{P}. \tag{6.6}$$

It is now easy to evaluate $A_m^{(k)}$ (mod P). Substituting in (6.5) we get the following result.

THEOREM ($p^n \neq 2$). *Let*

$$m = \sum_i a_i p^i, \qquad 0 \leq a_i < p.$$

Then if

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[15] Mathematische Annalen, vol. 51 (1899), pp. 196–226 (= Mathematische Werke II, Basel, 1933, pp. 342–373).

$$\sum_i a_i = nk(p-1), \qquad p^{nk}-1 \mid m, \tag{6.7}$$

is inconsistent, B_m is integral; while if (6.7) *is consistent, k is uniquely determined and*

$$B_m = G_m - e \sum_{\deg P = k} \frac{1}{P}, \tag{6.8}$$

where G_m is integral, the summation is over irreducible polynomials P of degree k, and

$$e = \frac{(-1)^{nk+dk}}{\prod_i (a_i!)}, \qquad d = \sum_{i,j} i a_{ink+j}.$$

The case $p^n = 2$ is covered by a supplementary theorem which we shall omit.

We remark that if the function $f(t)$ has an inverse of the form $\sum (D_i/L_i) t^{p^{ni}}$, where the D_i are integral, then for the coefficients of $t/f(t)$ there is a decomposition into partial fractions similar to (6.8). This result evidently generalizes the above theorem on B_m.

DUKE UNIVERSITY

DIFFERENTIAL EQUATIONS WITH GENERAL BOUNDARY CONDITIONS

WILLIAM M. WHYBURN

This paper concerns ordinary differential equations in the real domain. More specifically, it discusses systems of first order, nonsingular, equations of the form

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \cdots, y_n), \qquad i = 1, 2, \cdots, n, \tag{1}$$

together with boundary conditions at one, two, or more points of the interval of definition. System (1) is quite general and can be made to include the nonsingular equations of the nth order as special cases. The paper makes no attempt to generalize the equations beyond those that occur in standard treatments. Any elements of generality or novelty introduced appear in connection with the types of boundary conditions discussed. The mathematical literature of the past century contains many results for system (1) with boundary conditions at one point—the so-called fundamental existence theorems occupying central positions—and with boundary conditions at two points of the interval where second order linear systems have been of prime importance. These important bodies of results have been suveyed before the Society through addresses by Bôcher, Bliss, Pell-Wheeler, Reid and others, and therefore are given little mention in the present paper. We discuss results that have been obtained in cases where the boundary conditions apply to n or less points, to any finite number of points, to any infinite point set of the first species,[1] and where the boundary conditions contain integrals over an interval. The literature on differential systems with such boundary conditions is not extensive although many substantial results have been obtained and potential applications exist in a number of fields.

1. **Fundamental existence theorems.** Let the real functions $f_i(x, y_1, \cdots, y_n)$ in system (1) be defined over a domain

$$D: \begin{cases} (a, b): & a \leqq x \leqq b \\ R: & A_i < y_i < B_i \qquad (i = 1, \cdots, n), \end{cases}$$

An address delivered before the Pasadena meeting of the Society on November 22, 1941, by invitation of the Program Committee; received by the editors May 28, 1942.

[1] A set of the first species is one which has at most a finite number of nonvacuous derived sets. Any such set is at most denumerably infinite. See Hobson, *Theory of Functions of a Real Variable*, vol. 1, 2d edition, Cambridge, 1920, pp. 71 and 79.

of $(n+1)$-dimensional real space, where a and b are finite but any or all of the A_i's and B_i's may be infinite. By a solution of system (1) on an interval $\alpha \leqq x \leqq \beta$ we understand a set of absolutely continuous functions $y_i = g_i(x)$ $(i=1, 2, \cdots, n)$, such that the equations $g_i' = f_i[x, g_1(x), g_2(x), \cdots, g_n(x)]$ $(i=1, 2, \cdots, n)$, hold at all points of $\alpha \leqq x \leqq \beta$ except possibly for a set of points of measure zero. Classical fundamental existence theorems for this system are of "neighborhood" and "interval" types. The first of these uses light hypotheses but gives existence of solutions in neighborhoods of specified points only. The second or "interval" type uses strong hypotheses and yields existence throughout previously assigned intervals. A typical theorem of the first type follows:

THEOREM I. *Let the functions* $f_i(x, y_1, \cdots, y_n)$ *be continuous in* $(y_1, \cdots, y_n)$ *throughout* R *for each fixed* x *on* (a, b), *and measurable in* x *on* (a, b) *for fixed* $(y_1, \cdots, y_n)$ *in* R. *Let a Lebesgue integrable function* $M(x)$ *exist on* (a, b) *such that throughout* D, $|f_i(x, y_1, \cdots, y_n)| < M(x)$ $(i=1, 2, \cdots, n)$. *If* $P(c, k_1, k_2, \cdots, k_n)$ *is any point of* D, *there exists a neighborhood of* $x=c$ *on which equation* (1) *has at least one solution such that* $y_i(c) = k_i$ $(i=1, 2, \cdots, n)$.

Theorem I is due to Carathéodory[1] and contains the Peano type theorems where the functions $f_i(x, y_1, \cdots, y_n)$ are assumed continuous throughout D, as special cases. For the latter theorems, the uniform boundedness condition is a consequence of the assumed continuity. If the domain D is such that $A_1, \cdots, A_n$, are negatively infinite while $B_1, \cdots, B_n$ are positively infinite, then Theorem I becomes the following "interval" type theorem:

THEOREM II. *Let the functions* $f_i(x, y_1, \cdots, y_n)$ *be continuous in* $(y_1, \cdots, y_n)$ *for each fixed* x *on* (a, b) *and all real* $(y_1, \cdots, y_n)$ *and let these functions be measurable in* x *on* (a, b) *for any fixed* $(y_1, \cdots, y_n)$ *and, finally, let a Lebesgue integrable function* $M(x)$ *exist such that*

$$|f_i(x, y_1, \cdots, y_n)| < M(x), \qquad i = 1, \cdots, n,$$

for all x *on* (a, b) *and all real* $(y_1, \cdots, y_n)$. *If* $x=c$ *is a point of* (a, b) *and* $k_1, \cdots, k_n$ *are real numbers, there exists at least one solution of system* (1) *on* (a, b) *such that* $y_i(c) = k_i$ $(i=1, 2, \cdots, n)$.

Through use of *over* and *under* functions, Perron and others have obtained "interval" type existence theorems[2] which differ materially in hypotheses from those stated above and which frequently apply when Theorems I and II fail.

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[1] *Vorlesungen über reelle Funktionen*, Leipzig, 1928, p. 672.

[2] For discussions of these theorems, see W. M. Whyburn, University of California Publications in Mathematical and Physical Sciences, vol. 1 (1935), pp. 115–134.

The hypotheses of Theorem II exclude linear systems. Separate existence theorems may be proved for such systems or Theorem II may be modified in such a way that it includes them as special cases. The latter result[4] is obtained by replacing $f_i(x, y_1, \cdots, y_n)$ with $F_i(x, y_1, \cdots, y_n)+\sum_{j=1}^{n} G_{ij}(x, y_1, \cdots, y_n)y_j$ and requiring that F_i, G_{ij}, satisfy the same hypotheses originally stated for $f_i(x, y_1, \cdots, y_n)$.

Niccoletti[5] studied a single differential equation of the nth order together with initial conditions at more than one point of the interval. The actual number of such points did not exceed n, the order of the differential equation. Since this work seems to have been the first on differential systems with boundary conditions at more than two points, we indicate the results obtained and the nature of the methods used. Through the use of certain algebraic formulas, Niccoletti established the fact that the function

$$u = \frac{1}{(n-1)!} f(x) \sum_{i=1}^{k} \frac{\partial^{\alpha_i-1}}{\partial a_{1_i}^{\alpha_i-1}} \left\{ \frac{\alpha_i}{f^{(\alpha_i)}(a_i)(x-a_i)} \int_{a_i}^{x} (a_i - s)^{n-1}\phi(s)ds \right\},$$

where $f(x)=(x-a_1)^{\alpha_1}(x-a_2)^{\alpha_2} \cdots (x-a_k)^{\alpha_k}$, $\alpha_1+\alpha_2+\cdots+\alpha_k=n$, vanishes along with its first (α_i-1) derivatives at the point $x=a_i$ $(i=1, 2, \cdots, k)$. Furthermore $u^{(n)}=\phi(x)$. If $P_{n-1}(x)$ is a polynomial of degree $n-1$ which together with its first α_i-1 derivatives takes assigned values at $x=a_i$ $(i=1, 2, \cdots, k)$, then $y(x)=u(x)+P_{n-1}(x)$ is a function which satisfies the differential equation $y^{(n)}=\phi(x)$ together with boundary conditions which assign the values of $y(x)$ and its first α_i-1 derivatives at the points $x=a_i$ $(i=1, 2, \cdots, k)$. With this result and the Picard method of successive approximations, Niccoletti proved the following theorem:

THEOREM III. *In the differential equation*

$$y^{(n)} = \phi(x, y, y', \cdots, y^{(n-1)}) \tag{2}$$

let the function $\phi(x, y, y', \cdots, y^{(n-1)})$ *be continuous for all* x *on* (a, b) *and all real* $(y, y', \cdots, y^{(n-1)})$ *and let it satisfy a Lipschitz condition with respect to* $(y, y', \cdots, y^{(n-1)})$, *then there exists a unique solution of* (2) *which together with its first* α_i *derivatives takes on assigned values at* $x=a_i$ $(i=1, \cdots, k)$, *provided the points* $a_1, a_2, \cdots, a_k$ *all lie on a single subinterval of* (a, b) *that is sufficiently small.*

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[4] See W. M. Whyburn, Annals of Mathematics, (2), vol. 30 (1928), pp. 31–38.

[5] Atti della R. Accademie Delle Scienze Di Torino, vol. 33 (1897), pp. 746 ff.

Niccoletti indicated that his results could be extended to systems of differential equations and to partial differential equations, but it seems clear that the actual extension to systems was not accomplished. Otherwise, the restriction that $y, y', \cdots, y^{(\alpha-1)}$ be assigned at any point where $y^{(\alpha)}$ is given would have been unnecessary. Niccoletti stated that his solutions were unique and that they were continuous functions of the initial values.

Independently of the work of Niccoletti, W. M. Whyburn[6] established a theorem of the type of Theorem III for systems (1) with initial conditions at n or less points of the interval. This theorem requires that the points lie in a single subinterval of (a, b) of length equal to a definite positive number but establishes existence and uniqueness of the solution on the entire interval (a, b). The solution functions are continuous functions of the initial points and values. The essential character of the requirement that the points used in the initial conditions be contained in a restricted subinterval of (a, b) is shown by the example $y' = kz$, $z' = -ky$, $y(a_1) = \alpha$, $z(a_2) = \beta$, $\alpha^2 + \beta^2 \neq 0$. A solution exists here if $|a_1 - a_2| < \pi/2k$ but fails to exist when $a_2 - a_1 = \pi/2k$.

Theorems of the above type are important since they allow the study of differential systems under conditions which do not permit the simultaneous assignment of all of the initial values but which do admit this assignment over a relatively short range of the independent variable. One possible application would be to a study of rapid machine gun fire where the gun is moved about during the firing interval and the individual trajectories are not as important as the family of such paths generated over a relatively short interval of time. Here one would have a system of second order equations—in number equal to the number of times the gun is fired—together with $2n$ initial conditions assigned at the times $t_1, t_2, \cdots, t_n$, when the gun actually fires. The above theorems, together with approximation methods which grow out of them, would apply to such problems.

This section of the paper is closed with a brief consideration of linear systems in which the notation used in the remainder of the paper is indicated. With the adaptation mentioned above, fundamental existence theorems apply to a system of first order linear differential equations with coefficients which are Lebesgue integrable functions of x on (a, b). It is convenient to write these systems in matrix form[7]

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[6] Annals of Mathematics, (2), vol. 30 (1928), pp. 31–38.

[7] This notation has been used by many writers and is becoming quite standard in the literature. One of many references to its use is Birkhoff and Langer, Proceedings of the American Academy of Arts and Sciences, vol. 57 (1922), pp. 51–128.

$$L(Y) = Y' + P(x)Y = Q(x), \tag{3}$$

where $L(Y)$, P, Y, Q, and Y' are square matrices of n-rows. If $x=s$ is a point of (a, b) and if initial conditions are given at this point, the method of variation of parameters yields

$$Y(x) = U(x)\left[\int_s^x V(t)Q(t)dt + V(s)Y(s)\right], \tag{4}$$

where $U(x)$ is a nonsingular matrix solution of the homogeneous equation $L(Y)=0$ and $V(t)=U^I(t)$ is the inverse of $U(t)$. This formula is useful in the remainder of the paper and is important in its own right since it is an explicit expression for the general solution of equation (3) in terms of a solution of the homogeneous equation and the initial values at one point. It is especially useful in a study of the solution of (3) as a *functional* of the coefficients and initial values where it provides estimates of the effects on the solution produced by small variations in the coefficients of the system.[8]

2. **Boundary conditions at two points.** Extensive investigations have been made of system (1) with conditions at two points of the interval. By far the greater part of these studies have been devoted to second order linear systems with parameters in the coefficients. The present paper omits[9] discussion of these results for second order systems and mentions only one development for nth order linear systems—namely that of the Green's function[10] or matrix—which is of interest in connection with later parts of the paper.

Let system (3) be given along with boundary conditions

$$H(Y) = AY(a) + BY(b) = C \tag{5}$$

where A, B, C are nth order square matrices of constants. If the homogeneous system $L(Y)=0$, $H(Y)=0$, is incompatible, then the existence and properties of the Green's matrix may be deduced in the following manner. In formula (4) let $s=a$ and multiply the resulting equation on the left by $A\,U(a)\,V(x)$. Similarly, replace s in formula (4) by b and multiply the resulting equation on the left by $B\,U(b)\,V(x)$. Addition of these two equations followed by multiplication on the left by $U(x)H^I(U)$ yields

$$Y(x) = U(x)H^I(U)C + \int_a^b G(x, t)Q(t)dt, \tag{6}$$

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[8] See W. M. Whyburn, Transactions of this Society, vol. 32 (1930), pp. 502–508.

[9] For references on this literature, see Reid, this Bulletin, vol. 43 (1937), pp. 633–666.

[10] See Bounitsky, Journal de Mathématiques, vol. 5 (1909), pp. 65–125

where

$$G(x, t) = \begin{cases} U(x)H^{I}(U)A\,U(a)V(t) & \text{for} \quad t < x \\ -\,U(x)H^{I}(U)B\,U(b)V(t) & \text{for} \quad t > x. \end{cases} \tag{7}$$

$G(x, t)$ as given by (7) is the Green's matrix for the system and its properties of uniqueness, continuity, symmetry, and so on, are readily deduced from this explicit formula.

Our discussion of two point boundary conditions is concluded with the observation that, in general, the conditions apply to the end points of the interval under consideration and hence at the boundary of the simply-connected domain (interval) of definition. Cases where one or both of the points are interior to the interval are usually treated by studying the solution on the interval bounded by these points and extending this work to the larger interval through use of results for boundary conditions at one point.

3. Conditions at a finite number of points. In this section we consider nth order linear systems with boundary conditions which involve a finite number k of points of the interval. For the differential system (3) the boundary conditions have the form

$$H(Y) = \sum_{j=1}^{k} A_j Y(a_j) = C, \tag{8}$$

where $A_1, \cdots, A_k$, C, are constant matrices and $a_1 < a_2 < \cdots < a_k$ are k distinct points of (a, b). For the single linear differential equation of the nth order, the boundary conditions are linear combinations of the values of the solution and its first $(n-1)$ derivatives at the k points. M. Picone[11] seems to have been the first person to formulate boundary conditions of type. He also obtained some results for a special nth order differential equation with integral boundary conditions. Bôcher discussed such conditions in his Fifth International Congress address[12] of 1912 and in his Paris lectures[13] of 1913–1914. Bôcher indicated the importance of boundary value problems of this type but did not publish any investigations on them. The first systematic study of such problems was made by C. E. Wilder,[14] who treated the single nth order equation with conditions at k points. Wilder defined a

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[11] Rendiconti della Accademia dei Lincei, vol. 17 (1908), pp. 340–347.

[12] Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1912, vol. 1, pp. 163–195.

[13] Bôcher, *Leçons sur les Methodes de Sturm*, Paris, 1917, p. 18.

[14] Transactions of this Society, vol. 18 (1917), pp. 415–442, and vol. 19 (1918), pp. 157–166.

Green's function for his differential system and investigated the adjoint relationship. He then introduced a parameter into one of the coefficients of the equation and developed expansion theorems for the resulting system making use of the Green's function and methods first introduced by Birkhoff.[15] Existence and oscillation theorems for second order systems containing a parameter in a general fashion and with boundary conditions, one of which involved k points, were obtained by W. M. Whyburn.[16] The results here differed from those ordinarily obtained for two point conditions in that the characteristic numbers were not isolated but occurred in sets or in bands. This, in itself, is of interest since the occurrence of bands of characteristic numbers in spectral analysis is frequent. Toyada[17] studied the Green's function and adjoint relationship for nth order differential systems, apparently overlooking the earlier work of Wilder. His results are quite similar to those obtained by Wilder although his method of attack differed in that he did not raise the dimensionality of the space or increase the order of the differential system. Toyada used the skew-symmetry of the Green's matrix as a basis for building an adjoint system and arrived at such a system under the condition that a total of kn^2 identities hold. He attempted to generalize his results to systems with integral boundary conditions but seems to have overlooked difficulties that arise due to the above-mentioned identities. Wilder's previous results showed that, in general, an adjoint system in the ordinary sense would not exist. This result has recently been confirmed by Mansfield.[18]

The connections between k-point boundary value problems and the calculus of variations have been explored by Reid,[19] Denbow,[20] Smiley[21] and Mansfield.[18] This work starts from the relation between the Jacobi necessary condition on the one hand, and second order differential equations with boundary conditions at two points on the other hand, and proceeds to cases where more than two points appear in the boundary conditions. An important device used is a transformation due to Reid[19] which effectively reduces the k-point case to one that involves $k-1$ systems each of the two point type. Mansfield[18] made a systematic study of k-point boundary value problems through use of this device and applied his results to the calculus of variations and to differential systems in the complex domain.

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[15] Transactions of this Society, vol. 9 (1908), pp. 373–395.

[16] Transactions of the Society, vol. 30 (1928), pp. 630–640.

[17] Tohôku Mathematical Journal, vol. 38 (1933), pp. 343–355, and vol. 39 (1934), pp. 387–398.

[18] *Contributions to the Calculus of Variations*, vol. 4, University of Chicago Press, pp. 413–472.

[19] American Journal of Mathematics, vol. 57 (1935), pp. 69–93.

[20] *Contributions to the Calculus of Variations*, 1933–1937, Chicago, 1937, pp. 449–494.

[21] Ibid., pp. 527–566.

Much of the work mentioned above makes specific use of the fact that the number of points appearing in the boundary conditions is finite. In some cases, use is made of an increase in the dimensionality of the space while in other cases, the order of the differential system is raised. When it is observed that the introduction of interior points of the interval into the boundary conditions has the effect of changing the simply-connected domain of definition into one that is multiply-connected, it is not surprising that devices of the above-mentioned type are used. However, difficulties arise when attempts are made to generalize the differential system through having the boundary conditions contain integrals or infinitely many points. The next two sections discuss conditions of these types.

4. **Integral conditions.** When the number of points which enter the boundary conditions is no longer required to be finite, there are possibilities that the conditions will contain linear combinations of the solution values on a non-dense infinite point set or contain definite integrals of the solution over the interval. Picone[22] investigated relationships that exist between integral equations and certain special nth order differential equations with integral boundary conditions. von Mises[23] studied the special second order system

$$\left\{\begin{aligned} &\frac{d}{dx}\left[k\,\frac{du}{dx}\right] + (\lambda g - q)u = 0 \\ &\int_a^b A(x)u(x)dx = 0, \qquad \int_a^b B(x)u(x)dx = 0, \end{aligned}\right. \tag{9}$$

where k, g, q, A, B are continuous functions of x and λ is a parameter. von Mises was interested in this system because of its occurrence in certain problems of hydrodynamics.[24] His special case of hydrodynamic application had $k=q=1$, $g=x+c$, $A=1/B=e^x$, where c is a constant. With specializations in the direction of this application, von Mises used Sturm's method of passage to the limit from an algebraic system to obtain existence and oscillation theorems for system (9).

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[22] Loc. cit., pp. 340–347.

[23] *Festschrift Heinrich Weber*, Leipzig, 1912, pp. 252–282.

[24] See Sommerfeld, Proceedings of the Fourth International Congress of Mathematicians, Rome, 1909, vol. 3, p. 116.

Bôcher[25] mentioned integral boundary conditions for second order systems and indicated a transformation that could sometimes be applied to replace such conditions by conditions at two points. The transformation suggested by Bôcher depends upon the actual solution in closed form of a slightly modified form of the given differential equation and, for this reason, offers little probability of use in any particular problem. Bôcher observed that this transformation would replace system (9) by one which no longer involved the parameter in the simple linear fashion of its original occurrence. In general, systems with integral boundary conditions should be regarded as essentially different from those with two point or k-point conditions. It is clear that integral boundary conditions would arise quite naturally in hydrodynamics or indeed in any investigation where such things as fixed areas under solution curves, fixed lengths of curves, and so on enter.

Tamarkin[26] considered k-point and integral conditions for single nth order linear differential equations.

W. M. Whyburn[27] established existence and oscillation theorems for linear second order equations with one integral boundary condition and one condition of the two point type. The coefficients of the system contained a parameter in a general (non-linear) fashion. As in the case of k-point conditions, it was found that the characteristic numbers were not isolated but occurred in sets which might be continua. Toyada[28] set up a Green's matrix for nth order linear systems with integral boundary conditions and attempted to generalize the adjoint notion to systems of this type. His methods were extensions of ones that he had used for k-point conditions and such extensions were possible because these methods were largely independent of the number of points involved in the boundary conditions. In connection with his work on adjoint systems, Toyada imposed a set of conditions the number of which depended upon the number of points involved in the boundary conditions. These conditions become unruly when integral conditions are used.

Integral boundary conditions of a different type have been investigated by a number of people.[29] These conditions contain Stieltjes integrals and for the nth order linear system (3) have the form

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[25] Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1912, vol. 1, p. 167.

[26] *On Some General Problems of the Theory*, (in Russian), Petrograd, 1917.

[27] Transactions of this Society, vol. 30 (1928), pp. 630–640.

[28] Loc. cit., p. 391.

[29] See N. Cioranescu, Bulletinul, Facultătea de Stiinte din Cernăuti, vol. 5 (1931), p. 99 and Mathematische Zeitschrift, vol. 35 (1932), pp. 601–608. Also see A. Smorgorshewsky, Recueil Mathématique, (n.s.), vol. 7 (1940), pp. 179–196.

$$H(Y) = \int_a^b dF(x)Y(x) = C, \tag{10}$$

where $H(Y)$, C, $F(x)$, Y are square matrices of n-rows and the elements of $F(x)$ are of bounded variation on (a, b). Boundary conditions of this type are quite general and contain all of the conditions discussed above as special cases. For example, conditions (8) involving k points of the interval are obtained when $F(x)$ is defined as a matrix of step functions—constant on each of the subintervals $a_i < x < a_{i+1}$ but with discontinuities at the points $x = a_i$ given by $F(a_i^+) - F(a_i^-) = A_i$ $(i = 1, \cdots, n)$, where $F(a_1^-) = F(a)$, $F(a_k^+) = F(b)$. In this case

$$\int_a^b dF(x)Y(x) = \sum_{i=1}^{k} A_i Y(a_i).$$

Integral boundary conditions of the type $\int_a^b F(x)Y(x)dx = C$ are reduced to the above type when use is made of the relations between Stieltjes integrals and those of Riemann or Lebesgue types. Smorgorshewsky[30] developed the notions of Green's matrix and generalized Green's matrix for systems of type (3), (10), and investigated the properties of these matrices. The fact that linear boundary conditions of quite general type can be replaced by conditions containing Stieltjes integrals has been known for many years. However, partly due to the somewhat unwieldly character of Stieltjes integrals, few advantages have resulted from such replacement. The above-mentioned results for the Green's matrix represent one place, at least, where advances have been made through use of Stieltjes integrals.

5. **Boundary conditions at infinitely many points.** In this section nth order differential systems with boundary conditions at infinitely many points of the interval are discussed. The most general sets considered are those of the *first species*. If g is a point set of the first species and if $A_1, A_2, \cdots$ is an absolutely convergent[31] sequence of constant matrices, the boundary conditions for system (3) are of the form

$$H_1(Y) = \sum_{1}^{\infty} A_i Y(d_i) = C_1, \tag{11}$$

where $d_1, d_2, \cdots$ is some ordering of the elements of g into a countable sequence. It is a consequence of the absolute convergence of $\sum_1^\infty A_i$ that conditions (11) are equivalent to those obtained by any other countable ordering of the elements of g. More general boundary conditions are obtained if an integral term occurs in (11) as follows:

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[30] Loc. cit., pp. 179–196.

[31] That is to say, each element sequence converges absolutely.

$$H_2(Y) = \sum_1^\infty A_i Y(d_i) + \int_a^b F_2(x)Y(x)dx = C_2. \tag{12}$$

In a forthcoming article[32] the present author has shown that differential system (3) together with boundary conditions (12) is equivalent to a system consisting of (3) and conditions

$$H(Y) = AY(a) + BY(b) + \int_a^b F(x)Y(x)dx = C. \tag{13}$$

These systems are equivalent in the sense that any solution of the one system is also a solution of the other. Details of the proof are omitted here since they appear in the article cited. It is to be noted that the matrix function $F(x)$ is Lebesgue integrable on (a, b) provided $F_2(x)$ has this property. In general, $F(x)$ is not continuous and may fail to be Riemann integrable. A further formal reduction of boundary conditions (13) is accomplished if $F(x)$ is assumed to have properties that permit replacement of the integral and two point terms by a single Stieltjes integral. However, this replacement does not actually simplify the system and does lose certain advantages that result from the form of conditions (13). To illustrate these advantages, we develop the notion of Green's matrix for system (3), (13) and thus for system (3), (12).

Let $U(x)$ and $V(x)$ have the same meanings as in formula (4) and let $Y(x)$ denote the general solution of equation (3). Multiplication of equation (4) on the left by $F(s)U(s)V(x)ds$ followed by integration, once between the limits $s=a$ and $s=x$ and once between the limits $s=b$ and $s=x$ yields

$$\begin{aligned}\int_a^x F(s)U(s)V(x)Y(x)ds \\ = \int_a^x F(s)Y(s)ds + \int_a^x ds\int_s^x F(s)U(s)V(t)Q(t)dt,\end{aligned} \tag{14}$$

$$\begin{aligned}\int_b^x F(s)U(s)V(x)Y(x)ds \\ = \int_b^x F(s)Y(s)ds + \int_b^x ds\int_s^x F(s)U(s)V(t)Q(t)dt.\end{aligned} \tag{15}$$

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[32] Soon to appear in University of California Publications in Mathematics, University of California Press, Los Angeles.

When the double integrals in (14) and (15) are integrated by parts and the function

$$G_2(x, t) = \begin{cases} \displaystyle\int_a^t F(s)U(s)V(t)ds & \text{for } t < x \\ \displaystyle\int_b^t F(s)U(s)V(t)ds & \text{for } t > x \end{cases}$$

is introduced, equation (14) and (15) combine to yield

$$\text{(16)} \qquad \int_a^b F(t)U(t)V(x)Y(x)dt = \int_a^b F(t)Y(t)dt + \int_a^b G_2(x, t)Q(t)dt.$$

Let $s=a$ in equation (4) and multiply this equation on the left by $A\,U(a)\,V(x)$. Similarly, let $s=b$ in equation (4) and multiply this equation on the left by $B\,U(b)\,V(x)$. Addition of the two resulting equations to equation (16) yields

$$\text{(17)} \qquad H(U)V(x)Y(x) = H(Y) + \int_a^b [G_1(x, t) + G_2(x, t)]Q(t)dt,$$

where

$$G_1(x, t) = \begin{cases} A\,U(a)V(t) & \text{for } t < x \\ -\,BU(b)V(t) & \text{for } t > x. \end{cases}$$

If the homogeneous system $L(U)=0$, $H(U)=0$ is assumed incompatible, then $H(U)^I$ exists and equation (17) yields the unique solution of system (3), (13) in the form

$$\text{(18)} \qquad Y(x) = U(x)H(U)^I C + \int_a^b G(x, t)Q(t)dt,$$

where the Green's matrix $G(x, t)$ is given explicitly by

$$\text{(19)} \quad G(x, t) = \begin{cases} U(x)H(U)^I\left[A\,U(a) + \displaystyle\int_a^t F(s)U(s)ds\right]V(t) & \text{for } t<x \\ -\,U(x)H(U)^I\left[B\,U(b) + \displaystyle\int_t^b F(s)U(s)ds\right]V(t) & \text{for } t>x. \end{cases}$$

$G(x, t)$ as defined by equation (19) has the usual properties of continuity, uniqueness, and so on, of the Green's matrix for systems with less general boundary conditions. If the function $F(x)$ is taken as the zero matrix, the differential system (3), (13) becomes the two point system (3), (5) and formula (19) shows immediately that $G(x, t)$ becomes the Green's matrix defined by equation (7). Examination of formula (19) reveals the fact that symmetry of $G(x, t)$ in x and t is to be expected only in the degenerate two point case $F(x) \equiv 0$.

Although an adjoint system for (3), (13) in the ordinary sense does not exist in general, it is possible to set up boundary conditions similar to those given in (13) which can be taken with the adjoint equation

$$M(Z) = Z' - Z(x)P(x) = 0 \tag{20}$$

to form a very useful system associated with (3), (13). When $F(x) \equiv 0$ this system becomes the adjoint system for (3), (13).

Simultaneous study of the Green's matrices for (3), (13) and this associated system shows that they satisfy relations which readily degenerate into the well known symmetry relations for the case of two point systems.

6. **Conclusion.** This paper has attempted to review a body of literature on boundary value problems for ordinary differential equations which has not, to our knowledge, been summarized before. Although many of the results cited have appeared in print, it is believed that the paper is not without some novelty. It is hoped that presentation of the paper at this time will lead to applications of systems with more general boundary conditions in a number of fields of applied mathematics. The author does not believe that known results for such systems are adequate to handle many of these applications, but he does think that additional results can be obtained as needs for them arise in applied problems. It is also likely that many problems now treated as two point boundary value problems would receive better treatment if they were studied through the use of more general boundary conditions.

University of California at Los Angeles

ON THE LOGARITHMIC MEANS OF REARRANGED PARTIAL SUMS OF A FOURIER SERIES

OTTO SZÁSZ

Let $f(\theta)$ be a real, even and Lebesgue integrable function; let

$$f(\theta) \sim (1/2)a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta.$$

We write

$$s_0 = (1/2)a_0, \qquad s_n = (1/2)a_0 + a_1 + \cdots + a_n, \qquad n \geqq 1,$$

and denote by $s_0^*, s_1^*, \cdots, s_n^*$ the values of $|s_0|, |s_1|, \cdots, |s_n|$ rearranged in decreasing order. In 1935 Hardy and Littlewood [2][1] proved the following remarkable theorem:

THEOREM 1. *If*

$$f(\theta) = o\left(\log \frac{1}{\theta}\right)^{-1} \tag{1}$$

for small positive θ, *then*

$$\sum_{0}^{n} \frac{s_\nu^*}{\nu + 1} = o(\log n). \tag{2}$$

Hardy and Littlewood gave two applications of this theorem by proving:

THEOREM 2. *If* (1) *holds, then*

$$\sum_{1}^{n} | s_\nu |^q = o(n) \tag{3}$$

for every positive q.

THEOREM 3. *If* (1) *holds and if*

$$a_n > -An^{-\xi} \tag{4}$$

for a positive A *and* ξ, *then* $s_n \to 0$.

They have also proved [1, Theorem 9] that in Theorem 3 the

Presented to the Society, December 30, 1941 under the title *On a theorem of Hardy and Littlewood*; received by the editors November 26, 1941.

[1] Numbers in brackets refer to the bibliography at the end of this paper.

assumption (1) can be replaced by

$$(5)\qquad F(\theta) = \int_0^\theta |f(t)|\,dt = o\left(\frac{\theta}{\log(1/\theta)}\right),$$

but the proof requires a very difficult Tauberian theorem. Thus the question arose whether Theorem 1 remains true when (1) is replaced by (5).

In partial answer to this question we shall prove the following result.

THEOREM I. *If* (5) *holds and if for small positive* θ

$$(6)\qquad |f(\theta)| < \theta^{-c}, \qquad c \text{ a positive constant,}$$

then (2) *holds.*

Note that condition (6) permits $f(\theta)$ to become as large as any power of $1/\theta$ near $\theta = 0$.

For the proof of Theorem I we follow the device of Hardy and Littlewood. We suppose that $|s_m| = s^*_{v_m}$, $0 \le m \le n$, so that $v_0, v_1, \cdots, v_n$ is a permutation of $0, 1, \cdots, n$. We then have

$$S_n = \sum_0^n \frac{s_m^*}{m+1} = \frac{2}{\pi}\int_0^\pi \frac{f(\theta)}{2\sin\theta/2}\, g(\theta)\,d\theta,$$

where

$$g(\theta) = \sum_0^n \frac{\operatorname{sgn} s_m}{v_m+1}\sin(m+1/2)\theta.$$

We choose δ so that (6) holds for $0<\theta<\delta$, and write

$$S_n = \frac{1}{\pi}\left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_\delta^\pi\right)\frac{f(\theta)}{\sin\theta/2}\, g(\theta)\,d\theta$$
$$= I_1 + I_2 + I_3, \qquad \delta < 1 < n.$$

We first estimate I_1:

$$(7)\qquad \begin{aligned} |I_1| &\le \int_0^{1/n}\frac{|f(\theta)|}{\theta}|g(\theta)|\,d\theta \le \int_0^{1/n}|f(\theta)|\left(\sum_0^n \frac{m+1/2}{v_m+1}\right)d\theta \\ &\le (n+1/2)\left(\sum_{v=0}^n \frac{1}{v+1}\right)\int_0^{1/n}|f(\theta)|\,d\theta; \end{aligned}$$

hence, by (5)

$$I_1 = o(1) \qquad \text{as } n \to \infty.$$

Next

$$|I_3| = \left| \sum_{0}^{n} \frac{\operatorname{sgn} s_m}{v_m + 1} j_m \right|,$$

where

$$j_m = \frac{1}{\pi} \int_\delta^\pi \frac{f(\theta)}{\sin \theta/2} \sin (m + 1/2)\theta d\theta.$$

To any given $\epsilon > 0$ we can choose, by the Riemann-Lebesgue theorem, $m_0(\epsilon)$ so that $|j_m| < \epsilon$ for $m \geqq m_0$. Then

$$\tag{8} \begin{aligned} |I_3| &< \frac{1}{\pi} \frac{1}{\sin \delta/2} \int_\delta^\pi |f(\theta)| d\theta \cdot \sum_{0}^{m_0} \frac{1}{v_m + 1} + \epsilon \sum_{m_0}^{n} \frac{1}{v_m + 1} \\ &< \frac{m_0 + 1}{\pi \sin \delta/2} \int_\delta^\pi |f(\theta)| d\theta + \epsilon \log (n + 1), \end{aligned}$$

hence $I_3 = o$ (log n), as $n \to \infty$.

Theorem I is now reduced to showing that $I_2 = o$ (log n). We have from Young's inequality [2, p. 319]

$$uv \leqq lu \log^+ u + le^{(v-l)/l}$$

for $u > 0$, $l > 0$, and all real v. Here $\log^+ u = \max (0, \log u)$. We take $u = |f(\theta)/\sin \theta/2|$, $v = |g(\theta)|$, $l = 2$; then

$$\begin{aligned} |I_2| &< \frac{2}{\pi} \int_{1/n}^\delta \left| \frac{f(\theta)}{\sin \theta/2} \right| \log^+ \left| \frac{f(\theta)}{\sin \theta/2} \right| d\theta + \frac{1}{\pi} \int_{1/n}^\delta e^{|g(\theta)|/2} d\theta \\ &= K + L, \qquad \text{say.} \end{aligned}$$

Now from (6)

$$\log^+ \left| \frac{f(\theta)}{\sin \theta/2} \right| < \log \pi\theta^{-c-1} < (c + 1) \log \frac{\pi}{\theta},$$

hence

$$K \leqq 2(c + 1) \int_{1/n}^\delta |f(\theta)| \theta^{-1} \log \frac{\pi}{\theta} d\theta.$$

Integration by parts yields

$$\tag{9} K \leqq 2(c + 1) \left[F(\delta)\delta^{-1} \log \frac{\pi}{\delta} + \int_{1/n}^\delta F(\theta)\theta^{-2} \left\{ \log \frac{\pi}{\theta} + 1 \right\} d\theta \right],$$

and (5) gives $K = o$ (log n).

Finally, to estimate L, we use the following two lemmas (cf. Hardy and Littlewood [2]):

LEMMA A. *If*

$$\phi(\theta) = \sum_{-n}^{n} c_\nu e^{i\nu\theta}, \qquad \phi^+(\theta) = \sum_{-n}^{n} c_\nu^+ e^{i\nu\theta},$$

where c_ν^+ *are the numbers* $|c_\nu|$ *rearranged so that* $c_0^+ \geqq c_{-1}^+ \geqq c_1^+ \geqq c_{-2}^+ \geqq \cdots$; *then*

$$\int_{-\pi}^{\pi} e^{b|\phi(\theta)|} d\theta \leqq 2 \int_{-\pi}^{\pi} e^{b|\phi^+(\theta)|} d\theta, \qquad \textit{for every } b > 0.$$

LEMMA B. $\left|\sum_1^n \cos \nu\theta/(\nu+1)\right| < \log (1/|\theta|) + c$ *for all* θ *and* n, *and a constant* $c > 0$.

Now, if we put

$$\frac{\operatorname{sgn} s_m}{\nu_m + 1} = \rho_m,$$

we have

$$g(\theta) = \Im(e^{i\theta/2}\gamma(\theta)), \qquad |g(\theta)| \leqq |\gamma(\theta)|,$$

where

$$\gamma(\theta) = \sum_0^n \rho_m e^{im\theta}$$

is a polynomial of degree n whose coefficients have (in some order) the absolute values $1, 1/2, \cdots, 1/(n+1)$. Hence, in the notation of the above lemma.

$$\gamma^+(\theta) = 1 + e^{-i\theta}/2 + e^{i\theta}/3 + e^{-2i\theta}/4 + \cdots,$$

the last term having modulus $1/(n+1)$ and an argument depending on the parity of n. The imaginary part of $\gamma^+(\theta)$ is bounded in n and θ, and the modulus of the real part is less than $\log (1/|\theta|) + c$, c a positive constant, due to Lemma B; hence $|\gamma^+(\theta)| < \log (1/|\theta|) + c$. We now obtain

$$(10) \qquad L < \frac{1}{\pi}\int_0^\pi e^{|\gamma(\theta)|/2} d\theta < \int_0^\pi e^{|\gamma^+(\theta)|/2} d\theta < \int_0^\pi \theta^{-1/2} d\theta + c;$$

the theorem now follows from (7), (8), (9), and (10).

THEOREM II. *To a given* $\delta > 0$ *denote by* $\nu(\delta, n)$ *the number of values* m *for which* $m \leqq n$, $|s_m| > \delta$; *then* $\log \nu = o(\log n)$ *as* $n \to \infty$.

We have

$$\delta \log \nu < \sum_{0}^{n} \frac{s_m^*}{m+1} = o(\log n), \qquad \text{from (2);}$$

δ being fixed our theorem follows [**2**, Theorem 3].

THEOREM III. *If* (5) *holds then* $s_n = o(\log \log n)$.

The proof is straightforward, using

$$s_n = \frac{1}{\pi} \int_{0}^{\pi} \frac{f(\theta) \sin (n + 1/2)\theta}{\sin \theta/2} d\theta.$$

We have

$$|s_n| < \int_0^{\pi} \frac{| f(\theta) | \, | \sin (n + 1/2)\theta|}{\theta} d\theta = \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}$$

$$= A_1 + A_2 + A_3,$$

where we choose $\delta = \delta(\epsilon)$ so that

$$F(\theta) < \frac{\epsilon\theta}{\log 1/\theta} \qquad \text{for } 0 < \theta < \delta.$$

Now $A_1 < (n+1/2)F(1/n) = o(1)$ as $n \to \infty$;

$$A_2 < \int_{1/n}^{\delta} | f(\theta) | \theta^{-1} d\theta = F(\theta) \cdot \theta^{-1} \Big]_{1/n}^{\delta} + \int_{1/n}^{\delta} F(\theta) \cdot \theta^{-2} d\theta$$

$$< \frac{\epsilon}{\log 1/\delta} + \epsilon \int_{1/n}^{\delta} \frac{d\theta}{\theta \log 1/\theta} = \frac{\epsilon}{\log 1/\delta} + \epsilon \log \log \frac{1}{\theta} \Big]_{\delta}^{1/n}$$

$$< \epsilon/\log 1/\delta + \epsilon \log \log n.$$

Hence for fixed ϵ and δ

$$\limsup_{n \to \infty} \frac{A_2}{\log \log n} \leq \epsilon;$$

but ϵ is arbitrarily small, thus $A_2 = o(\log \log n)$. Finally

$$A_3 < \delta^{-1} \int_0^{\pi} | f(\theta) | d\theta = o(\log \log n);$$

this proves Theorem III.

THEOREM IV. *Denote by* $\phi(x)$ *a function satisfying the conditions* $\phi(0) = 0$, $\phi(x) \uparrow$, *as* $x \uparrow$; $\phi(x) < \exp(e^{cx})$ *for all* $x > 0$; c *a positive constant. Under the assumptions of Theorem* I, $\sum_1^n \phi(|s_m|) = o(n)$. *We may say the sequence* $\{s_n\}$ *is strongly summable relative to* $\phi(x)$.

We write for a given $\delta > 0$

$$\sum_{1}^{n} \phi(|s_m|) = \sum_{|s_m|<\delta} + \sum_{|s_m|\geqq\delta} = B_1 + B_2;$$

then $B_1 < n\phi(\delta)$. By Theorems II and III we may write

$$|s_\nu| = \epsilon_n \log\log n, \quad \log \nu = \eta_n \log n, \qquad \text{where } \epsilon_n \to 0,\ \eta_n \to 0.$$

Now

$$B_2 < \nu\phi(\epsilon_n \log\log n) < n^{\eta_n} \cdot \exp\left\{(\log n)^{\epsilon_n}\right\} = o(n),$$

hence

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{1}^{n} \phi(|s_m|) \leqq \phi(\delta),$$

letting $\delta \to 0$ our theorem follows.

THEOREM V. *The assumptions* (4), (5), *and* (6) *imply* $s_n \to 0$.

This follows from Theorem II; see the proof of Theorem 5 in [2].

Final remark. In the more general case, when

$$f(\theta) \sim (1/2)a_0 + \sum_{1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

and when the point under consideration is θ_0, we use the familiar reduction. We have

$$(1/2)\{f(\theta_0 + \theta) + f(\theta_0 - \theta)\} \sim (1/2)\alpha_0 + \sum_{1}^{\infty} \alpha_n \cos n\theta,$$

where

$$\alpha_n = a_n \cos n\theta_0 + b_n \sin n\theta_0, \qquad n \geqq 1.$$

Suppose there is an s, for which

$$\int_0^\theta |f(\theta_0 + t) + f(\theta_0 - t) - 2s|\, dt = o\left(\frac{\theta}{\log 1/\theta}\right), \qquad \text{as } \theta \to 0,$$

and

$$|f(\theta_0 + \theta) + f(\theta_0 - \theta) - 2s| < 2|\theta^{-c}| \quad \text{for small } |\theta|.$$

Then application of Theorem I yields:

$$\sum_{0}^{n} \frac{1}{\nu + 1} s_\nu^*(\theta_0) = o(\log n),$$

where $s_\nu^*(\theta_0), \nu=0, 1, \cdots, n$, is the sequence $|s_\nu(\theta_0)-s|$ in decreasing order.

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1. G. H. Hardy and J. E. Littlewood, *Some new convergence criteria for Fourier series*, Annali, Scuola Normale Superiore, Pisa, (2), vol. 3 (1934), pp. 43–62.

2. ———, *Notes on the theory of series* (XVIII): *On the convergence of Fourier series*, Proceedings of the Cambridge Philosophical Society, vol. 31 (1935), pp. 317–323.

3. A. Zygmund, *Trigonometrical Series*, 1935.

UNIVERSITY OF CINCINNATI

THE BASIC ANALOGUE OF KUMMER'S THEOREM[1]

J. A. DAUM

1. **Introduction.** About one hundred years ago, E. E. Kummer[2] proved the formula

$$(1)\qquad {}_2F_1\begin{bmatrix}a, & b; & -1\\ 1+a-b & & \end{bmatrix} = \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)}$$

which has since been known as Kummer's theorem. This appears to be the simplest relation involving a hypergeometric function with argument (-1).

All the relations in the theory of hypergeometric series ${}_rF_s$ which have analogues in the theory of basic series[3] are those in which the argument is $(+1)$. Apparently, there has been no successful attempt to establish the basic analogue of any formula involving a function ${}_rF_s(-1)$. Since Kummer's theorem is fundamental in the proofs of numerous relations between hypergeometric functions of argument (-1), it seemed desirable that an attempt be made to prove the basic analogue of Kummer's theorem and to investigate the possibility of obtaining new relations in basic series with arguments corresponding to the argument (-1) in the classical case.

In this paper, the basic analogue of Kummer's theorem is obtained

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Received by the editors November 27, 1941.

[1] The results presented in this paper are included in a dissertation for the doctorate, University of Nebraska, 1941.

[2] E. E. Kummer, *Ueber die hypergeometrische Reihe*, Journal für die reine und angewandte Mathematik, vol. 15 (1836), pp. 39–83.

[3] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tract, no. 32, chap. 8.

by a simple specialization of a result due to Bailey.[4] The existence of a basic analogue of Dixon's theorem[5] is discussed and a new type of relation connecting a modified basic series and one of the conventional type is presented.

2. **Derivation of the basic analogue of Kummer's theorem.** In the formula:[4]

$$ {}_6\Phi_5\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & d, & e, & f; & aq/def \\ & a^{1/2}, & -a^{1/2}, & aq/d, & aq/e, & aq/f & \end{matrix}\right] = \frac{\Omega(aq/d)\Omega(aq/e)\Omega(aq/f)\Omega(aq/def)}{\Omega(aq)\Omega(aq/ef)\Omega(aq/df)\Omega(aq/de)} $$

where[6]

$$ \Omega(z) = \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-zq^{n}}, $$

put $e=a^{1/2}$, $f=-a^{1/2}$, and $d=b$, and obtain

$$ {}_2\Phi_1\left[\begin{matrix} a, b; & -q/b \\ aq/b & \end{matrix}\right] = \frac{\Omega(aq/b)\Omega(qa^{1/2})\Omega(-qa^{1/2})\Omega(-q/b)}{\Omega(aq)\Omega(-q)\Omega(qa^{1/2}/b)\Omega(-qa^{1/2}/b)}. $$

This is the basic analogue of Kummer's theorem. It should be noted that it cannot be obtained from the basic analogue of Gauss's theorem.[7]

3. **Application.** The basic analogue of Kummer's theorem immediately suggests the existence of the basic analogue of Dixon's theorem. It has been shown by Watson[8] that Dixon's theorem, which sums a general well poised ${}_3F_2$ with unit argument, can be derived from the theorems of Gauss and Kummer by means of an interchange in the order of summation.

Unfortunately, this method, when applied to the basic analogues, does not lead to an analogue of Dixon's theorem. The difficulty is due to an essential difference in the properties of the omega and gamma functions. However, a slight modification of this method does lead to a new and curious result, namely,

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[4] W. N. Bailey, *Series of hypergeometric type which are infinite in both directions*, Quarterly Journal of Mathematics, Oxford, vol. 7 (1936), p. 105.

[5] A. G. Dixon, *Summation of certain series*, Proceedings of the London Mathematical Society, (1), vol. 35 (1903), pp. 285–289.

[6] N. A. Hall, *An algebraic identity*, Journal of the London Mathematical Society, vol. 11 (1936), p. 276.

[7] W. N. Bailey, *Generalised Hypergeometric Series*, Cambridge Tract, no. 32, p. 68.

[8] G. N. Watson, *Dixon's theorem on generalised hypergeometric functions*, Proceedings of the London Mathematical Society, (2), vol. 22 (1924), pp. xxxii–xxxiii (records for May 1923). See also W. N. Bailey, *Generalised Hypergeometric Series*, p. 13.

$$\sum_{n=0}^{\infty} \frac{\Omega(aq^n)\Omega(bq^n)\Omega(cq^n)}{\Omega(q^{n+1})\Omega(aq^{n+1}/b)\Omega(aq^{n+1}/c)} (aq/bc)^n q^{n(n+1)/2} \tag{2}$$
$$= \frac{\Omega(a)\Omega(b)\Omega(c)}{\Omega(aq)\Omega(aq/bc)} {}_3\Phi_2\left[\begin{matrix} b,\ c,\ -q;\ aq/bc \\ qa^{1/2},\ -qa^{1/2} \end{matrix}\right].$$

The argument requires that a certain double series be absolutely convergent.

The left member of (2) is a modified basic series somewhat similar to some of the q-hypergeometric series studied by Jackson.[9] Since Dixon's theorem sums a general well-poised ${}_3F_2$ with unit argument, (2) cannot be considered a basic analogue of Dixon's theorem. This result is interesting inasmuch as it expresses what may be called a pseudo-basic series in terms of a function ${}_3\Phi_2$. This seems to be the first result of its type to appear in the theory of basic series.

Agricultural and Mechanical College of Texas and
University of Nebraska

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[9] F. H. Jackson, *Summation of q-hypergeometric series*, Messenger of Mathematics, vol. 50 (1921), p. 101.

PERRON INTEGRALS

R. L. JEFFERY

At the recent conference on the theory of integration held at the University of Chicago Professor E. J. McShane remarked that formulas for integration by parts and for change of variable in the integrand have been established for the special Denjoy integral, and consequently for the Perron integral, since the two integrals are equivalent. He then raised the question as to the possibility of proving these formulas directly from the Perron definitions. Later, as we were waiting at the station for my train to leave, Professor Graves talked over with me the implications in McShane's point of view. The following solutions are the outcome of this conversation.

Integration by parts. *Let $g(x)$ be a continuous function on the interval (a, b) for which g' is finite except possibly for a denumerable set, and summable. Let the function $f(x)$ be integrable in the Perron sense and set*

$$F(x) = \int_a^x f(x)dx.$$

Then $f(x)g(x)$ is integrable in the Perron sense, and

$$\int_a^b f(x)g(x)dx = \left[F(x)g(x)\right]_a^b - \int_a^b F(x)g'(x)dx.$$

The conditions on g' make g absolutely continuous, and consequently the difference of two non-decreasing functions each of which is positive or zero. Hence we need consider only the case in which $g(x)$ is non-decreasing and positive or zero. Let $\psi(x)$ be a major function to $f(x)$. Consider $D_*(\psi g)$ which is the lower limit as $h\rightarrow 0$ of

$$\frac{\psi(x+h)g(x+h)-\psi(x)g(x)}{h}$$

$$= \frac{g(x+h)\{\psi(x+h)-\psi(x)\}}{h} + \frac{\psi(x)\{g(x+h)-g(x)\}}{h}.$$

From the continuity of g and the fact that both g and g' are not less than zero it follows that

$$D_*(\psi g) \geqq gD_*\psi + \psi g'$$

at every point for which g' exists. Then, since $D_*\psi > -\infty$ except for a

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Received by the editors September 16, 1941, and December 11, 1941.

denumerable set, and $g' \geqq 0$ and finite except for a denumerable set, we have $D_*(\psi g) > -\infty$ except for a denumerable set. Furthermore, from $g' \geqq 0$, $\psi \geqq F$, and $D_*\psi \geqq f$ almost everywhere we get

$$D_*(\psi g) \geqq fg + Fg'$$

almost everywhere. These considerations together with the continuity of ψ and g show that ψg is a major function to $fg + Fg'$. Furthermore, if ϕ is a minor function to f it can be shown in a similar way that ϕg is a minor function to $fg + Fg'$. Again, if the difference $\psi - \phi$ is sufficiently small the difference $\psi g - \phi g$ is arbitrarily small. Then, since the common bound of ψg and ϕg is Fg, we conclude that $fg + Fg'$ is Perron integrable to Fg. The function F is continuous, and g' is summable. Hence Fg' is summable and, a priori, Perron integrable. Then fg being the difference of two Perron integrable functions is itself Perron integrable, and

$$\int_a^b fg\,dx = [Fg]_a^b - \int_a^b Fg'\,dx, \tag{1}$$

which is the desired result.

In the most general formula so far obtained[1] the only restriction on $g(x)$ is that it be of bounded variation. Then fg is Perron integrable and

$$\int_a^b fg\,dx = [Fg]_a^b - \int_a^b F\,dg. \tag{2}$$

In this case the foregoing procedure cannot be used for two reasons: The function ψg is not continuous at the discontinuities of g; if $g = +\infty$ at the points of a more than denumerable set and ψ is negative at the points of this set then $D_*(\psi g) = -\infty$ at the points of this more than denumerable set. Dropping the requirement that g be continuous, we prove the following result.

If $g(x)$ is of bounded variation and g' is finite except for a denumerable set, then formula (2) *holds.*

At the points of discontinuity of g it is necessarily true that g' is infinite, but this set is not more than denumerable. Our restriction is therefore that g' cannot be infinite at a more than denumerable set other than discontinuities. As in the proof of (1) there is no loss of generality in taking $g \geqq 0$ and non-decreasing. Let ψ be a major function to F with $\psi - F < \epsilon$, and consider

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[1] Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 201.

$$R(x) = \psi(x)g(x) - F(a)g(a) - \int_a^x \psi dg.$$

Using the law of the mean for Stieltjes integrals we have

$$\begin{aligned} R(x+h) - R(x) &= \psi(x+h)\{g(x+h) - g(x)\} \\ &\quad + g(x)\{\psi(x+h) - \psi(x)\} \\ &\quad + \psi(\xi)\{g(x+h) - g(x)\} \\ &= \{\psi(x+h) - \psi(\xi)\}\{g(x+h) - g(x)\} \\ &\quad + g(x)\{\psi(x+h) - \psi(x)\} \end{aligned}$$

where ξ is on the interval defined by the points x and $x+h$. The continuity of R then follows from that of ψ. Again

$$\begin{aligned} \frac{R(x+h) - R(x)}{h} &= \{\psi(x+h) - \psi(\xi)\}\frac{g(x+h) - g(x)}{h} \\ &\quad + g(x)\frac{\psi(x+h) - \psi(x)}{h}. \end{aligned}$$

The lower limit of the second term on the right is not greater than $g(x)D_*\psi > -\infty$ except for a denumerable set. The lower limit of the first term on the right can be $-\infty$ at points for which $g' = +\infty$ (h and ξ taking values which make $\psi(x+h) - \psi(\xi)$ negative). But $g' = +\infty$ for at most a denumerable set. Hence we have

$$D_*R > -\infty$$

except for a denumerable set. At a point for which g' is finite the limit of the first term on the right is zero. Hence we have

$$D_*R = gD_*\psi \geqq fg$$

almost everywhere. From these considerations it follows that R is a major function to fg. Also from $\psi - F < \epsilon$ we get

$$\theta(x) = \int_a^x \psi dg - \int_a^x F dg = \int_a^x \eta dg$$

where $0 \leqq \eta < \epsilon$. Hence $0 \leqq \theta(x) < \epsilon\{g(b) - g(a)\}$. From this and $\psi \geqq F$ we see that by taking a sequence of values of ϵ tending to zero we get a set of major functions $R(x)$ whose lower bound is

$$F(x)g(x) - F(a)g(a) - \int_a^x F dg.$$

Working in a similar way with functions ϕ which are minor to f we arrive at a set of functions which are minor to fg, and whose upper bound is the same as the lower bound of the set R. It then follows that fg is Perron integrable, and

$$\int_a^b fg\,dx = [Fg]_a^b - \int_a^b F\,dg. \tag{2}$$

Formula (2) includes formula (1), but a separate proof of (1) is of interest because of its simplicity.

Change of variable. *Let $f(x)$ be Perron integrable on (a, b), and let $x(t)$ be a continuous non-decreasing function of t on the interval (α, β) where $a = x(\alpha)$, $b = x(\beta)$, and on this interval let $x'(t)$ be finite except perhaps for a denumerable set. Then the function $f\{x(t)\}x'(t)$ is Perron integrable on this interval (α, β), and*

$$\int_a^b f(x)dx = \int_\alpha^\beta f\{x(t)\}x'(t)dt.$$

Let $\psi(x)$ be a major function to $f(x)$, and let $\chi(t) = \psi\{x(t)\}$. Then $\psi(t)$ is continuous. Also, since $D_*\psi > -\infty$ except for a denumerable set, and $x'(t) \geqq 0$ and finite except for a denumerable set, we have

$$D_*\chi(t) = D_*\psi x'(t) > -\infty$$

except for a denumerable set. Again, since $D_*\psi(x) \geqq f(x)$ almost everywhere, and $x'(t) \geqq 0$, we have

$$D_*\chi(t) \geqq f(x)x'(t) = f\{x(t)\}x'(t)$$

almost everywhere. From these results it follows that $\chi(t)$ is a major function to $f\{x(t)\}x'(t)$. If $\phi(x)$ is a minor function to $f(x)$ it can be shown in a similar way that $\Theta(t) = \phi\{x(t)\}$ is a minor function to $f\{x(t)\}x'(t)$. Furthermore, since the common bound of the functions $\psi(x)$ and $\phi(x)$ is $F(x)$, the common bound of the functions $\chi(t)$ and $\Theta(t)$ is $F\{x(t)\}$. Thus the function $f\{x(t)\}x'(t)$ is Perron integrable to $F\{x(t)\}$ and

$$F(b) - F(a) = \int_a^b f(x)dx = \int_\alpha^\beta \{x(t)\}x'(t)dt.$$

ACADIA UNIVERSITY

ON PERRON INTEGRATION

E. J. McSHANE

The definition of integral here presented had its origin in an unsuccessful attempt to establish the theorem on integration by parts[1] (Theorem 4.1 of this note) for the Perron integral, without detouring through the special Denjoy integral. In order to avoid the difficulties which I could not overcome, I amended the definition of the integral; the resulting definition I presented at the Oslo congress in 1936. Recently I have found a proof that the "new" integral is actually equivalent to that of Perron.

As compared with Perron's definition, the new definition has the slight disadvantage that it requires four associated functions instead of two; in all other respects the proofs of theorems for the Perron integral carry over unaltered. It has the advantage that it permits us to prove the general theorem on integration by parts (Theorem 4.1) without recourse to the deep-lying equivalence with the special Denjoy integral. This theorem is important, not only in itself, but because it also contains the corollary that if $f(x)$ is Perron integrable and $g(x)$ of limited total variation, their product is Perron integrable. Also it leads at once to the second theorem of the mean for Perron integrals.

1. **Definition of the integral.** Let $f(x)$ be a function defined on an interval $[a, b]$ and assuming values which are real numbers or $+\infty$ or $-\infty$. A set of four functions $\phi^i(x)$ $(i=1, 2, 3, 4)$ is a *tetrad adjoined to* $f(x)$ *on* $[a, b]$ if the following conditions are satisfied.

(1.1a) *The* $\phi^i(x)$ *are continuous on* $[a, b]$ *and all vanish at* $x=a$.

(1.1b) *For all except at most a denumerable collection of values of* x *the relations*

$$-\infty \neq D_+\phi^1, \qquad -\infty \neq D_-\phi^2,$$
$$+\infty \neq D^-\phi^3, \qquad +\infty \neq D^+\phi^4$$

are valid.

(1.1c) *For all except at most a denumerable collection of values of* x *the relations*

Presented to the Society, December 31, 1941; received by the editors December 1, 1941.

[1] For a proof of the theorem with the added hypothesis that $g'(x)$ is finite except for a denumerable set, see R. L. Jeffery, *Perron integrals*, this Bulletin, vol. 48 (1942). pp. 714–717.

$$D_+\phi^1(x) \geqq f(x), \qquad D_-\phi^2(x) \geqq f(x),$$
$$D^-\phi^3(x) \leqq f(x), \qquad D^+\phi^4(x) \leqq f(x)$$

are valid.

In case the functions $\phi^i(x)$ also satisfy the conditions

$$|\phi^i(b) - \phi^j(b)| < \epsilon, \qquad i, j = 1, 2, 3, 4,$$

the tetrad will be said to be ϵ-adjoined to f. Functions satisfying the conditions imposed on $\phi^1, \phi^2, \phi^3, \phi^4$, respectively, will be called right majors, left majors, left minors, right minors, respectively.

(1.2) *The function $f(x)$ is P^*-integrable on $[a, b]$ if for every positive ϵ there is a tetrad ϵ-adjoined to f on $[a, b]$.*

Before defining the integral we establish an essential lemma.

LEMMA 1.1. *If ψ is a major function (right or left) for f, and ϕ is a minor function (right or left) for f, then $\psi-\phi$ is monotonic non-decreasing.*

Suppose first that ψ is a right major and ϕ a right minor. By (1.1b) and (1.1c), except on a denumerable set we have

$$D_+(\psi - \phi) \geqq D_+\psi - D^+\phi \geqq 0.$$

So[2] $\psi-\phi$ is non-decreasing. A similar proof applies if ψ is a left major and ϕ a left minor.

Suppose next that ψ is a right major and ϕ a left minor. By a theorem of G. C. Young,[3] the inequality $D_+\phi \leqq D^-\phi$ holds except at most on a denumerable set. So by (1.1b) and (1.1c) we have

$$D^+(\psi - \phi) \geqq D_+\psi - D_+\phi \geqq 0$$

except on a denumerable set, and[2] $\psi-\phi$ is non-decreasing.

A similar proof applies if ψ is a left major and ϕ a right minor.

COROLLARY 1.2. *If $f(x)$ is P^*-integrable on $[a, b]$, it is also P^*-integrable on $[a, x]$ for all x such that $a < x \leqq b$.*

For by Lemma 1.1 every tetrad which is ϵ-adjoined to f on $[a, b]$ is also ϵ-adjoined to f on $[a, x]$.

LEMMA 1.3. *If $f(x)$ is P^*-integrable on $[a, b]$, there is a function $F(x)$ on $[a, b]$ which is the* g.l.b. *of all right majors, the* g.l.b. *of all left majors, the* l.u.b. *of all left minors and the* l.u.b. *of all right minors. The function $F(x)$ is continuous.*

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[2] Saks, *Theory of the Integral*, p. 204.

[3] Hobson, *Theory of Functions of a Real Variable*, vol. 1, 1927, p. 392.

Let ϕ be any minor of f. If ϕ^1 is a right major, for every x in $[a, b]$ we have

$$\phi^1(x) - \phi(x) \geqq \phi^1(a) - \phi(a) = 0;$$

so $\phi(x)$ is a lower bound for right majors. If we define $F(x)$ to be the g.l.b. of all right majors, then $F(x) \geqq \phi(x)$. Here ϕ is any minor, so F is an upper bound for all minors. If $\epsilon > 0$, there is a tetrad ϕ^i which is ϵ-adjoined to f, so

$$(1.4) \qquad F(x) - \phi^4(x) \leqq \phi^1(x) - \phi^4(x) \leqq \phi^1(b) - \phi^4(b) < \epsilon,$$

and likewise $F(x) - \phi^3(x) < \epsilon$. That is, $F(x)$ is the l.u.b. of all right minors and also is the l.u.b. of all left minors. Interchanging majors and minors shows that F is also the g.l.b. of all left majors.

If we let ϵ approach 0 through a sequence of positive values, by (1.4) the corresponding right minors $\phi^4(x)$ converge uniformly to $F(x)$, which therefore is continuous.

DEFINITION 1.4. *If $f(x)$ is P^*-integrable on $[a, b]$, and $F(x)$ is the function defined in Lemma* 1.2, *then the P^*-integral of f over $[a, b]$ is defined to be*

$$(P^*) \int_a^b f(x)dx = F(b).$$

Henceforth we omit the P^* before the integral.

COROLLARY 1.5. *With the hypotheses of Definition* 1.4, *if $a < x \leqq b$ then*

$$\int_a^x f(x)dx = F(x).$$

2. **Properties of the integral.** We can now readily establish the elementary properties of the P^*-integral; if f_1 and f_2 are P^*-integrable, so is kf_1 and so is $f_1 \pm f_2$ if defined (that is, if never of the form $\infty - \infty$), and the integrals have the values to be expected; f_1 is P^*-integrable over every subinterval of $[a, b]$, and is an additive function of intervals; and the function $f_0(x) = f_1(-x)$ is P^*-integrable over $[-b, -a]$, and its integral is equal to

$$\int_a^b f_1(x)dx.$$

The proofs differ from the corresponding proofs for the Perron integral only in minor details.

Furthermore, we can readily show that every P^*-integrable function is finite almost everywhere, is measurable, and is almost everywhere equal to the derivative of its indefinite P^*-integral. The proof in Sak's *Theory of the Integral*, p. 202, needs only obvious changes.

An alternative form of definition is given in the next theorem.

THEOREM 2.1. *Let $f(x)$ be defined on an interval $[a, b]$. In order that $f(x)$ be P^*-integrable on $[a, b]$ it is necessary and sufficient that for every positive number ϵ there exist four functions $\psi^i(x)$ $(i=1, 2, 3, 4)$ which satisfy* (1.1a) *and* (1.1b), *satisfy the inequalities in* (1.1c) *for almost all x, and are such that*

$$|\psi^i(b) - \psi^j(b)| < \epsilon, \qquad i, j = 1, 2, 3, 4. \tag{2.1}$$

In this case, the P^*-integral of f over $[a, x]$ is the common g.l.b. *of all functions $\psi^1(x)$ and $\psi^2(x)$ and the common* l.u.b. *of all functions $\psi^3(x)$ and $\psi^4(x)$ which satisfy the conditions stated.*

The proof is essentially that given by Bauer[4] for the Perron integral. As an obvious corollary, if $f(x)$ is P^*-integrable on $[a, b]$ so is every function equivalent to f, and the integrals are the same.

3. **Substitution.** Next we establish the following theorem on change of independent variable.

THEOREM 3.1. *Let $f(x)$ be P^*-integrable on an interval $[a, b]$. Let $g(y)$ be a function defined on an interval $[\alpha, \beta]$ and possessing the following properties.*

(a) *$g(y)$ is continuous and non-decreasing on $[\alpha, \beta]$.*

(b) $g(\alpha)=a$, $g(\beta)=b$.

(c) *$\overline{D}g(y) < \infty$ except at most on a denumerable subset of $[\alpha, \beta]$. Then $f(g(y))\, g(y)$ is P^*-integrable over $[\alpha, \beta]$, and*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(y))g(y)dy.$$

For each positive ϵ there is a tetrad $[\phi^i]$ which is ϵ-adjoined to $f(x)$ on $[a, b]$. Define

$$\psi^i(y) = \phi^i(g(y)), \qquad i = 1, \cdots, 4; \alpha \leqq y \leqq \beta.$$

It is obvious that the ψ^i are continuous on $[a, b]$ and vanish at $y=\alpha$. Let E_1 consist of all y in $[\alpha, \beta]$ such that there exists a point y' with $y<y'\leqq\beta$ and $g(y')=g(y)$. The rest of the set $[\alpha, \beta]$ we

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[4] H. Bauer, *Der Perronsche Integralbegriff und seine Beziehung zum Lebesgueschen*, Monatshefte fur Mathematik und Physik, vol. 26 (1915), pp. 153–198; also Saks, op. cit., p. 203.

denote by E_2; if y is in E_2 then $g(y') > g(y)$ for all y' such that $y < y' \leqq \beta$.

If y is in E_1, the function $g(y)$ remains constant on an interval $[y, y']$, so each $\psi^i(y)$ also is constant on $[y, y']$. Therefore

$$(3.1) \qquad -\infty < D_+\psi^1(y) = 0 = f(g(y))g(y),$$

$$(3.2) \qquad +\infty > D^+\psi^4(y) = 0 = f(g(y))g(y).$$

If y is in E_2, for each y' such that $y < y' \leqq \beta$ we have

$$(3.3) \qquad \frac{\psi^1(y') - \psi^1(y)}{y' - y} = \frac{\phi^1(g(y')) - \phi^1(g(y))}{g(y') - g(y)} \cdot \frac{g(y') - g(y)}{y' - y}.$$

By definition (1.1), $D_+\phi^1 > -\infty$ except at most on a denumerable set X_0. If x is in X_0, the equation $g(y) = x$ can have at most one solution in E_2, for of two solutions lying in $[\alpha, \beta]$ the lesser is in E_1 by definition. So the set of such solutions is denumerable. To it we add the at most denumerable set on which $\overline{D}g = \infty$; the sum Y_0 is at most denumerable. If y is in $E_2 - Y_0$, we let $y' \to y^+$, and from (3.3) obtain

$$(3.4) \qquad D_+\psi^1 \neq -\infty.$$

Moreover, if $g'(y)$ exists, as it does almost everywhere in $[\alpha, \beta]$, the same limiting process yields

$$(3.5) \qquad D_+\psi^1(y) = D_+\phi^1(g(y))g(y).$$

Combining this with (3.1) and (1.1), we see that $D_+\psi^1(y) > -\infty$ except at most on a denumerable set and

$$D_+\psi^1(y) \geqq f(g(y))g(y)$$

on almost all of $[\alpha, \beta]$.

The other functions ψ^i can be discussed analogously; we thus see that they satisfy the first three conditions of Theorem 2.1. The last condition (2.1) is trivial, since

$$\psi^i(\beta) - \psi^j(\beta) = \left| \phi^i(b) - \phi^j(b) \right| < \epsilon.$$

So by Theorem 2.1 the product $f(g(y))g(y)$ is P^*-integrable on $[\alpha, \beta]$. Moreover, its integral over $[\alpha, \beta]$ lies between $\psi^4(\beta)$ and $\psi^1(\beta)$, that is, between $\phi^4(b)$ and $\phi^1(b)$. So does the integral of $f(x)$ over $[a, b]$. Since $\phi^4(b)$ and $\phi^1(b)$ differ by less than an arbitrary ϵ, the two integrals are equal.

Theorem 3.1 has an obvious analogue for monotonic non-increasing functions $g(y)$. Also, it is worth observing that the hypotheses on $g(y)$ force it to be absolutely continuous. For if we set $f\equiv 1$ in Theorem 3.1 we find that

$$g(\beta) - g(\alpha) = a - b = \int_a^b g(y)dy.$$

Since g is non-negative, the integral on the right reduces to a Lebesgue integral, as we can show by the usual proof for the Perron integral. This equation implies the absolute continuity of $g(y)$.

4. Integration by parts. The principal reason for devising Definition 1.1 was to enable us to prove the following theorem.

THEOREM 4.1. *Let $f(x)$ be P^*-integrable on $[a, b]$, and let $F(x)$ be its indefinite P^*-integral. If $g(x)$ is of limited total variation on $[a, b]$, then the product $f(x)g(x)$ is P^*-integrable, and*

$$\int_a^b f(x)g(x)dx = F(b)g(b) - \int_a^b F(x)dg(x), \tag{4.1}$$

the last integral being a Stieltjes integral.

The function $g(x)$ can be written in either of the forms

$$g(x) = g_1(x) - g_2(x),$$
$$g(x) = g_3(x) - g_4(x),$$

where the g_i are positive, g_1 and g_2 are monotonic non-increasing and g_3 and g_4 are monotonic non-decreasing. Let ϕ^i be a tetrad ϵ-adjoined to f on $[a, b]$. We define sixteen functions by the formulas

$$\psi^i_j(x) = \phi^i(x)g_j(x) - \int_a^x \phi^i(x)dg_j(x), \qquad i, j = 1, 2, 3, 4. \tag{4.2}$$

It is evident that these all vanish at $x=a$. If x and x' are in $[a, b]$, we can apply the first mean-value theorem to the integral in (4.2) to obtain

$$\begin{aligned} \psi^i_j(x') - \psi^i_j(x) &= \phi^i(x')g_j(x') - \phi^i(x)g_j(x) - \phi^i(\bar{x})[g_j(x') - g_j(x)] \\ &= g_j(x')[\phi^i(x') - \phi^i(x)] \\ &\quad - [g_j(x') - g_j(x)](\phi^i(\bar{x}) - \phi^i(x)), \end{aligned} \tag{4.3}$$

where $\bar{x}$ is between x and x' (inclusive). Since ϕ^i is continuous and g_j is bounded, this shows that $\psi^i_j(x)$ is continuous.

Suppose that $g'(x)$ is defined and finite; this is true for almost all x. We divide both numbers of (4.3) by $x'-x$, writing the $x'-x$

under the square-bracketed terms on the right. The second term on the right tends to zero, so

$$D^+\psi_j^i(x) = g_j(x)D^+\phi^i(x), \tag{4.4}$$

and likewise for the three other Dini derivates.

If $\bar{x} \neq x$ equation (4.3) yields

$$\begin{aligned}\frac{\psi_j^i(x') - \psi_j^i(x)}{x' - x} &= g_j(x')\frac{\phi^i(x') - \phi^i(x)}{x' - x} \\ &\quad + (g_j(x) - g_j(x'))\cdot\frac{\phi^i(\bar{x}) - \phi^i(x)}{\bar{x} - x}\cdot\frac{\bar{x} - x}{x' - x}.\end{aligned} \tag{4.5}$$

This is still valid if $\bar{x} = x$, provided that we assign a value, say 0, to the middle factor in the last term. Henceforth we assume that x is in the set on which the inequalities in (1.1b) hold; this rejects at most a denumerable set of x. Let x' approach x from the right. The last factor in (4.5) is non-negative and bounded; so are the factors involving g_i if $j=1$ or 2. If $i=1$ the fractions involving ϕ^i are bounded below, since $D_+\phi^i(x) > -\infty$; if $i=4$ they are bounded above. Hence the left member is bounded below if $i=1$, above if $i=4$, and

$$D_+\psi_j^1 > -\infty, \qquad D^+\psi_j^4 < \infty, \qquad j = 1, 2. \tag{4.6}$$

In a similar way, if $x' < x$ and $j=3$ or 4 we can establish

$$D_-\psi_j^2 > -\infty, \qquad D^-\psi_j^3 < \infty, \qquad j = 3, 4. \tag{4.7}$$

Now we define

$$\begin{aligned}\psi^1 &= \psi_1^1 - \psi_2^4, \\ \psi^2 &= \psi_3^2 - \psi_4^3, \\ \psi^3 &= \psi_3^3 - \psi_4^2, \\ \psi^4 &= \psi_1^4 - \psi_2^1.\end{aligned} \tag{4.8}$$

From (4.6) and (4.7) we find that these satisfy (1.1b). From (4.4), together with (1.1b), we find that the inequalities of (1.1c) hold for almost all x, and (1.1a) has already been established.

Since F is the g.l.b. of majors and the l.u.b. of minors, we find

$$\psi_j^4(b) \leqq F(b)g_j(b) - \int_a^b F(x)dg_j(x) \leqq \psi_j^1(b), \qquad j = 1, 2.$$

Hence by (4.8)

$$\psi^1(b) \geqq F(b)g(b) - \int_a^b F(x)dg(x) \geqq \psi^4(b). \tag{4.9}$$

Likewise

$$(4.10)\qquad \psi^2(b) \geqq F(b)g(b) - \int_a^b F(x)dg(x) \geqq \psi^3(b).$$

But

$$\left| \psi_j^1(b) - \psi_j^4(b) \right| = \left| [\phi^1(b) - \phi^4(b)]g_j(b) - \int_a^b [\phi^1(x) - \phi^4(x)]dg_j(x) \right|$$
$$\leqq \epsilon[\,| g_j(b) | + | g_j(b) - g_j(a) |\,],$$

and, denoting the coefficient of ϵ by K_j, this implies

$$\left| \psi^1(b) - \psi^4(b) \right| < \epsilon(K_1 + K_2).$$

Likewise

$$\left| \psi^2(b) - \psi^3(b) \right| < \epsilon(K_3 + K_4).$$

Since ϵ is arbitrary, this implies that with (4.9) and (4.10) all four numbers $\psi^i(b)$ lie arbitrarily close to

$$F(b)g(b) - \int_a^b F(x)dg(x).$$

This establishes the inequality (2.1) of Theorem 2.1, hence proves the P^*-integrability of $f(x)\,g(x)$, and also establishes equation (4.1).

5. **Equivalence of the P^*-integral and the Perron integral.** It is evident that every function which is[6] Perron integrable on an interval $[a, b]$ is also P^*-integrable, and the integrals are equal. For the Perron major functions serve simultaneously as right and left majors, and the Perron minors serve as right and left minors.

The converse is less evident. It requires slight generalizations of two known theorems.

LEMMA 5.1. *If $F(x)$ is a function which at all but at most a denumerable subset of a set M satisfies one of the inequalities*

$$D^+F < \infty, \qquad D^-F < \infty,$$
$$D_+F > -\infty, \qquad D_-F > -\infty,$$

then $F(x)$ is VBG_* on M.

The proof is essentially that given on p. 235 of Saks' *Theory of the Integral.* His equation (10.3) is replaced by

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[6] By the Perron integral we mean the integral originally defined by Perron and studied by Bauer (loc. cit.); this is called the $\mathcal{P}_0$ integral by Saks, and is equivalent to the $\mathcal{P}$ integral (Saks, op. cit., p. 252).

$$0 < t - x < 1/n \textit{ implies } [F(t) - F(x)]/(t - x) \leqq n;$$

the remainder of the proof is unaltered. The next lemma is a slight generalization of a theorem of Marcinkiewicz.

LEMMA 5.2. *If $f(x)$ is measurable on $[a, b]$, and has either a left major or a right major, and also has either a left minor or a right minor, then $f(x)$ is Perron integrable on $[a, b]$.*

The proof is that given by Saks, op. cit., p. 253; the principal change is that the reference to his Theorem 10.1 is replaced by a reference to our Lemma 5.1.

Since every P^*-integrable function $f(x)$ is measurable and has right majors and right minors, it is also Perron integrable by Lemma 5.2, and the equivalence of the integrals is established.

UNIVERSITY OF VIRGINIA

ON THE LEAST PRIMITIVE ROOT OF A PRIME

LOO-KENG HUA

It was proved by Vinogradow[1] that the least positive primitive root $g(p)$ of a prime p is $O(2^m p^{1/2} \log p)$ where m denotes the number of different prime factors of $p-1$. In 1930 he[2] improved the previous result to

$$g(p) = O(2^m p^{1/2} \log \log p),$$

or more precisely,

$$g(p) \leqq 2^m \frac{p-1}{\phi(p-1)} p^{1/2}.$$

It is the purpose of this note, by introducing the notion of the average of character sums,[3] to prove that if $h(p)$ denotes the primitive root with the least absolute value, mod p, then

$$| h(p) | < 2^m p^{1/2};$$

Received by the editors December 3, 1941.

[1] See, Landau, *Vorlesungen über Zahlentheorie*, vol. 2, part 7, chap. 14. The original papers of Vinogradow are not available in China.

[2] Comptes Rendus de l'Académie des Sciences de l'URSS, 1930, pp. 7–11.

[3] The present note may be regarded as an introduction of a method which has numerous applications.

and that for $p \equiv 1 \pmod 4$, we have

$$g(p) < 2^m p^{1/2},$$

while, for $p \equiv 3 \pmod 4$, we have

$$g(p) < 2^{m+1} p^{1/2}.$$

Since

$$\frac{p-1}{\phi(p-1)} \geqq 2,$$

the result is always better than that due to Vinogradow.

LEMMA 1. *Let* $p > 2$, $1 \leqq A < p$. *For each non-principal character*[4] $\chi(n)$, mod p, *we have*

$$\frac{1}{A+1}\left|\sum_{a=0}^{A}\sum_{n=-a}^{a}\chi(n)\right| \leqq p^{1/2} - \frac{A+1}{p^{1/2}}.$$

PROOF. Let $\epsilon = e^{2\pi i/p}$ and let

$$\tau(\chi) = \sum_{h=1}^{p-1}\chi(h)\epsilon^{h}.$$

It is known that

$$|\tau(\chi)| = p^{1/2}.$$

For $p \nmid n$, we have

$$\sum_{h=1}^{p-1}\bar{\chi}(h)\epsilon^{hn} = \chi(n)\sum_{h=1}^{p-1}\bar{\chi}(hn)\epsilon^{hn}$$

$$= \chi(n)\sum_{h=1}^{p-1}\bar{\chi}(h)\epsilon^{h} = \chi(n)\tau(\bar{\chi}).$$

The formula holds also for $p \mid n$, since $\chi(n) = 0$ for $p \mid n$ and

$$\sum_{h=1}^{p-1}\bar{\chi}(h) = 0.$$

Thus

$$\tau(\bar{\chi})\sum_{a=0}^{A}\sum_{n=-a}^{a}\chi(n) = \sum_{h=1}^{p-1}\bar{\chi}(h)\sum_{a=0}^{A}\sum_{n=-a}^{a}\epsilon^{hn}$$

$$= \sum_{h=1}^{p-1}\bar{\chi}(h)\left(\frac{\sin (A+1)\pi h/p}{\sin \pi h/p}\right)^2.$$

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[4] See, for example, Landau loc. cit., vol. 1, pp. 83–87.

Consequently

$$p^{1/2}\left|\sum_{a=0}^{A}\sum_{n=-a}^{a}\chi(n)\right| \le \sum_{h=1}^{p-1}\left(\frac{\sin (A+1)\pi h/p}{\sin \pi h/p}\right)^2$$
$$= \sum_{h=1}^{p-1}\sum_{a=0}^{A}\sum_{n=-a}^{a}\epsilon^{hn}$$
$$= \sum_{a=0}^{A}\sum_{n=-a}^{a}\left(\sum_{h=1}^{p}\epsilon^{hn} - 1\right)$$
$$= (A+1)p - (A+1)^2.$$

LEMMA 2. *Let* $p>2$, $1\leqq A<(p-1)/2$. *Then, for each non-principal character*, mod p, *we have*

$$\frac{1}{A+1}\left|\sum_{a=0}^{A}\sum_{n=A+1-a}^{A+1+a}\chi(n)\right| \leqq p^{1/2} - \frac{A+1}{p^{1/2}}.$$

PROOF. As in Lemma 1, we have

$$p^{1/2}\left|\sum_{a=0}^{A}\sum_{n=A+1-a}^{A+1+a}\chi(n)\right| = \left|\sum_{h=1}^{p-1}\bar{\chi}(h)e^{2\pi i h(A+1)/p}\left(\frac{\sin (A+1)\pi h/p}{\sin \pi h/p}\right)^2\right|$$
$$\le \sum_{h=1}^{p-1}\left(\frac{\sin (A+1)\pi h/p}{\sin \pi h/p}\right)^2$$
$$= (A+1)p - (A+1)^2.$$

LEMMA 3. *Let* $p>2$. *If* n *is not a primitive root*, mod p, *then*

$$\sum_{k\mid p-1}\frac{\mu(k)}{\phi(k)}\sum_{\chi^{(k)}}\chi^{(k)}(n) = 0,$$

where $\chi^{(k)}$ *runs over all characters* χ *satisfying the condition that* k *is the least positive integer such that* $(\chi)^k$ *is the principal character.*

(See Landau, loc. cit., p. 496. The condition $1\leqq n<p$ there mentioned is not necessary.)

THEOREM 1. *We have* $|h(p)|<2^m p^{1/2}$.

PROOF. Let $p>2$. By Lemma 3, we have

$$0 = \sum_{k\mid p-1}\frac{\mu(k)}{\phi(k)}\sum_{\chi^{(k)}}\sum_{a=0}^{|h(p)|-1}\sum_{n=-a}^{a}\chi^{(k)}(n).$$

For $k=1$, the right-hand side gives

$$\sum_{a=0}^{|h(p)|-1} \sum_{n=-a}^{a} \chi^{(1)}(n) = \sum_{a=0}^{|h(p)|-1} 2a.$$
$$= |h(p)|^2 - |h(p)|.$$

On the other hand, for $k \neq 1$, we have, by Lemma 1 with $A = |h(p)| - 1$,

$$\left| \sum_{a=0}^{|h(p)|-1} \sum_{n=-a}^{a} \chi^{(k)}(n) \right| \leqq |h(p)| p^{1/2} - \frac{|h(p)|^2}{p^{1/2}}.$$

Therefore

$$|h(p)|^2 - |h(p)| \leqq \left(|h(p)| p^{1/2} - \frac{|h(p)|^2}{p^{1/2}} \right) \sum_{k|p-1} \frac{|\mu(k)|}{\phi(k)} \phi(k)$$
$$= 2^m \left(|h(p)| p^{1/2} - \frac{|h(p)|^2}{p^{1/2}} \right).$$

Then

$$|h(p)| \leqq \frac{2^m p^{1/2} + 1}{1 + 2^m/p^{1/2}} < 2^m p^{1/2}.$$

COROLLARY. *For* $p \equiv 1 \pmod 4$, *we have* $g(p) = |h(p)| < 2^m p^{1/2}$.

PROOF. We have to show that $|h(p)|$ is a primitive root. Suppose it is not. Then $-|h(p)|$ is a primitive root and $|h(p)|$ belongs to an exponent l where $l | (p-1)$ and $l < p-1$, that is,

$$|h(p)|^l \equiv 1 \pmod p,$$
$$(h(p))^{2l} \equiv 1 \pmod p.$$

Thus $2l = p-1$ and $|h(p)|^{(p-1)/2} \equiv 1 \pmod p$ so that $|h(p)|$ is a quadratic residue. Since -1 is a quadratic residue, mod p, $-|h(p)|$ is also a quadratic residue and $\{-|h(p)|\}^{(p-1)/2} \equiv 1 \pmod p$. This contradicts the fact that $-|h(p)|$ is a primitive root.

REMARK. Sometimes Theorem 1 may be improved by the fact that

$$\sum_{n=-a}^{a} \chi^{(k)}(n) = 0,$$

for $\chi^{(k)}(-1) = -1$ and hence $\chi^{(k)}(n) = -\chi^{(k)}(-n)$. Thus for $p \equiv 3 \pmod 4$,

$$|h(p)| < 2^{m-1} p^{1/2}.$$

In fact, we have $g^{(p-1)/2} \equiv -1 \pmod p$ and $\chi^{(k)}(g) = e^{2\pi i \lambda/k}$. Since

$$-1 = \chi^{(k)}(g^{(p-1)/2}) = e^{\pi i (p-1)\lambda/k},$$

we have $2\nmid(p-1)\lambda/k$. The terms appearing in the formula of Lemma 3 are those with square-free k. Thus $\chi^{(k)}(-1)=-1$ holds only for the case $p\equiv 3 \pmod 4$, and $2\nmid\lambda$. Thus

$$\sum_{a=0}^{|h(p)|-1}\sum_{n=-a}^{a}\chi^{(k)}(n)=0 \qquad \text{for } 2\mid k.$$

Therefore

$$|h(p)|^2-|h(p)|\leqq\left(|h(p)|\,p^{1/2}-\frac{|h(p)|^2}{p^{1/2}}\right)\sum_{k|(p-1)/2}|\mu(k)|$$
$$=2^{m-1}\left(|h(p)|\,p^{1/2}-\frac{|h(p)|^2}{p^{1/2}}\right).$$

Then

$$|h(p)|\leqq\frac{2^{m-1}p^{1/2}+1}{1+2^{m-1}/p^{1/2}}<2^{m-1}p^{1/2}.$$

THEOREM 2. *We have* $g(p)<2^{m+1}p^{1/2}$.

PROOF. Let A be the greatest integer not exceeding $(g-1)/2$. Then

$$0=\sum_{k|p-1}\frac{\mu(k)}{\phi(k)}\sum_{\chi^{(k)}}\sum_{a=0}^{A}\sum_{n=A+1-a}^{A+1+a}\chi^{(k)}(n).$$

For $k=1$, the right-hand side gives

$$\sum_{a=0}^{A}\sum_{n=A+1-a}^{A+1+a}\chi^{(1)}(n)=\sum_{a=0}^{A}(2a+1)=(A+1)^2.$$

For $k\neq 1$, we have

$$\left|\sum_{a=0}^{A}\sum_{n=A+1-a}^{A+1+a}\chi^{(k)}(n)\right|\leqq(A+1)p^{1/2}-\frac{1}{p^{1/2}}(A+1)^2.$$

Therefore, as in the proof of Theorem 1, we have

$$(A+1)^2<2^m\left((A+1)p^{1/2}-\frac{1}{p^{1/2}}(A+1)^2\right),$$

$$(g-1)/2<A+1\leqq\frac{2^m p^{1/2}}{1+2^m/p^{1/2}},$$

that is,

$$g\leqq\frac{2^{m-1}p^{1/2}}{1+2^m/p^{1/2}}+1<2^{m+1}p^{1/2}.$$

NATIONAL TSING HUA UNIVERSITY

ON THE LEAST SOLUTION OF PELL'S EQUATION

LOO-KENG HUA

Let x_0, y_0 be the least positive solution of Pell's equation

$$x^2 - dy^2 = 4,$$

where d is a positive integer, not a square, congruent to 0 or 1 (mod 4). Let $\epsilon = (x_0 + d^{1/2}y_0)/2$. It was proved by Schur[1] that

$$\epsilon < d^{d^{1/2}}, \tag{1}$$

or, more precisely,

$$\log \epsilon < d^{1/2}((1/2)\log d + (1/2)\log\log d + 1). \tag{2}$$

He deduced (1) from (2) by the property that

$$d^{1/2}((1/2)\log d + (1/2)\log\log d + 1) < d^{1/2}\log d$$

for $d > 244.69 \cdots$, and, for $d \leqq 244$, (1) is established by direct computation. It is the object of the present note to establish a slightly better result that

$$\log \epsilon < d^{1/2}((1/2)\log d + 1). \tag{3}$$

Thus (1) follows immediately without any calculation. The method used is that described in the preceding paper.

Let $(d|r)$ be Kronecker's symbol. (We extend the definition to include negative values of r by the relation $(d|r_1) = (d|r_2)$ for $r_1 \equiv r_2 \pmod{d}$.)

Let f denote the fundamental discriminant related to d, that is,

$$d = m^2 f,$$

where f is not divisible by a square of odd prime and is either odd, or congruent to 8 or congruent to 12 (mod 16).

LEMMA 1. *For $d > 0$, we have*

$$\left(\frac{d}{r}\right) = \left(\frac{d}{-r}\right).$$

PROOF. Landau, *Vorlesungen über Zahlentheorie*, vol. 1, Theorem 101.

LEMMA 2. *We have*

Received by the editors December 3, 1941.

[1] Göttingen Nachrichten, 1918, pp. 30–36.

$$\sum_r \left(\frac{f}{r}\right) e^{2\pi i n r/f} = \left(\frac{f}{n}\right) f^{1/2},$$

where r runs over a complete residue system, mod f.

PROOF. Landau, loc. cit., Theorem 215.

LEMMA 3. *We have*

$$\frac{1}{A^*+1}\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right)\right| \leqq \frac{1}{2}\left(f^{1/2} - \frac{A^*+1}{f^{1/2}}\right),$$

where A^* is the least positive residue of A, mod f.

PROOF. (See Lemma 1 of the preceding paper.) We have, by Lemma 2,

$$\begin{aligned} f^{1/2}\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right) &= \frac{1}{2} f^{1/2}\sum_{a=0}^{A}\ \sum_{n=-a}^{a}\left(\frac{f}{n}\right) \\ &= \frac{1}{2}\sum_{a=0}^{A}\ \sum_{n=-a}^{a}\ \sum_{r=1}^{f}\left(\frac{f}{r}\right) e^{2\pi i n r/f} \\ &= \frac{1}{2}\sum_{r=1}^{f}\left(\frac{f}{r}\right)\sum_{a=0}^{A}\ \sum_{n=-a}^{a} e^{2\pi i n r/f}. \end{aligned}$$

Then

$$\begin{aligned} f^{1/2}\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right)\right| &\leqq \frac{1}{2}\sum_{r=1}^{f-1}\left|\sum_{a=0}^{A}\ \sum_{n=-a}^{a} e^{2\pi i n r/f}\right| \\ &= \frac{1}{2}\sum_{r=1}^{f-1}\left(\frac{\sin (A+1)\pi r/f}{\sin \pi r/f}\right)^2 \\ &= \frac{1}{2}\sum_{r=1}^{f-1}\left(\frac{\sin (A^*+1)\pi r/f}{\sin \pi r/f}\right)^2 \\ &= \frac{1}{2}\sum_{r=1}^{f-1}\sum_{a=0}^{A_*}\ \sum_{n=-a}^{a} e^{2\pi i n r/f} \\ &= \frac{1}{2}\left((A^*+1)f - (A^*+1)^2\right), \end{aligned}$$

since

$$\sum_{r=1}^{f-1} e^{2\pi i n r/f} = \sum_{r=1}^{f} e^{2\pi i n r/f} - 1 = \begin{cases} -1 & \text{if } f\nmid n, \\ f-1 & \text{if } f\mid n. \end{cases}$$

LEMMA 4. *For any discriminant $d>0$ and $A>d^{1/2}$, we have*

$$\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{d}{n}\right)\right| \leqq \frac{1}{2}Ad^{1/2}.$$

PROOF. It is well known that[2]

$$\left(\frac{d}{n}\right) = \left(\frac{f}{n}\right)\sum_{r\mid(m,n)}\mu(r).$$

Then

$$\begin{aligned}\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{d}{n}\right) &= \sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{f}{n}\right)\sum_{r\mid(m,n)}\mu(r)\\ &= \sum_{r\mid m}\mu(r)\sum_{a=1}^{A}\sum_{n=1,\, r\mid n}^{a}\left(\frac{f}{n}\right) = \sum_{r\mid m}\mu(r)\sum_{a=1}^{A}\sum_{n=1}^{[a/r]}\left(\frac{f}{rn}\right)\\ &= \sum_{r\mid m}\mu(r)\left(\frac{f}{r}\right)\sum_{a=1}^{A}\sum_{n=1}^{[a/r]}\left(\frac{f}{n}\right).\end{aligned}$$

Then, by Lemma 2,

$$\begin{aligned}\left|\sum_{a=1}^{A}\sum_{n=1}^{a}\left(\frac{d}{n}\right)\right| &\leqq \frac{1}{2}\sum_{r\mid m}\left|\sum_{a=1}^{A}\sum_{n=1}^{[a/r]}\left(\frac{f}{n}\right)\right|\\ &\leqq \frac{1}{2}\sum_{r\mid m}r\left|\sum_{b=1}^{[A/r]}\sum_{n=1}^{b}\left(\frac{f}{n}\right)\right|\\ &\leqq \frac{1}{2}\sum_{r\mid m}r\left(\left(\left[\frac{A}{r}\right]+1\right)f^{1/2} - \frac{1}{f^{1/2}}\left(\left[\frac{A}{r}\right]+1\right)^{2}\right)\\ &\leqq \frac{1}{2}\sum_{r\mid m}r\cdot\frac{A}{r}f^{1/2} \leqq \frac{1}{2}Af^{1/2}m = \frac{1}{2}Ad^{1/2},\end{aligned}$$

since we have $f^{1/2}r < f^{1/2}m < A$,

$$f^{1/2} - \frac{1}{f^{1/2}}\left(\left[\frac{A}{r}\right]+1\right)^{2} < f^{1/2} - \frac{1}{f^{1/2}}\cdot f = 0$$

and

$$\sum_{r\mid m}1 \leqq m.$$

LEMMA 5. *We have*

$$\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n} < \frac{1}{2}\log d + 1.$$

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[2] This follows from the fact that $\sum_{d\mid a}\mu(d) = 0$ or 1 according as $a > 1$ or $a = 1$.

PROOF. For $n \geqq 1$ let

$$S(n) = \sum_{a=1}^{n} \sum_{m=1}^{a} \left(\frac{d}{m}\right),$$

and let $S(0) = S(-1) = 0$. Then we have

$$S(n) - 2S(n-1) + S(n-2) = \left(\frac{d}{n}\right), \qquad n \geqq 1,$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n} &= \sum_{n=1}^{\infty} \{S(n) - 2S(n-1) + S(n-2)\} \frac{1}{n} \\ &= \sum_{n=1}^{\infty} S(n) \left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) \\ &= \sum_{n=1}^{\infty} \frac{2S(n)}{n(n+1)(n+2)}. \end{aligned}$$

We divide the series into two parts

$$S_1 = \sum_{n=1}^{A-1}, \qquad S_2 = \sum_{n=A}^{\infty}.$$

Since

$$|S(n)| \leqq \sum_{a=1}^{n} \sum_{m=1}^{a} 1 = \frac{n(n+1)}{2},$$

it follows that

$$|S_1| \leqq \sum_{n=1}^{A-1} \frac{1}{n+2}.$$

If $A > d^{1/2}$ we have by Lemma 4

$$|S_2| < \sum_{n=A}^{\infty} \frac{n d^{1/2}}{n(n+1)(n+2)} = \frac{d^{1/2}}{A+1}.$$

Hence

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n} \right| &\leqq \sum_{n=1}^{A-1} \frac{1}{n+2} + \frac{d^{1/2}}{A+1} \\ &= \sum_{m=1}^{A-1} \frac{1}{m} - 1 - \frac{1}{2} + \frac{1}{A} + \frac{1}{A+1} + \frac{d^{1/2}}{A+1} \\ &\leqq \log (A-1) - \frac{1}{2} + \frac{1}{A} + \frac{d^{1/2}+1}{A+1}. \end{aligned}$$

Taking $A = [d^{1/2}]+1$ we have

$$\left|\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n}\right| \leqq \log d^{1/2} - \frac{1}{2} + \frac{1}{d^{1/2}} + \frac{d^{1/2}+1}{d^{1/2}+1}$$
$$= \frac{1}{2}\log d + \frac{1}{2} + \frac{1}{d^{1/2}} < \frac{1}{2}\log d + 1$$

since $d \geqq 5$.

THEOREM 1. *We have*

$$\log \epsilon < d^{1/2}((1/2)\log d + 1).$$

PROOF. It is known that the number $h(d)$ of classes of non-equivalent quadratic forms with determinant $d>0$, is given by

$$h(d) = \frac{d^{1/2}}{\log \epsilon}\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n}\cdot$$

Since $h(d) \geqq 1$, we have the theorem.

THEOREM 2 (Schur). *We have*

$$\log \epsilon \leqq d^{1/2}\log d.$$

PROOF. For $d > e^2$, the theorem follows from Theorem 1. If $d < e^2$, then $d = 5$. Evidently $\epsilon = (3+5^{1/2})/2$ and

$$\log \epsilon < 5^{1/2}\log 5.$$

NATIONAL TSING HUA UNIVERSITY

A NOTE ON THE DIOPHANTINE PROBLEM OF FINDING FOUR BIQUADRATES WHOSE SUM IS A BIQUADRATE

J. O. PATTERSON

The problem is to assign integers satisfying the relation

$$x^4 = y^4 + a^4 + b^4 + c^4. \tag{1}$$

Granted that the five integers have no common divisor, it follows that x and one quantity on the right (here taken as y) are not multiples of 5; and it can be shown that (1) without loss of generality can be replaced by

$$x^4 = y^4 + 5^4t^4(\alpha^4 + \beta^4 + \gamma^4). \tag{2}$$

In this expression x is an odd number prime to y; y may be odd or even; and neither is divisible by 5, as stated above.

Let $(x^4-y^4)/5^4t^4=d$; which for values satisfying (2) will be an integer.

Evidently x and y must satisfy the congruence $x^4 \equiv y^4 \pmod{5^4}$. Hence, for values of x up to any required magnitude, there are corresponding values of y;[1] and the resultant values of d can be found and tabulated.

If (2) is satisfied, then $d=\alpha^4+\beta^4+\gamma^4$; and from the elementary properties of the sum of three biquadrates it is seen that many values of d may be rejected at once: for example, all those having for the final digit 0, 4, 5, or 9; and all those incapable of representing the sum of three square numbers. The remaining values of d will have for the final digit 1, 2, 3, 6, 7, or 8. Regarding these as separate cases, it is possible to devise a numerical test for each case. As an example, consider the values of d ending with $\cdots 3$. These must have the form $80k+3$ (sum of three odd biquadrates prime to 5) or the form $80k+33$ (sum of one odd and two even biquadrates prime to 5). Choosing the latter form,[2] seventy instances are found in a tabulation of values of d based on all admissible values of $x<700$. The first ten in order of magnitude are: 164833, 195313, 198593, 3029873, 4106193, 4590753, 5086913, 5693793, 5948193, 7424753. Testing these by a method (based upon the character of k), which need not be

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Received by the editors November 24, 1941.

[1] Solutions of the congruence in question can be read from C. J. G. Jacobi's *Canon Arithmeticus*, Berlin, 1839, pp. 230–231.

[2] This form was considered first because it includes Norrie's solution, which appears in The University of St. Andrew 500th Anniversary Memorial Volume, Edinburgh, 1911, p. 89.

described in this brief discussion, it is found that the third integer is the sum of three biquadrates, and gives Norrie's solution

$$d = 198593 = \frac{353^4 - 272^4}{15^4} = 21^4 + 8^4 + 2^4.$$

The seventh integer furnishes a second numerical solution

$$d = 5086913 = \frac{651^4 - 599^4}{10^4} = 43^4 + 34^4 + 24^4.$$

It is the intention of the writer to carry the investigation of each case as far as values of $x < 1250$ $(= 2 \cdot 5^4)$. The fact that an additional solution was obtained in the first case examined would seem to give some promise that other solutions exist among the lower integers.

GLENSIDE, PA.

ON THE INVERSION OF THE q-SERIES ASSOCIATED WITH JACOBIAN ELLIPTIC FUNCTIONS

A. N. LOWAN, G. BLANCH AND W. HORENSTEIN[1]

The elliptic functions sn (u, k), cn (u, k) and dn (u, k) may be computed from theta functions by well known methods outlined in standard texts. (See, for instance, Whittaker & Watson, *Modern Analysis*, 4th edition, p. 485.) Given the modulus k, and $k' = (1-k^2)^{1/2}$, there is associated with k, k' a function ϵ defined by

$$\epsilon = \frac{1}{2}\left[\frac{1-(k')^{1/2}}{1+(k')^{1/2}}\right] = \frac{1}{2}\,\frac{\vartheta_2(0, q^4)}{\vartheta_3(0, q^4)}\cdot$$

Values of theta functions for a given parameter q can be readily computed, and the Jacobi elliptic functions turn out to be ratios of the theta functions.

The series for ϵ in terms of q is given by

$$\epsilon = \sum_{k=0}^{\infty} q^{(2k+1)^2} \Big/ 1 + 2\sum_{k=1}^{\infty} q^{4k^2}. \tag{1}$$

Presented to the Society, February 28, 1942; received by the editors December 19, 1941.

[1] The above results were obtained in the course of work carried out by the Mathematical Tables Project, conducted by the Work Projects Administration for the City of New York under the sponsorship of the National Bureau of Standards.

Since, however, the known parameter in elliptic functions is ϵ and not q, the problem of inverting formula (1) arises. Weierstrass[2] gave the first four terms of this expansion, and Milne-Thomson[3] added two more, giving

$$q = \epsilon + 2\epsilon^5 + 15\epsilon^9 + 150\epsilon^{13} + 1707\epsilon^{17} + 20{,}910\epsilon^{21} + \cdots .$$

As is well known, the above series converges for $|\epsilon| < 1/2$; and it is rapidly convergent over a large range of ϵ. But when ϵ is greater than 0.3, only a few decimals in q may be obtained from the six terms given above. Because of the importance of this series in the computation of elliptic functions as well as in the Jacobian zeta functions, the above series has been extended to fourteen terms, as given below:

$$\begin{aligned} q = \epsilon[1 &+ 2u + 15u^2 + 150u^3 + 1707u^4 + 20{,}910u^5 + 268{,}616u^6 \\ &+ 3{,}567{,}400u^7 + 48{,}555{,}069u^8 + 673{,}458{,}874u^9 \\ &+ 9{,}481{,}557{,}398u^{10} + 135{,}119{,}529{,}972u^{11} \\ &+ 1{,}944{,}997{,}539{,}623u^{12} + 28{,}235{,}172{,}753{,}886u^{13} \cdots] \end{aligned} \tag{2}$$

where $u = \epsilon^4$.

The following accuracy may be obtained with the above series:

ϵ	Accuracy in q with 14 terms of (2)	Accuracy in q with 6 terms of (2)
	decimals	*decimals*
.4	7	4
.36	10	5
.30	14	6
.25	18	8

New York City

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[2] *Werke*, vol. 2 (1895), p. 276.

[3] *Ten decimal table of the nome q*, Journal of the London Mathematical Society, vol. 5 (1930) pp. 148–149.

TABLE OF THE ZEROS OF THE LEGENDRE POLYNOMIALS OF ORDER 1-16 AND THE WEIGHT COEFFICIENTS FOR GAUSS' MECHANICAL QUADRATURE FORMULA[1]

ARNOLD N. LOWAN, NORMAN DAVIDS AND ARTHUR LEVENSON

Gauss' method of mechanical quadrature has the advantage over most methods of numerical integration in that it requires about half the number of ordinate computations. This is desirable when such computations are very laborious, or when the observations necessary to determine the average value of a continuously varying physical quantity are very costly. Gauss' classical result[2] states that, for the range $(-1, +1)$, the "best" accuracy with n ordinates is obtained by choosing the corresponding abscissae at the zeros $x_1, \cdots, x_n$ of the Legendre polynomials $P_n(x)$. With each x_i is associated a constant a_i such that

$$(1) \qquad \int_{-1}^{1} f(x)dx \sim a_1f(x_1) + a_2f(x_2) + \cdots + a_nf(x_n).$$

The accompanying table computed by the Mathematical Tables Project gives the roots x_i for each $P_n(x)$ up to $n=16$, and the corresponding weight coefficients a_i, to 15 decimal places.

The first such table, computed by Gauss gave 16 places up to $n=7$.[3] More recently work was done by Nyström,[4] who gave 7 decimals up to $n=10$, but for the interval $(-1/2, +1/2)$. B. de F. Bayly has given the roots and coefficients of $P_{12}(x)$ to 13 places.[5]

The Gaussian quadrature formula for evaluating an integral with arbitrary limits (p, q) is given by

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Presented to the Society, October 25, 1941, under the title *Tables for Gauss' mechanical quadrature formula*; received by the editors December 18, 1941.

[1] The results reported here were obtained in the course of the work done by the Mathematical Tables Project conducted by the Work Projects Administration for New York City under the sponsorship of the National Bureau of Standards, Dr. Lyman J. Briggs, Director.

[2] *Methodus nova integralium valores per approximationem inveniendi*, Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores, vol. 3 (1814), or *Werke*, vol. 3, pp. 193–195.

[3] It may be found reproduced in Heine's *Kugelfunctionen*, vol. 2, 1881, p. 15, or Hobson, *Spherical Harmonics*, pp. 80–81.

[4] Nyström, Acta Mathematica, vol. 54 (1930), p. 191.

[5] B. de F. Bayly, Biometrika, vol. 30 (1938), pp. 193–194.

$$(2) \qquad \int_p^q f(x)dx = \frac{q-p}{2}\sum_{i=1}^{n} a_i f\left(x_i\frac{q-p}{2}+\frac{q+p}{2}\right)+R_n(f)$$

where $x_i = x_{i,n}$ is the ith root of $P_n(x)$ and

$$(3) \qquad a_i = a_{i,n} = \frac{1}{P_n'(x_i)}\int_{-1}^{1}\frac{P_n(x)}{x-x_i}dx.$$

We have

$$x_i = -x_{n-i+1}, \quad a_i = a_{n-i+1},$$

so that only half the values need be tabulated.

It is well known that if $f_{2n-1}(x)$ is an arbitrary polynomial of degree at most $2n-1$, then

$$R_n(f_{2n-1}) = 0,$$

that is, formula (1) is exact. Thus Gauss' formula with n ordinates provides an approximation as good as would have been obtained by using a polynomial of degree $2n-1$.

If $f(x)$ has a continuous derivative of order $2n$ in the interval (p, q), then[6]

$$R_n = \frac{f^{(2n)}(\xi)}{(2n)!k_n^2}$$

where ξ is a point in the interval (p, q) and k_n is the normalizing factor for $P_n(x)$, which is equal to

$$\left(\frac{2n+1}{q-p}\right)^{1/2}.$$

The roots were calculated by successive approximations, combining synthetic division with Newton's tangent formula. They were checked by using the relations between roots and coefficients.

By suitable transformations of formula (3), the computation of the weight coefficients a_i was made to depend on that of the roots, in particular on the values of the successive quotients obtained in the process of synthetic division. These weight coefficients were checked by putting $f(x)=1$ in formula (1), giving

$$2 = a_1 + a_2 + \cdots + a_n.$$

The sum of the a_i's was required not to differ from 2 by more than one unit in the 17th place.

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[6] This expression is due to Markoff. See Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, 1939, p. 369.

TABLE

$n=2$

$x_1=0.577350269189626$	$a_1=1.000000000000000$

$n=3$

$x_0=0.000000000000000$	$a_0=0.888888888888889$
$x_1=0.774596669241483$	$a_1=0.555555555555556$

$n=4$

$x_1=0.339981043584856$	$a_1=0.652145154862546$
$x_2=0.861136311594053$	$a_2=0.347854845137454$

$n=5$

$x_0=0.000000000000000$	$a_0=0.568888888888889$
$x_1=0.538469310105683$	$a_1=0.478628670499366$
$x_2=0.906179845938664$	$a_2=0.236926885056189$

$n=6$

$x_1=0.238619186083197$	$a_1=0.467913934572691$
$x_2=0.661209386466265$	$a_2=0.360761573048139$
$x_3=0.932469514203152$	$a_3=0.171324492379170$

$n=7$

$x_0=0.000000000000000$	$a_0=0.417959183673469$
$x_1=0.405845151377397$	$a_1=0.381830050505119$
$x_2=0.741531185599394$	$a_2=0.279705391489277$
$x_3=0.949107912342759$	$a_3=0.129484966168870$

$n=8$

$x_1=0.183434642495650$	$a_1=0.362683783378362$
$x_2=0.525532409916329$	$a_2=0.313706645877887$
$x_3=0.796666477413627$	$a_3=0.222381034453374$
$x_4=0.960289856497536$	$a_4=0.101228536290376$

$n=9$

$x_0=0.000000000000000$	$a_0=0.330239355001260$
$x_1=0.324253423403809$	$a_1=0.312347077040003$
$x_2=0.613371432700590$	$a_2=0.260610696402935$
$x_3=0.836031107326636$	$a_3=0.180648160694857$
$x_4=0.968160239507626$	$a_4=0.081274388361574$

$n=10$

$x_1=0.148874338981631$	$a_1=0.295524224714753$
$x_2=0.433395394129247$	$a_2=0.269266719309996$
$x_3=0.679409568299024$	$a_3=0.219086362515982$
$x_4=0.865063366688985$	$a_4=0.149451349150581$
$x_5=0.973906528517172$	$a_5=0.066671344308688$

$n=11$

$x_0=0.000000000000000$	$a_0=0.272925086777901$
$x_1=0.269543155952345$	$a_1=0.262804544510247$
$x_2=0.519096129110681$	$a_2=0.233193764591990$
$x_3=0.730152005574049$	$a_3=0.186290210927734$
$x_4=0.887062599768095$	$a_4=0.125580369464905$
$x_5=0.978228658146057$	$a_5=0.055668567116174$

TABLE—*Continued*

$n=12$	
$x_1=0.125333408511469$	$a_1=0\ 249147045813403$
$x_2=0.367831498918180$	$a_2=0.233492536538355$
$x_3=0.587317954286617$	$a_3=0.203167426723066$
$x_4=0.769902674194305$	$a_4=0.160078328543346$
$x_5=0.904117256370475$	$a_5=0.106939325995318$
$x_6=0.981560634246719$	$a_6=0.047175336386512$
$n=13$	
$x_0=0.000000000000000$	$a_0=0.232551553230874$
$x_1=0.230458315955135$	$a_1=0.226283180262897$
$x_2=0.448492751036447$	$a_2=0.207816047536889$
$x_3=0.642349339440340$	$a_3=0.178145980761946$
$x_4=0.801578090733310$	$a_4=0.138873510219787$
$x_5=0.917598399222978$	$a_5=0.092121499837728$
$x_6=0.984183054718588$	$a_6=0.040484004765316$
$n=14$	
$x_1=0.108054948707344$	$a_1=0.215263853463158$
$x_2=0.319112368927890$	$a_2=0.205198463721296$
$x_3=0.515248636358154$	$a_3=0.185538397477938$
$x_4=0.687292904811685$	$a_4=0.157203167158194$
$x_5=0.827201315069765$	$a_5=0.121518570687903$
$x_6=0.928434883663574$	$a_6=0.080158087159760$
$x_7=0.986283808696812$	$a_7=0.035119460331752$
$n=15$	
$x_0=0.000000000000000$	$a_0=0.202578241925561$
$x_1=0.201194093997435$	$a_1=0\ 198431485327111$
$x_2=0.394151347077563$	$a_2=0.186161000015562$
$x_3=0.570972172608539$	$a_3=0.166269205816994$
$x_4=0.724417731360170$	$a_4=0.139570677926154$
$x_5=0.848206583410427$	$a_5=0.107159220467172$
$x_6=0.937273392400706$	$a_6=0.070366047488108$
$x_7=0.987992518020485$	$a_7=0.030753241996117$
$n=16$	
$x_1=0.095012509837637$	$a_1=0.189450610455069$
$x_2=0.281603550779259$	$a_2=0\ 182603415044924$
$x_3=0.458016777657227$	$a_3=0.169156519395003$
$x_4=0.617876244402644$	$a_4=0.149595988816577$
$x_5=0.755404408355003$	$a_5=0.124628971255534$
$x_6=0.865631202387832$	$a_6=0.095158511682493$
$x_7=0.944575023073233$	$a_7=0.062253523938648$
$x_8=0.989400934991650$	$a_8=0\ 027152459411754$

EXAMPLE. Let it be desired to find $I=\int_1^2 dx/x$. By formula (2)

$$I \sim \sum \frac{a_k}{x_k^2}\cdot$$

Taking $n=10$, we have

$$
\begin{aligned}
I \sim {} & \frac{0.295524224714753}{2.851125661018369} + \frac{0.269266719309996}{2.566604605870753} \\
& + \frac{0.219086362515982}{2.320590431700976} + \frac{0.149451349150581}{2.134936633311015} \\
& + \frac{0.066671344308688}{2.026093471482828} + \frac{0.295524224714753}{3.148874338981631} \\
& + \frac{0.269266719309996}{3.433395394129247} + \frac{0.219086362515982}{3.679409568299024} \\
& + \frac{0.149451349150581}{3.865063366688985} + \frac{0.066671344308688}{3.973906528517172} \\
= {} & 0.693147180559947.
\end{aligned}
$$

This agrees to fourteen places with the actual value

$$\log 2 = 0.693147180559945.$$

Grateful acknowledgement is extended to Mr. V. Galin and Mr. A. Grossman, and especially to Miss M. Robinson for her assistance in the preparation of the manuscript.

NEW YORK CITY

SOME REMARKS ON RELATIVELY COMPLETE FIELDS

IRVING KAPLANSKY AND O. F. G. SCHILLING

In this note we discuss some properties of fields which are relatively complete with respect to valuations of rank one. We show that relatively complete fields have in many respects simpler properties than complete fields. This note arose as a consequence of a discussion of an erroneous statement in a note of one of the authors.[1]

Let K be a field on which a valuation V of rank one is defined. Denote the ring of integers in K with respect to V by O and let P be the prime ideal of nonunits in O.

DEFINITION[2] 1. *The field K is termed relatively complete with respect to V if every polynomial congruence*

$$f(x) \equiv g_0(x)h_0(x) \pmod{P}$$

with $f(x)$ in $O[x]$ and $(g_0(x), h_0(x)) \equiv 1 \pmod{P}$ implies

$$f(x) = g(x)h(x)$$

where $g(x) \equiv g_0(x) \pmod{P}$ and $h(x) \equiv h_0(x) \pmod{P}$.

We remark[3] that there exist relatively complete fields which are not complete with respect to the congruence topology induced by V. To find examples for such fields it suffices to consider infinite algebraic extensions of a field which is complete with respect to a discrete valuation of rank one.

DEFINITION[4] 2. *Two polynomials $a(x)$ and $b(x)$ over a field K are said to have the same decomposition type if $a(x)=\prod_{i=1}^{s} p_i(x)^{m_i}$, $b(x)=\prod_{i=1}^{r} q_i(x)^{n_i}$ when the $p_i(x)$ and $q_i(x)$ are irreducible polynomials and $s=r$, $m_i=n_i$ and $p_i(x)$ and $q_i(x)$ have the same degree for a suitable arrangement of the prime factors.*

We are now able to generalize certain of the results of F. K.

Presented to the Society, December 31, 1941; received by the editors January 9, 1942.

[1] O. F. G. Schilling, *Remarks on a special class of algebras*, American Journal of Mathematics, vol. 62 (1940), pp. 346–352.

[2] A. Ostrowski, *Untersuchungen zur arithmetischen Theorie der Körper*, Mathematische Zeitschrift, vol. 39 (1935), pp. 269–404.

[3] A. Ostrowski, *Über einige Fragen der allgemeinen Körpertheorie*, Journal für die reine und angewandte Mathematik, vol. 143 (1913), pp. 255–284.

[4] F. K. Schmidt, *Mehrfach perfekte Körper*, Mathematische Annalen, vol. 108 (1933), pp. 1–25.

Schmidt. We first state without proof a simple consequence of Ostrowski's criterion for relative completeness.[5]

LEMMA 1. *Let K be a field which is relatively complete with respect to a rank one valuation V and let $\bar{K}$ be the completion of K. Then the decomposition type of any separable polynomial $f(x)$ with coefficients in K is the same in K as in $\bar{K}$.*

LEMMA[6] 2. *Let $\bar{K}$, $\bar{K}_1$ be the completions of K with respect to the inequivalent rank one valuations V, V_1 of K, and let the fields L, L_1 be relatively complete with respect to V and V_1 such that $K \leqq L \leqq \bar{K}$, $K \leqq L_1 \leqq \bar{K}_1$. Then there exists for any two separable polynomials $g(x)$, $g_1(x)$ of the same degree in L, L_1 a third polynomial $h(x)$ in K which has the same decomposition types as $g(x)$, $g_1(x)$ in L, L_1.*

PROOF. According to a result of F. K. Schmidt[7] there exists a polynomial $h(x)$ in K whose decomposition types in $\bar{K}$, $\bar{K}_1$ coincide with those of $g(x)$, $g_1(x)$. Then, by Lemma 1, the decomposition types of $h(x)$ and $g(x)$ are the same in L. Similarly, the decomposition types of $h(x)$ and $g_1(x)$ are the same in L_1.

Next we prove the following theorem.[8]

THEOREM 1. *If the field K is relatively complete with respect to the rank one valuation V and if, for the inequivalent rank one valuation V_1, the field L_1 is a relatively complete field containing K and contained in the completion of K, then L_1 is separably algebraically closed.*

PROOF. Let $g_1(x)$ be an irreducible separable polynomial of degree n in L_1 and let $g(x)$ be the product of any n distinct linear factors in K. Then, according to Lemma 2, the polynomials $g(x)$ and $g_1(x)$ have the same type of decomposition in K and L_1. Hence $g_1(x)$ must be a product of linear polynomials; in other words, the field L_1 is separably algebraically closed.

THEOREM 2. *A field K is relatively complete with respect to two inequivalent rank one valuations if and only if it is separably algebraically closed.*

PROOF. The necessity of the statement follows at once from Theorem 1. On the other hand Ostrowski's criterion implies that a separably algebraically closed field is relatively complete with respect to every valuation of rank one.

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[5] Ostrowski's criterion states that a field is relatively complete if and only if it is separably algebraically closed in its complete closure. Loc. cit., p. 316.

[6] F. K. Schmidt, loc. cit., lemma on p. 16.

[7] Loc. cit., p. 16.

[8] Theorem 1 is the analogue of Schmidt's Theorem 1, p. 18, loc. cit.

THEOREM 3. *If a field K is complete with respect to a rank one valuation and relatively complete with respect to a rank one valuation which is inequivalent to the first valuation, then K is a multiply complete field.*

PROOF. By Theorem 2 the field K is separably algebraically closed. Since K is complete with respect to the second valuation, it follows that K is algebraically closed.[9] Moreover the cardinal number $\mathfrak{c}$ of K satisfies the equation $\mathfrak{c}^{\aleph_0}=\mathfrak{c}$ for K is a complete field. Hence K is a multiply complete field.

We should like to point out that the above theorems do not require conditions on the cardinal number of the field K as in the theory of F. K. Schmidt for complete fields.

In Theorem 2 it was proved that the separable algebraic closure of a field K is always relatively complete with respect to every valuation extending any non-trivial rank one valuation of K. We then may ask whether already a finite algebraic extension L of K is relatively complete with respect to some rank one valuation though K itself is not relatively complete with respect to any rank one valuation.

We shall give a partial answer to this problem in this theorem.

THEOREM 4. *Suppose that K is a field which has for every integer n exactly one separable extension of degree n and suppose that K is not relatively complete with respect to any non-trivial rank one valuation. Then there exists no finite algebraic extension L of K which is relatively complete with respect to a rank one valuation.*

PROOF. Suppose that, contrary to the theorem, there exists a finite extension L of degree m over K which is relatively complete with respect to some rank one valuation V_L. Since L is relatively complete with respect to V_L there is, by Ostrowski's criterion, a one-to-one correspondence between the algebraic extensions of L and its completion $\bar{L}$ with respect to V_L. In other words, if Z_n is an arbitrary extension of degree n over L then $[Z_n\bar{L}:\bar{L}]=n$. Consider next the valuation V on K which is induced by V_L. Let $\bar{K}$ be the completion of K with respect to V. We assert that the Galois group of the algebraic closure $\bar{K}'$ of $\bar{K}$ (relative to $\bar{K}$) is cyclic. For a proof it suffices to show that the decomposition groups for the prolongations V_i of V to any extension $M>K$ are cyclic.[10] Since M/K is cyclic all decomposition fields coincide and are cyclic. The field $\bar{L}$ is then equivalent to a subfield of $\bar{K}'$; without loss of generality we may suppose $\bar{K}' > \bar{L} \geqq \bar{K}$. The degree $[\bar{L}:\bar{K}] = \bar{m}$ is a divisor of m. Consequently $[Z_n\bar{L}:\bar{K}] = [Z_n\bar{L}:\bar{L}][\bar{L}:\bar{K}] = n\bar{m}$. By the Galois theory there is then for every integer n an extension Z_n^* of degree n over $\bar{K}$. The defining equation $f^*(x) = 0$ of $Z_n^*/\bar{K}$ now may be approximated by an irreducible equation $f(x) = 0$ of degree n with coefficients in K so that Z_n^* is generated by the roots of $f(x) = 0$. The root field of $f(x) = 0$ over K is the cyclic extension Z_n' of degree n over K. Hence $Z_n^* = Z_n'\bar{K}$ for all n, contrary to the assumption that K is not relatively complete with respect to any rank one valuation.

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[9] F. K. Schmidt, loc. cit. Lemma 2, p. 18.

[10] M. Deuring, *Verzweigungstheorie bewerteter Körper*, Mathematische Annalen, vol. 105 (1931), pp. 277–307.

HARVARD UNIVERSITY AND
UNIVERSITY OF CHICAGO

A DIFFERENTIAL GEOMETRY PROBLEM USING TENSOR ANALYSIS

ATHERTON H. SPRAGUE

1. **Introduction.** The problem at hand was worked out in attempting to apply tensors to a much more general problem in classical differential geometry. The results obtained in a general coordinate system reduce readily to classical results of Eisenhart. An interesting interpretation of Christoffel symbols appears.

2. *R* **net.** A rectilinear congruence in 3-space is called a *W*-congruence if the asymptotic lines on the two focal surfaces correspond. If the tangents to both families of curves of a conjugate net on a surface form *W*-congruences the net is called an *R* net.[1] We derive the analytic conditions that must obtain in order that a given conjugate net on a surface shall be an *R* net.

3. **Equations for an** *R* **net.** Let S_1 be one focal surface of a *W*-congruence, the vector equation of the surface being

$$(3.1) \qquad \overset{\alpha}{s_1} = \overset{\alpha}{s_1}(\overset{i}{x}), \qquad \alpha = 1, 2, 3;\ i = 1, 2.$$

Received by the editors November 11, 1941.

[1] Tzitzeica, Comptes Rendus de l'Académie des Sciences, Paris, vol. 152 (1911), p. 1077.

Let S_2 be the other focal surface of the congruence with vector z_2^α so that we have

$$z_2^\alpha = z_1^\alpha + \rho_1 \xi_1^\alpha \tag{3.2}$$

where ξ_1^α is a unit vector tangent to S_1, and ρ_1 is an invariant.

Then if $\lambda^i_{1/}$ are the components of this vector in the x's we have

$$\xi_1^\alpha = z^\alpha_{1/,i} \lambda^i_{1/} \tag{3.3}$$

where in this case

$$z^\alpha_{1/,i} = \frac{\partial z_1^\alpha}{\partial x^i}$$

since z_1^α being an invariant for a transformation of coordinates in the x's, the ordinary derivative of z_1^α with respect to x^i is the same as the covariant derivative with respect to g_{ij}, the fundamental tensor of S_1. Then substituting (3.3) in (3.2) we have

$$z_2^\alpha = z_1^\alpha + \rho_1 z^\alpha_{1/,i} \lambda^i_{1/}. \tag{3.4}$$

Similarly by the property of focal surfaces we have

$$z_1^\alpha = z_2^\alpha + \rho_2 z^\alpha_{2/,i} \lambda^i_{2/} \tag{3.5}$$

where ρ_2 is an invariant, and $\lambda^i_{2/}$ is a unit vector tangent to S_2.

Adding (3.4) and (3.5) we have

$$\rho_1 \lambda^i_{1/} z^\alpha_{1/,i} + \rho_2 \lambda^i_{2/} z^\alpha_{2/,i} = 0. \tag{3.6}$$

From (3.6) we have

$$\begin{aligned} \rho_1 &= \bar{e} \rho_2, \\ \lambda^i_{1/} z^\alpha_{1/,i} &= e \lambda^i_{2/} z^\alpha_{2/,i} \end{aligned} \tag{3.6'}$$

where $\bar{e} = 1$ if $e = -1$, and $\bar{e} = -1$ if $e = 1$, and conversely.

We differentiate (3.4) covariantly and have

$$z^\alpha_{2/,k} = z^\alpha_{1/,k} + \rho_{1/,k} \lambda^i_{1/} z^\alpha_{1/,i} + \rho_1 \lambda^i_{1/,k} z^\alpha_{1/,i} + \rho_1 \lambda^i_{1/} z^\alpha_{1/,ik}. \tag{3.7}$$

Let η_1^α be the unit normal to S_1.

We multiply (3.7) by $\lambda^k_{2/}$, sum for k, multiply by η_1^α, sum for α and we have

$$0 = \rho_1 b_{1/ij} \lambda^i_{1/} \lambda^j_{2/} \tag{3.8}$$

the first three terms on the right vanishing because η_1^α is perpendicular to S_1, the term on the left vanishing because of this fact and (3.6).

and the last term becoming $\rho_1\eta_1^\alpha\cdot z^\alpha_{1/,ij}\lambda^i_{1/}\lambda^j_{2/}$ the second factor of which is denoted by $b_{1/ij}\lambda^i_{1/}\lambda^j_{2/}$.[2]

Hence *the directions* $\lambda^i_{1/}$ *and* $\lambda^j_{2/}$ *are conjugate on* S_1.[3]

Similarly they are conjugate on S_2.

We next differentiate (3.7) covariantly with respect to g_{ij}, the fundamental tensor of S_1,

$$\begin{aligned}\frac{\partial^2 z^\alpha_2}{\partial x^k\partial x^j}-z^\alpha_{2/,m}\begin{Bmatrix} m\\ kj\end{Bmatrix}_{g_{ij}} &= z^\alpha_{1/,kj}+\rho_{1/,kj}\lambda^i_{1/}z^\alpha_{1/,i}+\rho_{1/,k}\lambda^i_{1/,j}z^\alpha_{1/,i}\\ &\quad+\rho_{1/,k}\lambda^i_{1/}z^\alpha_{1/,ij}+\rho_{1/,j}\lambda^i_{1/,k}z^\alpha_{1/,i}+\rho_1\lambda^i_{1/,kj}z^\alpha_{1/,i}+\rho_1\lambda^i_{1/,k}z^\alpha_{1/,ij}\\ &\quad+\rho_{1/,j}\lambda^i_{1/}z^\alpha_{1/,ik}+\rho_1\lambda^i_{1/,j}z^\alpha_{1/,ik}+\rho_1\lambda^i_{1/}z^\alpha_{1/,ikj}.\end{aligned}\tag{3.9}$$

Multiply by η_2^α (the unit normal to S_2), sum for α and we have

$$\begin{aligned}b_{2/kj} &= b_{1/kj}\eta^\alpha_1\cdot\eta^\alpha_2+\rho_{1/,kj}\lambda^i_{1/}z^\alpha_{1/,i}\cdot\eta^\alpha_2+\rho_{1/,k}\lambda^i_{1/,j}z^\alpha_{1/,i}\cdot\eta^\alpha_2\\ &\quad+\rho_{1/,k}\lambda^i_{1/}b_{1/ij}\eta^\alpha_1\cdot\eta^\alpha_2+\rho_{1/,j}\lambda^i_{1/,k}z^\alpha_{1/,i}\cdot\eta^\alpha_2+\rho_1\lambda^i_{1/,kj}z^\alpha_{1/,i}\cdot\eta^\alpha_2\\ &\quad+\rho_1\lambda^i_{1/,k}b_{1/ij}\eta^\alpha_1\cdot\eta^\alpha_2+\rho_{1/,j}\lambda^i_{1/}b_{1/ik}\eta^\alpha_1\ \eta^\alpha_2+\rho_1\lambda^i_{1/,j}b_{1/ik}\eta^\alpha_1\cdot\eta^\alpha_2\\ &\quad+\rho_1\lambda^i_{1/}b_{1/ik,j}\eta^\alpha_1\cdot\eta^\alpha_2+b_{1/ik}\rho_1\lambda^i_{1/}\eta^\alpha_{1/,j}\cdot\eta^\alpha_2.\end{aligned}\tag{3.10}$$

In the future, since unless otherwise stated covariant differentiation is with respect to the fundamental tensor of S_1, we shall note the covariant derivative of $z^\alpha_{2/,k}$ by $z^\alpha_{2/,kj}$.

We evaluate $z^\alpha_{1/,i}\cdot\eta^\alpha_2$ as follows.

We differentiate (3.5) covariantly giving

$$z^\alpha_{1/,i}=z^\alpha_{2/,i}+\rho_{2/,i}\lambda^s_{2/}z^\alpha_{2/,s}+\rho_2\lambda^s_{2/,i}z^\alpha_{2/,s}+\rho_2\lambda^s_{2/}z^\alpha_{2/,si}.\tag{3.11}$$

Multiply by η_2^α and sum for α, giving

$$z^\alpha_{1/,i}\cdot\eta^\alpha_2=\rho_2\lambda^s_{2/}b_{2/si}.\tag{3.12}$$

Substituting this value for $z^\alpha_{1/,i}\cdot\eta_2^\alpha$ in (3.10) we have

$$\begin{aligned}b_{2/kj} &= (\eta^\alpha_1\cdot\eta^\alpha_2)(b_{1/kj}+b_{1/ij}\rho_{1/,k}\lambda^i_{1/}+b_{1/ij}\rho_1\lambda^i_{1/,k}+b_{1/ik}\rho_{1/,j}\lambda^i_{1/}\\ &\quad+b_{1/ik}\rho_1\lambda^i_{1/,j}+b_{1/ik,j}\rho_1\lambda^i_{1/})+\rho_1\lambda^i_{1/}b_{1/ik}\eta^\alpha_{1/,j}\cdot\eta^\alpha_2\\ &\quad+\rho_{1/,k}\lambda^i_{1/,j}\rho_2\lambda^s_{2/}b_{2/si}+\rho_{1/,j}\lambda^i_{1/,k}\rho_2\lambda^s_{2/}b_{2/si}\\ &\quad+\rho_1\lambda^i_{1/,kj}\rho_2\lambda^s_{2/}b_{2/si}.\end{aligned}\tag{3.13}$$

Since the asymptotic lines on S_1 and S_2 are to correspond we have

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[2] L. P. Eisenhart, *Riemannian Geometry*, 1926, Equation 56.2, p. 189.

[3] Ibid., Equation 56.3, p. 189.

$$b_{2/ij} = \mu b_{1/ij}. \tag{3.14}$$

We determine μ as follows. Differentiate the second of (3.6′) covariantly, giving

$$\overset{i}{\lambda}_{1/,j}\overset{\alpha}{z}_{1/,i} + \overset{i}{\lambda}_{1/}\overset{\alpha}{z}_{1/,ij} + e(\overset{i}{\lambda}_{2/,j}\overset{\alpha}{z}_{2/,i} + \overset{i}{\lambda}_{2/}\overset{\alpha}{z}_{2/,ij}) = 0. \tag{3.15}$$

Multiply by η^{α}_{2}, sum for α and use (3.12). This becomes

$$\overset{i}{\lambda}_{1/}b_{1/ij}(\overset{\alpha}{\eta}_1\cdot\overset{\alpha}{\eta}_2) + \rho_2\overset{i}{\lambda}_{1/,j}\overset{k}{\lambda}_{2/}b_{2/ki} + e\overset{i}{\lambda}_{2/}b_{2/ij} = 0. \tag{3.16}$$

Now multiply by $\lambda^{j}_{1/}$ and sum for j giving

$$(\overset{\alpha}{\eta}_1\cdot\overset{\alpha}{\eta}_2)b_{1/ij}\overset{i}{\lambda}_{1/}\overset{j}{\lambda}_{1/} + b_{2/ik}\rho_2\overset{i}{\lambda}_{1/,j}\overset{k}{\lambda}_{2/}\overset{j}{\lambda}_{1/} = 0. \tag{3.17}$$

But since $b_{2/ik}=\mu b_{1/ik}$ we have

$$\mu = -\frac{(\overset{\alpha}{\eta}_1\cdot\overset{\alpha}{\eta}_2)b_{1/ij}\overset{i}{\lambda}_{1/}\overset{j}{\lambda}_{1/}}{\rho_2 b_{1/ik}\overset{i}{\lambda}_{1/,j}\overset{k}{\lambda}_{2/}\overset{j}{\lambda}_{1/}}. \tag{3.18}$$

Substituting in (3.13) and using the fact that[3]

$$\overset{\alpha}{\eta}_{1/,j} = -\,b_{1/sj}g^{sm}\overset{\alpha}{z}_{1/,m}$$

we have

$$\begin{aligned}
&\rho_2 b_{is}\overset{s}{\lambda}_{2/}\overset{i}{\lambda}_{1/,k}\overset{k}{\lambda}_{1/}[b_{kj} + \rho_{1/,k}\overset{i}{\lambda}_{1/}b_{ij} + \rho_1\overset{i}{\lambda}_{1/,k}b_{ij} + \rho_{1/,j}\overset{i}{\lambda}_{1/}b_{ik}\\
&\quad + \rho_1\overset{i}{\lambda}_{1/,j}b_{ik} + \rho_1\overset{i}{\lambda}_{1/}b_{ik,j}] + b_{kr}\overset{k}{\lambda}_{1/}\overset{r}{\lambda}_{1/}[b_{kj} - \rho_{1/,k}\overset{i}{\lambda}_{1/,j}\rho_2\overset{s}{\lambda}_{2/}b_{is}\\
&\quad - \rho_1\rho_2\overset{i}{\lambda}_{1/,kj}\overset{s}{\lambda}_{2/}b_{is} + b_{ik}\rho_1\rho_2\overset{i}{\lambda}_{1/}b_{sj}g^{sm}b_{ms}\overset{s}{\lambda}_{2/}\\
&\quad - \rho_{1/,j}\overset{i}{\lambda}_{1/,k}\rho_2\overset{s}{\lambda}_{2/}b_{is}] = 0
\end{aligned} \tag{3.19}$$

where the b's are all those of S_1.

To evaluate ρ_1, multiply (3.16) by $\lambda^{j}_{2/}$ and sum for j, giving, by (3.14)

$$\rho_1 b_{1/ki}\overset{k}{\lambda}_{2/}\overset{i}{\lambda}_{1/,j}\overset{j}{\lambda}_{2/} + \overset{i}{\lambda}_{2/}\overset{j}{\lambda}_{2/}b_{1/ij} = 0. \tag{3.20}$$

Similarly, we have

$$\rho_2 b_{1/ki}\overset{k}{\lambda}_{1/}\overset{i}{\lambda}_{2/,j}\overset{j}{\lambda}_{1/} + \overset{i}{\lambda}_{1/}\overset{j}{\lambda}_{1/}b_{1/ij} = 0 \tag{3.21}$$

where $\lambda^{i}_{2/,j}$ is with respect to $\bar{g}_{ij}$ of S_2. (It should be remarked that ρ_2 may be expressed entirely in terms of elements of S_1 by means by (3.6) and (3.7) and differentiation.)

Equations (3.19) with ρ_1 and ρ_2 determined by (3.20) and (3.21), respectively, constitute the condition that must obtain in order for the tangents to the curves of direction $\lambda^{i}_{1/}$ on S_1 to form a W-congruence. An equation similar to (3.19) obtains for the direction $\lambda^{j}_{2/}$. These two equations must hold in order for the net with directions $\lambda^{i}_{1/}$ and $\lambda^{j}_{2/}$ to be an R net.

In particular we consider the case where $\lambda^{i}_{1/}$ and $\lambda^{j}_{2/}$ are tangent to the u and v parametric curves, respectively. Then[4]

$$
\begin{aligned}
&\lambda^{1}_{1} = 1, && b_{11} = D, \\
&\lambda^{2}_{1} = 0, && b_{12} = b_{21} = D' = 0, \\
&\lambda^{1}_{2} = 0, && b_{22} = D'', \\
&\lambda^{2}_{2} = 1,
\end{aligned}
\tag{3.22}
$$

and it is easily shown that

$$\lambda^{i}_{a/,j} = \begin{Bmatrix} i \\ aj \end{Bmatrix}$$

for any *fixed* i, a, j, which is an interesting interpretation of the Christoffel symbols in this case.

In this case (3.19) reduces to

$$2\frac{\partial}{\partial v}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial u}\left(\begin{Bmatrix} 2 \\ 22 \end{Bmatrix} - \frac{D''}{D}\begin{Bmatrix} 2 \\ 11 \end{Bmatrix}\right), \tag{3.23}$$

the equation obtained by Eisenhart.[5]

The equation similar to (3.19) reduces to

$$2\frac{\partial}{\partial u}\begin{Bmatrix} 1 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial v}\left(\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} - \frac{D}{D''}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix}\right) \tag{3.24}$$

and these two equations constitute the condition that the parametric curves of a surface S form an R net.

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[4] L. P. Eisenhart, *Differential Geometry*, 1909, p. 115.

[5] L. P. Eisenhart, *Transformation of Surfaces*, 1923, p. 106.

ON THE NILPOTENCY OF THE RADICAL OF A RING

RICHARD BRAUER[1]

1. **Introduction.** A few years ago, it was shown by C. Hopkins[2] that the structure theory of noncommutative rings[3] can be based on the assumption of only the minimum condition for left-ideals. Before Hopkins, a maximum condition for ideals had also been used in order to prove that the radical of the ring is nilpotent. Actually this last fact is a special case of the maximum condition, for example, the existence of a maximal nilpotent (two-sided) ideal, and this makes Hopkins' result appear rather surprising.

In this note, I give a short and simple proof for Hopkins' theorem. I also show that it is sufficient to assume only the minimum condition for sets of two-sided nil-ideals (that is, ideals consisting only of nilpotent elements) in order to prove the nilpotency of the radical. The later sections are concerned with the existence of idempotents and primitive left-ideals contained in a given regular left-ideal. Here the assumptions concerning the ring R are those on which Köthe[4] and Deuring[5] based their treatment of noncommutative rings. As was shown by Köthe, these assumptions are equivalent to the validity of the structure theory, so that it is natural to work with them. Once the results of the later sections have been established, there is no difficulty in developing the theory with the usual methods.[6]

2. **Preliminaries.** A ring R is a set of elements for which an addition and a multiplication are defined such that the elements form an abelian group under addition and that the associative law of multiplication and both distributive laws hold. We may also have a set K of operators. Then the product $t\alpha = \alpha t$ of any α in R with any t in K must be defined as an element of R, and the following rules are to hold (α, β in R, t in K)

Received by the editors December 26, 1941.

[1] Guggenheim Fellow.

[2] Charles Hopkins, Duke Mathematical Journal, vol. 4 (1938), p. 664; cf. also J. Levitzki, Compositio Mathematica, vol. 7 (1939), p. 214.

[3] E. Artin, Hamburger Abhandlungen, vol. 5 (1928), p. 251; B. L. van der Waerden, *Moderne Algebra*, vol. 2; M. Deuring, *Algebren*, Ergebnisse der Mathematik, vol. 4, 1935; A. A. Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939.

[4] G. Köthe, Mathematische Zeitschrift, vol. 32 (1930), p. 161.

[5] Loc. cit.

[6] The treatment thus obtained seems to me simpler than Deuring's treatment.

$$(1) \qquad (\alpha+\beta)t = \alpha t + \beta t, \qquad (\alpha\beta)t = \alpha(\beta t) = (\alpha t)\beta.$$

We then say that R is a K-ring. However, for some purposes, these postulates are not suitable, for example, it is easy to see that it is not always possible to imbed a K-ring R in a K-ring R^* which has a 1-element. We may modify the definition of a K-ring R in the following manner: If t lies in K and α lies in R, then αt and $t\alpha$ both are defined as elements of R. For α, β in R, and for t in K, we have

$$(2) \qquad \begin{gathered} (\alpha+\beta)t = \alpha t + \beta t, \qquad t(\alpha+\beta) = t\alpha + t\beta, \\ (\alpha\beta)t = \alpha(\beta t), \qquad (\alpha t)\beta = \alpha(t\beta), \qquad t(\alpha\beta) = t(\alpha\beta). \end{gathered}$$

We admit the possibility that $\alpha t \neq t\alpha$. A K-ring R in this sense can always be imbedded in a K-ring R^* which has a 1-element. It does not mean an essential restriction to assume that K itself is a ring which has a 1-element e such that: (a) $\alpha e = e\alpha = \alpha$ for all α in R. (b) If $\alpha t = 0$ for a fixed t in K and all α in R, then $t = 0$. The same holds, if all $t\alpha = 0$. (c) For the elements of R and for the elements of K, all possible associative and distributive laws hold. (This includes the equations (2).) A left-ideal (abbreviated l-ideal) of the K-ring R is a subset $\mathfrak{a}$ of R which satisfies the following conditions: (1) If α and β lie in $\mathfrak{a}$, then $\alpha \pm \beta$ lies in $\mathfrak{a}$. (2) If α lies in $\mathfrak{a}$, then $\rho\alpha$ and $t\alpha$ lie in $\mathfrak{a}$ for any ρ in R and any t in K. In the case of a right-ideal (r-ideal), (2) has to be replaced by: (2′) If α lies in $\mathfrak{a}$, then $\alpha\rho$ and αt lie in $\mathfrak{a}$ for any ρ in R and any t in K. A set $\mathfrak{a}$ is an ideal, if $\mathfrak{a}$ is both l-ideal and r-ideal.

For the following, it does not make any difference which definition of a K-ring is used.

3. **The radical.** An element ν of the ring R is a *radical-element*, if it belongs to at least one nilpotent ideal. Since every nilpotent l-ideal and every nilpotent r-ideal is contained in a nilpotent ideal,[7] the elements of nilpotent l-ideals and r-ideals are radical-elements. The sum of two nilpotent ideals is a nilpotent ideal;[8] the same holds for any finite number of nilpotent ideals. It follows readily that the set of all radical-elements forms an ideal N, the *radical* of R. It is easy to give examples of rings R whose radical N is not nilpotent. Hence we have to make a further assumption.

ASSUMPTION (A). *If Σ is a nonvacuous set of ideals $\mathfrak{a}$ which consist of nilpotent elements of R, then there exists at least one minimal ideal of Σ.*

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[7] Cf. A. A. Albert, loc. cit., p. 22.

[8] Cf. A. A. Albert, loc. cit., p. 23.

We now prove this theorem.

THEOREM 1. *If the ring R satisfies this assumption* (A), *then its radical N is nilpotent.*

PROOF. (a) Let us first suppose that the ring R even satisfies the assumption (A) when the word "ideal" in it is replaced by the word "left-ideal."[9,10]

We have

$$N \supseteq N^2 \supseteq N^3 \supseteq \cdots . \tag{3}$$

Since all these ideals consist of nilpotent elements of R, there exists a minimal ideal $N^k = T$ of the set (3). If $T = 0$, we are finished. Assume $T \neq 0$. Then

$$T^2 = T. \tag{4}$$

Consider the set Σ of all l-ideals $\mathfrak{a}$ contained in T for which $T\mathfrak{a} \neq 0$. This set is not empty, since it contains $\mathfrak{a} = T$. Let $\mathfrak{a}$ be a minimal l-ideal of Σ. Since $T\mathfrak{a} \neq 0$, there exists an element α in $\mathfrak{a}$ such that $T\alpha \neq 0$. Then $T\alpha \subseteq \mathfrak{a} \subseteq T$ and $T(T\alpha) = T^2\alpha = T\alpha \neq 0$. Hence $T\alpha$ itself belongs to Σ. Since $\mathfrak{a}$ was minimal, we have

$$\mathfrak{a} = T\alpha. \tag{5}$$

In particular, the element α of $\mathfrak{a}$ belongs to $T\alpha$. We can find an element τ of T such that $\alpha = \tau\alpha$. This implies $\alpha = \tau\alpha = \tau^2\alpha = \tau^3\alpha = \cdots$. However, τ as an element of $T = N^k$ is nilpotent. Hence $\tau^l\alpha = 0$ for a suitable l, and we obtain $\alpha = 0$ which contradicts $T\alpha \neq 0$. This proves Theorem 1 under our present assumption.

(b) If we assume that R satisfies the assumption (A) in its original form, we have to replace the set Σ by the set Σ' of all ideals $\mathfrak{a}$ contained in T for which $T\mathfrak{a}T \neq 0$. Again, the ideal T belongs to the set. If $\mathfrak{a}$ is a minimal ideal of Σ', we can find an element α of $\mathfrak{a}$ such that $T\alpha T \neq 0$. Then $T\alpha T$ belongs to Σ', and the minimal property of $\mathfrak{a}$ gives

$$\mathfrak{a} = T\alpha T. \tag{6}$$

Consequently, the element α of $\mathfrak{a}$ belongs to $T\alpha T$. This means that there exist elements $\tau_1, \tau_2, \cdots, \tau_n, \tau_1', \tau_2', \cdots, \tau_n'$ in T such that

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[9] For the proof of the theorem, it is not necessary to deal with this case separately. However, the proof becomes somewhat simpler when we make the stronger assumption. The minimum condition for l-ideals of R, implies this stronger assumption.

[10] Added July 5, 1942: The proof in (a) was found independently by Reinhold Baer.

$$\alpha = \sum_{i=1}^{n} \tau_i \alpha \tau_i'.$$

On replacing α on the right side by $\sum \tau_j \alpha \tau_j'$ and continuing in this manner, we obtain

$$\alpha = \sum_{i} \tau_i \alpha \tau_i' = \sum_{i,j} \tau_i \tau_j \alpha \tau_j' \tau_i' = \sum_{i,j,k} \tau_i \tau_j \tau_k \alpha \tau_k' \tau_j' \tau_i' = \cdots . \tag{7}$$

The radical element τ_i belongs to a nilpotent ideal $\mathfrak{n}_i$. Hence the sum $\mathfrak{q}$ of the n ideals $\mathfrak{n}_i$ is a nilpotent ideal containing all τ_i. If $\mathfrak{q}^r = 0$, then the rth of the sums in (7) vanishes since all products of r factors τ_i $(1 \leq i \leq n)$ will vanish. Hence $\alpha = 0$, which contradicts the condition $T\alpha T \neq 0$. This proves the theorem.

4. **Existence of idempotents.** For the last two sections, we make the following assumptions concerning the ring R:

(I) *The radical N of R is nilpotent.*

(II) *If Σ is a nonvacuous set of l-ideals $\mathfrak{a} \supseteq N$, there exists at least one minimal l-ideal of Σ.*

The condition (A), §3, implies the condition (I) as is shown by Theorem 1. If R satisfies the minimum condition for l-ideals, then certainly (A) and (I) hold, that is, (I) and (II) hold.

We say that an l-ideal is *regular*, if it is not nilpotent. An l-ideal $\mathfrak{a}$ is *primitive*, if $\mathfrak{a}$ is regular while every l-ideal $\mathfrak{b}$ with $\mathfrak{b} \subset \mathfrak{a}$ is nilpotent.

LEMMA 1. *Every regular l-ideal $\mathfrak{m}$ contains an element η with $\eta^2 \equiv \eta$, $\eta \not\equiv 0 \pmod{N}$.*

PROOF.[11] (a) Assume first that $\mathfrak{m} \supset N$. Using the assumption (II), we obtain an l-ideal $\mathfrak{a}$ with $\mathfrak{m} \supseteq \mathfrak{a} \supset N$ such that no l-ideal lies between $\mathfrak{a}$ and N. If $\mathfrak{a}\alpha \subseteq N$ for all α in $\mathfrak{a}$, we have $\mathfrak{a}^2 \subseteq N$ which would imply that $\mathfrak{a}^2$ is nilpotent. But then $\mathfrak{a}$ is nilpotent, that is, $\mathfrak{a} \subseteq N$. Hence for a suitable α in $\mathfrak{a}$, the l-ideal $\mathfrak{a}\alpha$ does not belong to N. Then $N \subset N + \mathfrak{a}\alpha \subseteq \mathfrak{a}$.[12] It follows that

$$\mathfrak{a} = N + \mathfrak{a}\alpha.$$

This implies that α can be written in the form $\alpha = \nu + \eta\alpha$ with ν in N and η in $\mathfrak{a}$. Then $\eta\alpha \equiv \alpha \pmod{N}$, and hence $\eta^2\alpha \equiv \eta\alpha$, $(\eta^2 - \eta)\alpha \equiv 0$

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[11] If the minimum condition for l-ideals is satisfied in R, this proof can be simplified as follows: The l-ideal $\mathfrak{m}$ contains a primitive l-ideal $\mathfrak{a}$. As in the proof, we may choose an α in $\mathfrak{a}$ such that $\mathfrak{a}\alpha$ does not lie in N. Then $\mathfrak{a}\alpha = \mathfrak{a}$. This gives the existence of an η in $\mathfrak{a}$ for which $\eta\alpha = \eta$. As in the proof, we can conclude $\eta^2 \equiv \eta \not\equiv 0 \pmod{N}$.

[12] We use the $+$ sign, even if the sum of the l-ideals is not direct.

(mod N). The elements x of $\mathfrak{a}$ for which $x\alpha\equiv 0$ (mod N) form an l-ideal $\mathfrak{b}$ with $N\subseteq\mathfrak{b}\subseteq\mathfrak{a}$. However, η does not lie in $\mathfrak{b}$, since $\eta\alpha\equiv 0$ (mod N) would imply $\alpha\equiv 0$ (mod N) and $\mathfrak{a}\alpha\subseteq N$. Hence $\mathfrak{b}\neq\mathfrak{a}$, that is, $\mathfrak{b}=N$. The element $\eta^2-\eta$ lies in $\mathfrak{b}$, which gives $\eta^2\equiv\eta$ (mod N). If we had $\eta\equiv 0$ (mod N), then again $\alpha\equiv\eta\alpha\equiv 0$ (mod N), which was impossible. Hence $\eta\not\equiv 0$, $\eta^2\equiv\eta$ (mod N) and η lies in $\mathfrak{a}$.

(b) If $\mathfrak{m}$ does not contain N, set $\mathfrak{m}^*=\mathfrak{m}+N$. Then, as shown in (a), the l-ideal $\mathfrak{m}^*$ contains an element η^* with $\eta^{*2}\equiv\eta^*\not\equiv 0$ (mod N). However, every η^* of $\mathfrak{m}+N$ is congruent to an element η of $\mathfrak{m}$, and this η will satisfy the conditions of Lemma 1.

LEMMA 2. *If r is a given positive integer, we may find a polynomial $f(x)$ with rational integral coefficients such that*

$$f(x)\equiv 0 \pmod{x^{r+1}},\qquad f(x)\equiv 1 \pmod{(1-x)^r}. \tag{8}$$

PROOF. Expand the square bracket on the right side of $1=1^{2r}=[x+(1-x)]^{2r}$ according to the binomial theorem. If $f(x)$ is the sum of the terms containing x at least to the power x^{r+1}, then $f(x)$ satisfies the congruences (8).

THEOREM 2. *Every regular l-ideal $\mathfrak{m}$ contains an idempotent ϵ.*

PROOF. Construct η according to Lemma 1. Then $(\eta-\eta^2)^r=0$ for some r. The element $\epsilon=f(\eta)$ is well defined, as $f(x)$ has no constant term. It follows from (8) that we have an equation $f(x)^2-f(x)=(x-x^2)^r g(x)$ where $g(x)$ is a polynomial with rational integral coefficients such that $g(x)$ has no constant term. If we replace x by η, we obtain $\epsilon^2-\epsilon=0$. If we had $\epsilon=f(\eta)=0$, we could multiply the second congruence (8) by x^{r+1} and replace x by η. This would give $0=\eta^{r+1}$ which contradicts the congruences $\eta\equiv\eta^2\equiv\eta^3\equiv\cdots$, $\eta\not\equiv 0$ (mod N). Hence ϵ is an idempotent belonging to $\mathfrak{a}$.

COROLLARY. *An element ν of R is a radical element, if it is properly nilpotent, that is, if $\alpha\nu$ is nilpotent for every α in R.*

PROOF. If ν belongs to the nilpotent ideal $\mathfrak{n}$, then $R\nu\subseteq\mathfrak{n}\subseteq N$, and all $\alpha\nu$ are nilpotent. If ν is properly nilpotent, then $\mathfrak{a}=R\nu$ cannot contain an idempotent. Hence $R\nu\subseteq N$. The set of all ν for which $R\nu\subseteq N$, forms an ideal $\mathfrak{n}$ which again cannot contain an idempotent. Hence $\mathfrak{n}\subseteq N$; in particular, ν belongs to N.

5. **Primitive l-ideals contained in regular l-ideals.**[13] We prove the following theorems.

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[13] If the minimum condition for l-ideals is assumed, Theorem 4 becomes trivial.

THEOREM 3. *Let $\mathfrak{a}$ be an l-ideal with $\mathfrak{a}\supseteq N$, such that no l-ideal lies between $\mathfrak{a}$ and N. If ϵ is an idempotent belonging to $\mathfrak{a}$, then $R\epsilon$ is a primitive l-ideal contained in $\mathfrak{a}$.*

PROOF. Suppose $\mathfrak{b}$ is a regular l-ideal with $\mathfrak{b}\subset R\epsilon$. Then $\mathfrak{b}$ contains an idempotent ϵ' and we have

$$R\epsilon' \subset R\epsilon. \tag{9}$$

Since ϵ' belongs to $R\epsilon$, we have $\epsilon'\epsilon=\epsilon'$. Set $\xi=\epsilon-\epsilon\epsilon'$. Then $\xi\epsilon=\epsilon^2-\epsilon\epsilon'\epsilon=\epsilon-\epsilon\epsilon'=\xi$, $\xi\epsilon'=0$. Hence $\xi^2=\xi\epsilon-\xi\epsilon\epsilon'=\xi$.

If $\xi\neq0$, it is an idempotent contained in $R\epsilon$. Then $N\subseteq R\xi+N\subseteq\mathfrak{a}$. Since no l-ideal lies between $\mathfrak{a}$ and N and $R\xi$ contains $\xi^2=\xi\not\equiv0$ (mod N), we have $R\xi+N=\mathfrak{a}$. This implies $\mathfrak{a}\epsilon'=R\xi\epsilon'+N\epsilon'=N\epsilon'\subseteq N$. However, $\mathfrak{a}\epsilon'$ contains $\epsilon'^2=\epsilon'$ which does not lie in N; we have a contradiction.

Hence $\xi=0$, that is, $\epsilon\epsilon'=\epsilon$. Then $R\epsilon'$ contains $\epsilon\epsilon'=\epsilon$, and $R\epsilon'\supseteq R\epsilon$. This contradicts (9), and the theorem is proved.

THEOREM 4. *Every regular l-ideal $\mathfrak{m}$ contains a primitive l-ideal.*

PROOF. Let $\mathfrak{a}$ be an l-ideal such that $N\subset\mathfrak{a}\subseteq\mathfrak{m}+N$ and that no l-ideal lies between N and $\mathfrak{a}$. Then $\mathfrak{a}$ contains an idempotent ϵ_0, and $\epsilon_0=\eta+\nu$ with η in $\mathfrak{m}$ and ν in N. Hence $\eta^2\equiv\epsilon_0^2=\epsilon_0\equiv\eta$ (mod N), $\eta\equiv\epsilon_0\not\equiv0$ (mod N). Using Lemma 2 as in the proof of Theorem 2, we obtain an idempotent $\epsilon=f(\eta)$ which belongs to $\mathfrak{m}$. Then $R\epsilon\subseteq\mathfrak{m}$. Since $\eta=\epsilon_0-\nu$ lies in $\mathfrak{a}+N=\mathfrak{a}$, the element $\epsilon=f(\eta)$ lies in $\mathfrak{a}$. Theorem 3 shows that $R\epsilon$ is primitive.

We can now prove this theorem.

THEOREM 5. *Every l-ideal $\mathfrak{m}$ is a direct sum of primitive l-ideals $R\epsilon_i$ and a nilpotent l-ideal $\mathfrak{n}$:*

$$\mathfrak{m} = R\epsilon_1 + R\epsilon_2 + \cdots + R\epsilon_n + \mathfrak{n}.\tag{10}$$[14]

Here the ϵ_i can be taken as idempotents such that

$$\epsilon_i\epsilon_j = 0 \quad \textit{for} \quad i\neq j, \qquad \epsilon_i^2=\epsilon_i, \qquad \mathfrak{n}\epsilon_i = 0.\tag{11}$$

PROOF. Because of the assumption (II), §4, we may assume that the theorem is correct for all regular r-ideals $\mathfrak{m}'$ with $\mathfrak{m}'+N\subset\mathfrak{m}+N$. Let $R\epsilon_n$ be a primitive l-ideal contained in $\mathfrak{m}$, ϵ_n an idempotent, and apply the Peirce decomposition. Then $\mathfrak{m}$ is a direct sum

$$\mathfrak{m} = \mathfrak{m}' + R\epsilon_n \tag{12}$$

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[14] If $\mathfrak{m}$ is nilpotent, the terms $R\epsilon_i$ are missing.

where $\mathfrak{m}'$ consists of those elements μ of $\mathfrak{m}$ for which $\mu\epsilon_n = 0$. This implies $\mathfrak{m}'+N \subset \mathfrak{m}+N$, as $(\mathfrak{m}'+N)\epsilon_n = N\epsilon_n \subseteq N$ while $(\mathfrak{m}+N)\epsilon_n$ contains ϵ_n. Then Theorem 5 holds for $\mathfrak{m}'$. If $\mathfrak{m}' = R\epsilon_1 + \cdots + R\epsilon_{n-1} + \mathfrak{n}$ is the corresponding representation, then (12) gives the representation (10) of $\mathfrak{m}$. However, we obtain the formula (11), $\epsilon_i\epsilon_j = 0$, only for $i \neq n$. We must replace ϵ_i $(i = 1, 2, \cdots, n-1)$ by $\epsilon_i - \epsilon_n\epsilon_i$ in order to have $\epsilon_n\epsilon_i = 0$. As is easily seen, these new elements satisfy all the conditions.

THEOREM 6. *If Theorem* 5 *is applied to* $R = \mathfrak{m}$, *then* $\zeta = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ *is a* 1-*element* (mod N), *that is,* $\alpha\zeta \equiv \zeta\alpha \equiv \alpha$ (mod N) *for all* α *in* R. *If* R *has a* 1-*element* 1, *then* $\zeta = 1$, *and in the representation* (10) *of* $\mathfrak{m} = R$ *no term* $\mathfrak{n}$ *appears.*

PROOF. If we represent an element μ of R in accordance with (10) for $\mathfrak{m} = R$ we obtain easily from (11) that $\mu\zeta \equiv \mu$ (mod N). For any α in R, we then have $\mu(\zeta\alpha - \alpha) \equiv 0$ (mod N) for every μ in R. Consequently, $\zeta\alpha - \alpha$ is properly nilpotent, that is, $\zeta\alpha \equiv \alpha$ (mod N). If R contains a 1-element 1, then $\mu(1-\zeta) \equiv \mu - \mu \equiv 0$ (mod N) which proves that $1-\zeta$ is properly nilpotent. Since $(1-\zeta)^2 = 1 - \zeta - \zeta + \zeta^2 = 1 - \zeta$, the element $1-\zeta$ is either 0 or an idempotent. The latter case is excluded, hence $\zeta = 1$. Finally, $\mu = \mu 1 = \mu\zeta = \sum \mu\epsilon_i$ which shows that no term $\mathfrak{n}$ appears in this case.

Theorems 5 and 6 form the basis for the structure theory of rings, and for the theory of representations of rings.

UNIVERSITY OF TORONTO AND
INSTITUTE FOR ADVANCED STUDY

ON AN ELEMENTARY ANALOGUE OF THE RIEMANN-MANGOLDT FORMULA

AUREL WINTNER

Ramanujan's unsuccessful approach to the Prime Number Theorem, published only recently, is based on the power series

$$\sum_{n=1}^{\infty} \Lambda(n)x^n = \sum_{p} \log p \sum_{k=1}^{\infty} x^{p^k} \quad \text{and} \quad \sum_{n=1}^{\infty} p^n x^{p^n}, \qquad 0 < x < 1,$$

where p denotes a prime and p is 2 in the last series. In his discussion of Ramanujan's failure in case of the latter series, a series impracticable as $x \to 1$, Hardy gives for the function represented by the series another expansion, one exhibiting the critical "wobbles," as follows:[1]

$$\sum_{n=1}^{\infty} p^n \exp(-p^n s) = \{\quad\}/\log p, \qquad p = 2, \tag{1}$$

where $\{\quad\}$ is the expression

$$\begin{aligned} \{\ \} = \frac{1}{s} - \frac{1}{s} \sum_{k=-\infty}^{\infty}{}' \Gamma\left(\frac{1+2\pi k i}{\log p}\right) s^{-2\pi k i/\log p} \\ - \log p \sum_{n=0}^{\infty} \frac{(-1)^n p^{n+1}}{p^{n+1}-1} \frac{s^n}{n!}; \end{aligned} \tag{2}$$

it being understood that $\sum_{k=-\infty}^{\infty\prime} = \sum_{k=-\infty}^{-1} + \sum_{k=1}^{\infty}$ and $-\log x = s > 0$.

It will be seen later on that Hardy's result (2) contains two errors. However, the purpose of this note is not calculation of the corrections necessary, which are of a trivial nature, but the presentation of a short approach which seems to be of methodical and historical interest.

First, (2) is of the same type as the "explicit formula" of Riemann-Mangoldt[2] (the two sums representing the contributions of the "nontrivial" and "trivial" zeros, respectively). Correspondingly, Hardy's

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Received by the editors January 12, 1942.

[1] G. H. Hardy, *Ramanujan*, Cambridge, 1940, chap. II, formulae (2.9.1) and (2.11.2).

[2] Cf., for example, A. E. Ingham, *The Distribution of Primes*, Cambridge Tracts, no. 30 (1932), chap. IV.

Relevant for the comparison is only the "Abelian" form (instead of the deeper "Cesàro" form) of the Riemann-Mangoldt formula; cf. G. H. Hardy and J. E. Littlewood, Acta Mathematica, vol. 41 (1918), pp. 119–196.

proof, omitted loc. cit.,[1] is based on a contour integration.[3] However, since the "non-trivial" zeros occurring in the part $\sum'$ of (2) are equidistant, the "wobbly" terms now become *purely* periodic, indicating the possibility of a trivial approach.[4]

It turns out that such an approach has been found already by Dedekind. What makes it of particular interest is the fact that Dedekind's purpose was precisely the explicit illustration of the illegitimacy of the argument to occur decades later in Ramanujan's attempted proof of the Prime Number Theorem. Inasmuch as Dedekind's result, published in a certainly conspicuous work,[5] has never[6] been quoted, it is possible that his indications were found to be hard to follow. All of this seems to warrant a detailed and straightforward deduction of the explicit formula.

To this end, put

$$f(s) = \sum_{n=-\infty}^{\infty} c^n \exp(-c^n s), \tag{3}$$

where c is any constant satisfying $c>1$. Then (3) obviously represents a regular function[7] for $s>0$. It is also seen that (3) remains unchanged if n is replaced by $n+1$ in the exponentials; so that[8] $sf(s)$ remains unaltered if s is replaced by cs. It follows that, if $\log c>0$ and $s>0$,

$$f(s)\log c = s^{-1}\sum_{k=-\infty}^{\infty}\Gamma(1+Cki)s^{-Cki}, \qquad \text{where } C = 2\pi/\log c \tag{4}$$

(it being understood that $s^{\alpha i}$ is meant to be defined in terms of the principal branch of $\log s$).

In fact, since $sf(s)=csf(cs)$, the function $e^u f(e^u)$ of the real variable $u=\log s$ has the period $\log c$. Since this function, being regular, is continuous and of bounded variation, the Fourier expansion

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[3] G. H. Hardy, *Lectures on the Mathematical Work of Ramanujan*, The Institute for Advanced Study, 1936, p. 14.

[4] Cf. Ingham's comments on $\sum' \sin 2\pi nx/n$, loc. cit., p. 81.

[5] Supplement IX of Dirichlet's *Vorlesungen ueber Zahlentheorie* (pp. 385–386 of the 3rd edition (1879)).

The non-differentiable function of Weierstrass depends on a limiting case of Dedekind's series. In this connection, cf. G. H. Hardy and J. E. Littlewood, Proceedings of the National Academy of Sciences, vol. 2 (1916), pp. 583–586.

[6] Cf. Hardy's paper of 1907, quoted loc. cit., Footnote 1, p. 47, §2.11.

[7] It will always be assumed that $s>0$, the transition from the half-line $s>0$ to the half-plane $R(s)>0$ being trivial.

[8] This remark of Dedekind corresponds to the more elaborate argument (loc. cit., Footnote 1, p. 39) which Hardy's paper of 1907 attributes to Wedderburn (cf. loc. cit., Footnote 6).

$$e^u f(e^u) = \sum_{k=-\infty}^{\infty} \alpha_k e^{-Ckiu},$$

where

$$\alpha_k = \frac{1}{\log c}\int_0^{\log c} e^{Ckiu} e^u f(e^u)\,du, \qquad C = \frac{2\pi}{\log c},$$

is valid for every $u = \log s$. Hence, in order to prove (4), it is sufficient to verify that the last integral is $\Gamma(1+Cki)$. But the uniformity of the convergence of the series (3) assures that this integral is

$$\sum_{n=-\infty}^{\infty} c^n \int_0^{\log c} e^{Ckiu} e^u \exp(-c^n e^u)\,du$$

or, since $\int_0^a F(u)du = \int_{na}^{(n+1)a} F(u-na)du$ and $C = 2\pi/\log c$,

$$\sum_{n=-\infty}^{\infty} c^n \int_{n\log c}^{(n+1)\log c} e^{Ckiu} e^{u-n\log c} \exp(-c^n e^{u-n\log c})\,du,$$

that is,

$$\sum_{n=-\infty}^{\infty} c^n \int_{n\log c}^{(n+1)\log c} e^{Ckiu} e^u c^{-n} \exp(-c^n e^u c^{-n})\,du$$

or simply

$$\sum_{n=-\infty}^{\infty} \int_{n\log c}^{(n+1)\log c} e^{(1+Cki)u} \exp(-e^u)\,du = \int_{-\infty}^{\infty} e^{(1+Cki)u} \exp(-e^u)\,du.$$

Since the last integral is reduced by $u = \log s$ to the integral defining $\Gamma(1+z)$ for $z = Cki$, the proof of the Dedekind expansion (4) is complete.

While the (real) series (3) converges only[9] for $s > 0$, its "non-principal part," that is, the series

$$g(s) = \sum_{n=1}^{\infty} c^{-n} \exp(-c^{-n}s), \tag{5}$$

converges for every s, since $c > 1$. Furthermore, the function (5) has the Taylor series

$$g(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{c^{n+1}-1}\,\frac{s^n}{n!} \tag{6}$$

---

[9] Incidentally, the function (3) of the complex variable s, where $R(s) > 0$, has the line $R(s) = 0$ as natural boundary. In fact, (3) and (5) show that $f(s) - g(s)$ is a lacunary Dirichlet series for every $c > 1$. On the other hand, (5) or (6) implies that $g(s)$ is an entire function.

for every s. In fact, substitution of the geometric progression

$$(c^{n+1}-1)^{-1} = \sum_{m=1}^{\infty} c^{-(n+1)m}, \qquad c > 1,$$

transforms (6) into an iterated series which, being absolutely convergent, can be rearranged into

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^n c^{-(n+1)m} s^n/n!.$$

But this may be written in the form

$$\sum_{m=1}^{\infty} c^{-m} \sum_{n=0}^{\infty}(-c^{-m}s)^n/n!,$$

which is identical with (5).

According to (3) and (5),

$$\sum_{n=0}^{\infty} c^n \exp(-c^n s) = f(s) - g(s). \tag{7}$$

If (4) and (6) are substituted into (7), the product of the Dirichlet series (7) into s appears as decomposed into a Fourier series in $\log s$ and an entire power series in s.

If the power series of $\exp(-s)$ is added to (5) and (6), then (7) appears in the form

$$\sum_{n=1}^{\infty} c^n \exp(-c^n s) = f(s) - \sum_{n=0}^{\infty} \frac{(-1)^n c^{n+1}}{c^{n+1}-1}\,\frac{s^n}{n!}. \tag{8}$$

Hence, (4) implies Hardy's explicit formula (2) for (1), if $-\sum'$ is corrected to $+\sum'$ and

$$\Gamma\left(\frac{1+2\pi ki}{\log p}\right) \quad \text{to} \quad \Gamma\left(1+\frac{2\pi ki}{\log p}\right).$$

THE JOHNS HOPKINS UNIVERSITY

A NOTE ON THE LOWER SEMI-CONTINUITY OF DOUBLE INTEGRALS IN THE PARAMETRIC FORM

WILLIAM SCOTT

Consider a double integral problem of the calculus of variations in parametric form, that is, a problem involving an integral of the form

$$\int\int_B f(x, X)du$$

where x stands for the three coordinate functions, $x^1(u^1, u^2)$, $x^2(u^1, u^2)$, $x^3(u^1, u^2)$, and X stands for their three Jacobians.

Various theorems of the following general type have been proved by McShane,[1] Caccioppoli,[2] Cimmino,[3] and Radó:[4]

If (i) the triples of functions $x^i_n(u^1, u^2)$, $i=1, 2$; $n=0, 1, 2, \cdots$, satisfy certain conditions; (ii) the surfaces defined by $x^i=x^i_n(u^1, u^2)$, $i=1, 2$; $n=1, 2, \cdots$, converge in some prescribed sense to the surface defined by $x^i_0=x^i_0(u^1, u^2)$, $i=1, 2$; (iii) $f(x, X)$ satisfies certain conditions; then

$$\liminf_n \int\int_{B_n} f(x_n(u), X_n(u))du \geqq \int\int_{B_0} f(x_0(u), X_0(u))du.$$

The conditions on $f(x, X)$ usually contain a condition on its Weierstrass E-function

$$E(x, X, \overline{X}) = f(x, \overline{X}) - \overline{X}^\alpha f_\alpha(x, X),$$

the summation convention being used for convenience in writing.

McShane[1] requires either $f>0$, $E\geqq 0$, or $f\geqq 0$, $E>0$. Caccioppoli[2] makes the same requirements. Cimmino[3] requires only $f\geqq 0$, $E\geqq 0$, but works with smooth surfaces.

Presented to the Society, April 11, 1941 under the title *A lemma on the Weierstrass E-function*; received by the editors January 17, 1942.

[1] McShane, *On the semi-continuity of double integrals in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 460–484.

[2] Caccioppoli, *Gli integrali doppi di forma parametrica nel calcolo delle variazioni*, Arri del Reale Istituto Veneto di Scienze, Lettere ed Arti, Anno Accademico, 1933-1934, vol. 93, Part 2.

[3] Cimmino, *Sulle condizione necessarie e sufficienti per la semi-continuita degle integrali doppi di forma parametrica*, Annali di Mathematica Pura ed Applicata, vol. 15 (1936), pp. 159–173. The author wishes to thank Professor McShane for calling his attention to this paper.

[4] Radó, *On the semi-continuity of double integrals in the parametric form*, Transactions of this Society, vol. 51 (1942), pp. 336–361. The author wishes to thank Professor Radó for access to the manuscript of this paper before it was published.

Radó[4] proves the semi-continuity theorems under the same restrictions on $f(x, X)$ as McShane and Caccioppoli, but for a more general class of surfaces; namely, the class of surfaces that admit of a parametric representation for which the Lebesgue area is given by the usual integral formula.

The question arises as to whether we cannot prove the semi-continuity theorem for the class of surfaces which Radó considers with the requirements that $f \geqq 0$, $E \geqq 0$. We shall first prove a lemma on the Weierstrass E-function, then prove a stronger form of Lemma 1.5 in Radó.[4] The remainder of the proof as given by Radó will then prove the semi-continuity theorem with $f \geqq 0$, $E \geqq 0$ if the revised form of Lemma 1.5 is used.

We shall use the notations and definitions in Radó[4] throughout the remainder of the paper.

LEMMA 1. *If f is admissible and x_0, X_0 are such that*

(1) $f(x_0, X_0) > 0$,

(2) $f(x_0, X) \geqq 0$,

(3) $E(x_0, X_0, X) \geqq 0$,

for every X, then there exists a $\delta > 0$ such that if $\overline{X}^\alpha f_\alpha(x_0, X_0) > 0$ then $f(x_0, \overline{X}) > \delta \|\overline{X}\|$.

PROOF. Since

$$X_0^\alpha f_\alpha(x_0, X_0) = f(x_0, X_0) > 0$$

at least one of $f_1(x_0, X_0), f_2(x_0, X_0), f_3(x_0, X_0)$ is different from 0. Hence the set of points $\overline{X}$ satisfying $\overline{X}^\alpha f_\alpha(x_0, X_0) > 0$, $\|\overline{X}\| = 1$, is an open hemisphere H^0. Since $E(x_0, X_0, \overline{X}) \geqq 0$ we have $f(x_0, \overline{X}) > 0$ for every $\overline{X} \in H^0$. If we deny the lemma, then there exists a sequence $\{\overline{X}_n\}$, $\overline{X}_n \in H^0$ such that $f(x_0, \overline{X}_n) \to 0$. Hence there exists a point $\overline{X}_0 \in H - H^0$ such that $f(x_0, \overline{X}_0) = 0$.

Let $H - H^0 = C$, and let the plane containing C be P. Let $d(\overline{X}, P)$ denote the distance of $\overline{X}$ from P. Then if $\overline{X} \in H$ we have

$$(*) \qquad \overline{X}^\alpha f_\alpha(x_0, X_0) = d(\overline{X}, P)\left(\sum_{i=1}^{3} f_i^2(x_0, X_0)\right)^{1/2}.$$

Let C' be the great circle with center at $X = 0$, unit radius, passing through $\overline{X}_0$, and perpendicular to C. Let the direction cosines of the tangent to C' at $\overline{X}_0$ toward H^0 be $\cos \phi_1$, $\cos \phi_2$, $\cos \phi_3$.

Let $\partial f(x_0, \overline{X}_0)/\partial \nu$ be the directional derivative of f along this tangent. Then we have

$$\frac{\partial f(x_0, \overline{X}_0)}{\partial \nu} = f_\alpha(x_0, \overline{X}_0) \cos \phi_\alpha.$$

On the one hand, we have from (*)

$$\frac{\partial f(x_0, \bar{X}_0)}{\partial \nu} \geqq \left(\sum_{i=1}^{3} f_i^2(x_0, X_0) \right)^{1/2} > 0,$$

and on the other hand, since $f(x_0, \bar{X}_0) = 0$, $f(x_0, X) \geqq 0$ we have

$$\frac{\partial f(x_0, \bar{X}_0)}{\partial \nu} = 0.$$

This is a contradiction. Hence the lemma is true.

LEMMA 2. *Let there be given an admissible* f *and a set of six constants* $(x_0^1, x_0^2, x_0^3, X_0^1, X_0^2, X_0^3) = (x_0, X_0)$ *such that*

(a) $f(x_0, X_0) > 0$,
(b) $f(x_0, X) \geqq 0$,
(c) $E(x_0, X_0, X) \geqq 0$,

for all X. *Then there exist two positive constants* δ_1, δ_2 *such that the following holds. If* η, $\bar{x}^1$, $\bar{x}^2$, $\bar{x}^3$, $\bar{X}^1$, $\bar{X}^2$, $\bar{X}^3$ *satisfy the conditions*

(α) $0 < \eta < \delta_1$,
(β) $\bar{X}^\alpha f_\alpha(x_0, X_0) > 0$,
(γ) $|f(\bar{x}, \bar{X}) - f(x_0, \bar{X})| \leqq \eta \|\bar{X}\|$,

then

$$f(\bar{x}, \bar{X}) - f(x_0, \bar{X}) + E(x_0, X_0, \bar{X}) \geqq -\eta\delta_2 \bar{X}^\alpha f_\alpha(x_0, X_0).$$

PROOF. The conditions of Lemma 1 are satisfied. Hence there exists a positive constant δ as described there. Let $\delta_1 = \delta/2$, $\delta_2 = 2/\delta$.

Case 1. $\bar{X}^\alpha f_\alpha(x_0, X_0) \geqq (\delta/2)\|\bar{X}\|$. Then

$$\begin{aligned} f(\bar{x}, \bar{X}) - f(x_0, \bar{X}) + E(x_0, X_0, \bar{X}) &\geqq -\eta\|\bar{X}\| \\ &\geqq -\eta\delta_2 \bar{X}^\alpha f_\alpha(x_0, X_0). \end{aligned}$$

Case 2. $\bar{X}^\alpha f_\alpha(x_0, X_0) < (\delta/2)\|\bar{X}\|$. Then

$$\begin{aligned} f(\bar{x}, \bar{X}) - f(x_0, \bar{X}) &+ E(x_0, X_0, \bar{X}) \\ &= f(\bar{x}, \bar{X}) - \bar{X}^\alpha f_\alpha(x_0, X_0) \\ &\geqq f(x_0, \bar{X}) - \eta\|\bar{X}\| - \bar{X}^\alpha f_\alpha(x_0, X_0) \\ &\geqq 0 \\ &\geqq -\eta\delta_2 \bar{X}^\alpha f_\alpha(x_0, X_0). \end{aligned}$$

Therefore the lemma is true.

Let K_2 be the class of oriented surfaces whose Lebesgue area is given by the usual integral formula. Let ${}_oS_0$, ${}_oS_n$ be oriented surfaces in the class K_2 such that:

(I) ${}_oS_n \rightarrow {}_oS_0$.

(II) If $T_0: x_0^i = x_0^i(u^1, u^2)$, $(u^1, u^2) \in B_0$, $i = 1, 2, 3$ is a representation of ${}_oS_0$ then there exists, in x-space, a closed bounded set A such that $\sum(T_0, B_0) \subset A^0$, and $f(x, X) \geqq 0$ for $x \in A$ and for every vector X.

(III) For a. e. point $u \in B_0^0$ such that $X_0(u)$ exists and is different from 0 we have $E(x_0(u), X_0(u), \bar{X}) \geqq 0$ for every vector $\bar{X} \neq 0$.

As we mentioned above the proof given by Radó will now lead to the following result.

THEOREM. *If f is admissible and if conditions* (I), (II), *and* (III) are *satisfied then*

$$\liminf_n \iint_{B_n} f(x_n(u), X_n(u))du \geqq \iint_{B_0} f(x_0(u), X_0(u))du.$$

This theorem is final in a definite sense. We cannot enlarge the class of surfaces if we use ordinary Jacobians. Counterexamples have been given which show the necessity of the conditions $E \geqq 0$, $f \geqq 0$. Moreover if f is only assumed to be continuous, convex, and non-negative, then we do not necessarily have lower semi-continuity. For, let

$$f(x, X) = \left| X^3 + g(x^1, x^3)X^1 \right|$$

where

$$g(x^1, x^3) = \frac{\cos 16x^1}{2} \qquad \text{if } x^3 > 5/8,$$

$$g(x^1, x^3) = \frac{\cos n^4x^1}{n} \qquad \text{if } 1/n - 1/n^3 \leqq x^3 \leqq 1/n + 1/n^3,\ n \geqq 2,$$

$$\begin{aligned} g(x^1, x^3) = {} & \frac{\cos (n+1)^4x^1}{n+1} \\ & + \frac{x^3 - (1/(n+1) + 1/(n+1)^3)}{(1/n - 1/n^3) - (1/(n+1) + 1/(n+1)^3)} \\ & \cdot \left[\frac{\cos n^4x^1}{n} - \frac{\cos (n+1)^4x^1}{n+1}\right] \\ & \text{if } 1/(n+1) + 1/(n+1)^3 < x^3 < 1/n - 1/n^3, \end{aligned}$$

$$g(x^1, x^3) = 0 \qquad \text{if } x^3 \leqq 0.$$

Then f satisfies all the conditions stated. Further let

$$x_0^1 = u^1, \qquad x_n^1 = u^1, \qquad 0 \leqq u^1 \leqq \pi,$$

$$\overset{2}{x}_0 = \overset{2}{u}, \qquad \overset{2}{x}_n = \overset{2}{u}, \qquad\qquad 0 \leqq \overset{2}{u} \leqq 1,$$

$$\overset{3}{x}_0 = 0, \qquad \overset{3}{x}_n = \frac{1}{n} + \frac{\sin n^4 u^1}{n^3}.$$

Then we have

$$\begin{aligned}\liminf_n \iint_{B_n} f(x_n, X_n)du &= \liminf_n \int_0^\pi \int_0^1 \left| 1 - \cos^2 n^4 u^1 \right| du^2 du^1 \\ &= \frac{\pi}{2} < \pi = \int_0^\pi \int_0^1 du^2 du^1 \\ &= \iint_{B_0} f(x_0, X_0)du.\end{aligned}$$

THE OHIO STATE UNIVERSITY

A NON-INVOLUTORIAL SPACE TRANSFORMATION ASSOCIATED WITH A $Q_{1,n}$ CONGRUENCE

M. L. VEST

1. **Introduction.** The involutorial transformation associated with the congruence of lines meeting a curve of order m and an $(m-1)$-fold secant has been studied by DePaolis,[1] and Vogt[2] has studied the non-involutorial transformations for a linear congruence and bundle of lines. Cunningham[3] has recently studied some non-involutorial transformations associated with a $Q_{1,2}$ congruence. In the present paper a non-involutorial transformation associated with the congruence of lines on a plane curve of order n having an $(n-1)$-point and a secant through that point is considered. The bundle of lines through the multiple point is not considered as belonging to the congruence. The tangents to the curve at the point are considered to be distinct.

Received by the editors November 26, 1941.

[1] DePaolis, *Alcuni particolari trasformazioni involutori dello spazio*, Rendiconti dell'Academia dei Lincei, Rome, (4), vol. 1 (1885), pp. 735–742, 745–758.

[2] Vogt, *Zentrale und windschiefe Raum-Verwandtschaften*, Jahresbericht der Schlesischen Gesellschaft für Vaterlandische Kultur, class 84, 1906, pp. 8–16.

[3] Cunningham, *Non-involutorial space transformations associated with a $Q_{1,2}$ congruence*, this Bulletin, vol. 47 (1941), pp. 309–312.

Given the plane n-ic r, a line s meeting r at an $(n-1)$-point A, and two projective pencils of surfaces $|F_m|: s^{m-1}g_{2m-1}$ and $|F'_{m'}|: s^{m'-1}g_{2m'-1}$. Through a generic point $P(y)$ there passes a single F of $|F|$. The unique line t through $P(y)$, s, r meets the associated F' of $|F'|$ in one residual point $P'(x)$ the image of $P(y)$ under the transformation thus defined. The residual base curves of $|F|$ and $|F'|$, other than s, have been denoted by g and g', respectively. Through a point $O_{g'}$ on g' there is a unique line t' of the congruence, this line lying upon one surface of $|F'|$. The associated surface of $|F|$ meets t' in a point $\bar{P}$ which generates a curve $\bar{g}$. Similarly, beginning with a point O_g on g, a point $\bar{P}'$ generating a curve $\bar{g}'$ is found. It will be shown that r, s, g, g', $\bar{g}$, $\bar{g}'$ are fundamental curves of the transformation, and that the point A is a fundamental point of the second kind.

2. **Equations of the transformation.** Let us take the equations of r and s, respectively, as

$$x_3[cx_1x_2] - [dx_1x_2] = 0, \qquad x_4 = 0, \qquad x_1 = x_2 = 0 \tag{1}$$

where

$$[cx_1x_2] = \sum_{i=0}^{n-1} c_{i,n-i-1}x_1^i x_2^{n-i-1}, \quad [dx_1x_2] = \sum_{j=0}^{n} d_{j,n-j}x_1^j x_2^{n-j}, \tag{2}$$

and the pencils of surfaces as

$$|F_m| \equiv U - uV = 0, \qquad |F'_{m'}| \equiv U' - uV' = 0 \tag{3}$$

where

$$U = (ax)\{ex_1x_2\} - (\alpha x)\{fx_1x_2\}, \qquad V = (bx)\{gx_1x_2\} - (\beta x)\{hx_1x_2\},$$
$$U' = (a'x)\{e'x_1x_2\} - (\alpha' x)\{f'x_1x_2\}, \quad V' = (b'x)\{g'x_1x_2\} - (\beta' x)\{k'x_1x_2\};$$

$$\{ex_1x_2\} = \sum_{p=0}^{m-1} e_{p,m-p-1}x_1^p x_2^{m-p-1}, \quad \{e'x_1x_2\} = \sum_{p=0}^{m'-1} e'_{p,m'-p-1}x_1^p x_2^{m'-p-1} \tag{4}$$

$$(ax) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4,$$

and so on.

Through a generic point $P(y)$ there is an F of $|F|$ with parameter $u = U(y)/V(y)$, and to this corresponds the F' of $|F'|$ having equation

$$U'(x)V - V'(x)U = 0. \tag{5}$$

The unique transversal t through P, r, s meets (5) in the point P' having coordinates

$$(6) \qquad \overset{-1}{T}_{m+m'+n}: \left| \begin{aligned} \sigma x_1 &= Ry_1 + Ky_1[cy_1y_2] = y_1S, \\ \sigma x_2 &= Ry_2 + Ky_2[cy_1y_2] = y_2S, \\ \sigma x_3 &= Ry_3 + K[dy_1y_2], \\ \sigma x_4 &= Ry_4, \end{aligned} \right.$$

where

$$(7) \qquad \begin{aligned} K_{m+m'} &= U'V - V'U, \qquad R_{m+m'+n-1} = UW' - VZ', \\ W'_{m'+n-1} &= (b'z)\{g'y_1y_2\} - (\beta'z)\{h'y_1y_2\}, \\ Z'_{m'+n-1} &= (a'z)\{e'y_1y_2\} - (\alpha'z)\{f'y_1y_2\}, \\ (a'z) &= (a_1'y_1 - a_2'y_2)[cy_1y_2] + a_3'[dy_1y_2], \end{aligned}$$

and so on. Equations (6) are those of the inverse transformation.

In a similar manner the equations of the direct transformation are found to be

$$(8) \qquad T_{m+m'+n}: \left| \begin{aligned} \tau y_1 &= R'x_1 + K'x_1[cx_1x_2] = x_1S', \\ \tau y_2 &= R'x_2 + K'x_2[cx_1x_2] = x_2S', \\ \tau y_3 &= R'x_3 + K'[dx_1x_2], \\ \tau y_4 &= R'x_4, \end{aligned} \right.$$

where

$$(9) \qquad \begin{aligned} K'_{m+m'} &= UV' - U'V = -K, \quad R'_{m+m'+n-1} = U'W - V'Z, \\ W_{m+n-1} &= (bz)\{gx_1x_2\} - (\beta z)\{hx_1x_2\}, \\ Z_{m+n-1} &= (az)\{ex_1x_2\} - (\alpha z)\{fx_1x_2\}. \end{aligned}$$

3. **Images of fundamental curves and elements.** The transformations T^{-1} and T applied to an F' and F of $|F'|$ and $|F|$, respectively, give $U' \sim (T^{-1})US^{m'-1}G$, $U \sim (T)U'S'^{m-1}G'$ where

$$(10) \qquad \begin{aligned} G_{2m'+n-1} &= W'U' - V'Z', \\ G'_{2m+n-1} &= WU - VZ, \\ S_{m+m'+n-1} &= UN' - VM', \\ S'_{m+m'+n-1} &= U'N - V'M, \\ M'_{m'+n-1} &= (a'w)\{e'y_1y_2\} - (\alpha'w)\{f'y_1y_2\}, \\ M_{m+n-1} &= (aw)\{ex_1x_2\} - (\alpha w)\{fx_1x_2\}, \\ N'_{m'+n-1} &= (b'w)\{g'y_1y_2\} - (\beta'w)\{h'y_1y_2\}, \\ N_{m+n-1} &= (bw)\{gx_1x_2\} - (\beta w)\{hx_1x_2\}, \\ (a'w) &= a_3'[dy_1y_2] - (a_3'y_3 + a_4'y_4)[cy_1y_2]. \end{aligned}$$

Here U and U' are the corresponding surfaces of $|F|$ and $|F'|$ and $g'\sim(T^{-1})G$, $s'\sim(T^{-1})S$, $g\sim(T)G'$, $s\sim(T)S'$.

Similarly

$$
\begin{aligned}
&K'\sim(T^{-1})\, K'S^{m+m'-2}GG', && K\sim(T)\, KS'^{m+m'-2}GG',\\
&K'\sim(T)\, K'S'^{m+m'-2}GG', && K\sim(T^{-1})\, KS^{m+m'-2}GG',\\
&G'\sim(T^{-1})\, RS^{2m+n-2}G', && G\sim(T)\, R'S'^{2m'+n-2}G,\\
&G'\sim(T)\, R'S'^{2m+n-2}G', && G\sim(T^{-1})\, RS^{2m'+n-2}G,\\
&R'\sim(T^{-1})\, S^{m+m'+n-2}GG', && R\sim(T)\, S'^{m+m'+n-2}GG',\\
&R'\sim(T)\, S'^{m+m'+n-2}[R'^2+K'(W'Z-WZ')], &&\\
&R\sim(T^{-1})\, S^{m+m'+n-2}[R^2+K(W'Z-WZ')], &&\\
&S'\sim(T^{-1})\, RS^{m+m'+n-3}GG', && S\sim(T)\, R'S'^{m+m'+n-3}GG',\\
&S'\sim(T)\, S'^{m+m'+n-3}\{R'^2S'+R'GG'+S'[GG'+K'(W'Z-WZ')]\}, &&\\
&S\sim(T^{-1})\, S^{m+m'+n-3}\{R^2S+RGG'+S[GG'+K(W'Z-WZ')]\}. &&
\end{aligned}
\tag{11}
$$

Through a point O_r on r there is a pencil of transversals through s. O_r determines an F' and the associated F cuts the pencil in s and a line l. The line l generates the ruled surface R, the image of r.

From a point O_s on s there is an n-ic cone of transversals to r, to each line of which corresponds one F of $|F|$ cutting that line in one point. The locus of all such points is a curve k which generates the surface S, the image of s. The order of k, determined by the intersection of S and a homaloidal surface, is $m+m'+n-2$.

Through a point $O_{g'}$ on g' there is a unique line t of the congruence, but every F of $|F|$ passes through $O_{g'}$, hence $O_{g'}\sim(T^{-1})t$. The ruled surface G generated by t is the image of g' under T. Furthermore, every point P' of the line determines the same F' and t meets the associated F in one point $\bar{P}$ so that $\bar{P}\sim(T)t$. The locus of points $\bar{P}$ is the curve $\bar{g}$ and $\bar{g}\sim(T)G$. The order of $\bar{g}$, determined by the intersection of two homaloidal surfaces, is $m+3m'+2n-3$. In a similar manner we find a curve $\bar{g}'$, of order $3m+m'+2n-3$, such that

$$\bar{g}'\sim(T^{-1})G'.$$

The multiple point A is a fundamental point of the second kind and has as an image $n-1$ lines $c_{1,i}$, $i=1, \cdots, n-1$, other than s lying one in each of the $n-1$ planes determined by s and the tangent lines to r at A.

We can now write the following correspondences:

$$
(12)\quad
\begin{aligned}
&r \sim (T)\ R' : s^{m+m'+n-2} g' \bar{g}' c'_{1,1} \cdots c'_{1,n-1},\\
&r \sim (T^{-1})\ R : s^{m+m'+n-2} g \bar{g} c_{1,1} \cdots c_{1,n-1},\\
&s \sim (T)\ S' : rs^{m+m'+n-2} g' \bar{g}' c'_{1,1} \cdots c'_{1,n-1},\\
&s \sim (T^{-1})\ S : rs^{m+m'+n-2} g \bar{g} c_{1,1} \cdots c_{1,n-1},\\
&g \sim (T)\, G' : rs^{2m+n-2} g\bar{g}', \qquad g' \sim (T^{-1})\, G : rs^{2m'+n-2} g'\bar{g},\\
&\bar{g}' \sim (T)\, G' : rs^{2m+n-2} g\bar{g}', \qquad \bar{g} \sim (T)\, G : rs^{2m'+n-2} g'\bar{g}.
\end{aligned}
$$

4. **Invariant and homaloidal surfaces.** The eliminant of the parameter from $|F|$ and $|F'|$ is the pointwise invariant surface K. The plane $x_4=0$ and the planes determined by s and the tangent lines to r at A are also invariant, but not pointwise invariant.

Generic planes subjected to the transformations give

$$
\begin{aligned}
\pi' &\equiv (A'x) \sim (T^{-1})\ R(A'y) + K(A'z) = \phi_{m+m'+n},\\
\pi &\equiv (Ay) \sim (T)\ R'(Ax) + K'(Az) = \phi'_{m+m'+n},
\end{aligned}
$$

where the ϕ's are homaloidal surfaces of the transformations. Further,

$$
\phi \sim (T)(A'x)R'S'^{m+m'+n-2}GG', \qquad \phi' \sim (T^{-1})\ (Ay)RS^{m+m'+n-2}GG',
$$

hence the homaloidal webs are

$$
\infty^3 \left| \phi \right| : rs^{m+m'+n-2} g\bar{g}, \qquad \infty^3 \left| \phi' \right| : rs^{m+m'+n-2} g'\bar{g}'.
$$

The intersection of two homaloidal surfaces gives the homaloidal net

$$
\begin{aligned}
H' &\equiv [\phi'\phi'] : rs^{m^2+m'^2+n^2+2mm'+2m'n-4m-4m'-4n+4} g'\bar{g}' k_{m+m'+n}\\
H &\equiv [\phi\phi] : rs^{m^2+m'^2+n^2+2mm'+2mn+2m'n-4m-4m'-4n+4} g\bar{g} k_{m+m'+n}.
\end{aligned}
$$

We now write the additional correspondences:

$$
(13)\quad
\begin{aligned}
&\pi \sim (T)\ \phi' : rs^{m+m'+n-2} g'\bar{g}', \qquad \pi' \sim (T^{-1})\ \phi : rs^{m+m'+n-2} g\bar{g},\\
&K \sim (T)\ K' : s^{m+m'-2} gg'\bar{g}\bar{g}', \qquad K' \sim (T^{-1})\ K : s^{m+m'-2} gg'\bar{g}\bar{g}'.
\end{aligned}
$$

The jacobian of the transformation is $J = RGG'S$.

5. **Tangency along** s. The projectivity $y_1=x_1$, $y_2=x_2$, $y_3=kx_3$, $y_4=x_3+x_4$ is applied to the fundamental surfaces of the transformation and an examination of the coefficients of the highest powers of x_3 shows K and S to have

$$
\begin{aligned}
D_{m+m'-2} \equiv {} & [(a'_3 k + a'_4)\{e'x_1x_2\} - (\alpha'_3 k + \alpha'_4)\{f'x_1x_2\}]\\
&\qquad\qquad \cdot [(b_3 k + b_4)\{gx_1x_2\} - (\beta_3 k + \beta_4)\{hx_1x_2\}]\\
&- [(b'_3 k + b'_4)\{g'x_1x_2\} - (\beta'_3 k + \beta'_4)\{h'x_1x_2\}]\\
&\qquad\qquad \cdot [(a_3 k + a_4)\{ex_1x_2\} - (\alpha_3 k + \alpha_4)\{fx_1x_2\}] = 0
\end{aligned}
$$

as common tangent surface along s. The same surface is tangent to K' and S' also along s.

6. **Intersection table.** Referring to (12), (13) and §5 we can now write the following intersection table:

$$[R'S']: s^{m^2+m'^2+n^2+2mm'+2mn+2m'n-5m-5m'-5n+6} g'\bar{g}' c_{1,1} \cdots c_{1,n-1}$$

$$[R'G']: s^{2m^2+n^2+2mm'+3mn+m'n-6m-4n-2m'+4} \bar{g}'$$

$$[R'\phi']: s^{m^2+m'^2+n^2+2mm'+2mn+2m'n-4m-4m'-4n+4} g'\bar{g}' l_{1,1} \cdots l_{1,n}$$

$$[R'K']: s^{m^2+m'^2+2mm'+mn+m'n-4m-4m'-2n+4} g'\bar{g}'$$

$$[S'G']: rs^{2m^2+n^2+2mm'+3mn+m'n-8m-2m'-5n+6} \bar{g}' k_{1,1} \cdots k_{1,2m-2}$$

$$[S'\phi']: rs^{m^2+m'^2+n^2+2mm'+2mn+2m'n-5m-5m'-5n+6} g'\bar{g}' k_{m+m'+n-2}$$

$$[S'K']: s^{m^2+m'^2+2mm'+mn+m'n-5m-5m'-2n+6+(m+m'-2)d} g'\bar{g}'$$

$$[G'\phi']: rs^{2m^2+n^2+2mm'+3mn+m'n-6m-4n-2m'+4} \bar{g}' j_{1,1} \cdots j_{1,2m-1}$$

$$[G'K']: s^{2m^2+mn+2mm'+m'n-6m-2m'-2n+4} g\bar{g}'$$

$$[\phi'K']: S^{m^2+m'^2+2mm'+mn+m'n-4m-4m'-2n+4} g'\bar{g}' k_{m+m'}.$$

7. **The T_2 in a plane through s.** A plane $\pi \equiv x_1 - \sigma x_2$ cuts the surfaces of $|F_m|$ and $|F'_{m'}|$ in residual pencils of lines

$$|l| \equiv u - \mu v = 0, \qquad |l'| \equiv u' - \mu v' = 0$$

where $u = (ax)\{e\sigma\} - (\alpha x)\{f\sigma\}$, $u' = (a'x)\{e'\sigma\} - (\alpha' x)\{f'\sigma\}$, $v = (bx)\{g\sigma\} - (\beta x)\{h\sigma\}$, $v' = (b'x)\{g'\sigma\} - (\beta' x)\{h'\sigma\}$, $(ax) = (a_1\sigma + a_2)x_2 + a_3x_3 + a_4x_4$, $\{e\sigma\} = \sum_{p=0}^{m-1} e_{p,m-p-1}\sigma^p$, $\{e'\sigma\} = \sum_{p=0}^{m'-1} e'_{p,m'-p-1}\sigma^p$, and so on. The n-ic r intersects π in one residual point P: $(\sigma[c\sigma], [c\sigma], [d\sigma], 0)$ where $[c\sigma] = \sum_{i=0}^{n-1} c_{i,n-i-1}\sigma^i$, $[d\sigma] = \sum_{j=0}^{n} d_{j,n-j}\sigma^j$, and the vertices of $|l|$ and $|l'|$ are designated as Γ and Γ'.

Through a generic point $P(y)$ of π passes one l of $|l|$ having parameter $\mu = u(y)/v(y)$ and to this corresponds the line $u'v - uv' = 0$ which is met by l in a point $P'(x)$, image of P under T. The T_2^{-1} in π is thus

$$(14) \qquad x_2 = \rho y_2 + \kappa[c\sigma], \qquad x_3 = \rho y_3 + \kappa[d\sigma], \qquad x_4 = \rho y_4,$$

where $\rho_1 \equiv uw' - vz'$, $\kappa_2 = u'v - v'u$, $z' = (a'q)\{e'\sigma\} - (\alpha' q)\{f'\sigma\}$, $w' = (b'q)\{g'\sigma\} - (\beta' q)\{h'\sigma\}$, $(a'q) = (a_1'\sigma + a_2')[c\sigma] + a_3'[d\sigma]$, and so on.

The direct transformation T is

$$(15) \qquad y_2 = \rho' x_2 + \kappa'[c\sigma], \qquad y_3 = \rho' x + \kappa'[d\sigma], \qquad y_4 = \rho' x_4$$

where $\rho_1' \equiv u'w - v'z$, $\kappa' = -\kappa$.

The conic $\kappa:\Gamma\bar{\Gamma}\Gamma'\bar{\Gamma}'$ is pointwise invariant under the transformation.

Through the point Γ there is one line γ' through P and every line of $|l'|$ corresponds to Γ, hence γ' is the image of Γ under T. Moreover, every point of γ' determines the same line of $|l'|$ which intersects γ' in a point $\bar{\Gamma}'$ whose image under T^{-1} is also γ'. Similarly, a γ of $|l|$ is the image of Γ' and $\bar{\Gamma}$ under T^{-1} and T, respectively.

The point P is the vertex of a pencil of transversals. Moreover, through P there passes one line of $|l|$, hence there corresponds one line of $|l'|$. This line is met in every point by a line of the pencil, hence is the image of P under T.

Thus the points $P, \Gamma, \bar{\Gamma}$ and $P, \Gamma', \bar{\Gamma}'$ are fundamental under T_2 and T_2^{-1}, respectively, so that we have $P\sim(T)\ \rho':\Gamma'\bar{\Gamma}'$, $\Gamma\sim(T)\ \gamma':P\Gamma\bar{\Gamma}'$, $\bar{\Gamma}\sim(T)\ \gamma:P\Gamma'\bar{\Gamma}$ and $P\sim(T)\ \rho:\Gamma\bar{\Gamma}$, $\bar{\Gamma}'\sim(T^{-1})\ \gamma:P\Gamma'\bar{\Gamma}$, $\bar{\Gamma}'\sim(T^{-1})$ $\gamma':P\Gamma\bar{\Gamma}'$.

The homaloidal nets of T and T^{-1} are

$$\infty^2|f_2'|:P\Gamma\bar{\Gamma}', \qquad \infty^2|f_2|:P\Gamma\bar{\Gamma}$$

while the jacobian of T_2 is $j=\rho'\gamma\gamma'$ and of T_2^{-1} is $j'=\rho\gamma\gamma'$.

As the plane π generates the pencil on s the T generates the space $T_{m+m'+n}$ whose equations may be obtained from (14) and (15) by replacing $u, u', v, v', w, w', z, z'$ and σ by $U, U', V, V', W, W', Z, Z'$ and x_1/x_2, respectively.

Since the point P is the section of r by π, r may be represented by $x_1=\sigma[c\sigma]$, $x_2=[c\sigma]$, $x_3=[d\sigma]$, $x_4=0$.

8. **r having $(n-1)$-point with coincident tangents.** In case k of the tangents to r at A are coincident the transformation will be identical with the above except for the image of A, the correspondences involved then being

$$r\sim(T)\ R':s^{m+m'+n-2}g'\bar{g}'c_{1,1}'^{k},\ c_{1,k+1}'\cdots c_{1,n-1}',$$
$$s\sim(T)\ S':rs^{m+m'+n-3}g'\bar{g}'c_{1,1}',\ c_{1,k+1}'\cdots c_{1,n-1}',$$

and the intersection

$$[R'S']:s^{m^2+m'^2+n^2+2mm'+2mn+2m'n-5m-5m'-5n+6}g'\bar{g}'c_{1,1}'^{k},\ c_{1,k+1}'\cdots c_{1,n-1}'.$$

West Virginia University

HAUSDORFF MEANS INCLUDED BETWEEN $(C, 0)$ AND $(C, 1)$

HERBERT J. GREENBERG AND H. S. WALL

In this paper we show that if $\phi(u)$ is any function of bounded variation on the interval $0 \leqq u \leqq \infty$ and $\phi(\infty)-\phi(0)=1$, then the function $\alpha(z)=\int_0^\infty d\phi(u)/(1+zu)$ is a regular moment function; and we show that when $\phi(u)$ is further restricted to be monotone then the Hausdorff mean determined by $\alpha(z)$ is included between $(C, 0)$ and $(C, 1)$. Conditions under which this mean is equivalent to $(C, 0)$ or to $(C, 1)$ are obtained which are analogous to the conditions found by Scott and Wall[1] for the special case where $\phi(u)=1$ for $u \geqq 1$, $\phi(0)=0$. In §1 we give an elementary development of the notion of Hausdorff summability; §2 contains a proof that $\alpha(z)$ is a regular moment function; §3 contains the above mentioned inclusion theorems; and §4 contains examples and a discussion of some transformations of moment functions which are suggested by the earlier developments.

1. Hausdorff summability. Let $A=(a_{ij})$ be any matrix in which $a_{ii}\neq 0$ and $a_{ij}=0$ for $j>i$, $i, j=0, 1, 2, \cdots$, and consider the system of equations

$$
\begin{aligned}
&a_{00}q_0 &&= c_0(a_{00}p_0),\\
&a_{10}q_0 + a_{11}q_1 &&= c_1(a_{10}p_0 + a_{11}p_1),\\
&a_{20}q_0 + a_{21}q_1 + a_{22}q_2 &&= c_2(a_{20}p_0 + a_{21}p_1 + a_{22}p_2),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}
\tag{1.1}
$$

These equations constitute a linear transformation of the sequence $\{p_n\}$ into the sequence $\{q_n\}$, the transformation depending upon the matrix A and the sequence $\{c_n\}$. If $\lim q_n=p$, we shall say that the sequence $\{p_n\}$ is $[A, c_n]$-*summable* to the limit p. A sequence $\{c_n\}$ such that $[A, c_n]$ sums every convergent sequence to its proper limit will be called A-*regular*. The following statements are almost obvious consequences of the above definitions:

(i) If $[A, c_n]$ transforms $\{p_n\}$ into $\{q_n\}$, and $[A, d_n]$ transforms $\{q_n\}$ into $\{r_n\}$, then $[A, c_nd_n]$ transforms $\{p_n\}$ into $\{r_n\}$.

(ii) If $\{c_n\}$, $\{d_n\}$ are A-regular, then $[c_nd_n]$ is A-regular.

(iii) If $[A, c_n]$ sums $\{p_n\}$ to the limit p, then $[A, kc_n]$ sums $\{p_n\}$ to the limit kp.

Presented to the Society, February 28, 1942; received by the editors January 15, 1942.

[1] W. T. Scott and H. S. Wall, *Transformation of series and sequences*, Transactions of this Society, vol. 51 (1942), pp. 255–279.

(iv) If $\{c_n\}$, $\{d_n\}$ are A-regular, and $g+h=1$, then $\{gc_n+hd_n\}$ is A-regular.

(v) If $\{a_n\}$, $\{b_n\}$ are A-regular, and $b_n\neq 0$, $n=0, 1, 2, \cdots$, then $[A, a_n]\supset[A, b_n]$, that is, every sequence summable $[A, b_n]$ is summable $[A, a_n]$ to the same limit, if and only if $\{a_n/b_n\}$ is A-regular.

Hausdorff summability[2] is the particular $[A, c_n]$-summability obtained by taking $A=H=((-1)^jC_{ij})$ where $C_{ij}=i!/j!(i-j)!$, $0\leqq j\leqq i$; $C_{ij}=0$, $j>i$. In this case the equations (1.1) may be written

$$\Delta^m q_0 = c_m\Delta^m p_0, \qquad m = 0, 1, 2, \cdots, \tag{1.2}$$

where $\Delta^i x_j = x_j - C_{i1}x_{j+1} + C_{i2}x_{j+2} - \cdots + (-1)^i C_{ii}x_{j+i}$; or, as may be readily shown:

$$q_m = \sum_{n=0}^{m} C_{mn}\Delta^{m-n}c_n\cdot p_n, \qquad m = 0, 1, 2, \cdots. \tag{1.3}$$

Inasmuch as we shall be dealing exclusively with Hausdorff summability from now on, we shall write *regular* instead of *H-regular*, whenever the occasion arises.

If $\{p_n\}$ is the sequence 1, 1, 1, $\cdots$, then (1.2) becomes $\Delta^m q_0=0$, $m>0$, $\Delta^0 q_0=q_0=c_0$, or $q_m=c_0$, $m=0, 1, 2, \cdots$; and if $\{p_n\}$ is the sequence 1, 0, 0, $\cdots$ then $\Delta^m q_0=c_m$, $q_m=\Delta^m c_0$, $m=0, 1, 2, \cdots$. It follows that *necessary* conditions for $\{c_n\}$ to be regular are:

$$c_0 = 1, \tag{a}$$

$$\lim_{m\to\infty} \Delta^m c_0 = 0. \tag{b}$$

Another necessary condition obtained by applying one of the well known conditions for the regularity of sequence transformations to (1.3) is:

$$\sum_{n=0}^{m} C_{mn}\left|\Delta^{m-n}c_n\right| \leqq M, \qquad m = 0, 1, 2, \cdots, \tag{c}$$

where M is independent of m. These necessary conditions are also *sufficient*.[3]

Hausdorff showed that (c) holds if and only if $\{c_n\}$ is a *moment sequence*, that is,

$$c_n = \int_0^1 u^n d\phi(u), \qquad n = 0, 1, 2, \cdots, \phi(u) \in BV[0, 1]; \tag{c'}$$

---

[2] F. Hausdorff, *Summationsmethoden und Momentfolgen*, I and II, Mathematische Zeitschrift, vol. 9 (1921), pp. 74–109, 280–299.

[3] Hausdorff, loc. cit.

and that (a), (b) are equivalent, respectively, to the conditions:

$$\text{(a')} \qquad \phi(1) - \phi(0) = 1,$$

$$\text{(b')} \qquad \phi(u) \text{ continuous at } u = 0.$$

We recall that $(C, 0)$, or convergence, is the Hausdorff method $[H, c_n]$ where $c_n = 1$, $n = 0, 1, 2, \cdots$. Also, $(H, \alpha) = [H, (n+1)^{-\alpha}]$, $(E, r) = [H, r^n]$, $0 < r < 1$, are the Hölder method of order α and the Euler-Knopp method, respectively. The former is equivalent to $(C, \alpha) = [H, 1/C_{n+\alpha,n}]$, the Cesàro method of order α.

2. Proof that $\{\alpha(n)\}$ is a regular sequence. We define the function $\alpha(z)$ by the equation

$$(2.1) \qquad \alpha(z) = \int_0^\infty \frac{d\phi(u)}{1 + zu}, \qquad \phi(u) \in BV[0, \infty], \phi(\infty) - \phi(0) = 1,$$

and shall prove that the sequence $\{\alpha(n)\}$, $n = 0, 1, 2, \cdots$, is a moment sequence. To do this, it suffices to show that $\Delta^m\alpha(n) \geqq 0$, $m, n = 0, 1, 2, \cdots$, when $\phi(u)$ is monotone non-decreasing. But this is obvious from the formula

$$\Delta^m\alpha(n) = \int_0^\infty \frac{u^m m! d\phi(u)}{(1 + nu)(1 + [n + 1]u) \cdots (1 + [n + m]u)},$$

To show that $\{\alpha(n)\}$ is regular we must show that $\alpha(0) = 1$, and that $\Delta^m\alpha(0) \to 0$ as $m \to \infty$. That $\alpha(0) = 1$ follows from the hypothesis $\phi(\infty) - \phi(0) = 1$. Write $\Delta^m\alpha(0)$ as the sum of three integrals J_1, J_2, J_3, with the limits of integration 0 to 1, 1 to k, k to ∞, respectively, $k > 1$. Since the integrand

$$1 \Big/ \left[\left(1 + \frac{1}{u}\right)\left(1 + \frac{1}{2u}\right) \cdots \left(1 + \frac{1}{mu}\right)\right] < 1$$

it follows that if $\phi(u)$ is monotone then $0 \leqq J_3 \leqq \phi(\infty) - \phi(k) < \epsilon/3$, $\epsilon > 0$, if $k \geqq K_\epsilon$, ($K = K_\epsilon$ independent of m). If k is fixed, then

$$0 \leqq J_2 \leqq 1 \Big/ \left[\left(1 + \frac{1}{K}\right)\left(1 + \frac{1}{2K}\right) \cdots \left(1 + \frac{1}{mK}\right)\right] \leqq \frac{\epsilon}{3},$$

for all m sufficiently large, inasmuch as $\prod_{n=1}^{\infty}(1 + 1/nK)$ diverges to ∞. Also, $\lim_{m\to\infty} J_1 = 0$ inasmuch as the integrand tends monotonically to 0 for each u, $0 \leqq u \leqq 1$. Hence $\lim_{m\to\infty} \Delta^m\alpha(0) = 0$ if $\phi(u)$ is monotone. If $\phi(u) \in BV[0, \infty]$ it can be written as a linear combination of bounded monotone functions with coefficients ± 1, $\pm i$, and therefore $\lim_{n\to\infty} \Delta^n\alpha(0)=0$ in any case. We have completed the proof of the following theorem:

THEOREM 2.1. *The sequence* $\{\alpha(n)\}$, *where* $\alpha(z)$ *is given by* (2.1), *is a regular sequence.*

3. **An inclusion relation for $[H, \alpha(n)]$ when $\phi(u)$ is monotone.** In the integral defining $\alpha(z)$ replace z by $z+1$ and u by $v/(1-v)$, and we obtain:

$$\alpha(z+1) = \int_0^1 \frac{(1-v)d\phi_1(v)}{1+zv}, \qquad \phi_1(v) = \phi\left(\frac{v}{1-v}\right).$$

If $\phi(u)$ is monotone and $\phi(\infty)-\phi(0)=1$, then $\phi_1(v)$ is monotone, $0\leq v\leq 1$, and $\phi_1(1)-\phi(0)=1$. Consequently,[4] we have an expansion of the form

$$\alpha(z+1) = \frac{g_1}{1+}\,\frac{(1-g_1)g_2z}{1+}\,\frac{(1-g_2)g_3z}{1}+\cdots,$$

where $0\leq g_n\leq 1$, $n=1, 2, 3, \cdots$, and where the continued fraction is to be terminated with the first identically vanishing partial quotient in case some g_n is 0 or 1. Excepting in the trivial case $g_1=1$, $\alpha(z)\equiv 1$, we then have:

$$\frac{g_1}{\alpha(z+1)} = 1+(1-g_1)z\left[\frac{g_2}{1+}\,\frac{(1-g_2)g_3z}{1+}\,\frac{(1-g_3)g_4z}{1}+\cdots\right]$$
$$= 1+(1-g_1)z\alpha^*(z+1),$$

where

$$\alpha^*(z+1) = \int_0^1 \frac{(1-v)d\phi_1^*(v)}{1+zv},$$

a function of the same kind as $\alpha(z+1)$. That is, $\phi_1^*(v)$ is monotone; and $\phi_1^*(1)-\phi_1^*(0)=1$, since[5]

$$1 = \phi_1(1)-\phi_1(0) = 1-\left[1+\sum_{m=1}^{\infty}\frac{g_1g_2\cdots g_m}{(1-g_1)(1-g_2)\cdots(1-g_m)}\right]^{-1}$$

and $\phi_1^*(1)-\phi_1^*(0)$ is equal to the same expression with the subscripts of all the g's advanced by unity.

On replacing z by $z-1$ we therefore have the identity

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[4] H. S. Wall, *Continued fractions and totally monotone sequences*, Transactions of this Society, vol. 48 (1940), pp. 165–184; p. 179 and p. 182.

[5] H. S. Wall, *A class of functions bounded in the unit circle*, Duke Mathematical Journal, vol. 7 (1940), pp. 146–153; p. 147.

$$\frac{\alpha(1)}{\alpha(z)} = 1 + [1 - \alpha(1)](z - 1)\alpha^*(z), \qquad \alpha(1) < 1, \tag{3.1}$$

where

$$\alpha^*(z) = \int_0^\infty \frac{d\phi^*(u)}{1 + zu}, \qquad \phi^*(u) \text{ monotone, } \phi^*(\infty) - \phi^*(0) = 1.$$

We are now prepared to prove the inclusion relation:

$$(C, 0) \subset [H, \alpha(n)] \subset (C, 1). \tag{3.2}$$

The left half is of course a restatement of the regularity of $[H, \alpha(n)]$, and (3.2) is obviously true if $\alpha(n) \equiv 1$. To prove the right half when $\alpha(1) < 1$ we write, by (3.1),

$$\frac{1}{(n+1)\alpha(n)} = \frac{1}{\alpha(1)}\left[\frac{1}{n+1} + [1 - \alpha(1)]\alpha^*(n) - 2[1 - \alpha(1)]\frac{\alpha^*(n)}{n+1}\right].$$

We have expressed the ratio $[1/(n+1)]:\alpha(n)$ as a linear combination of regular sequences where the constants of combination add up to unity. Hence by (iv), (v) of §1 it follows that $[H, \alpha(n)] \subset (C, 1)$.

If $\alpha(z)$ has a convergent Stieltjes[6] continued fraction expansion $1/a_1 + z/a_2 + z/a_3 + z/a_4 + \cdots$, where $a_1 = 1$, $a_n > 0$, $\sum a_n$ diverges, then there is the relation

$$1/\alpha(z) = 1 + z a_2^{-1} \alpha^*(z), \tag{3.3}$$

where $\alpha^*(z) = a_2(1/a_2 + z/a_2 + z/a_3 + z/a_4 + \cdots)$, with the aid of which the inclusion $[H, \alpha(n)] \subset (C, 1)$ can be established. However, if $\sum a_n$ converges so that the continued fraction diverges, there is no assurance that a relation of the form (3.3) exists. The difficulty disappears as soon as one has (3.1).

We shall now proceed to obtain conditions under which one or the other of the inclusion symbols in (3.2) may be replaced by the equivalence symbol. In order to have $(C, 0) \approx [H, \alpha(n)]$ it is necessary and sufficient that $\{1/\alpha(n)\}$ be a regular sequence. In particular, $1/\alpha(n)$ must be bounded. Now $\lim_{n\to\infty} \alpha(n) = \phi(+0) - \phi(0)$, and therefore a *necessary* condition for $\{1/\alpha(n)\}$ to be regular is that $\phi(u)$ be discontinuous at $u = 0$. This condition is also *sufficient*. For, by (3.1) we may write, if $\alpha(1) < 1$:

$$\frac{1}{\alpha(n)} = \frac{1}{\alpha(1)}\left[1 - [1 - \alpha(1)]\alpha^*(n) + [1 - \alpha(1)]\int_0^\infty \frac{d\phi^*(u)}{(1/n) + u}\right].$$

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[6] T. J. Stieltjes, *Oeuvres*, vol. 2, pp. 402–566.

Supposing $\phi(u)$ to be discontinuous at $u=0$, so that $1/\alpha(n)$ has a finite limit for $n=\infty$, this relation shows that $\int_0^\infty d\phi^*(u)/u<\infty$. Hence, if we put $\int_0^t d\phi^*(u)/u=\theta(t)$ we have:

$$\frac{1}{\alpha(n)}=\frac{1}{\alpha(1)}\left[1-[1-\alpha(1)]\alpha^*(u)+[1-\alpha(1)][\theta(\infty)-\theta(0)]-\int_0^\infty\frac{d\theta(u)}{1+nu}\right].$$

On applying (iv) of §1 we therefore see that $\{1/\alpha(n)\}$ is a regular sequence. In case $\alpha(1)=1$ so that $\alpha(n)\equiv 1$, the function $\phi(u)$ is discontinuous at $u=0$ and in this case $[H,\alpha(n)]\approx(C,0)$.

In order to have $[H,\alpha(n)]\approx(C,1)$ it is necessary and sufficient that $\{(n+1)\alpha(n)\}$ be a regular sequence. Since

$$(n+1)\alpha(n)=\int_0^\infty\frac{d\phi(u)}{1+nu}+\int_0^\infty\frac{d\phi(u)}{(1/n)+u},$$

we see as in the preceding that $\{(n+1)\alpha(n)\}$ is regular if and only if $\int_0^\infty d\phi(u)/u<\infty$. We have completed the proof of the following theorem:

THEOREM 3.1. *If* $\alpha(n)=\int_0^\infty d\phi(u)/(1+nu)$ *where* $\phi(u)$ *is monotone and* $\phi(\infty)-\phi(0)=1$, *then* $(C,0)\subset[H,\alpha(n)]\subset(C,1)$; *and* $(C,0)\approx[H,\alpha(n)]$ *if and only if* $\phi(u)$ *is discontinuous at* $u=0$, *while* $[H,\alpha(n)]\approx(C,1)$ *if and only if* $\int_0^\infty d\phi(u)/u<\infty$.

In case $\alpha(z)$ has a convergent Stieltjes continued fraction we have this theorem:

THEOREM 3.2. *If* $\alpha(z)=1/a_1+z/a_2+z/a_3+\cdots$, $a_1=1$, $a_n>0$, $\sum a_n$ *diverges, then* $[H,\alpha(n)]\approx(C,0)$ *if and only if* $\sum a_{2n+1}$ *converges; and* $[H,\alpha(n)]\approx(C,1)$ *if and only if* $\sum a_{2n}$ *converges.*

This follows from Theorem 3.1, together with the fact that $\phi(+0)-\phi(0)=1/\sum a_{2n+1}$,[7] and the fact that $\int_0^\infty d\phi(u)/u<\infty$ if and only if $\sum a_{2n}$ converges.[8]

In case the continued fraction for $\alpha(z)$ diverges, that is, $\sum a_n$ converges, then the odd approximants have one limit $\alpha_1(z)$ and the even approximants another limit $\alpha_2(z)$, one of which may equal $\alpha(z)$. In any case, $\alpha_1(z)$ and $\alpha_2(z)$ have integral representations of the form to which Theorem 2.1 applies. It is not difficult to show that $(C,0)\approx[H,\alpha_1(n)]$, while $[H,\alpha_2(n)]\approx(C,1)$. For[9] the integrals may be written as infinite series of the form $\sum_{i=0}^{\infty} M_i/(1+zu_i)$, $0\leqq u_0<u_1<\cdots$, $M_i>0$, where $u_0=0$ in the case of $\alpha_1(z)$ whereas $u_0>0$ in the case of $\alpha_2(z)$. Hence it follows that $\phi(u)$ is discontinuous at $u=0$ in the case of $\alpha_1(z)$, while $\int_0^\infty d\phi(u)/u<\infty$ in the case of $\alpha_2(z)$.

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[7] Stieltjes, loc. cit., p. 510.

[8] H. S. Wall, *On extended Stieltjes series*, Transactions of this Society, vol. 31 (1929), pp. 771–781; p. 774.

[9] Stieltjes, loc. cit., p. 403.

4. **Some transformations of moment sequences.** If

$$\alpha(z)=\int_0^{\infty'}\frac{d\phi(u)}{1+zu}$$

where $\phi(u)$ is monotone and $\phi(\infty)-\phi(0)=1$, then

$$\frac{1-\alpha(z)}{c_1 z}=\int_0^\infty\frac{(u\,d\phi(u)/c_1)}{1+zu},\qquad c_1=\int_0^\infty u\,d\phi(u),$$

defines a method of summation which is equivalent to $(C, 1)$ inasmuch as $\int_0^\infty (u\,d\phi(u)/c_1)/u<\infty$. It is noteworthy that there are other moment functions which are not of the form considered above and which have the property that the transformation

$$\alpha_1(z)=\frac{1-\alpha(z)}{c_1 z},$$

where c_1 is a normalizing factor such that $\alpha_1(0)=1$, carries a regular moment function $\alpha(z)$ into another regular moment function $\alpha_1(z)$ such that $[H, \alpha_1(n)]\approx(C, 1)$. For instance, if $\alpha(z)=(1+z)^{-k}$, $k=1, 2, 3, \cdots$, we find that

$$\alpha_1(z)=\frac{1}{k}\left(\frac{1}{(1+z)}+\frac{1}{(1+z)^2}+\cdots+\frac{1}{(1+z)^k}\right);$$

and it is not difficult to show that $[H, \alpha_1(n)]\approx(C, 1)$.

If $\alpha(z)=r^z$, $0<r<1$, so that $[H, \alpha(n)]$ is Euler-Knopp summability (E, r), then

$$\alpha_1(z)=\frac{1-r^z}{z\log(1/r)}=\int_0^1 u^z d\phi_1(u)$$

where

$$\phi_1(u)=\begin{cases}0, & 0\leqq u\leqq r,\\ 1-(\log u/\log r), & r<u\leqq 1.\end{cases}$$

Since $\phi_1(u)$ is obviously a regular mass function it follows that $\alpha_1(z)$ is a regular moment function. *Although* $(C, 1)$ *and* (E, r) *are not comparable methods, nevertheless* $[H, \alpha_1(n)]\supset(C, 1)$. For,

$$\alpha_1(z):(1+z)^{-1}=\int_0^1 u^z d\phi_2(u),$$

where

$$\phi_2(u) = \begin{cases} 0, & 0 \leqq u < r, \\ (1/\log r) - (\log u/\log r) + 1, & r \leqq u < 1, \\ 1, & u = 1, \end{cases}$$

and is a regular mass function. *In this case*, $(C, 1)$ *does not include* $[H, \alpha_1(n)]$. To see this, put $(1+z)^{-1}:\alpha_1(z) = \beta(z) = [\log (1/r)/2(1-r)] \cdot\beta_1(z-1)$, where

$$\beta_1(z) = \frac{2(1+z)}{(2+z)}\cdot\frac{1-r}{1-r^{z+1}}.$$

Inasmuch as

$$\tag{4.1} \begin{aligned} \frac{2(1+z)}{2+z} &= \int_0^1 u^z d\theta_1(u), \\ \theta_1(u) &= \begin{cases} 1, & u = 1, \\ -u^2, & 0 \leqq u < 1; \end{cases} \\ \frac{1-r}{1-r^{z+1}} &= \int_0^1 u^z d\theta_2(u), \\ \theta_2(u) &= \begin{cases} 1, & u = 1, \\ r^k, & r^k < u < r^{k-1}; \qquad k = 1, 2, 3, \cdots, \end{cases} \end{aligned}$$

$\theta_2(u) = (1/2)[\theta_2(u+0)+\theta_2(u-0)]$ if $0<u<1$, we see that $\{\beta_1(n)\}$ is a regular sequence, being the product of regular sequences (cf. (ii) of §1). By means of the composition formula[10]

$$\theta(u) = \theta_1(u) + \int_0^1 \theta_2(u/v)d\theta_1(v),$$

we find that

$$\beta_1(z) = \int_0^1 u^z d\theta(u),$$

$$\theta(u) = \begin{cases} 1, & u = 1, \\ r^{k+1} - (r^{-k} - r)u^2, & r^{k+1} < u < r^k; \quad k = 0, 1, 2, \cdots, \end{cases}$$

where $\theta(u) = (1/2)[\theta(u+0)+\theta(u-0)]$, $0<u<1$.

If $\{\beta(n)\}$ is a regular sequence, then we must have $\beta(n) = \int_0^1 u^n d\pi(u)$, $n = 0, 1, 2, \cdots$, and at the same time we would have:

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[10] H. L. Garabedian, Einar Hille and H. S. Wall, *Formulations of the Hausdorff inclusion problem*, Duke Mathematical Journal, vol. 8 (1941), pp. 193–213; p. 196.

$$\beta(n+1) = \frac{\log (1/r)}{2(1-r)}\int_0^1 u^n d\theta(u), \quad n = 0, 1, 2, \cdots .$$

This implies that

$$\frac{2(1-r)}{\log (1/r)}\cdot \pi(u) = \int_0^u \frac{d\theta(t)}{t}.$$

This is impossible, for the integral on the right does not exist. In fact:

$$\int_{r^{k+1}-0}^1 \frac{d\theta(u)}{u} = 2(1-r) + 2r(1-r^{k+1}),$$

$$\int_{r^{k+1}+0}^1 \frac{d\theta(u)}{u} = 2r(1-r^{k+1}),$$

so that $\int_t^1 d\theta(u)/u$ does not have a limit as $t \to +0$. We have proved that $\{\beta(n)\}$ is not a regular sequence, and therefore $(C, 1)$ does not include $[H, \alpha_1(n)]$.

Another transformation which is suggested by Theorem 2.1 is

$$\alpha_1(z) = \frac{c + \alpha(z)}{c+1}, \qquad c > 0.$$

If $\alpha(z)$ is as in Theorem 2.1, then $\alpha_1(z)$ is a function of the same kind; and since the effect of the transformation is to add a discontinuity to $\phi(u)$ at $u=0$ we must have $(C, 0) \approx [H, \alpha_1(n)]$.

One of the most important problems in the Hausdorff theory is the problem of recognizing whether or not a given sequence is a moment sequence. One way of contributing to this problem is to find transformations which carry moment sequences into moment sequences. *It is perhaps of interest to point out that any Hausdorff transform of a moment sequence is a moment sequence.* For, if $\{p_n\}$ is any moment sequence, and $[H, c_n]$ carries $\{p_n\}$ into $\{q_n\}$, then the relation (1.2) holds. Hence, inasmuch as $\{\Delta^n p_0\}$ is a moment sequence, and the product of two moment sequences is a moment sequence, it follows that $\{\Delta^n q_0\}$ is a moment sequence. Therefore $\{q_n\}$ is a moment sequence, as was to be proved. *The sequence* $\{q_n\}$ *is regular if and only if* $p_0c_0 = 1$ *and either* $c_n \to 0$ *or* $\Delta^n p_0 \to 0$. For then and only then are the conditions (a), (b) of §1 satisfied by $\{q_n\}$. *In particular,* $\{q_n\}$ *is regular if* $\{p_n\}$, $\{c_n\}$ *are both regular.*

As an example, let $c_n = 1/(n+1)$, $p_n = r^n$, $0 < r < 1$. Then $[H, q_n]$, where

$$q_n = \frac{1 + r + r^2 + \cdots + r^n}{n+1} = \frac{1 - r^{n+1}}{(n+1)(1-r)},$$

is a regular Hausdorff mean. By (4.1) we see that $[H, q_n] \supset (C, 1)$. It is easy to show that $(C, 1) \supset [H, q_n]$, thereby proving that $[H, q_n] \approx (C, 1)$. If $c_n = p_n = 1/(n+1)$, then

$$q_n = \frac{1 + 2^{-1} + 3^{-1} + \cdots + n^{-1}}{n+1}$$

and $[H, q_n]$ is a regular Hausdorff mean. This mean does not include $(C, 1)$ inasmuch as $q_n : (n+1)^{-1}$ is unbounded.

NORTHWESTERN UNIVERSITY

WHIRL-SIMILITUDES, EUCLIDEAN KINEMATICS, AND NON-EUCLIDEAN GEOMETRY

J. M. FELD

1. Introduction. The geometry of whirls and whirl-motions in the plane had its origin in a paper by E. Kasner [6],[1] was subsequently developed in a series of papers by Kasner and DeCicco [3, 7, 8, 9], adapted to the sphere by Strubecker [10], and to 3-space by Feld [4]. In this paper we shall, by adjoining three involutory transformations, extend Kasner's whirl-motion group G_6 to a mixed group Γ_6—the *complete whirl-motion group*—composed of eight mutually exclusive, six-parameter families; these families will in turn be extended to seven-parameter families comprising the mixed group Γ_7—the *complete whirl-similitude group*. The principal results obtained are the extension of Kasner's G_6 and two representations of Γ_7: a *kinematic representation on the plane*, §6, and a *representation in quasi-elliptic 3-space*, §7.

2. Slides, turns, and whirls. Let the point of an oriented lineal element E have the rectangular coordinates x, y, and let the inclination of E to the x-axis be the angle θ, $0 \leqq \theta < 2\pi$. Let $z = x + iy$, $\bar{z} = x - iy$, $\zeta = e^{i\theta}$. We shall call z, ζ the *element coordinates* of E (x, y, θ), which, henceforth, shall be represented by the symbol (z, ζ).

DEFINITIONS. *A slide S_s is a lineal element transformation that translates the point of each element along its line the same distance s.*

Presented to the Society, April 13, 1940; received by the editors January 9, 1942.

[1] The numbers in brackets refer to the bibliography at the end of the paper.

A turn T_α is a lineal element transformation that rotates the line of every lineal element around its point through the same angle arg α.

A direct whirl[2] *is a product $T_\alpha S_s T_\beta$.*

From these definitions it follows that the slide S_s, $(z, \zeta)\rightarrow(z^*, \zeta^*)$, is given by the equations $z^*=z+s\zeta$, $\zeta^*=\zeta$, $(s=\bar{s})$; the turn T_α is given by the equations: $z^*=z$, $\zeta^*=\alpha\zeta$, $(|\alpha|=1)$; and the direct whirl $T_\alpha S_s T_\beta \equiv W^+_{c,\gamma}$ is given by $z^*=z+c\zeta$, $\zeta^*=\gamma\zeta$, where $c=\alpha s$, $\gamma=\alpha\beta$, $|\gamma|=1$.

Evidently $W^+_{a,\alpha}W^+_{b,\beta}=W^+_{c,\gamma}$ where $c=a+\alpha b$, and $\gamma=\alpha\beta$. The direct whirls form a three-parameter group $\mathfrak{W}_3^+$. The direct whirls $W^+_{c,\gamma}$ such that $c+\bar{c}=0$ and $\gamma=\pm 1$ are dilatations.

3. **Flat fields and turbines.** A *flat field* has been defined by Kasner as a totality of ∞^2 oriented lineal elements that lie on the ∞^1 cycles (oriented circles) containing a given element called the center of the flat field. If s, σ are the coordinates of the center of the flat field F, the equation of F in current lineal element coordinates z, ζ is

$$z - s = (\bar{z} - \bar{s})\sigma\zeta. \tag{3.1}$$

Inasmuch as the parameters s, σ completely characterize F, we shall call them the coordinates of F, and use the symbol $\{s, \sigma\}$ to represent the flat field whose center is (s, σ) To distinguish the coordinates of a flat field from those of a lineal element, the former shall be referred to as ***dual coordinates*** and the latter as ***element coordinates***. A flat field $\{s, \sigma\}$ and a lineal element (z, ζ) shall be said to be ***incident*** if their respective coordinates satisfy (3.1). Evidently flat fields and lineal elements can be regarded as duals of one another, and (3.1), accordingly, can be interpreted both as the equation of $\{s, \sigma\}$ in element coordinates and as the equation of (z, ζ) in dual coordinates.

Direct whirls convert flat fields into flat fields. Let $W^+_{a,\alpha}$ convert $\{s, \sigma\}$ into $\{s^*, \sigma^*\}$; then we obtain for the equations of $W^+_{a,\alpha}$ expressed in dual coordinates:

$$s^* = s - \bar{a}\sigma, \qquad \sigma^* = \bar{\alpha}\sigma. \tag{3.2}$$

Two flat fields $\{a, \alpha\}$ and $\{b, \beta\}$, $\alpha\neq\beta$, have in common ∞^1 lineal elements (z, ζ) whose coordinates satisfy the equation

$$z - l = r\zeta, \tag{3.3}$$

where $l=(a\beta-b\alpha)/(\beta-\alpha)$, $r=-(\bar{a}-\bar{b})/(\bar{\alpha}-\bar{\beta})$. The points of the lineal elements (z, ζ) on the locus (3.3) lie on the circle $(z-l)(\bar{z}-\bar{l})=r\bar{r}$ —the *outer circle* of the locus—and their oriented lines envelope a concentric *inner cycle*. Such a locus of lineal elements has been named a *turbine* by Kasner [6]. Evidently the turbine

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[2] The term *direct whirl* is used in place of Kasner's term *whirl* to distinguish between his and another kind of whirl which will be introduced below.

$$z - l = -\bar{r}\zeta \tag{3.4}$$

has the same outer circle and inner cycle as (3.3). Turbines (3.3) and (3.4) shall be called *conjugate*. A necessary and sufficient condition that the turbine (3.3) be self-conjugate is that $r+\bar{r}=0$; self-conjugate turbines, and only such turbines, are cycles.

By duality, two lineal elements (a, α) and (b, β), $\alpha\neq\beta$, have in common ∞^1 flat fields $\{s, \sigma\}$ whose coordinates satisfy the equation

$$s - \frac{a\beta - b\alpha}{\beta - \alpha} = -\frac{\bar{a} - \bar{b}}{\bar{\alpha} - \bar{\beta}}\sigma. \tag{3.5}$$

This equation represents the turbine determined by (a, α) and (b, β), $\alpha\neq\beta$, as a locus of flat fields.

Since there exists a (1, 1) correspondence between turbines and the ordered pairs of complex parameters l, r, we shall let the symbol $[l, r]$ represent the turbine whose equation in element coordinates is (3.3); l and r shall be called the *turbine coordinates* of $[l, r]$. The equation of $[l, r]$ in the dual coordinates s, σ is

$$s - l = -\bar{r}\sigma. \tag{3.6}$$

The turbine $[l, r]$ and the lineal element (z, ζ) shall be said to be *incident* if their coordinates satisfy the equation (3.3); likewise $[l, r]$ and $\{s, \sigma\}$ shall be said to be incident if their coordinates satisfy (3.6).

4. The complete group of whirls $\mathfrak{W}_3$. Let $\mathfrak{J}_1$ represent the involutory lineal element transformation $(z, \zeta)\rightarrow(z^*, \zeta^*)$, given by the equations

$$z^* = \bar{z}, \qquad \zeta^* = \bar{\zeta}. \tag{4.1}$$

The transformations $W^-_{a,\alpha} = \mathfrak{J}_1 W^+_{a,\alpha}$ shall be called *opposite whirls*. Evidently $W^+_{a,\alpha}\mathfrak{J}_1 = \mathfrak{J}_1 W^+_{\bar{a},\bar{\alpha}}$. Opposite whirls convert flat fields into flat fields and turbines into turbines. In dual coordinates, s, σ, transformation $\mathfrak{J}_1$ assumes the form

$$s^* = \bar{s}, \qquad \sigma^* = \bar{\sigma} \tag{4.2}$$

and in turbine coordinates, l, r, the form

$$l^* = \bar{l}, \qquad r^* = \bar{r}. \tag{4.3}$$

The direct and opposite whirls constitute a mixed three-parameter group of which $\mathfrak{W}_3^+$ is an invariant subgroup.

Let $\mathfrak{J}_2$ designate the transformation $(z, \zeta)\rightarrow\{s^*, \sigma^*\}$ given by the equations

$$s^* = z, \qquad \sigma^* = \zeta. \tag{4.4}$$

It is easy to ascertain that the ∞^1 lineal elements (z, ζ) incident to a given flat field $\{s, \sigma\}$ are converted by $\mathfrak{J}_2$ into the ∞^1 flat fields $\{s^*, \sigma^*\}$ incident to a unique lineal element (z^*, ζ^*); the latter shall, accordingly, be regarded as the image of $\{s, \sigma\}$ produced by $\mathfrak{J}_2$. We find that $z^*=s$, $\zeta^*=\sigma$; $\mathfrak{J}_2$ is consequently an involutory transformation. The ∞^3 transformations $C_{a,\alpha}^+\equiv\mathfrak{J}_2W_{a,\alpha}^+$ shall be called *direct correlations*, and the ∞^3 transformations $C_{a,\alpha}^-\equiv\mathfrak{J}_2W_{a,\alpha}^-$ shall be called *opposite correlations*. The direct and the opposite correlations transform turbines into turbines. In turbine coordinates l, r the involution $\mathfrak{J}_2$ assumes the form

$$l^* = l, \qquad r^* = -\bar{r}. \tag{4.5}$$

Consequently, $\mathfrak{J}_2$ *converts turbines into their conjugates.*

The four continuous, mutually exclusive, three-parameter families: $\mathfrak{W}_3^+$, $\mathfrak{W}_3^-$ (opposite whirls), $\mathfrak{C}_3^+$ (direct correlations), and $\mathfrak{C}_3^-$ (opposite correlations) constitute a mixed group—namely, the *complete group of whirls* $\mathfrak{W}_3$.

5. **Whirl-motions and whirl-similitudes.** Let the euclidean point displacement

$$z^* = \alpha z + a, \qquad |\alpha| = 1,$$

be extended to represent a displacement of lineal elements, as follows

$$z^* = \alpha z + a, \qquad \zeta^* = \alpha\zeta, \qquad |\alpha| = 1. \tag{5.1}$$

Let $D_{a,\alpha}^+$ denote the displacement of lineal elements given by (5.1), and let $\mathfrak{D}_3^+$ denote the group of such displacements. Furthermore, let $D_{a,\alpha}^-\equiv\mathfrak{J}_1D_{a,\alpha}^+$ and $\mathfrak{D}_3^-\equiv\mathfrak{J}_1\mathfrak{D}_3^+$; $\mathfrak{D}_3^-$ evidently denotes the family of euclidean symmetries operating on lineal elements. The group of euclidean motions $\mathfrak{D}_3$ $(=\mathfrak{D}_3^+ + \mathfrak{D}_3^-)$ converts flat fields into flat fields and turbines into turbines. The product of a direct whirl by a displacement is commutative. Let such a product be called a *direct whirl-motion*. The direct whirl-motions constitute a continuous, six-parameter group which shall be denoted by $\mathfrak{G}_6^1$.

Let

$$\mathfrak{G}_6^2 \equiv \mathfrak{W}_3^+\mathfrak{D}_3^-, \qquad \mathfrak{G}_6^3 \equiv \mathfrak{C}_3^+\mathfrak{D}_3^+, \qquad \mathfrak{G}_6^4 \equiv \mathfrak{C}_3^-\mathfrak{D}_3^+.$$

Evidently $\mathfrak{G}_6^2=\mathfrak{J}_1\mathfrak{G}_6^1$; $\mathfrak{G}_6^3=\mathfrak{J}_2\mathfrak{G}_6^1$; $\mathfrak{G}_6^4=\mathfrak{J}_1\mathfrak{G}_6^3$. The four families $\mathfrak{G}_6^i$ are mutually exclusive and constitute a mixed six-parameter group $\mathfrak{G}_6$, which shall be called the *group of proper whirl-motions*. The two families $\mathfrak{G}_6^1$, $\mathfrak{G}_6^2$ form a mixed invariant subgroup of $\mathfrak{G}_6$.

Let $\mathfrak{J}_3$ denote the involutory lineal element transformation

$$(5.2)\qquad z^* = z\bar{\zeta}, \qquad \zeta^* = \bar{\zeta}.$$

Evidently $\mathfrak{J}_3$ transforms flat fields into flat fields and turbines into turbines. In dual coordinates $\mathfrak{J}_3$ is given by the equations

$$(5.3)\qquad s^* = -\bar{s}\sigma, \qquad \sigma^* = \sigma,$$

and in turbine coordinates l, r, by the equations

$$(5.4)\qquad l^* = r, \qquad r^* = l.$$

Let $\mathfrak{H}_6^i\equiv\mathfrak{J}_3\mathfrak{G}_6^i$, $i=1, 2, 3, 4$. The four, mutually exclusive, continuous families of turbine-preserving transformations $\mathfrak{H}_6^i$ comprise the *improper whirl-motions*. The totality of proper and improper whirl-motions constitute a mixed group composed of eight continuous families, which shall be called the *complete group of whirl-motions*, and denoted by Γ_6.

Let $\mathfrak{M}$ denote the magnification $(z, \zeta)\rightarrow(z^*, \zeta^*)$ given by

$$(5.5)\qquad z^* = kz, \qquad \zeta^* = \zeta,$$

where k is real and positive. Let $\Gamma_7\equiv\mathfrak{M}\Gamma_6$. Evidently Γ_7 is a mixed seven-parameter group of turbine-preserving transformations comprising the eight continuous families: $\mathfrak{G}_7^i\equiv\mathfrak{M}\,\mathfrak{G}_6^i$, $\mathfrak{H}_7^i\equiv\mathfrak{M}\,\mathfrak{H}_6^i$, $i=1, 2, 3, 4$; Γ_7 shall be called the *complete group of whirl-similitudes*.

6. The kinematic representation. Owing to the (1, 1) correspondence subsisting between turbines $\mathfrak{T}$: $[l, r]$ and ordered pairs of complex numbers l, r, $\mathfrak{T}$ can be mapped upon the ordered pair of points on the Gauss plane represented by l and r in this order. This mapping, $\mathfrak{T}\rightarrow(l, r)$, of the ∞^4 turbines upon the ∞^4 ordered point pairs shall be called a *kinematic representation*. With the aid of equations (3.3) and (3.6), which entail necessary and sufficient conditions for the incidence of turbine and lineal element in the former case, and for the incidence of turbine and flat field in the latter, the following two theorems can now be established.

THEOREM 1. *The kinematic* (1, 1) *representation of turbines by ordered pairs of real points on a plane, namely* $\mathfrak{T}\leftrightarrow(l, r)$, *induces a* (1, 1) *correspondence between lineal elements* (z, ζ) *and planar euclidean displace-*

ments $l\rightarrow r$. *To the elements* $(\mathfrak{z}, \zeta)$ *such that* $\zeta=-1$ *correspond the translations* $r=l-\mathfrak{z}$, *and to those such that* $\zeta\neq-1$ *correspond the rotations around the point* $\mathfrak{z}/(\zeta+1)$ *through the angle* arg $(-\bar{\zeta})$.

THEOREM 2. *The kinematic* (1, 1) *representation* $\mathfrak{T}\leftrightarrow(l, r)$ *induces a* (1, 1) *correspondence between flat fields* $\{s, \sigma\}$ *and planar euclidean symmetries* $l\rightarrow r$. *To the flat fields* $\{s, \sigma\}$ *such that* $\sigma\bar{s}+s=0$ *correspond reflections* $l\rightarrow r$ *in the line* $\bar{s}\mathfrak{z}+s\bar{\mathfrak{z}}=s\bar{s}$; *and to those such that* $\sigma\bar{s}+s\neq 0$ *correspond symmetries compounded, in either order, of a reflection in the axis* $\mathfrak{z}-s/2=\sigma(\bar{\mathfrak{z}}-\bar{s}/2)$ *and a translation (parallel to the axis) given by* $r=l-(\sigma\bar{s}+s)/2$.

The proper whirl-similitudes—that is, those constituting the four families $\mathfrak{S}_\gamma^i$—regarded as turbine transformations $\mathfrak{T}\rightarrow\mathfrak{T}^*$, are mapped by means of the kinematic representation upon pairs of similitudes $l\rightarrow l^*$, $r\rightarrow r^*$ having the same ratio of magnification. These pairs of similitudes form four continuous families corresponding to the $\mathfrak{S}_\gamma^i$; their equations are as follows:

$$\begin{aligned}
&\mathfrak{S}_\gamma^1\colon\quad l^* = \alpha k l + a, &&r^* = \beta k r + b,\\
&\mathfrak{S}_\gamma^2\colon\quad l^* = \alpha k \bar{l} + a, &&r^* = \beta k \bar{r} + b,\\
&\mathfrak{S}_\gamma^3\colon\quad l^* = \alpha k l + a, &&r^* = \beta k \bar{r} + b,\\
&\mathfrak{S}_\gamma^4\colon\quad l^* = \alpha k \bar{l} + a, &&r^* = \beta k r + b,\\
&k \text{ real and positive},\ |\alpha| = |\beta| = 1.
\end{aligned}$$

By interchanging l and r in the right-hand members of the equations of the similitudes corresponding to $\mathfrak{S}_\gamma^i$, we obtain the equations of the kinematic image of the family of improper whirl-similitudes $\mathfrak{H}_\gamma^i$. The kinematic images of the eight families comprising Γ_8 are the pairs of euclidean motions to which the above pairs of similitudes reduce when $k=1$.

7. **The quasi-elliptic representation.** Let a point $\mathfrak{p}$ (X, Y, Z) in S_3 have the homogeneous coordinates $X=p_2/p_1$, $Y=p_3/p_1$, $Z=p_0/p_1$. Let a line C, together with the pair of conjugate imaginary points on C, namely:

$$\begin{aligned}
&c_l, &&p_0\colon p_1\colon p_2\colon p_3 = 0\colon 0\colon i\colon 1,\\
&c_r, &&p_0\colon p_1\colon p_2\colon p_3 = 0\colon 0\colon 1\colon i,
\end{aligned}$$

and the pair of conjugate imaginary planes through C:

$$\gamma_l,\quad p_0 - ip_1 = 0, \qquad \gamma_r,\quad p_0 + ip_1 = 0$$

be the Cayley absolute underlying a non-euclidean space Q_3, called *quasi-elliptic* by Blaschke [1, 2]. The proper (real) points of Q_3 are those that do not lie on C; the proper lines are those that do not intersect C; and the proper planes are those that do not pass through C.

By means of a method independently discovered by J. Grünwald [5] and W. Blaschke [1, 2], the ∞^4 proper (real) lines $\mathfrak{g}$ of Q_3 can be mapped continuously and (1, 1) upon the ∞^4 pairs of ordered real points (l, r) of the euclidean ground plane $Z=0$ in a manner such that the following two properties subsist: (1) the two fields of points l and r formed by the ∞^2 pairs (l, r) that correspond to the ∞^2 proper real lines $\mathfrak{g}$ passing through a proper real point $\mathfrak{p}$ are such that a unique euclidean displacement $l \rightarrow r$ will transform the points l into their associated points r; (2) the two fields l and r formed by the ∞^2 pairs (l, r) that correspond to the ∞^2 proper real lines $\mathfrak{g}$ lying in a proper real plane π are such that a unique euclidean symmetry $l \rightarrow r$ will transform the points l into their associated points r. By virtue of these two properties a (1, 1) correspondence exists between the points $\mathfrak{p}$ of Q_3 and the euclidean displacements in the ground plane on the one hand, and between the planes π and the euclidean symmetries in the ground plane on the other hand.

Let us associate with every proper real line $\mathfrak{g}$ in Q_3 a turbine $\mathfrak{T}$ in such a manner that $\mathfrak{g}$ is the B-G (Blaschke-Grünwald) image of that pair of points l, r as that of which $\mathfrak{T}$ is the kinematic image. The ensuing correspondence $\mathfrak{g} \leftrightarrow \mathfrak{T}$ is (1, 1) and continuous. Furthermore, let us establish a (1, 1) correspondence between lineal elements $\mathcal{E}$ and proper points $\mathfrak{p}$ in Q_3 such that $\mathcal{E}$ and $\mathfrak{p}$ are images—kinematic and B-G, respectively,—of the same euclidean displacement $l \rightarrow r$; and, finally, let us establish a third (1, 1) correspondence between flat fields $\mathcal{J}$ and proper planes π such that $\mathcal{J}$ and π are images of the same symmetry $l \rightarrow r$. These three (1, 1) correspondences define the *quasi-elliptic representation* of whirl-similitude geometry. Incidences among lineal elements, turbines, and flat fields imply corresponding incidences among their quasi-elliptic images.

Inasmuch as the group $\overline{\mathfrak{G}}_7^1$ of collineations in Q_3 that leave the points c_l, c_r and the planes γ_l, γ_r of the absolute individually invariant corresponds, by virtue of the B-G mapping, to the group of transformations $(l, r) \rightarrow (l^*, r^*)$ such that $l \rightarrow l^*$, $r \rightarrow r^*$ are two similitudes having the same ratio of magnification [1, 2], we see that, by virtue of the results given in §6, $\overline{\mathfrak{G}}_7^1$ is simply isomorphic to $\mathfrak{G}_7^1$. The totality of automorphisms (collineations and correlations) of Q_3 constitutes a mixed group $\overline{\Gamma}_7$, composed of eight continuous families, $\overline{\mathfrak{G}}_7^i$ and $\overline{\mathfrak{H}}_7^i$, $i=1, 2, 3, 4$, which are isomorphic to the families comprising the complete group of whirl-similitudes Γ_7. To characterize the isomorphism

$\Gamma_\gamma \leftrightarrow \Gamma_\gamma$, it is sufficient to indicate the quasi-elliptic representations of $\mathfrak{J}_1$, $\mathfrak{J}_2$, and $\mathfrak{J}_3$. These are, in terms of the involutory transformations effected on the absolute, as follows

$$\begin{array}{lllll} \mathfrak{J}_1: & c_l \rightarrow c_r, & c_r \rightarrow c_l, & \gamma_l \rightarrow \gamma_r, & \gamma_r \rightarrow \gamma_l; \\ \mathfrak{J}_2: & c_l \rightarrow \gamma_l, & c_r \rightarrow \gamma_r, & \gamma_l \rightarrow c_l, & \gamma_r \rightarrow c_r; \\ \mathfrak{J}_3: & c_l \rightarrow c_l, & c_r \rightarrow c_r, & \gamma_l \rightarrow \gamma_r, & \gamma_r \rightarrow \gamma_l. \end{array}$$

To $\mathfrak{G}_6^1$ (Kasner's whirl-motion group) there corresponds in quasi-elliptic space, by virtue of the quasi-elliptic representation, the group of *quasi-motions* $\overline{\mathfrak{G}}_6^1$. The quasi-motions can be uniquely resolved into products of what Blaschke calls, because of their analogy to Clifford's left and right translations in elliptic space, *quasi-left and quasi-right* translations in Q_3 [1]; the former are the quasi-elliptic images of the displacements $\mathfrak{D}_3^+$, and the latter are the images of the direct whirls $\mathfrak{W}_3^+$.

By means of the quasi-elliptic representation, many of the results obtained by Kasner and DeCicco for the geometry of whirl-motions can be easily identified with results obtained independently by Blaschke and Grünwald for quasi-elliptic space.

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QUEENS COLLEGE AND
BROOKLYN COLLEGE

A PROPOSED JOURNAL OF APPLIED MATHEMATICS

During the past two years a school for advanced instruction in applied mechanics has been conducted at Brown University with notable success. One result has been to form a group of men from many different centers who desire to make a permanent place for advanced applied mathematics in this country. To this end, it has now been decided to publish a *Journal of Applied Mathematics*, to appear quarterly, and to form a volume of 300–400 pages annually. The first issue will be as for January, 1943, although actual publication may be delayed beyond that month.

From time to time over the past two decades, a journal of this type has been proposed, but the project has never been brought to a focus. This journal, as now planned, is to fill the gap between purely engineering journals and purely mathematical ones. It will provide a place where the various subjects can be regarded as sciences, and where any piece of work which fills a gap in this structure or extends it, can be accepted without having to show immediate application to engineers or fundamentally new ideas of mathematical interest to mathematicians.

Not only must applied mathematics serve the war effort, as it is now doing in increasing measure, but, looking ahead to the post-war period, it appears essential that America should be strong in the whole field of Science in order that it serve as a world center for advanced instruction and research. The new journal will aid in bringing America to the forefront in applied mechanics and in other branches of applied mathematics. It is hoped that many mathematicians will approve and will give this project both moral and financial support.

The Board of Editors will be:

Dr. H. L. Dryden, Physicist, National Bureau of Standards.
Dr. T. C. Fry, Mathematical Research Director, Bell Telephone Laboratories.
Dr. W. Prager, Professor of Applied Mathematics, Brown Univ.
Dr. I. S. Sokolnikoff, Professor of Mathematics, Univ. of Wisconsin.
Dr. J. L. Synge, Professor of Applied Mathematics, Univ. of Toronto.
Dr. Th. von Kármán, Professor of Aeronautical Engineering, Calif. Inst. of Tech.; Consultant, Northrop Aircraft, Inc.

These names will carry a message of confidence of high standards to all who know the field. Some fifteen others, eminent in their fields, will be asked to accept appointments as collaborators.

E. R. HEDRICK

OCCUPATIONAL CLASSIFICATION OF MATHEMATICIANS

General Lewis B. Hershey, Director of Selective Service, has issued Occupational Bulletin No. 23 which certifies that the educational services of instructors and professors of mathematics in colleges, universities, or professional schools are activities essential to the war effort. This Bulletin supplements Occupational Bulletin No. 10 which was distributed to department chairmen by the Secretary's office on June 30, 1942. Bulletin No. 10 and Bulletin No. 23 are given at the end of this note.

This memorandum is published with the realization that the whole situation with respect to our colleges and universities will undoubtedly be fundamentally altered by the legislation concerned with the lowering of the draft age. Nevertheless, we do believe that plans will be developed whereby the government will use the machinery already existing in our colleges and universities in order to train young men in the mathematics and science which are the necessary equipment of Army and Navy officers. The need for instructors in mathematics is bound to increase as the tempo of the war is increased.

The Secretary of War has stated that teachers in fields that are essential to the war effort are doing the job that the country wants them to do and are performing their full duty in the war effort. With this in mind, every effort should be made to conserve the supply of mathematicians; this supply is already greatly depleted. Mathematicians should not be discouraged from assuming positions in fields other than teaching, if these offer greater opportunities for service to the nation in the war effort. On the other hand, mathematics teachers should not feel that their services will shortly be unnecessary and consequently rush into positions where their contribution is less vital than that of teaching men who are preparing for commissions in the armed forces. Mathematics at the college level will have to be taught to prospective officers. We hope that this teaching will be done in our colleges and universities. Whether it is done there or in schools set up by the government, mathematicians will be needed and the supply should be conserved.

I. REQUESTS FOR OCCUPATIONAL CLASSIFICATION

In view of the recent Bulletin, we wish to offer the following suggestions for securing the proper classification of mathematicians:

(a) *For those who have completed their training, i.e., instructors and men of professorial grade.* Requests for continued occupational deferment or for reclassification should be based on Occupational Bulletin No. 23 in which the Manpower Commission has certified as essential to the war effort professors and instructors engaged in full-time instruction and research in mathematics. This Bulletin should be attached to every request. In addition, Occupational Bulletin No. 10, which certifies that there is a serious shortage of mathematicians, should be attached. Local boards must be supplied with all the necessary information; this should include evidence that the registrant cannot be replaced. The request for reclassification should be made on Form 42A which can be obtained from the registrant's local board.

(b) *For those in training, i.e., graduate assistants and undergraduate majors in mathematics.* The officer requesting the deferment should file Form 42A directly with the local board with the ordinary questionnaire filled out by all registrants. Reference should be made to Occupational Bulletin No. 10. It is urged that a copy of this Bulletin be attached to Form 42A when it is filed.

II. APPEALS

Having made request for occupational deferment, the person making the request should be prepared to exert every effort to secure that deferment, including the making of an appeal. The appeal may be made by the registrant, his employer, or the government appeal agent. When it is clear that the end of negotiations with the local board has been reached, without having modified the registrant's 1A classification, the appeal should be carried through. *Such an appeal must be made within ten days of the date when the local board mailed to the registrant a notice of his classification.* The form for appeals appears at the end of the questionnaire.

III. ADDITIONAL STEPS

National Roster of Scientific and Specialized Personnel, 10th and U Streets, N. W., Washington, D. C.

Any mathematician who has not already done so should register immediately with the National Roster. This may be done by writing a letter to the Roster at the above address, and asking for appropriate forms in mathematics. In this communication, the writer should indicate whether he is subject to the Selective Training and Service Act of 1940.

The National Roster may take appropriate action in the case of professional mathematicians and graduate students in mathematics.

Under a cooperative plan with the Selective Service System, the Roster may send to the National Headquarters of the Selective Service System appropriate information about professionally trained men of military age, and that office in turn may then forward letters about these men through the various State Directors to the individual local boards, in order to assist in proper classification.

It is essential that a man (or his employing institution) should *not* wait until he is placed in Class 1A before writing to the Roster. The Roster should be kept informed as to any changes in his Selective Service status or in his employment. In communicating with the Roster, a man should give the following information: (1) the name, number, and address of his Selective Service local board; (2) his own order number; (3) a description of his specific duties, especially as related to the war effort.

Office of Scientific Personnel, National Research Council,
2101 Constitution Avenue, Washington, D. C.

Dean Homer L. Dodge is now Director of the Office of Scientific Personnel. In cases where an appeal has been made and developments indicate that a proper decision may not be reached, it is advisable to contact Dean Dodge. He must be informed as to the applicant's order number and the name, number, and address of the local board under which he is registered. Dean Dodge has offered to consult with the State Director concerned or with Selective Service Headquarters as the urgency of the case demands.

However, the Office of Scientific Personnel must start where the person making the original request leaves off and can do little to make up for deficiencies in the presentation of the case by the employer. Unless there is presented evidence that a genuine, but unsuccessful, attempt has been made to secure a replacement for the particular registrant in question, it will be impossible for the Office of Scientific Personnel to be of assistance.

Copies of Occupational Bulletin No. 10 and of Occupational Bulletin No. 23 will be furnished, on request, by the office of the Secretary.

MARSTON MORSE, *President*
J. R. KLINE, *Secretary*

November 2, 1942.

National Headquarters
SELECTIVE SERVICE SYSTEM
21st Street and C Street, N. W.
Washington, D. C.

COPY September 30, 1942

OCCUPATIONAL BULLETIN NO. 23

EFFECTIVE: IMMEDIATELY
SUBJECT: EDUCATIONAL SERVICES

1. The War Manpower Commission has certified that educational services are essential to the support of the war effort.

2. This bulletin covers the following essential activities which are considered as included within the list attached to Local Board Release No. 115, as amended:

(a) *Educational services*: Public and private industrial vocational training; elementary, secondary and preparatory schools; junior colleges, colleges, universities and professional schools; educational and scientific research agencies; and the production of technical and vocational training films.

3. In considering registrants engaged in educational services there must be taken into consideration the following:

(a) The kind of institution in which the registrant is engaged;

(b) the occupation of the registrant in that institution; and

(c) the classroom studies under the registrant's instruction, supervision, or administration jurisdiction. Attached is a list of occupations by institutions and classroom studies in educational services which require a reasonable degree of training, qualification, or skill to perform the duties involved. It is the purpose of this list to set forth by institutions and classroom studies the important occupations in educational services which must be filled by persons capable of performing the duties involved in order that the essential portions of the activity may be maintained. Item 4 of the list does not include classroom studies but occupations which shall be considered in the same manner as any other occupations. The entire list is confined to occupations which require more than six months of training and preparation.

4. In classifying registrants employed in these activities, consideration should be given to the following:

(a) The training, qualification, or skill required for the proper discharge of the duties involved in his occupation;

(b) the training, qualification, or skill of the registrant to engage in his occupation; and

(c) the availability of persons with his qualifications or skill, or who can be trained to his qualification, to replace the registrant and the time in which such replacement can be made.

LEWIS B. HERSHEY, *Director*

CRITICAL OCCUPATIONS

Educational Services

1. Elementary, Secondary and Preparatory Schools

(a) Superintendents of elementary, secondary and preparatory school systems; and
(b) teachers who are engaged in full-time instruction in one or more of the following subjects:

Aeronautics
Biology
Chemistry
Mathematics
Physics
Radio

2. Junior Colleges, Colleges, Universities and Professional Schools, Educational and Scientific Research Agencies

(a) Presidents, Deans, and Registrars in junior colleges, colleges, universities and professional schools; and
(b) professors and instructors engaged in full-time instruction and research in one or more of the following subjects:

Agricultural Sciences
Architecture, Naval
Astronomy
Bacteriology
Biology
Chemistry
Dentistry
Engineering Sciences
Geology
Industrial Management
Mathematics
Medicine and Surgery
Metallurgy
Meteorology
Navigation, Aerial and Marine
Oceanography
Pharmacy
Physics
Physiology
Veterinary Sciences

3. Public and Private Industrial Vocational Training

(a) Superintendents of public and private industrial vocational training; and
(b) teachers who are engaged in full-time instruction in one or more of the following subjects designated to prepare students for war activities:
Trade, Vocational and Agricultural subjects (such as, Machine Shop practice, Aircraft, Sheetmetal Work, and similar subjects) and in Vocational Rehabilitation.

4. Production of Technical and Vocational Training Films

(a) Persons engaged full-time and exclusively in the production of technical and vocational training films for the Army, Navy and war production industries.

Animator
Camerman
Cutter
Film Editor
Project Supervisor
Technical Consultant
Technical Writer
Sound Engineer

National Headquarters
SELECTIVE SERVICE SYSTEM
21st Street and C Street, N. W.
Washington, D. C.

COPY

June 18, 1942

OCCUPATIONAL BULLETIN (NO. 10)

SUBJECT: SCIENTIFIC AND SPECIALIZED PERSONNEL
EFFECTIVE: IMMEDIATELY
DISTRIBUTION: STATE DIRECTORS
BOARD OF APPEAL MEMBERS
LOCAL BOARD MEMBERS
GOVERNMENT APPEAL AGENTS

PART I

1. There are certain persons trained, qualified, or skilled in scientific and specialized fields who, if engaged in the practice of their respective professions, are in a position to perform a vital service in activities necessary to war production and in activities essential to the support of the war effort.

PART II

1. The National Roster of Scientific and Specialized Personnel has certified to the Director of Selective Service that in activities necessary to war production and in activities essential to the support of the war effort, there are certain "critical occupations" which for the proper discharge of the duties involved require a high degree of training, qualification, or skill in scientific and specialized fields. The critical occupations in these scientific and specialized fields, as certified to the Director of Selective Service, are listed on page 4 attached to this bulletin.

2. All of these critical occupations, as listed, require highly specialized periods of training of two years or more. The critical occupations on the attached list exist within the provisions of Part V, Memorandum to All State Directors (I-405).

PART III

1. The National Roster of Scientific and Specialized Personnel has certified to the Director of Selective Service that there are serious shortages of persons trained, qualified, or skilled to engage in these critical occupations in activities necessary to war production and in activities essential to the support of the war effort. These shortages exist within the provisions of Part VII, Memorandum to All State Directors (I-405), and accordingly careful consideration for occupational classification should be given to all persons trained, qualified, or skilled in these critical occupations and who are engaged in activities necessary to war production or essential to the support of the war effort.

PART IV

1. There are many registrants who are in training and preparation to acquire the qualification or skill to engaged in these critical occupations. Normally the period of training and preparation to acquire the necessary qualification or skill in these scientific and specialized fields extends over a period of four academic years in a recognized academic, professional, or technical college or university. In many instances, however,

it is necessary for persons to have additional study in a recognized academic, professional, or technical college or university in order to acquire the more highly specialized qualification or skill necessary for the performance of particular services in activities necessary to war production or essential to the support of the war effort. Persons engaging in further studies in addition to the four academic years normally required are referred to as graduate or postgraduate students.

2. A registrant who is in training and preparation for one of these scientific and specialized fields may not be considered for occupational classification until the close, or approximately the close, of his second or sophomore year in a recognized college or university.

3. A registrant who is in training and preparation for one of these scientific and specialized fields may be considered for occupational deferment at the close, or approximately at the close, of his second or sophomore year in a recognized college or university if he is pursuing a course of study upon the successful completion of which he will have acquired the necessary training, qualification, or skill, and if he gives promise of continuing and will be acceptable for continuing such course of study and will undertake actual further classroom work within a period of not to exceed four months from the close of his second year.

4. A registrant who is in training and preparation for one of these scientific and specialized fields shall be considered for occupational classification during his third and fourth years in a recognized college or university, provided that he gives promise of the successful completion of such course of study and the acquiring of the necessary degree of training, qualification, or skill.

5. A graduate or postgraduate student who is undertaking further studies for these scientific and specialized fields, following the completion of the normal four academic years, may be considered for occupational classification if, in addition to pursuing the additional studies, he is also acting as "graduate assistant" in a recognized college or university or is engaged in scientific research related to the war effort and which is supervised by a recognized Federal agency. A graduate assistant is a student in postgraduate studies who, in addition, is engaged in the teaching and instruction of undergraduate students in these scientific and specialized fields.

6. When a registrant has completed his training and preparation in a recognized college or university and has acquired a high degree of training, qualification, or skill in one of these scientific and specialized fields, such registrant should then be given the opportunity to become engaged in the practice of his profession in an activity necessary to war production or essential to the support of the war effort. In many instances following graduation from a recognized college or university, a certain period of time will be required in the placing of trained, qualified, or skilled personnel in an essential activity. When a registrant has been deferred as a necessary man in order to complete his training and preparation, it is only logical that his deferment should continue until he has an opportunity to use his scientific and specialized training to the best interest of the nation. Accordingly, following graduation from a recognized college or university in any of these scientific and specialized fields, a registrant should be considered for further occupational classification for a period of not to exceed 60 days in order that he may have an opportunity to engage in a critical occupation in an activity necessary to war production or essential to the support of the war effort, provided that during such period the registrant is making an honest and diligent effort to become so engaged.

LEWIS B. HERSHEY, *Director*

CRITICAL OCCUPATIONS

SCIENTIFIC AND SPECIALIZED PERSONNEL

- Accountants
- Chemists
- Economists
- Engineers:
  - Aeronautical Engineers
  - Automotive Engineers
  - Chemical Engineers
  - Civil Engineers
  - Electrical Engineers
  - Heating, Ventilating, Refrigerating, and Air Conditioning Engineers
  - Marine Engineers
  - Mechanical Engineers
  - Mining and Metallurgical Engineers, including Mineral Technologists
  - Radio Engineers
  - Safety Engineers
  - Transportation Engineers—Air, Highway, Railroad, Water
- Geophysicists
- Industrial Managers
- Mathematicians
- Meteorologists
- Naval Architects
- Personnel Administrators
- Physicists, including Astronomers
- Psychologists
- Statisticians

THE SUMMER MEETING IN POUGHKEEPSIE

The forty-eighth Summer Meeting of the Society and the twenty-fourth Colloquium were held at Vassar College, Tuesday to Thursday, September 8–10, 1942. The Mathematical Association of America met on Monday morning and afternoon. The Institute of Mathematical Statistics met Wednesday morning and jointly with the Society Tuesday afternoon and Wednesday afternoon. About three hundred persons attended the meetings among whom were the following one hundred ninety-eight members of the Society:

C. R. Adams, R. B. Adams, R. P. Agnew, Warren Ambrose, R. C. Archibald, K. J. Arnold, L. A. Aroian, K. J. Arrow, W. L. Ayres, F. E. Baker, G. M. Bareis, Walter Bartky, Stefan Bergman, Felix Bernstein, Henry Blumberg, J. W. Bower, R. W. Brink, A. B. Brown, J. A. Bullard, F. J. H. Burkett, Hobart Bushey, J. H. Bushey, S. S. Cairns, W. D. Cairns, B. H. Camp, W. B. Carver, W. F. Cheney, D. E. Christie, J. A. Clarkson, L. W. Cohen, Nancy Cole, Esther Comegys, T. F. Cope, A. H. Copeland, Richard Courant, C. C. Craig, J. H. Curtiss, D. R. Davis, F. F. Decker, R. P. Dilworth, J. L. Doob, M. E. Elveback, Benjamin Epstein, Paul Erdös, W. K. Feller, M. M. Flood, L. R. Ford, R. M. Foster, K. O. Friedrichs, R. E. Fullerton, Abe Gelbart, B. P. Gill, R. E. Gilman, Michael Goldberg, Cornelius Gouwens, H. S. Grant, R. E. Greenwood, T. N. E. Greville, C. C. Grove, V. G. Grove, E. J. Gumbel, Theodore Hailperin, D. W. Hall, O. G. Harrold, O. C. Hazlett, C. M. Hebbert, G. A. Hedlund, E. R. Hedrick, Edward Helly, L. S. Hill, Einar Hille, T. R. Hollcroft, M. W. Hopkins, G. M. Hopper, Harold Hotelling, S. E. Hotelling, E. M. Hull, C. C. Hurd, Witold Hurewicz, W. A. Hurwitz, Dunham Jackson, Fritz John, R. A. Johnson, A. W. Jones, B. W. Jones, F. B. Jones, H. A. Jordan, Wilfred Kaplan, Irving Kaplansky, Edward Kasner, S. Y. Keyser, B. F. Kimball, J. R. Kline, A. E. Landry, R. E. Langer, G. A. Larew, Solomon Lefschetz, Charles Loewner, C. I. Lubin, Eugene Lukacs, N. H. McCoy, C. C. MacDuffee, Saunders MacLane, H. F. MacNeish, H. B. Mann, Morris Marden, A. J. Maria, D. H. Maria, M. H. Martin, M. P. Martin, A. E. Meder, F. H. Miller, E. B. Mode, E. C. Molina, Deane Montgomery, C. N. Moore, E. M. Morenus, Valdimir Morkovin, Richard Morris, D. S. Morse, Marston Morse, M. E. Munroe, Otto Neugebauer, A. V. Newton, K. L. Nielsen, R. E. O'Connor, E. G. Olds, Oystein Ore, Gordon Pall, C. R. Phelps, H. B. Phillips, Everett Pitcher, Hillel Poritsky, G. B. Price, M. H. Protter, A. L. Putnam, S. E. Rasor, L. L. Rauch, Mina Rees, C. F. Rehberg, W. T. Reid, R. G. D. Richardson, J. F. Ritt, M. S. Robertson, H. A. Robinson, S. L. Robinson, R. E. Root, R. A. Rosenbaum, J. H. Rosenbloom, P. C. Rosenbloom, Raphael Salem, Hans Samelson, Arthur Sard, F. E. Satterthwaite, L. J. Savage, Henry Scheffé, Abraham Schwartz, G. E. Schweigert, R. G. Simond, A. J. Smith, Gertrude Smith, Ernst Snapper, Virgil Snyder, D. E. Spencer, V. E. Spencer, Abraham Spitzbart, A. H. Sprague, H. W. Steinhaus, J. J. Stoker, R. R. Stoll, A. H. Stone, S. R. R. Struik, Otto Szász, Gabor Szegö, J. D. Tamarkin, C. J. Thorne, A. W. Tucker, J. W. Tukey, D. F. Votaw, Abraham Wald, R. M. Walter, Louis Weisner, B. A. Welch, M. E. Wells, E. T. Welmers, George Whaples, H. S. White, G. T. Whyburn, R. L. Wilder, S. S. Wilks, W. L. G. Williams, E. W. Wilson, Jack Wolfe, Jacob Wolfowitz, E. R. Worthington, C. H. Yeaton, J. W. T. Youngs, Antoni Zygmund.

The Colloquium Lectures on the subject *Topology of manifolds* were delivered by Professor R. L. Wilder of the University of Michigan on Tuesday morning and afternoon and Wednesday and Thursday mornings. The presiding officers at the four lectures were, in order, President Marston Morse, Professor Solomon Lefschetz, Professor G. T. Whyburn, and Vice President J. D. Tamarkin.

There was a short business meeting Wednesday morning at the beginning of the general session, President Marston Morse presiding. Secretary J. R. Kline reported briefly concerning actions of the Council and President Morse spoke about the present status with respect to military service of teachers and students of mathematics. Lieutenant C. C. Hurd described the training program at the United States Coast Guard Academy at New London, Connecticut.

On Wednesday at 2 P.M., Professor W. L. Ayres of Purdue University gave an address entitled *Transformations with periodic properties*. At 3:15 P.M., a symposium on *The applicability of mathematical statistics to war efforts*, was held jointly with the Institute of Mathematical Statistics. Addresses were given on *Statistical prediction with special reference to the problem of tolerance limits* by Professor S. S. Wilks of Princeton University, and *On the nature of mathematical statistics in quality control* by Dr. W. E. Deming of the U. S. Department of Commerce. The discussion leaders were Professor J. H. Curtiss of Cornell University, and Dean Walter Bartky of the University of Chicago.

The presiding officers for the sessions of short papers were: Analysis, Tuesday morning, Professor R. E. Langer; Algebra, Geometry, and Topology, Tuesday morning, Professor A. W. Tucker; Mathematical Statistics (a joint session with the Institute of Mathematical Statistics), Tuesday afternoon, Professor C. C. Craig; Topology, Thursday morning, Professor R. L. Wilder; Analysis and Applied Mathematics, Thursday morning, Professor Gabor Szegö. Professor J. R. Kline presided at the general session at 2 P.M., Wednesday and Lieutenant Colonel L. E. Simon at the symposium held jointly with the Institute of Mathematical Statistics.

Headquarters for the meeting were in Main Building. Rooms in this building were made available to members of the three organizations and their guests.

On Wednesday evening in Main Building there was a dinner for members of the Society, the Association, and the Institute. Professor H. S. White was toastmaster. President H. N. MacCracken of Vassar College gave an address of welcome. Professor R. E. Langer spoke of the impetus being given to mathematics in our secondary schools and colleges. Professor W. T. Reid presented a resolution of thanks to the President, the Department of Mathematics, the Local Committee, and everyone concerned at Vassar College, for the competent arrangements and cordial hospitality. The resolution was adopted unanimously. Following the dinner members of the Department of Music of Vassar College gave an excellent concert in Belle Skinner Hall.

On Monday afternoon in Ely Hall, a tea was given for members of the three organizations and their guests by the Department of Mathematics of Vassar College.

The Council met on Tuesday, September 8, at 8:00 P.M. in Main Building at Vassar College.

The Secretary announced the election of the following twenty persons to membership in the Society:

Dr. William D. Berg, State University of Iowa;
Mr. Howard Jesse Eckweiler, New York University;
Mr Joseph Owen Evans, Shreveport Aeronautical Institute, Shreveport, La.;
Mother Ethelyn Fox, Barat College of the Sacred Heart, Lake Forest, Ill.;
Mr. William J. Fry, Naval Research Laboratory, Washington, D. C.;
Mr. Cecil Hastings, Jr., University of Florida;
Mr. Cyrus Giles Hill, J. G. Wray and Co., Engineers, Chicago. Ill.;
Professor Truman Lee Kelley, Graduate School of Education, Harvard University;
President Eugene McDermott, Geophysical Service, Inc., Dallas, Tex.;
Dr. Leonard McFadden, Virginia Polytechnic Institute;
Mr. Frank Albert McMahon, Brooklyn College Evening Session;
Dr. Henry Malin, College of the City of New York;
Mr. William Edwin Manring, American Cyanamid and Chemical Corporation, New York, N. Y.;
Professor Philip B. Norman, Tusculum College, Greenville, Tenn.;
Professor Edward Russell Stabler, Hofstra College, Hempstead, N. Y.;
Dr. Domina Eberle Spencer, Massachusetts Institute of Technology;
Dr. Robert L. Swain, University of Wisconsin;
Mr. Hsio-Fu Tuan, Princeton University;
Mr. David Freeman Votaw, Jr., Princeton University;
Mr. Hyman Joseph Zimmerberg, University of Chicago.

It was also announced that Texas Technological College, Lubbock, Texas, had been elected to institutional contributing membership.

Professor Michel Plancherel of the Ecole Polytechnique Fédérale, Zurich, Switzerland, was admitted to the Society in accordance with the reciprocity agreement with the Swiss Mathematical Society.

The following appointments by President Marston Morse were reported: as representative of the Society at the Inaugural Ceremony of the Polish Institute of Arts and Sciences in America on May 15, 1942, Professor C. C. MacDuffee; as representative of the Society at the inauguration of Harry Noble Wright as President of The College of the City of New York on September 30, 1942, Professor C. C. MacDuffee; as a member of the Committee on Places of Meetings, to replace Professor P. R. Rider (term to expire December 31, 1944), Professor E. W. Chittenden.

It was announced that reports were sent in August to the Carnegie Corporation and Rockefeller Foundation regarding their grants to Mathematical Reviews. The Secretary also announced that the problem of the importation of journals from European Axis and Axis-controlled countries had been solved for Mathematical Reviews. Under an arrangement concluded with the American Library Association, Mathematical Reviews is permitted to purchase all the journals to which it has been subscribing, thus continuing full coverage of the mathematical literature of the scientific world.

Certain invitations to give invited addresses were announced: Professor J. L. Synge for the October, 1942, meeting in New York; Professors Harry Bateman, L. M. K. Boelter, J. Holmboe, and H. U. Sverdrup for the November, 1942, meeting in Los Angeles; Professor Deane Montgomery for the 1942 Annual Meeting in New York; and Professor Emil Artin for the April, 1943, meeting in Chicago.

On recommendation of the Committee on Places of Meetings (Professor A. B. Coble, chairman), the Annual Meeting of 1943 was set at Cleveland, Ohio, in connection with the meetings of the American Association for the Advancement of Science. Times and places of meetings during 1943 were set as follows: in New York City on February 27; in New York City on April 23–24; in Chicago on April 23–24; at Stanford University on April 24; in Corvallis, Oregon, on June 16–17 (with A.A.A.S.); in New York City on October 30.

Because of travel difficulties and other problems created by the war, the Council voted to withdraw its acceptance of the invitation to meet at the University of Colorado for the Summer Meeting of 1943. It is the intention of the Society to meet at Colorado as soon as conditions permit. Arrangements are under way for holding a summer meeting at a more centrally-located institution.

The eighteenth Josiah Willard Gibbs Lecture is to be given by Professor Harry Bateman at the Annual Meeting of 1943 in Cleveland, Ohio.

On recommendation of the Colloquium Editorial Committee, the Council voted to invite Professor Einar Hille to deliver a series of Colloquium Lectures in 1944.

Professor Marston Morse was appointed representative on the National Research Council for a period of three years beginning July, 1943.

The Secretary announced the provisions of the will of the late Dr. Robert Henderson who was, for many years, a Trustee of the Society and deeply interested in all of its activities. The Society will share the income of his estate with two other beneficiaries, and, upon the death of the other legatees, the Society will receive the full principal of the estate for the Endowment Fund.

President Morse appointed the following committees: Committee on the Award of the Bôcher Prize, to be awarded at the Annual Meeting of 1943, for papers in Analysis published during the period 1938–1942, Professors T. H. Hildebrandt (chairman), Philip Franklin, and John von Neumann; Committee to Select Gibbs Lecturers for 1944 and 1945, Professors J. H. Van Vleck (chairman), B. O. Koopman, and E. J. McShane.

The Council voted to continue the practice, inaugurated two years ago, of presenting on the ballot twice as many candidates for vice-president and for members-at-large of the Council as there are positions to be filled. It was also voted that the ballot continue to have a place where members may present suggestions for trustees, officers, and members of the Council, these suggestions to be considered by the Nominating Committee for the following year. Members are urged to make more use of this method of placing names before the Nominating Committee.

Titles and cross references to the abstracts of papers read are given below. The papers were read as follows: papers 1–6 in the section for Analysis on Tuesday morning; papers 7–12 in the section for Algebra, Geometry, and Topology on Tuesday morning; papers 13–22 in the section for Mathematical Statistics on Tuesday afternoon; papers 23–27 in the general session Wednesday morning; papers 28–31 in the section for Topology Thursday morning; papers 32–34 in the section for Analysis and Applied Mathematics Thursday morning; and papers 35–89, whose abstract numbers are followed by the letter *t*, were read by title. Dr. Spencer was introduced by Professor D. J. Struik, Mr. Kincaid by Dr. D. W. Hall, Miss Marth by Professor Francis Regan, and Dr. Schwerdtfeger by Dr. Peter Scherk. Paper 2 was presented by Professor Clarkson, paper 3 by Professor Adams, paper 5 by Dr. Salem, paper 9 by Professor MacLane, paper 17 by Dr. Mann, paper 21 by Dr. Struik, paper 23 in the absence of Professor Whitney by Professor Saunders MacLane, paper 24 by Professor Hille, paper 30 by Dr. Samelson, and paper 33 by Dr. Poritsky.

1. M. E. Munroe: *On the finite dimensionality of certain Banach spaces*. (Abstract 48-7-241).

2. J. A. Clarkson and Paul Erdös: *On the approximation of continuous functions by polynomials.* (Abstract 48-9-258.)

3. C. R. Adams and A. P. Morse: *On approximating certain integrals by sums.* (Abstract 48-9-255.)

4. Otto Szász: *On the partial sums of Fourier series at points of discontinuity.* (Abstract 48-9-264).

5. Raphael Salem and D. C. Spencer: *The influence of gaps on density of integers.* (Abstract 48-7-245.)

6. P. C. Rosenbloom: *The zeros of the partial sums of power series representing entire functions.* (Abstract 48-11-321.)

7. C. J. Thorne: *An Appell subset.* (Abstract 48-9-265.)

8. Domina E. Spencer: *Geometric figures in affine space.* (Abstract 48-9-281.)

9. Saunders MacLane and Samuel Eilenberg: *Functions of groups as generalized tensors.* (Abstract 48-11-341.)

10. Gordon Pall: *The weight of an n-ary genus of quadratic forms.* (Abstract 48-11-307.)

11. Saly R. R. Struik: *Axiomatics of affine geometry.* (Abstract 48-11-334.)

12. R. P. Dilworth: *On the decomposition theory of modular lattices.* (Abstract 48-11-301.)

13. Henry Scheffé: *On the theory of testing composite hypotheses with one constraint.* (Abstract 48-7-248.)

14. Jacob Wolfowitz: *On the consistency of a class of non-parametric statistics.* (Abstract 48-11-338.)

15. E. J. Gumbel: *Graphical controls based on serial numbers.* (Abstract 48-9-283.)

16. D. S. Villars: *Significance tests for multivariate distributions.* (Abstract 48-9-289.)

17. H. B. Mann and Abraham Wald: *On the choice of the number of class intervals in the application of the chi-square test.* (Abstract 48-9-285.)

18. F. E. Satterthwaite: *Generalized Poisson distribution.* (Abstract 48-9-286.)

19. L. A. Aroian: *The relationship of Fisher's z distribution to Student's t distribution.* (Abstract 48-9-282.)

20. Abraham Wald: *On a statistical problem arising in the classification of an individual in one of two groups.* (Abstract 48-11-337.)

21. Saly R. R. Struik and Miriam van Waters: *Modern statistical methods in penology.* (Abstract 48-9-288.)

22. T. N. E. Greville: *Regularity of label-sequences under configuration transformations.* (Abstract 48-11-335.)

23. Hassler Whitney: *The self-intersections of an M^n in E^{2n}*. (Abstract 48-9-297.)

24. Einar Hille and Gabor Szegö: *On the complex zeros of the Bessel functions.* (Abstract 48-11-313.)

25. Edward Kasner: *The inverse problem of dynamics.* (Abstract 48-9-279.)

26. L. W. Cohen: *Integration on hypersurfaces.* (Abstract 48-11-311.)

27. W. T. Reid: *Some results on the growth of solutions of differential systems.* (Abstract 48-11-319.)

28. D. W. Hall: *On a partial solution of a problem of J. R. Kline.* (Abstract 48-9-290.)

29. J. W. T. Youngs: *A note on separation axioms and their application in the theory of a locally connected topological space.* (Abstract 48-11-347.)

30. Deane Montgomery and Hans Samelson: *Groups transitive on spheres.* (Abstract 48-9-293.)

31. F. B. Jones: *A characterization of the sphere.* (Abstract 48-11-340.)

32. Vladimir Morkovin: *On the deflection of anisotropic thin plates.* (Abstract 48-9-272.)

33. W. R. Foote, Hillel Poritsky and J. J. Slade: *The motion of a rotor with unequal shaft flexibilities mounted in bearings of unequal flexibility.* (Abstract 48-11-327.)

34. Stefan Bergman: *Operators in the theory of partial differential equations and their application.* II. (Abstract 48-9-271.)

35. G. E. Albert: *Criteria for the closure of systems of orthogonal functions.* (Abstract 48-9-256-*t*.)

36. Reinhold Baer: *Rings with duals.* (Abstract 48-7-230-*t*.)

37. Stefan Bergman: *Operators in the theory of differential equations and their application.* I. (Abstract 48-9-270-*t*.)

38. H. K. Brown: *The resolution of boundary value problems by means of the finite Fourier transformation.* (Abstract 48-7-246-*t*.)

39. Leonard Carlitz: *Representation of a polynomial in certain forms.* (Abstract 48-11-299-*t*.)

40. Leonard Carlitz: *Some formulas for the composition of numerical functions.* (Abstract 48-11-300-*t*.)

41. H. S. M. Coxeter: *A geometrical background for the description of de Sitter's world.* (Abstract 48-9-273-*t*.)

42. J. J. DeCicco: *The application of turbine geometry to the inverse problem of dynamics.* (Abstract 48-9-275-*t*.)

43. C. J. Everett: *Affine geometry of vector spaces over rings*. (Abstract 48-7-247-*t*.)

44. Willy Feller: *On some geometric inequalities*. (Abstract 48-11-312-*t*.)

45. B. H. Gere and David Zupnik: *On the construction of curves of constant width*. (Abstract 48-9-276-*t*.)

46. W. H. Gottschalk: *On k-to-1 mappings*. (Abstract 48-7-249-*t*.)

47. O. G. Harrold: *A higher dimensional analogue of a theorem of plane topology*. (Abstract 48-11-339-*t*.)

48. Mark Kac: *On the average number of roots of a random algebraic equation*. (Abstract 48-9-284-*t*.)

49. Mark Kac: *On the distribution of values of trigonometric sums with linearly independent frequencies*. (Abstract 48-9-259-*t*.)

50. Edward Kasner and J. J. DeCicco: *Transformation theory of isogonal trajectories of isothermal families*. (Abstract 48-9-280-*t*.)

51. A. J. Kempner: *Periodic decimals to any base, and quadratic residues*. (Abstract 48-11-305-*t*.)

52. W. M. Kincaid: *On non-cut sets of locally connected continua*. (Abstract 48-9-291-*t*.)

53. D. M. Krabill: *On extension of Wronskian matrices*. (Abstract 48-11-316-*t*.)

54. D. H. Lehmer: *Recurrence formulas for certain divisor functions*. (Abstract 48-9-252-*t*.)

55. Walter Leighton and W. J. Thron: *On the convergence of continued fractions to meromorphic functions*. (Abstract 48-7-237-*t*.)

56. Lee Lorch: *Lebesgue constants for Borel summability of Fourier series*. (Abstract 48-11-317-*t*.)

57. R. G. Lubben: *Mappings of spaces H Fréchet on completely regular spaces*. (Abstract 48-9-292-*t*.)

58. Szolem Mandelbrojt and F. E. Ulrich: *On a generalization of the problem of quasi-analyticity*. (Abstract 48-7-239-*t*.)

59. Ella Marth: *On Garvin's F-series*. Preliminary report. (Abstract 48-9-260-*t*.)

60. Karl Menger: *Projective generalization of metric geometry*. I. (Abstract 48-11-330-*t*.)

61. Karl Menger: *Projective generalizations of metric geometry*. II. (Abstract 48-11-331-*t*.)

62. Josephine M. Mitchell: *On double Sturm-Liouville series*. (Abstract 48-9-261-*t*.)

63. Deane Montgomery and Hans Samelson: *Groups transitive on the n-dimensional torus*. (Abstract 48-9-294-*t*.)

64. Deane Montgomery and Leo Zippin: *A class of transformation groups in E_n.* (Abstract 48-9-295-*t*.)

65. R. L. Moore: *Concerning a continuum and its boundary.* (Abstract 48-11-342-*t*.)

66. R. L. Moore: *Concerning intersecting continua.* (Abstract 48-11-343-*t*.)

67. Marston Morse: *Unstable minimal surfaces bounded by a rectifiable contour.* (Abstract 48-7-240-*t*.)

68. K. L. Nielsen and B. P. Ramsay: *On particular solutions of linear partial differential equations.* (Abstract 48-9-262-*t*.)

69. Gordon Pall: *The distribution of integers represented by binary quadratic forms.* (Abstract 48-11-306-*t*.)

70. C. N. Reynolds: *A calculus of finite topological differences with application to the four color problem.* (Abstract 48-11-344-*t*.)

71. Mary De Pazzi Rochford: *Completely non-integrable pfaffians.* (Abstract 48-7-242-*t*.)

72. Mary De Pazzi Rochford: *Differentiability properties of certain functions of two variables.* (Abstract 48-7-243-*t*.)

73. Raphael Salem: *On a theorem of Zygmund.* (Abstract 48-9-263-*t*.)

74. Henry Scheffé: *On the ratio of the variances of two normal populations.* (Abstract 48-9-287-*t*.)

75. A. R. Schweitzer: *On a class of ordered $(n+1)$-ads relevant to the algebra of logic.* III. (Abstract 48-11-333-*t*.)

76. H. Schwerdtfeger: *A complete parametrization of the symplectic group.* (Abstract 48-7-233-*t*.)

77. Max Shiffman: *Unstable extremal surfaces for double integral problems in the calculus of variations.* (Abstract 48-11-322-*t*.)

78. Dorothy M. Smiley: *A note on Burnside's problem.* (Abstract 48-9-253-*t*.)

79. M. F. Smiley: *A comparison of algebraic, metric, and lattice betweenness.* (Abstract 48-5-190-*t*.)

80. Saly R. R. Struik: *Emmy Noether and Maria Gaetana Agnesi, two outstanding women mathematicians.* (Abstract 48-11-332-*t*.)

81. H. S. Thurston: *The solution of p-adic equations.* (Abstract 48-9-254-*t*.)

82. A. D. Wallace: *Separation spaces.* II. (Abstract 48-7-251-*t*.)

83. Hermann Weyl: *On Hodge's theory of harmonic integrals.* (Abstract 48-9-266-*t*.)

84. Hassler Whitney: *Complexes of manifolds.* (Abstract 48-9-296-*t*.)

85. Hassler Whitney: *Differentiability of the remainder term in Taylor's formula.* (Abstract 48-9-267-*t*.)

86. Hassler Whitney: *Note on differentiable even functions.* (Abstract 48-9-268-*t*.)

87. Hassler Whitney: *The general type of singularity of a set of* $2n-1$ *smooth functions of* n *variables.* (Abstract 48-9-269-*t*.)

88. Hassler Whitney: *The singularities of an* M^n *in* E^{2n-1}. (Abstract 48-9-298-*t*.)

89. G. T. Whyburn: *Homotopy classes of mappings into the circle.* (Abstract 48-11-346-*t*.)

T. R. HOLLCROFT,
Associate Secretary

BOOK REVIEWS

What is Mathematics? By Richard Courant and Herbert Robbins. London, New York and Toronto, Oxford University Press, 1941. 19+521 pp. $5.00.

Professor Courant is too expert a hand at mathematics to attempt a formal definition of the subject. Instead, he and his collaborator offer "an elementary approach to ideas and methods," from which the reader who persists to the end may draw his own conclusions. It will be interesting to American teachers and students to forecast what a few of those conclusions may be.

First, as to the material presented. Roughly, the topics treated cover the usual superior undergraduate course in mathematics through the calculus, with glimpses of such things as Riemannian geometry and Plateau's problem beyond the customary curriculum. It is a misapprehension, under which some authors not acquainted at first hand with mathematics as taught in the better colleges labor, that mathematics for the American undergraduate ends with the calculus, or at farthest with an introduction to differential equations. It does not; quite detailed courses in the theory of algebraic equations, the theory of numbers, projective geometry, both synthetic and analytic, modern plane elementary geometry, also occasionally non-Euclidean geometry, are offered and taken; and in some of the undergraduate schools of the larger universities a course in the theory of functions of a complex variable is a commonplace, as also is a thorough introduction to mechanics. So the average student or graduate of the better college courses in mathematics will not be wholly unprepared to appreciate the numerous illuminating sidelights which this book offers on what he already knows; nor will he be entirely blind to the attractions of things which he may see here for the first time. All this sums up to the opinion that the book is one for inspirational collateral reading, rather than a detailed manual for the mastery of any one of the topics it treats. The hypothetical layman who remembers a little of what he learned in college ten to fifty years ago will find the book both stimulating and demanding.

In the method of presentation, at least, there is something new and interesting for nearly every topic, if only a brief note calling attention to significant progress made within the past decade, or an excellent drawing making textual comment all but superfluous. A sound pedagogical strategy ensnares the reader at the outset (pp. 1–51) in the serenely useless properties of the common whole numbers, catching his attention with simple but singularly recondite problems, solved and unsolved, while painlessly indoctrinating him with such useful things as the binomial theorem and geometrical progressions. The number system of analysis is then developed, culminating (p. 72) in the Dedekind cut as an alternative to the method of nested intervals previously developed in the text. After a brief excursus on the basic principle of analytic geometry, the discussion of the continuum is resumed in "the mathematical analysis of infinity," where the paradoxes and the present disorderly state of the foundations of mathematics are fairly and fearlessly discussed. Teachers who still believe such things too scandalous to be mentioned in the presence of the young, may find here just the temperate account they have been seeking. After all, college students are old enough not to be unduly shocked by the discovery that the Queen of the Sciences is no better than she should be.

So far, mathematics has appeared as a rather gentlemanly pursuit, interesting enough to those who like the sort of thing mathematics is (even if they do not know, and are not told, what it is), but essentially useless. Indeed, the emphasis throughout the book is on principles rather than on trite imitations of the genuinely practical applications of mathematics such as a modern engineer or theoretical physicist must know. In this connection, differentials (p. 435) as misused to great advantage by physicists, engineers and other practical men who get tangible things done, come in for some bad moments, and the hapless engineer who imagines his differentials as little bits of reality is left without a d to his x. Cannot something be done to span the chasm dividing unsound mathematics that helps to produce things that make money, from sound mathematics that barely pays a beggarly living wage to instructors in our colleges and universities? Rebuking the engineers and physicists for their differential and infinitesimal sins may relieve a pure mathematician's outraged emotions, but it will pay nobody's grocery bill, unless the mathematician is lucky enough to make a smash-hit with a best-selling text, as we hope this one will be.

Off the familiar highway are such things as items from topology in Chapters IV, V, where, incidentally, the Jordan curve theorem is described, perhaps to disabuse anyone who still takes too seriously the dictum of Gauss that "mathematics is the science of the eye." The balance between intuition and logical rigor, here as elsewhere, is justly preserved, and bigots for either intuition or logic as the sole guarantor of mathematical truth will find but little support for their exclusive dogmas. Interesting details that will be new to a majority of the readers to whom the book is addressed are Brouwer's fixed point theorem, the genus of a surface, Klein's tantalizing bottle (the drunkard's deliverance), and the five color problem. To an algebraist it may seem somewhat antiquated to present only the analytic version of the fundamental theorem of algebra. Topology appears again in connection with minimax points; and generally the discussion of maxima and minima is as unhackneyed as anything in the book. Even Dirichlet's principle is described, possibly as a danger signal that intuition alone is not always enough, even in applied mathematics. Surely this must be an innovation in a college text. As the senior author has worked on Plateau's problem, he indulges his fancy for soap films in an interesting section suggesting that what mathematics slaves through pages of intricate analysis to accomplish, is sometimes done effortlessly by nature in a single gesture.

Enough has been sampled to indicate that here is an unusually rich text, full of suggestions for further reading, and replete with interesting and beautiful things not found in other books at the same level. The authors have reworked some of the best from each of many sources, due acknowledgement being made in the text and in the bibliography; the welding of it all into a homogeneous unity is exclusively their own. Sometimes the style is reminiscent of Klein; at other times Hilbert is plainly visible in the background; but all in all the work is an expression of independent individuality.

There is one aspect of the book which may be the most significant of all for young students looking forward to a career in mathematics. The emphasis of the recent past in American mathematics has been heavily on the side of axiomatics and abstraction; this book reminds us that intuition also has its part in mathematical creation.

Within the last decade several texts aiming to break away from the deadening drill in formal manipulation and theorems that are now only museum pieces have been written for American students. Some have favored mathematics as a logical discipline; others have presented it as a fine art; and yet others have described the power of mathematics in science and daily affairs. Each has had its definite function to perform, and all have contributed to a long overdue enlightenment in the teaching of mathematics at the college level. The present book differs in many respects from its predecessors; the points of difference make it a most welcome addition to the library of modernized teaching in mathematics.

E. T. BELL

Tables of the Moments of Inertia and Section Modulus of Ordinary Angles, Channels and Bulb Angles with Certain Plate Combinations. New York, Works Projects Administration, 1941. 13+197 pp. $1.25.

These tables are a result of a suggestion of the U. S. Bureau of Marine Inspection and Navigation, which is now a part of the Navy Department. The most important entries in these tables are the moments of inertia and the section moduli of rectangular areas with adjacent L-, Γ- or C-shaped areas (called, respectively, "angles" for the L- and Γ-shape and "channels" for the C-shape, the word "bulb angle" being reserved for an L-shaped area with a lateral circular swelling at the top). Both moments of inertia and section moduli are taken with respect to the neutral axis of the whole area parallel to the larger side of the rectangle. These entries were calculated by means of the parallel axis theorem, and the tables of the U. S. Steel Corporation were used for the moments of inertia and for the position of the centroid of the "angles" and "channels." The dimensions of the areas (or "sections" as they are technically called) which we find in these tables correspond to the commercial types, and—as one would expect—the tabular entries are not appropriate for difference checking. However, the accuracy of the values was insured through independent computing by two groups of workers, and was checked by graphing the entries. The calculations are carried out to two decimal places for small entries and are limited to integer digits for large entries (the units being in.^{4} and in.^{3}). The arrangement of the tables is simple and handy, and the photo-offset reproduction gives them a very clear appearance. This publication will certainly be appreciated by many engineers.

I. OPATOWSKI

The Mathematical Papers of Sir William Rowan Hamilton. Vol. II. *Dynamics.* Edited for the Royal Irish Academy by A. W. Conway and A. J. McConnell. (Cunningham Memoir, no. 14.) Cambridge University Press; New York, Macmillan, 1940. 15+656 pp.

This volume contains Hamilton's work on dynamics and some investigations in optics. The period covered is, with two or three minor exceptions, that from 1833 to 1839. Except for a few short abstracts of papers read by Hamilton, most of the material presented has never before been published, and consists in the main, of a transcript of Hamilton's original work books and some scientific correspondence. The editors have added a few footnotes and some short explanatory material at the end of the volume.

The material is divided into three parts. A short introduction sketches the plan of the work and the nature of the problems investigated. The two hundred and eighty pages of Part I are devoted to dynamics and correspondence with Lubbock. Part II, consisting of some hundred pages, is devoted to the calculus of principal relations. Part III concerns itself with optical investigations and correspondence, and comprises some hundred and seventy pages. Also included in this volume are a thirty-two page appendix, contributed by the editors, and a very complete index.

The underlying idea in Part I is the determination of a single function whose partial derivatives will yield complete information about the trajectories. Hamilton uses at first the action function $\int 2T dt$ and applies it to the three body problem. In order to obviate the bothersome transformations that arise in considerations of the time, he perceives the advantage of using the principal function $S=\int L dt$ and indicates other alternative functions. It is in these investigations that the notion of varying action, the so-called canonical equations, and the Hamilton-Jacobi equation first come in as observations on the special dynamical situations treated, though, curiously enough, Hamilton makes no use of the canonical equations.

However, Hamilton requires that S satisfy a system of two partial differential equations, namely, the usual Hamilton-Jacobi equation and the corresponding equation where the differentiating variables are the initial coordinates and initial time. Now, at least when the force potential is independent of the time, the second equation is automatically satisfied whenever the first is, and Hamilton's lack of realization of this fact, quite obviously prevents him from making full application of the method. It is of course well known that it was only with Jacobi's recognition of this defect that the Hamiltonian method established its fundamental significance. (Apart from methodological novelty, Hamilton's dynamical and astronomical applications succeed only in verifying previously known computations.) Nevertheless in special cases Hamilton obtains an explicit principal function S and indicates an interesting method for this purpose. A complete integral $V(\cdots x_i \cdots ; \cdots a_i \cdots , t)$ of the Hamilton-Jacobi equation is a principal function if

$$V(\cdots \overset{0}{x}_i \cdots ; \cdots a_i \cdots ; t_0) = 0 \tag{1a}$$

and

$$L_{t\to t_0} \frac{V}{t - t_0} = L_{t\to t_0}\left(L\left(\cdots a_i \cdots , \cdots \frac{x_i - a_i}{t - t_0} \cdots ; t_0\right)\right) \tag{1b}$$

$$\text{essentially } L_{t\to t_0}\left(\frac{d}{dt}V - L\right) = 0.$$

Hence suppose

$$V = \sum V_i \tag{2}$$

where V is homogeneous of degree i in $x_i - a_i$, $t - t_0$. We may substitute (2) in the H.J. equation and expand $H(\cdots x_i \cdots, \partial V/\partial x_i \cdots, t)$ about x_i^0, t_0. On equating the linear and the quadratic terms, and so on, separately to zero, there results an infinite system of equations. The equation corresponding to terms of order n contains only V_j, $i \leqq j \leqq n$. On making use of (1) it is not difficult to show that the succeeding V_n's may be calculated step by step in terms of V_1. There are also some developments of perturbation and approximation methods for the determination of S.

The most ambitious appendix, quoted from a paper published by the editors, is devoted to the determination of the principal function from the knowledge of a complete integral of the H.J. equation. This is actually of slight importance, however, for it is well known that *any* complete integral determines the trajectories. Indeed (1a) suggests writing $S = V - V_0$, $V_0 = V(\cdots x_i^0 \cdots, \cdots a_i \cdots, t_0)$, and then the modification of the usual procedure replacing $b_i = \partial V/\partial a_i$ by $\partial V_0/\partial a_i = \partial V/\partial a_i$ for the elimination of the a_i's yields the principal function $S(\cdots x_i \cdots, \cdots x_i^0 \cdots, t, t_0)$.

Also in Part I is an ingenious derivation of the angular dependence of the resultant of two forces. The problem is reduced to the determination of an additive function f on the real numbers. Hamilton's argument to show $f(x)$ is linear and actually x overlooks the Hamel functions, but can readily be made rigorous.

The extensions of the dynamical methods, utilizing a principal function or relation, to arbitrary first and higher order partial differential and Mongean equations and systems, constitute the calculus of principal relations taken up in Part II. The following is typical. Consider

$$f_\sigma\left(x_1, \cdots \frac{d^{k_1}x^1}{dt^{k_1}}; \cdots ; x_i \cdots \frac{d^{k_i}x_i}{dt^{k_i}}, \cdots ; \cdots\right) = 0, \tag{3}$$

$$\sigma = 1, \cdots m;\ i = 1, \cdots n.$$

We associate with (3) the auxiliary equations of the calculus of variations. For instance with $\sigma = 1$, $k_i = k$ we have for a suitable choice of λ

$$(4)\qquad \lambda\frac{\partial f}{\partial x_i}-\frac{d}{dt}\lambda\frac{\partial f}{\partial \dot{x}_i}+\cdots+(-1)^k\frac{d^k}{dt^k}\lambda\frac{\partial f}{\partial x_i^k}=0,\qquad x_i^k=\frac{d^kx_i}{dt^k}\cdot$$

Consider now

$$(5)\qquad 0=\int_{t_0}^{t}\left(\delta f-\frac{df}{dt}\,\delta t\right)dt=I.$$

In view of (4) we may express the right side of (5) as a sum of terms of the form

$$(6)\qquad \left\{(-1)^\alpha\frac{d^\alpha}{dt^\alpha}\left(\lambda\,\frac{\partial f}{\partial x_i^{\beta+1}}\right)\frac{d^{\beta-\alpha}}{dt^{\beta-\alpha}}(\delta x_i-\dot{x}_i\delta t)\right\}\Bigg|_{t_0}^{t}.$$

If $F(\cdots;x_i\cdots d^{k-1}x_i/dt^{k-1};\ \cdots;\ \cdots;a_i\cdots a_i^{(k-1)};\ \cdots t,t_0)=0$ is the principal relation, then $\delta F=0$ is a consequence of (5) and hence

$$(7)\qquad \delta F=\mu I=0,$$

the central relation of this calculus. On equating the coefficients of the independent variations in (7) to 0 making use of (4), there arise auxiliary equations denoted by (8) for the partial derivatives of F with respect to $x_i^{(j)}$ and $x_i^{0,(j)}$ involving λ and μ also. When $k=1$ elimination of λ, μ, $\dot{x}_i$, $\dot{x}_i^0$ from these equations and (1) yields a differential equation for F and the equations (8) lead essentially to the bicharacteristics. For $k>1$ Hamilton indicates merely how the principal relation may be obtained. Thus to the equations (4) may be added a set of mn further equations obtained by replacing the left side of (4) by ith derivatives ($i=1,\cdots,n$) with respect to t. Then by successive eliminations we arrive at a single relation of the form specified above. Hamilton apparently did not attempt to determine the partial differential equation satisfied by F for $k>1$ and indeed the utility of the principal relation for higher order partial and Mongean equations is somewhat obscure. It would seem of interest to pursue these investigations.

The title assigned by the editors to Hamilton's treatment of higher order partial differential equations implies use of the principal relation but this is not clear from the text. He is led to relations obviously connected with the characteristic and bicharacteristic equations usually ascribed to Cauchy, Backlund, Goursat and Beudon and there seems to be some contact with Hamburger's investigations. However apart from the first order systems the results are incomplete and no account is taken of the elementary distinction between the elliptic, hyperbolic and parabolic cases.

The material on optics is perhaps of greatest interest. The propagation of disturbances may be taken up from two viewpoints. The first considers a compatible wave train already existing in the entire medium. Hamilton's interest is in the second viewpoint, namely that of the spreading of an originally localized disturbance. The typical problem is that of a collection of equi-spaced particles arranged over the entire x-axis and it is supposed that each particle acts only on its two adjacent neighbors.

The mathematical equivalent is the equation in mixed differences

$$y_{tt} - a^2 \frac{\Delta^2}{1+\Delta} y = 0, \qquad \Delta y(x, t) = y(x+1, t) - y(x, t)$$

subject to the initial conditions

$$\begin{aligned} y(x, 0) &= 0 \text{ or } \eta(1 - \cos 2\nu x) \\ y_t(x, 0) &= 0 \text{ or } -2a\eta \sin \nu \sin 2\nu x \end{aligned}$$

for $x<0$ and $x \geqq 0$, respectively. If instead $y(x, 0)$ and $y_t(x, 0)$ vanish identically outside a finite interval, the conclusions are essentially unaffected. In treating these problems Hamilton makes incidental use of tools of modern appearance. For instance he uses the Fourier integral with the Sommerfeld weighting factor, the Abel summability of the Laplace integral, the Heaviside operator p^{-1}, and various operational manipulations, the Riemann-Lebesgue lemma, the method of stationary phase in conjunction with repeated use of the Dirichlet discontinuous integral (and incidentally obtains the leading term in the asymptotic expansion of the Bessel's function). If a, η are 1, then the nub of the analysis is the investigation of the properties of

$$\frac{1}{2\pi} (\sin \nu)^2 \int_0^{\pi} \frac{\sin (2x\theta - 2t \sin \theta)}{\sin \theta(\cos \theta - \cos \nu)} d\theta.$$

Assuming t large and positive, the results may be roughly summarized as follows. Let M and N be certain sufficiently large constants. Then for $x \geqq t + (1/2)Mt^{1/3}$, y is sensibly 0. As x decreases y takes on a pure displacement whose value is $(1/3)/(\cos \nu/2)^2$ for $x=t$ and rises to $(\cos \nu/2)^2$ for I: $t\cos \nu + (1/2)N(t\sin\nu)^{1/2} \leqq x \leqq t - (1/2)M_1 t^{1/3}$. The displacement of amount $(1/3)(\cos \nu)^2$ travels with unit velocity. As x decreases still further an oscillatory disturbance sets in. Thus in the range II: $-t + (1/2)Mt^{1/3} \leqq x \leqq t\cos \nu - (1/2)N(t\sin \nu)^{1/2}$, $y = (1 - \cos \phi - (\sin \nu/2)^2$, $\phi = 2\nu x - 2t \sin \nu$. This is a copy of the initial disturbance shifted to a new mean position. In the transition region linking I and II the disturbance is a shifted, uniformly attenuated copy of the original. With further decrease in x, the effect is merely to change the mean position of the vibrations and for III: $x \leqq -t-(1/2)Mt^{1/3}$, $y = \cos\phi$. Hamilton adopts the view that it is the region of oscillatory disturbance that is significant. He chooses the value $x = t \cos\nu$ in the transition region T between I and II for the wave front. Here $y = (1/2)(1-\cos\phi+\cos\nu)$ and $\cos\nu$ is a dispersive (group) velocity. This is not altogether capricious, for with increasing t the velocity throughout T approaches $\cos\nu$, and the ratio of the size of T to the time approaches zero.

It has not been realized that the modern work on the transient phenomena in the propagation of waves in continuous media by Sommerfeld, Brillouin, and Colby is anticipated in these researches. (Even the reference to the published abstract of Hamilton's work, in Havelock's well known Cambridge tract, overlooks this fact.) There is moreover a marked parallelism in the methods, the stationary phase calculations being superseded by the more delicate saddle point evaluations. Actually, the adoption of a discrete rather than a continuous medium has certain advantages in indicating more intuitively the nature and genesis of the conclusions. Hamilton extends his work to two-dimensional and three-dimensional problems using the now familiar Cauchy-Fourier integral methods, and gives the explicit general formula for the group velocity.

The editors have obviously presented a most felicitous and careful rendering of Hamilton's work. However, it would seem desirable in collections of hitherto unpublished material to give a succinct summary of the *details* of (a) methods and (b) results in modern terminology and, wherever possible, (c) some indication of the present status of the field covered. The editors have made an attempt in this direction for (a) and (b), particularly as regards the dynamical papers, but the appendices and footnotes, could, with profit, be expanded considerably. The viewpoint and incomplete character of many of the investigations may well attract the attention of mathematicians ordinarily unconcerned with the historical development of mathematics.

D. G. BOURGIN

NOTES

The organization called VOKS (or Society for Cultural Relations with Foreign Countries, B Gruzinskaya 17, Moscow 56, U.S.S.R.) is able to locate institutions evacuated because of the war and has as its purpose the continuation of scientific contact between Russia and other countries. The Embassy advises us to send letters to individual scientists through VOKS.

Professor R. H. Fowler of the University of Cambridge and Dr. R. P. Paranjpye of Poona, India, have been knighted.

Dean L. P. Eisenhart of Princeton University has been appointed Executive Officer of the American Philosophical Society to succeed Professor Emeritus E. G. Conklin of Princeton University, who is now President of the Society.

Professor H. B. Evans of the University of Pennsylvania has retired with the title professor emeritus.

Professor A. C. Lunn of the University of Chicago has retired with the title professor emeritus.

Professor C. B. Upton of Teachers College, Columbia University, has retired with the title professor emeritus.

Dr. J. S. de Wet of the University of Pretoria has been promoted to a professorship.

The following members of the American Mathematical Society are in residence at the Institute for Advanced Study for all or part of this academic year: Dr. Valentine Bargmann, Dr. I. S. Cohen, Dr. Kurt Gödel, Dr. Audrey W. McMillan, Dr. W. R. Transue, Dr. L. I. Wade and Mr. J. E. Wilkins.

The following persons are also to be at the Institute for Advanced Study for all or part of this academic year: Dr. Shuichi Kusaka, Professor Wolfgang Pauli, formerly of the Technical School in Zurich, and Professor C. L. Siegel, formerly of the University of Göttingen.

Assistant Professor H. W. Bailey of the University of Illinois has been promoted to an associate professorship.

Professor B. L. Beegle of Seattle Pacific College has been appointed dean of the college.

Assistant Professor P. G. Bergmann of Black Mountain College, Black Mountain, North Carolina, has been appointed to an assistant professorship of physics at Lehigh University.

Dr. Elizabeth M. Boggs of Cornell University has been appointed to a lectureship in physics at the University of Pittsburgh.

Dr. H. E. Burns of Indiana University Extension, East Chicago, Indiana, has been promoted to an assistant professorship.

Associate Professor H. V. Craig of the University of Texas has been promoted to a professorship.

Assistant Professor M. W. Dehn of the University of Idaho has been appointed visiting lecturer at the Illinois Institute of Technology.

Dr. H. C. Dieckmann of Occidental College has been promoted to an assistant professorship.

Professor J. L. Dorroh of Ouachita College, Arkadelphia, Arkansas, has been appointed to a visiting assistant professorship at Louisiana State University.

Associate Professor H. P. Evans of the University of Wisconsin has been promoted to a professorship.

Assistant Professor Willy Feller of Brown University has been promoted to an associate professorship.

Associate Professor I. H. Fenn of the Polytechnic Institute of Brooklyn has been promoted to a professorship.

Dr. A. D. Fialkow of Brooklyn College has been appointed to a lectureship in the School of Pharmacy at Columbia University.

Dr. F. A. Ficken of Cornell University has been appointed to a professorship at the University of Tennessee.

Assistant Professor H. A. Giddings of the Illinois Institute of Technology has been appointed to an associate professorship at the University of New Hampshire.

Dr. G. L. Gross of the Agricultural and Mechanical College of Texas has been promoted to an assistant professorship.

Dr. Margaret Gurney of Wykeham Rise School, Washington, Connecticut, has been appointed associate economist in the Bureau of the Budget, Washington.

Mr. M. G. Harrison has accepted a position as consultant at Republic Aviation Corporation.

Dr. O. G. Harrold of Northwestern University has been appointed to an assistant professorship at Louisiana State University.

Assistant Professor Frances Harshbarger of Kent State University has been promoted to an associate professorship.

Dr. G. E. Hay of the University of Michigan has been promoted to an assistant professorship.

Associate Professor E. A. Hedberg of Baylor University is on leave at the United States Naval Pre-Flight School in Athens, Georgia.

Assistant Professor M. R. Hestenes of the University of Chicago has been promoted to an associate professorship.

Dr. Aughtum S. Howard of the University of Kentucky has been appointed to an associate professorship at Kentucky Wesleyan College.

Mr. J. D. Hutchinson of the University of Houston has been promoted to an associate professorship.

Dr. William Karush of the University of Chicago has accepted a position at Carnegie Institution of Washington.

Dr. M. W. Keller of Purdue University has been promoted to an assistant professorship.

Dr. J. C. Knipp of the University of Pittsburgh has been promoted to an assistant professorship.

Dr. C. F. Kossack of the University of Oregon has been promoted to an assistant professorship.

Associate Professor V. S. Lawrence of Cornell University has been commissioned major in the United States Army and will teach mathematics at the United States Military Academy at West Point.

Dr. Charles Loewner of the University of Louisville has been promoted to an assistant professorship.

Dr. Eugene Lukacs has been appointed to an assistant professorship at Illinois College.

Dr. Ingo Maddaus of the University of Oklahoma has been appointed to an assistant professorship at the University of Oregon.

Dr. P. T. Maker of Rutgers University has been appointed to an assistant professorship at Duke University.

Dr. Herman Meyer of the Illinois Institute of Technology has been appointed to an assistant professorship at the University of Miami.

Dr. Harlan C. Miller of Winthrop College has been appointed to an associate professorship at North Texas State Teachers College.

Mr. B. C. Moore of the Agricultural and Mechanical College of Texas has been promoted to an assistant professorship.

Associate Professor Sara L. Nelson of Georgia State College for Women has been promoted to a professorship.

Dr. E. N. Nilson of the University of Maryland has been appointed to an assistant professorship at Mount Holyoke College.

Reverend R. E. O'Connor has been appointed to a professorship at Loyola College, Montreal, Quebec.

Dr. H. L. Olson of Itasca Junior College has been appointed to an assistant professorship at Southwestern University, Georgetown, Texas.

Dr. W. T. Puckett of the University of California at Los Angeles has been promoted to an assistant professorship.

Assistant Professor W. T. Reid of the University of Chicago has been promoted to an associate professorship.

Dr. Haim Reingold of Our Lady of Cincinnati College is on leave at the Illinois Institute of Technology.

Professor W. J. Rusk of Grinnell College has retired.

Professor Emeritus Virgil Snyder of Cornell University has been appointed to a visiting professorship at Brown University.

Dr. Domina E. Spencer has been appointed to an assistant professorship at the American University.

Assistant Professor D. V. Steed of the University of Southern California has been promoted to an associate professorship.

Dr. Alfred Tarski has been appointed lecturer at the University of California in Berkeley.

Dr. R. M. Thrall of the University of Michigan has been promoted to an assistant professorship.

Dr. F. A. Valentine of the University of California at Los Angeles has been promoted to an assistant professorship.

Dr. R. K. Wakerling of Texas Technical College has been appointed to an assistant professorship at Fresno State College, Fresno, California.

Professor Evelyn P. Wiggin of Randolph Macon Woman's College is on leave as lecturer at Wellesley College.

The following appointments to instructorships are announced: Angleton Public Schools, Angleton, Texas: Mr. S. K. Bright; Bowdoin College: Dr. D. E. Christie; University of California: Dr. Dorothy L. Bernstein; Case School of Applied Science: Dr. L. J. Green; University of Cincinnati: Mr. Joshua Barlaz; Cornell University: Dr. Fred Kiokemeister, Dr. L. J. Savage; University of Florida: Dr. G. B. Lang; Hamilton College: Mr. L. B. Williams; Houghton College: Dr. R. R. R. Luckey; John Carroll University: Dr. P. H. Anderson; University of Kansas: Dr. G. K. Kalisch; Massachusetts Institute of Technology: Dr. P. R. Wallace; Michigan State College: Dr. D. E. Kibbey, Dr. W. L. Mitchell; U. S. Naval Air Station, Pensacola, Florida: Mr. C. C. Dearman; U. S. Naval Pre-Flight School, Iowa City, Iowa: Dr. Howard Levi; Northwestern University: Dr. Zenon Szatrowski; University of Oklahoma: Mr. W. J. Dixon; Pennsylvania State College: Mr. E. H. Umberger; Princeton University: Dr. Brockway McMillan, Dr. E. W. Montroll; Purdue University: Dr. Leonidas Alaoglu, Dr. J. H. Giese, Dr. Michael Golomb, Dr. G. S. Young; Randolph Field, Texas: Dr. C. L. Seebeck; St. Olaf College: Mr. W. S. Erickson; U. S. Military Academy: Mr. B. M. Ingersoll; University of Wisconsin: Mr. P. B. Norman; Yale University: Dr. Edward Begle, Mr. A. W. Jones.

Professor Emeritus A. R. Forsyth of the Imperial College of Science and Technology, South Kensington, died on June 2, 1942, at the age of eighty-three years.

The death of Professor W. H. Young, formerly professor of mathematics at the University of Wales, has been announced.

Professor Emeritus Eleanor C. Doak of Mt. Holyoke College died August 27, 1942, at the age of seventy-two years. She became a member of the Society in 1916.

Professor Emeritus W. A. Harshbarger of Washburn University died on July 17, 1942, at the age of seventy-nine years. He had been a member of the Society since 1909.

Dean T. M. Putnam of the University of California died September 22, 1942. He had been a member of the Society since 1902 and had served as Associate Secretary in charge of meetings on the Pacific Coast since 1930.

Professor J. F. Reilly of the State University of Iowa died August 18, 1942. He had been a member of the Society since 1914.

Dr. W. M. Strong, who retired about a year ago as vice president and actuary of Mutual Life Insurance Company of New York, died March 30, 1942, at the age of seventy-one years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

299. Leonard Carlitz: *Representation of a polynomial in certain forms.*

Explicit formulas are found for the representation of an irreducible polynomial P in $GF(p^n)$ in certain factorable forms of degree k in k indeterminates. For example, when $k=2$, $p\neq 2$, the form may be taken as $U^2-\alpha V^2$, where α is not a square in $GF(p^n)$. Here U is determined by the formula $2A(x)\equiv(-1)^k[1][3]\cdots[2k-1]$ (mod $P(x)$), where P is of degree $2k$, and $[i]=x^{p^{ni}}-x$ (Duke Mathematical Journal, vol. 9 (1942), pp. 234–243, p. 242). Similar results are obtained for the general case. (Received August 3, 1942.)

300. Leonard Carlitz: *Some formulas for the composition of numerical functions.*

This note is concerned with the sum $h(M)=\sum f(U)g(V)$ extended over polynomials of fixed degree k in $GF(p^n)$ such that $(\alpha+\beta)M=\alpha U+\beta V$, α, β in $GF(p^n)$; it is assumed that $f(U)$ and $g(V)$ are of the form $\sum e(D)$, where D runs through the divisors of U of degree at most $k/2$. It is found that $h(M)$ is of the same general form. In various special cases $h(M)$ can be expressed in quite simple form. In particular the formulas of the present paper include formulas occurring in the problem of representing a polynomial as the sum of an even number of squares. (Received August 3, 1942.)

301. R. P. Dilworth: *On the decomposition theory of modular lattices.*

The Kurosch-Ore decomposition theorem asserts that if an element a of a modular lattice has two reduced decompositions into irreducibles $a=p_1\cap\cdots\cap p_m=q_1\cap\cdots\cap q_n$, then each p_i may be replaced by a suitably chosen q_j. However, this leaves unanswered questions like the following: Does each q_j replace some p_i? Can different p's be replaced by different q's? The following precise result is proved: If $a=p_1\cap\cdots\cap p_m=q_1\cap\cdots\cap q_n$, then the q's can be renumbered in such a way that each p_i may be replaced by q_i. This theorem requires a much deeper analysis of the arithmetical structure of a modular lattice. For this purpose extensive use is made of the concept of an *over-divisor* which is defined as follows: a is said to *over-divide* b, or to be an *over-divisor* of b if $x\cap a=b$ implies $x=b$. (Received August 4, 1942.)

302. C. J. Everett: *Sequence completion of lattice moduls.*

A lattice modul L (G. Birkhoff, *Lattice ordered groups*, Annals of Mathematics, (2), vol. 43) is linear if and only if $a>0$, $b>0$ implies $a\wedge b>0$; hence the neighborhood topology $N_\epsilon(a)=(x;\ |x-a|<\epsilon,\ \epsilon>0)$ defines a topological group satisfying the crucial Hausdorff space intersection axiom if and only if L is linear. o-convergence and o-regular sequences are defined, and necessary and sufficient conditions given for every o-regular sequence to o-converge. Such an L is called o-complete. The first fundamental extension L' of L by classes of regular sequences imbeds L with preservation of $+$, unrestricted $\vee$, $\wedge$, and o-convergence. If L admits diagonalization (G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25. 1940, p. 28), then L' is o-complete. The MacNeille completion $\mathfrak{C}$ of the lattice L by cuts contains a maximal lattice submodul $\mathfrak{M}\supseteq L$ of all elements of $\mathfrak{C}$ having inverses. $\mathfrak{M}$ is o-complete and imbeds L with the same preservations as for L'. L' is isomorphic to the set L_1 of the elements of $\mathfrak{M}$ which are o-limits of L-sequences relative to L similarly L'', L''', and so on. A sequence is o-regular in L if and only if it o-converges in $\mathfrak{M}$ relative to L. If L is linear, o-convergence is equivalent to sequence convergence relative to $N_\epsilon(a)$ topology; L_1 is o-complete, indeed $L_1=\mathfrak{M}$ whenever there exist nonzero null sequences in L. (Received October 1, 1942.)

303. Irving Kaplansky: *Systems of congruences in a valuation ring.*

Infinite systems of linear congruences, introduced by Krull in his general definition of completeness of fields with valuations, are here used to obtain a characterization of maximality. As an application, a classical result is generalized; under the hypothesis that the field K is maximal in a valuation, it is shown that the degree of any extension of K is equal to the product of the ramification order and residue class degree. (Received September 29, 1942.)

304. Irving Kaplansky: *The direct product of rings.*

Whitney's tensor product $G\circ H$ of abelian groups (Duke Mathematical Journal, vol. 4 (1938), pp. 495–528), reduced with respect to a ring of operators R, provides a natural definition of the direct product of rings. If G and H are imbedded in a larger ring, the problem arises of deducing the usual criterion that the product GH be direct; this can be done if certain restrictions are made, as indicated by Dorroh in the case where R is the ring of integers (Annals of Mathematics, (2), vol. 36 (1935), pp. 882–885). This general notion of direct product is used to obtain a variety of results, some generalizations of known theorems, others new. (Received September 29, 1942.)

305. A. J. Kempner: *Periodic decimals to any base, and quadratic residues.*

Despite the large literature on the periods of decimal fractions there does not seem to exist any unified theory of this very special field. Restriction to the base 10 or to any individual base q is one reason for this deficiency. In order to gain a broader basis it is necessary to consider the representation of *all* proper fractions for *all* bases. In this enlarged formulation of the problem there is room for a satisfying theory. To bring out the structure of the system it is of advantage to place emphasis on the system of remainders $a_1, \cdots, a_\mu$ rather than on the system of digits $\alpha_1, \cdots, \alpha_\mu$ in the algorithm for the expansion of a/b to base q $(a\cdot q=\alpha_1\cdot b+a_1,\ a_1\cdot q=\alpha_2\cdot a_1+a_2,\ \cdots,\ a_{\lambda-1}\cdot q=\alpha_\lambda\cdot a_{\lambda-1}+a_\lambda,\ \cdots)$. The a's possess a period of the same length μ as do the α's, but have the advantage of being all distinct in a period, thus leading to a more incisive classification than is possible with the digits. (Received August 4, 1942.)

306. Gordon Pall: *The distribution of integers represented by binary quadratic forms.*

The formula due to R. D. James (American Journal of Mathematics, vol. 60 (1938), pp. 737–744) for the number of integers m prime to d represented by binary quadratic forms of discriminant d, is here freed of the restriction that m be prime to d. (Received August 3, 1942.)

307. Gordon Pall: *The weight of an n-ary genus of quadratic forms.*

The formula for the weight of a genus of integral positive quadratic forms in n variables is obtained in an explicit, useful form. The calculation of the factor for $p=2$ is greatly facilitated by the use of a much simplified system of invariants. There are numerous applications. (Received August 3, 1942.)

308. J. F. Ritt: *Bezout's theorem and algebraic differential equations.*

The intersection of the general solutions of two differential polynomials in two unknowns is examined with respect to the numbers of arbitrary constants on which its various irreducible components can depend. (Received August 6, 1942.)

309. Ernst Snapper: *The resultant of a linear set.*

The m-dimensional vector space $\overline{V}_m$ consists of vectors having, as components, m polynomials of the ring $P[y_1 \cdots y_n]$ where P is a field. The linear subsets of $\overline{V}_m$ are generated by the columns of $m \times s$ matrices with elements in $P[y_1 \cdots y_n]$. The ideal theory of $P[y_1 \cdots y_n]$, given by Hentzelt and Noether (Mathematische Annalen vol. 88 (1922), pp. 53–79), holds for these linear sets. By a linear, invertable transformation of the variables $y_1, \cdots, y_n$, which involves adjoining new variables γ_{ij} to P, the linear subsets of $\overline{V}_m$ become "transformed" linear sets of the vector space V_m over $P(\gamma)[x_1 \cdots x_n]$. Every transformed linear set L of V_m has a resultant $\rho \in P(\gamma)[x_1 \cdots x_n]$, which vanishes for, and only for, the zeros of the ideal $L/\overline{L}$. (See Snapper, Transactions of this Society, vol. 52 (1942), pp. 258–259 for the definitions of $\overline{L}$ and $L/\overline{L}$.) If $L_1 \subseteq L_2$, then $L_1 = L_2$, if and only if they have equal ranks and resultants. This gives a criterion for the existence of a polynomial solution of simultaneous linear equations with polynomials as coefficients. For $n=1$, the resultant becomes the highest dimensional determinantal factor of L. (Received September 29, 1942.)

ANALYSIS

310. M. A. Basoco: *On the Fourier developments of a certain class of theta quotients.*

This paper is concerned with the functions $\phi_\alpha^k(z) = \{\vartheta_\alpha'(z, q)/\vartheta({}_\alpha z, q)\}^k$ $(\alpha = 0, 1, 2, 3)$ where $\vartheta_\alpha(z, q)$ is a Jacobi theta function and k is a positive integer. The Fourier expansions of these functions are investigated and their arithmetical form is obtained for the cases $k = 1, 2, 3$. Using these results and certain simple identities, there is obtained using the method of paraphrase (E. T. Bell, Transactions of this Society, vol. 22 (1921), pp. 1–30 and 198–219; *Algebraic Arithmetic*, American Mathematical Society Colloquium Publications, vol. 7, 1927, chap. 3), a series of theorems on numerical functions. Among these there appears a theorem due to Liouville (Journal de Mathématiques Pures et Appliquées, (2), vol. 3 (1858), p. 247), which has only recently been proved by elementary means by J. V. Uspensky (*Elementary Number Theory*, New York, 1939, p. 462). An application of these theorems leading to relationships between infinite series of the Lambert type is indicated. This paper will appear shortly in this Bulletin. (Received September 26, 1942.)

311. L. W. Cohen: *Integration on hypersurfaces.*

An invariant measure is developed for m-dimensional surfaces Σ_m which yields proofs of the theorems of Gauss and Stokes in terms of Lebesgue integrals. A surface Σ_m is a complex of m-cells $R^m_x = X(R^m_u)$ where R^m_u is an m-dimensional interval and $X = X(U)$ is a 1-1 mapping of R^m_u on an arbitrary set. To each $X \subset R^m_x$ are assigned numbers $x_1, \cdots, x_n$, $n \geqq m$, which are almost everywhere differentiable functions of the coordinates of $U \subset R^m_u$. The measurable sets $E_x \subset R^m_x$ are the images of the measurable sets $E_u \subset R^m_u$. The measure of E_x is the integral of $D^n_m(U)$ over E_u where $(D^n_m(U))^2$ is the Gram function of the $\partial x_i/\partial u_j$. The measure on Σ_m is determined by the measure on its several cells. For the Gauss theorem, Σ_m need not be orientable and may have singularities on its boundary while the functions are summable on the boundary and have finite summable derivatives on Σ_m. The Stokes theorem states the invariance of a certain integral on a family of homologous surfaces. (Received August 12, 1942.)

312. Willy Feller: *On some geometric inequalities.*

The following problem was originally formulated by R. Salem and D. C. Spencer in connection with a number-theoretical investigation. Consider a domain Γ contained in the unit sphere of R_n; suppose that the intersection of Γ with any straight line has a measure not exceeding a fixed constant $\delta < 1$. The problem is to determine the maximum measure of Γ. In the present paper this and a more general analytical problem are solved. At the same time an extremely simple proof is given for the well known inequality for the measure of convex sets with the above property (due to Bieberbach and Kubota). (Received August 3, 1942.)

313. Einar Hille and Gabor Szegö: *On the complex zeros of the Bessel functions.*

This paper contains a new proof of the theorem of A. Hurwitz according to which $z^{\beta/2}J_{-\beta}(2z^{1/2})$ has exactly $[\beta]$ non-positive zeros when $\beta \geqq 0$. The proof is based upon the fact that $n^{\beta}L_n^{(-\beta)}(z/n)$ tends to the function in question and the number and position of the non-positive zeros of Laguerre polynomials of negative order are easily discussed. (Received August 3, 1942.)

314. Abraham Hillman and H. E. Salzer: *Complex roots of* $\sin z = z$.

The first ten nonzero roots of $\sin z = z$ in the first quadrant were computed to seven decimal places. Obviously the roots are symmetrically situated in the four quadrants. The imaginary part, y, of the nth root satisfies the equation $x_1(y) - x_2(y) = 0$, where $x_1 = \coth y(\sinh^2 y - y^2)^{1/2}$ and $x_2 = 2n\pi + \text{arc cos } (y/\sinh y)$. The y of the root was found by calculating $x_1 - x_2$ in the neighborhood of G. H. Hardy's approximation $y = \log (4n+1)\pi$ and then using inverse interpolation. The real part, x, was then taken as $x_2(y)$. The x and y were checked by the relation $x = x_1(y)$. (Received September 30, 1942.)

315. Ervand Kogbetliantz: *Inequality for definite integrals.* Preliminary report.

It is shown that the inequality for definite integrals $\int_a^b f(x)g(x)dx$ recently established by Shohat admits of a simple geometrical interpretation. This permits the extension of the inequality to the case where first n moments of $f(x)$ and $g(x)$ in $[a, b]$ vanish. (Received September 30, 1942.)

316. D. M. Krabill: *On extension of Wronskian matrices.*

The Wronskian matrix of order $n-1$ for a given set of solutions of a differential equation of the type (H): $y^{(n)}+f_1(x)y^{(n-1)}+\cdots+f_n(x)y=0$ with the $f_i(x)$ real and continuous in an interval J, has constant rank r in J and every column is a linear combination with constant coefficients of some r of the columns. The author gives a sufficient condition that an arbitrary set of functions having suitable class properties be solutions of an equation of type (H). The principal theorem states: If $k\leqq s\leqq t$ and the Wronskian matrix is of order s, $M_s(u_1, \cdots, u_k)$, for k arbitrary functions of class $C^{(t)}$ in J has constant rank k, there exists a function u_{k+1} of class $C^{(t)}$ with the extended matrix $M_s(u_1, \cdots, u_{k+1})$ of rank $k+1$ throughout J. A theorem of Curtiss (Mathematische Annalen, vol. 65 (1908), pp. 282–298, Theorem 4) on the zeros of the Wronskian determinant $W(u_1, \cdots, u_k)$ is utilized. From the extension property of Wronskian matrices is obtained the result: If the Wronskian matrix of order $n-1$ for k functions of class $C^{(t)}$ $(k\leqq n\leqq t)$ in an interval J has constant rank k, then the u's are linearly independent solutions of an equation of type (H) with the $f_i(x)$ of class $C^{(t-n)}$ in J. (Received August 3, 1942.)

317. Lee Lorch: *Lebesgue constants for Borel summability of Fourier series.*

Let $L_B(n)$ denote the nth Lebesgue constant for Borel summability of Fourier series. C. N. Moore (Proceedings of the National Academy of Sciences, vol. 11 (1925)) remarked that $L_B(n)$ becomes logarithmically infinite with n. Here it is shown that $L_B(n)=(2/\pi^2)\log n-(2/\pi^2)\log(\pi^2/2)-2C/\pi^2-(2/\pi^2)\int_0^\pi\psi(t/\pi)\sin t\,dt+O(n^{-1/2})$ as n becomes infinite, where $\psi(t)=\Gamma'(t)/\Gamma(t)$ and C is the Euler-Mascheroni constant. (Received August 3, 1942.)

318. Szolem Mandelbrojt: *Quasi-analyticity and properties of flatness of entire functions.*

The author proves a general theorem which may be regarded as a theorem concerning quasi-analyticity as well as the behaviour of entire functions on a set of points. This theorem relates an integrable function to an entire function. Each time that an entire function is chosen there results a theorem on quasi-analyticity, and when a suitable integrable function is given a theorem concerning entire functions follows. This paper will appear in the Duke Mathematical Journal. (Received September 14, 1942.)

319. W. T. Reid: *Some results on the growth of solutions of differential systems.*

In this paper elementary differential and integral inequalities are used to prove various results on the growth of solutions of differential systems, both linear and nonlinear, and involving functions of a real independent variable. In particular, the methods herein employed afford very brief proofs and improved forms of many of the results recently established by Trjitzinsky (Transactions of this Society, vol. 50 (1941), pp. 252–294). The paper also contains applications of an improved form of a theorem on matrix transformations due to Perron (Mathematische Zeitschrift, vol. 32 (1930), pp. 465–473). (Received September 7, 1942.)

320. R. M. Robinson: *Bounded analytic functions.*

(1) Suppose $g(z)$ is regular and satisfies $|g(z)| \leqq 1$ for $|z| < 1$. The relations between $g(0)$, z_0, $g(z_0)$, and $g'(z_0)$ are studied. For example, it is shown that for given values of $g(0)$ and z_0, the possible values of $g'(z_0)$ fill a closed convex set, which is bounded by a circle, a limaçon, or partly by one and partly by the other. The relations between the absolute values of the four quantities mentioned are also considered. (2) Suppose $f(z)$ satisfies the same conditions as $g(z)$, and also $f(0) = 0$. The relations between $f'(0)$, z_0, $f(z_0)$, $f'(z_0)$ are studied. The relations between their absolute values are also considered. For example, the sharp lower and upper bounds for $|f'(z_0)|$ in terms of $|f'(0)|$ and $|z_0|$ are found. The lower bound is given by $|f'(z_0)| \geqq [|f'(0)|(1+|z_0|^2) - 2|z_0|]/(1-|f'(0)z_0|)^2$ provided the right side is positive, and is 0 otherwise; the upper bound is given by four formulas, valid for different values of $|f'(0)|$ and $|z_0|$. The relations between $|f'(0)|$, $|z_0|$, $|f(z_0)|$, and $|f'(z_0)|$, for univalent functions, were studied by the author in an earlier paper (Transactions of this Society, vol. 52 (1942), pp. 426–449). (Received September 16, 1942.)

321. P. C. Rosenbloom: *The zeros of the partial sums of power series representing entire functions.*

Let $f(z)$ be an entire function of order $0 < \rho \leqq \infty$. Consider the zeros of the nth partial sum $s_n(z)$ for large n. If $\rho = \infty$, it is proved that there are a sequence $\{n_i\}$ and a sequence of radii $\rho_i \rightarrow \infty$ such that if ϵ, $\delta > 0$, then there is an $i_0 = i_0(\epsilon, \delta, f)$ for which the number of zeros of $s_{n_i}(z)$ in the ring $\rho_{n_i}(1-\delta) \leqq |z| \leqq \rho_{n_i}(1+\delta)$ is $n_i(1-\eta)$ with $0 \leqq \eta < \epsilon$. The arguments of the zeros of $s_{n_i}(z)$ are equally distributed in the interval $0 \leqq \theta \leqq 2\pi$. If R_{n_i} is the maximum modulus of the zeros of $s_{n_i}(z)$, then lim sup $R_{n_i}/\rho_{n_i} \leqq 2$. The radius ρ_{n_i} can be chosen as $|a_{n_i}|^{-1/n_i}$ where a_{n_i} is the nth coefficient in the Taylor series for $f(z)$. These results constitute a sharpening of some theorems of Carlson (Comptes Rendus de l'Académie des Sciences, Paris, 1924), whose proofs have never been published (so far as we know). These results bear a striking analogy to the results of Jentzsch (*Untersuchungen zur Theorie der Folgen analytischen Funktionen*, Dissertation, Berlin, 1914) and Szegö (*Über die Nullstellen der Polynomen, die in einem Kreise gleichmässig konvergieren*, Sitzungsberichte der Berliner Mathematische Gesellschaft, vol. 21 (1922)) for power series with a finite radius of convergence. For functions of finite order, lim sup $R_{n_i}/\rho_{n_i} \leqq 2$ exp $(1/\rho)$ for a certain sequence $\{n_i\}$. The paper also includes a detailed investigation of certain functions of finite order such as the error function, with results similar to Szegö's, for the exponential function (*Über eine Eigenschaft der Exponentialreihe*, Sitzungsberichte der Berliner Mathematische Gesellschaft, vol. 23 (1924)). (Received August 31, 1942.)

322. Max Shiffman: *Unstable extremal surfaces for double integral problems in the calculus of variations.*

Theories of the absolute minimum for double integral problems have been established in recent years. In this paper the author develops a theory of unstable extremal surfaces, proving the general Morse relations. The integrals to be made stationary have the form $\iint\{f(X, Y, Z)+k(X^2+Y^2+Z^2)^{1/2}\}\,du\,dv$ where X, Y, Z are the Jacobians $(y_u, z_u; y_v, z_v)$, and so forth, k is a positive constant, and f satisfies the following four conditions: (1) f is positively homogeneous in X, Y, Z; (2) $e=f(X, Y, Z)-\{Xf_{X_0}+Yf_{Y_0}+Zf_{Z_0}\}\geqq 0$; (3) $f(X, Y, Z)\geqq m>0$ for $X^2+Y^2+Z^2=1$; and (4) $f(X, Y, Z)<k$ for $X^2+Y^2+Z^2=1$. The first three are usual conditions to be expected; the main restriction is (4) which asserts that the integral is dominantly an area integral. The problem is to find extremal surfaces bounded by a given rectifiable curve. The integral is replaced by $\iint f(X, Y, Z)\,du\,dv+kD[x]$ where $D[x]$ is the Dirichlet integral, and the extremal surfaces found are given in isometric representation. (Received August 5, 1942.)

323. J. A. Shohat: *An inequality for definite integrals.* Preliminary report.

Starting with a simple case of the Schwarz inequality, another inequality is derived in a very elementary manner. This inequality seems to be new and may be considered an improvement over the Schwarz inequality. In fact, it is shown that the new inequality yields better results than the Schwarz inequality (it reduces to the latter, in some special cases).—Applications are indicated to orthogonal functions and to polynomials. (Received September 17, 1942.)

APPLIED MATHEMATICS

324. Lipman Bers and Abe Gelbart: *On solutions of the differential equations of gas dynamics.* I.

By a transformation due to Chaplygin and Busemann the equations of the steady two-dimensional irrotational flow of a gas can be reduced, in the subsonic case, to the form (1): $u_x=\tau(y)v_y$, $u_y=-\tau(y)v_x$, where τ is a given function. Equations of the same form occur in other branches of mechanics of continua. The class of complex-valued functions $f(x+iy)=u+iv$, where u and v satisfy (1) is shown to have many properties in common with analytic functions. Two processes, one the inverse of the other, are introduced: τ-differentiation and τ-integration. The τ-derivative and the τ-integral of f belong to the class. By τ-integrating a complex constant, a_n, n times a function $a_n\cdot Z\cdot{}^{n}(z)$, of the class is obtained. From these "powers," "polynomials" and "power series" are formed. A "polynomial" of the nth degree possesses n zeros. Any function, f, of the class can be developed in a unique manner in a power series, the "coefficients" a_n, being given by the nth τ-derivatives of f. Domains of convergence, mapping properties, and particular functions are considered. Analogous methods are applied to more general types of equations; in particular to the system $u_x=\tau_1(y)v_y$, $u_y=-\tau_2(y)v_x$ which describes the subsonic as well as supersonic flow. (Received August 27, 1942.)

325. Garrett Birkhoff: *A reversibility paradox in hydrodynamics.*

It is shown that every known differential equation and general boundary condition describing compressible non-viscous flow is reversible, in the sense that it is compatible with reversal of the direction of flow without change in pressure. This is even true in the presence of a conservative force-field. It follows that, contrary to a widespread impression, any theory of non-viscous flow (airfoil theory, resistance to projectiles, or resistance to surface craft in water) which purports to be based entirely on general considerations, must predict that a flow and its reverse give identical pressure distributions, hence identical lift and drag. This reversibility paradox partly overlaps the d'Alembert paradox, but it is far simpler, and more general in that it applies to modern airfoil theory and to ships. Special methods for avoiding it under particular circumstances, and their comparative validity, are then discussed. (Received August 27, 1942.)

326. Nathaniel Coburn: *The linear yield condition in the plane plasticity problem.*

The curvilinear nets consisting of curves of maximum and minimum shearing stress have been studied, extensively. However, all of these studies have assumed that the material under investigation follows the von Mises yield condition. In this paper, the above nets are investigated for materials which follow a linear yield condition. It is shown that two cases exist: (1) corresponding to lines of slope ± 1; (2) corresponding to lines of slope other than 0, ± 1 (the case of slope 0 is that due to von Mises). For the first case, it is shown that the two families of curves of the net possess the following two properties: (1) they constitute an equiareal system; (2) at any given point, the magnitudes of their radii of curvature are equal. Analytically, the functional relation is determined which the metric coefficients satisfy. For the second case, it is shown that a generalization of the von Mises diagonal property is valid. Analytically, the determination of the metric coefficients is shown to depend upon the solution of a nonlinear second order partial differential equation. Special solutions are studied. (Received September 28, 1942.)

327. W. R. Foote, Hillel Poritsky, and J. J. Slade: *The motion of a rotor with unequal shaft flexibilities mounted in bearings of unequal flexibility.*

In this paper is discussed the motion of a rotor possessing different flexibilities in two mutually perpendicular directions and mounted in bearings which likewise possess different flexibilities. The equations of motion of such a rotor revolving at constant speed form a fourth order linear system with periodic coefficients. The passage from stable to unstable solution is investigated in detail. For a fourth order system, unlike the second order Mathew equation, this passage may (and does) occur not only through periodic solutions when the characteristic exponent vanishes (or is equal to i) but also by having some of the characteristic exponents equal to each other. What are believed to be new and more convenient methods of calculation are developed. Some of these readily carry over to any order system with periodic coefficients which are trigonometric polynomials. (Received August 4, 1942.)

328. R. P. Isaacs: *Application of polygenic functions to two-dimensional elasticity.*

Let σ and τ be the tensile and shearing stresses in a two-dimensional body on a lineal element at z and inclined at θ with the real axis. If all internal parts of the body are in equilibrium, $\sigma+i\tau$ is the derivative of a polygenic function g which satisfies $I[G_z]=0$, and the derivative of any such function is a stress distribution in equilibrium. The derivative circle of g is the Mohr circle. $G_{zz}=0$ is necessary and sufficient for the body to obey Hooke's law. (Received September 12, 1942.)

GEOMETRY

329. Edward Kasner and J. J. DeCicco: *A generalized theory of dynamical trajectories.*

The differential geometry of the dynamical trajectories of positional fields of force has been completely developed in the *Differential-Geometric Aspects of Dynamics*, American Mathematical Society Colloquium Publications, vol. 3, 1913 (also Transactions of this Society, vols. 7–9 (1906–1910)). The fields of force, considered heretofore, depend only upon the position of the point. In their new work, the authors begin the study of the geometry of the generalized dynamical trajectories of fields of force which depend not only upon the position of the point but also upon the direction through the point. There are ∞^3 generalized dynamical trajectories. Any such system is completely characterized by the geometric Property I. The complete relationship between the rest trajectories and the lines of force is obtained (including the generalization of the theorem about one-third the curvature). Other systems of curves, such as velocity systems, systems S_1, pressure curves are defined for our generalized fields of force. These three new systems also possess the Property I. Complete geometric discussions of these systems are given. (Received August 24, 1942.)

330. Karl Menger: *Projective generalization of metric geometry.* I.

An F-projective space is called a set P if with each quadruple of elements p, q, r, s no three of which are identical an element (p, q, r, s) of a field F or ∞ is associated and $(p, q, r, s)=(q, p, s, r)=(r, s, p, q)$ and $(p, q, r, s)+(p, r, q, s)=(p, q, r, s)\cdot(p, s, r, q)=1$. An F-projective space, called $P_1(F)$, is obtained if for the elements of F the cross-ratio is defined as usual. In order that P be imbeddable into $P_1(F)$ it is necessary and sufficient that P satisfy the relation $(p, q, r, s)\cdot(p, s, r, t)=(p, q, r, t)$ and contain no pseudo-degenerate quadruple (four distinct elements with cross-ratio 0). The only non-imbeddable triples and pairs are pseudo-harmonic (that is, have cross-ratio -1). (Received August 3, 1942.)

331. Karl Menger: *Projective generalizations of metric geometry.* II.

P is called an $|F|$-projective space if with each quadruple p, q, r, s no three of whose elements are identical, a pair $(f, -f)$ of elements of F with the sum 0, denoted by $|p, q, r, s|$, is associated and $|p, q, r, s|=|q, p, s, r|=|r, s, p, q|$, $|p, q, r, s|\cdot|p, s, r, q|=|p, q, r, s|\cdot|p, r, s, q|\cdot|p, s, q, r|=1$, and $1+(f^2-g^2)^2=2(f^2+g^2)$ for $f=|p, q, r, s|$ and $g=|p, r, q, s|$. $P_1|F|$ is called the set F if $|p, q, r, s|=|(p, q, r, s)|$. In order that P be imbeddable into $P_1|F|$ it is necessary and sufficient that P satisfy the condition $|p, q, r, s|\cdot|p, s, r, t|=|p, q, r, t|$ and does not contain pseudo-degenerate quadruples or pseudo-harmonic triples or pairs, that is, distinct points with $|p, q, r, s|=|1|$ or not distinct points with $|p, q, r, s|=|2|$. (Received August 3, 1942.)

332. Saly R. R. Struik: *Emmy Noether and Maria Gaetana Agnesi, two outstanding women mathematicians.*

The work and life of these two mathematicians are evaluated and compared. (Received August 6, 1942.)

Logic and Foundations

333. A. R. Schweitzer: *On a class of ordered* $(n+1)$*-ads relevant to the algebra of logic.* III.

The author analyzes the formal sum (now not necessarily reflexive) of $(n+1)$-ads of generalized constituent type into two types of ordered dyads: $\xi\cdot\eta$, multiplicative, and $(\xi)+(\eta)$, additive, where $(\xi)=\xi_1\xi_2\cdots\xi_{n+1}$. Addition is assumed commutative and associative; multiplication is associative but not necessarily commutative. Formal addition is extended to elements ξ by assuming the formal equivalence: $(\alpha_1+\lambda_1)(\alpha_2+\lambda_2)\cdots(\alpha_{n+1}+\lambda_{n+1})=\alpha_1\alpha_2\cdots\alpha_{n+1}+\lambda_1\alpha_2\cdots\alpha_{n+1}+\cdots\lambda_1\lambda_2\cdots\lambda_{n+1}$. This assumption effects a gradual transition from the author's theory to formal developments based on the distributive property such as Grassmann's extensive algebra, the algebra of logic, Frobenius' calculus of group elements, number systems (fields) and so on. Grassmann's algebra contrasts with the latter in reference to closure properties. These are defined in terms of completeness (extension, invariance, attainment of limit of development) of the fundamental set of elements with reference to adjunction of operations on elements ξ, for example, $\xi\cdot\eta$, $\xi+\eta$. Finally the author discusses applications of restricted distributive properties such as $(x+y)y=xy+y^2$, $(x+I)y=xy+Iy$, where $Iy=y$ and addition and multiplication may or may not be reflexive. Reference is made to a paper reported in this Bulletin, abstract 48-3-135. (Received August 15, 1942.)

334. Saly R. R. Struik: *Axiomatics of affine geometry.*

The axiomatics are developed for two and three dimensions separately. The theorem of Desargues is used as an axiom in two dimensions. (Received August 6, 1942.)

Statistics and Probability

335. T. N. E. Greville: *Regularity of label-sequences under configuration transformations.*

There is developed a class of transformations on sequences of arbitrary labels in terms of which a wide variety of problems in the theory of probability can be formulated. It is shown that, with mild restrictions on the transformations used and on the measure function assumed on the label-space, almost every label-sequence produces a transform having the frequency distribution expected. The class of transformations considered is shown to include as special cases the four fundamental operations of von Mises: place selection, partition, mixing, and combination. (Received August 4, 1942.)

336. H. B. Mann: *The construction of orthogonal Latin squares.*

A Latin square is based on a group G if in the reduced form its rows form a regular representation of G. A set of orthogonal squares is based on G if all the squares of the set are based on the same representation of G. The mappings $S_1, S_2, \cdots, S_r$ of G onto itself are r-fold complete if every element of G is of the form $X^{S_i+S_{i+1}+\cdots+S_{i+k}}$ for every i and k with $1\leqq i\leqq r-k$ where X^{S_i} is the image of X under the mapping S_i and $X^{S_i+S_j}=X^{S_i}X^{S_j}$. A set of r orthogonal squares based on G exists if and only if G admits an r-fold complete mapping and vice versa. No $4n+2$ sided Graeco-Latin square based on a group exists. The orthogonal set is constructed by the automorphism method if for every i the mapping $S_1+S_2+\cdots+S_i$ is an automorphism of G. If

c_q is the number of classes of elements of order q in G, then not more than min c_q orthogonal squares can be constructed from G by the automorphism method. (Received September 30, 1942.)

337. Abraham Wald: *On a statistical problem arising in the classification of an individual in one of two groups.*

Let π_1 and π_2 be two p-variate normal populations which have a common covariance matrix. A sample of size N_i is drawn from the population π_i $(i=1, 2)$. Denote by $x_{i\alpha}$ the αth observation on the ith variate in π_1, and by $y_{i\beta}$ the βth observation on the ith variate in π_2. Let z_i $(i=1, \cdots, p)$ be a single observation on the ith variate drawn from a population π where it is known that π is equal either to π_1 or to π_2. The parameters of the populations π_1 and π_2 are assumed to be unknown. It is shown that for testing the hypothesis $\pi=\pi_1$ a proper critical region is given by $U \geqq d$ where $U=\sum\sum s^{ij} z_i(\bar{y}_j-\bar{x}_j)$, $\|s^{ij}\|=\|s_{ij}\|^{-1}$, $s_{ij}=[\sum_\alpha(x_{i\alpha}-\bar{x}_i)(x_{j\alpha}-\bar{x}_j)+\sum_\beta(y_{i\beta}-\bar{y}_i)(y_{j\beta}-\bar{y}_j)]/(N_1+N_2-2)$, $\bar{x}_i=(\sum_\alpha x_{i\alpha})/N_1$, $\bar{y}_i=(\sum_\beta y_{i\beta})/N_2$ and d is a constant. The large sample distribution of U is derived and it is shown that U is a simple function of three angles in the sample space whose exact joint sampling distribution is derived. (Received August 7, 1942.)

338. Jacob Wolfowitz: *On the consistency of a class of non-parametric statistics.*

Let X and Y be two stochastic variables about whose distribution nothing is known except that they are continuous and let it be required to test whether their distribution functions are the same. Let V be the observed sequence of zeros and ones constructed as described elsewhere (Wald and Wolfowitz, Annals of Mathematical Statistics, vol. 11 (1940), p. 148). Suppose that the statistic $S(V)$ used to test the hypothesis is of the form $S(V)=\sum\phi(l_j)$, where l_j is the length of the jth run and $\phi(x)$ a suitable function defined for all positive integral x. The notion of consistency, originated by Fisher for parametric problems, has already been extended to the nonparametric case (loc. cit., p. 153). The author now proves that, subject to reasonable conditions on $\phi(x)$ and statistically unimportant restrictions on the alternatives to the null hypothesis, statistics of the type $S(V)$ are consistent. In particular, a statistic discussed by the author (Annals of Mathematical Statistics, vol. 12 (1942)) and for which $\phi(x)=\log(x^x/x!)$ belongs to the class covered by the theorem. (Received August 7, 1942.)

TOPOLOGY

339. O. G. Harrold: *A higher dimensional analogue of a theorem of plane topology.*

Since the carriers of a Vietoris cycle may have a dimensionality far removed from that of the cycle, it is of interest to determine a class of spaces for which the bounding cycles have membranes of dimensionality exceeding that of the cycle by unity. An example is known of an $1c^1$ carrying an essential 1-cycle which has a 1-dimensional carrier but bounds only on a 3-dimensional set. A similar example is constructed in the Euclidean space E_4. That such cannot happen in certain Euclidean spaces is indicated by the following theorem, which is a generalization of a known result for $n=0$. Let X be a compact $1c^n$ subset of E_{n+2}. Denote by F the frontier of X relative to E_{n+2}. There exists in X a compact subset X_0 which is $1c^n$ such that $X_0 \supset F$ and dim $X_0 \leqq n+1$. (Received August 4, 1942.)

340. F. B. Jones: *A characterisation of the sphere.*

Let M denote a compact metric continuous curve such that (1) the sum of no two points separates M and (2) every simple closed curve in M separates M. Kline has raised the question as to whether M is a simple closed surface or not. The main result of this paper shows that it is if M contains no skew curve of type 1. (Received August 5, 1942.)

341. Saunders MacLane and Samuel Eilenberg: *Functions of groups as generalized tensors.*

Let Hom $\{G, H\}$ denote the group of all homomorphisms θ of the group G into the group H, and let α_1, β_1 be fixed homomorphisms of G into G_1, H into H_1, respectively. Then $\theta \in \text{Hom } \{G_1, H\}$ "induces" $\beta_1\theta\alpha_1 \in \text{Hom } \{G, H_1\}$, and the correspondence $\theta \rightarrow \beta_1\theta\alpha_1$ is itself a homomorphism $T(\alpha_1, \beta_1)$ of the first group (of θ's) into the second group Hom $\{G, H_1\}$. This function T has the property $T(\alpha_2\alpha_1, \beta_2\beta_1) = T(\alpha_1, \beta_2)T(\alpha_2, \beta_1)$. A function of two groups G, H, like the function "Hom," associated with such a function T of mappings of these groups, is called a tensor "covariant" in H, "contravariant" in G. The intuitive notion of a "natural isomorphism" between two groups can be defined exactly as an isomorphism between tensor functions of groups. This paper extends the same concepts from groups to topological spaces, to rings, to complexes, and to other types of mathematical systems. A number of general properties of tensors and their isomorphisms are established. (Received August 3, 1942.)

342. R. L. Moore: *Concerning a continuum and its boundary.*

If H, K and T are subsets of the continuum M then T is said to shield H from K in M provided it contains no point of K but intersects every connected subset of M which intersects both H and K. Among other things, it is shown that if, in a metric space, A and B are points of β, the boundary of the compact continuum M, and for each component x of $M-\beta$ there is a continuum β_x lying in β and shielding $A+B$ from x then A and B belong to the same component of β. (Received September 10, 1942.)

343. R. L. Moore: *Concerning intersecting continua.*

One of the results obtained is the theorem that if, in a metric space, β is the boundary of the compact continuum M and K is a compact continuum intersecting M and, for every component D of $M-\beta$ and component E of the complement of $\bar{D}$, the boundary of E is a connected subset of β, then $M \cdot K$ is a subset of a component of $M \cdot K+\beta$. (Received September 10, 1942.)

344. C. N. Reynolds: *A calculus of finite topological differences with application to the four color problem.*

A topological derivative of any function with respect to a well defined transformation of the neighborhood of an arbitrary boundary of a map is defined and is so applied to the four color problem as to reduce it to an equivalent problem in the theory of functions of a real variable. (Received August 4, 1942.)

345. A. D. Wallace: *A characterization of unicoherence.*

Let S be a Peano space. If $S=A+B+C$ is a decomposition into continua such that $AB, BC, CA \neq 0$ then assuming S to be unicoherent it is readily seen that $ABC \neq 0$. For $S=A+(B+C)$ so that, since $B+C$ is a continuum, so also is $A(B+C)=AB+AC$. The summands in the right member being non-void and closed it follows that they must intersect. The converse of this is also true and the proof is contained implicitly in the proof of Theorem 1 of *Monotone coverings and monotone transformations*, Duke Mathematical Journal, vol. 6 (1940), p. 32. (Received September 15, 1942.)

346. G. T. Whyburn: *Homotopy classes of mappings into the circle.*

In this paper it is shown primarily that any continuous mapping of a locally connected continuum X into a circle S is homotopic to a mapping f which admits factorization into the form $f=f_2f_1$, where f_1 is monotone and f_2 is interior and light (thus f is quasi-monotone). Used in the proof of this main theorem are auxiliary results to the effect that (1) any continuous mapping f of X into S is interior at every $x \in X$ which is a component of $f^{-1}f(x)$ and which is not contained in arbitrarily small neighborhoods U with $F(U) \subset f^{-1}(y)$ for some $y \in S-f(x)$; (2) a light mapping f of X into S is interior if and only if, for any $y \in S$, every component of $X-f^{-1}(y)$ maps onto $S-y$, (3) any non-alternating mapping f of X into S factors into the form $f=f_2f_1$, where f_1 is monotone and f_2 is non-alternating, interior and light; thus any non-alternating light mapping of X into S is interior; (4) any continuous mapping f of X into S such that, for any $y \in S$, each component of $X-f^{-1}(Y)$ maps onto $S-y$ is quasi-monotone. (Received August 5, 1942.)

347. J. W. T. Youngs: *A note on separation axioms and their application in the theory of a locally connected topological space.*

This note is in the nature of a sequel to a paper by G. E. Albert and the author (*The structure of locally connected topological spaces*, Transactions of this Society, vol. 51 (1942), pp. 637–654). A new separation axiom is introduced. T: If $x \cdot y=0$, then $\bar{x} \cdot \bar{y}=0$. In strength this lies between the T_0 and T_1 axioms. Using the terminology of the paper mentioned above it is shown that the class of locally connected T-spaces is hereditary. (Received August 3, 1942.)

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ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

R. P. BOAS, JR.

1. **Introduction.** An entire function $f(z)$ is said to be of exponential type if it is of order one and mean type; that is, if for some non-negative c and every positive ϵ there is a number $A(\epsilon)$ such that

$$|f(z)| < A(\epsilon)e^{(c+\epsilon)|z|} \tag{1}$$

for all z. The smallest c which can be used in (1) is called the type of $f(z)$. An alternative definition[1] is

$$c = \limsup_{n\to\infty} |f^{(n)}(z)|^{1/n}; \tag{2}$$

it is immaterial which value of z is used in (2). If (1) holds in a region of the z-plane, for example in an angle, $f(z)$ is said to be of exponential type c in that region.

Functions of exponential type have been extensively studied, both for their own sake and for their applications. I shall discuss here a selection of their properties, chosen to illustrate how the restriction (1) on the growth of a function restricts its behavior in other ways.[2]

2. **Representations.** Various formulas are available for representing functions of exponential type. Some of these representations are useful for deriving results of the kind discussed later in this report; and they are of considerable interest for their own sake.

If $f(z)$ is an entire function satisfying (1), there is a function $\phi(w)$, analytic in $|w| > c$, such that

$$f(z) = \int_C e^{zw}\phi(w)dw, \tag{3}$$

where C is any contour containing $|w| = c$ in its interior.[3] The function $\phi(w)$ is defined by either of the equivalent formulas

$$\phi(w) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{n!a_n}{w^{n+1}} \quad \text{if} \quad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and

An address delivered before the New York meeting of the Society on April 4, 1942, by invitation of the Program Committee; received by the editors July 28, 1942.

[1] See [10, p. 241].

[2] For a report on functions of exponential type from another point of view, see [10].

[3] See [25, pp. 578–586].

$$\phi(w) = \frac{1}{2\pi i}\int_0^\infty f(x) e^{-wx} dx, \qquad \Re(w) > c.$$

More precisely, C in (3) can be any curve surrounding the set of singular points of $\phi(w)$; the configuration of this set depends on the growth of $f(z)$ in various directions. In this report, however, I shall usually neglect the more refined results in which the shape of the set of singular points of $\phi(w)$ is taken into consideration. Any result involving the type of $f(z)$ will have refinements involving the growth of $f(z)$ in two or more directions.

However, the case in which $\phi(w)$ has singularities only along a line segment is of particular interest. If $f(z)$ belongs to L^2 on a line (for definiteness, the real axis), then

$$(4) \qquad f(z) = \int_{-c}^{c} e^{izt} F(t) dt, \qquad F(t) \in L^2(-c, c);$$

and conversely.[4] This result can be extended to other L-classes, though with some loss of elegance. If $f(z) \in L^p(-\infty, \infty)$ $(1 < p < 2)$, then $f(z)$ has the representation (4) with $F(t) \in L^{p/(p-1)}$, but the converse is false.[5] A necessary and sufficient condition [23, p. 111] for $f(z)$ to belong to L^p $(p > 1)$ is that

$$f(z) = z\int_{-c}^{c} [F(c) - F(t)] e^{izt} dt - 2F(c) z^{-1} \sin cz$$

where $F(t)$ is a continuous function of period $2c$ whose Fourier coefficients c_n satisfy

$$\sum_{-\infty}^{\infty} | n c_n |^p < \infty.$$

When $p = 1$, we have the necessary and sufficient condition [1, p. 283]

$$f(z) = \int_{-c}^{c} G(t) e^{izt} dt,$$

where $G(t)$ has an absolutely convergent Fourier series and $G(c-) = G(-c+) = 0$.

For functions $f(z)$ bounded on the real axis, we have the representations (necessary but not sufficient)[6]

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[4] [21, p. 13]. A sharper result is given by Plancherel and Pólya [23, p. 228].

[5] [1, p. 280]. The result is implicit in [23, p. 243].

[6] [3, p. 151]. To obtain (5), apply (4) to $z^{-1}[f(z) - f(0)]$.

$$f(z) = f(0) + z\int_{-c}^{c} e^{izt}F(t)dt, \qquad F(t) \in L^2; \tag{5}$$

and also [2]

$$f(z) = \int_{-c}^{c} e^{izt}\frac{d^2F(t)}{dt}; \tag{6}$$

the notation in (6) is purely formal; what is meant is that the 2-transform $F(t)$ (in the sense of Bochner [9]) of $f(z)$ is necessarily linear outside $(-c, c)$. Both (5) and (6) have been applied in the study of functions of exponential type [19, 3].

For a function of exponential type in an angle, Macintyre [20] has given a useful representation by a contour integral.

3. **Operators.** A large number of the results mentioned in this report have to do with relations between the behavior of an entire function $f(z)$ and that of the functions obtained by operating on $f(z)$ with a sequence of operators L_n. In many cases, the L_n are the powers L^n of a single operator. The most interesting—certainly the most easily dealt with—operators L appear to be those which are distributive and (in a suitable sense) continuous, which transform an entire function of exponential type less than c into another function of type less than c, and which are permutable with differentiation. Associated with any such operator L there is a function $\lambda(w)$, analytic in $|w| < c$, such that if $f(z)$ has the representation (3), then[7]

$$L[f(z)] = \int_C e^{zw}\phi(w)\lambda(w)dw; \tag{7}$$

correspondingly, L may be regarded as the differential operator of infinite order, $\lambda(d/dz)$. Examples which have been frequently discussed are $L[f(z)] = f(z+1)$, $L[f(z)] = f'(z)$, $L[f(z)] = f(z+1) - f(z)$. The corresponding functions $\lambda(w)$ are e^w, w, and $e^w - 1$.

Corresponding to any sequence of operators L_n of this character, there is a uniqueness problem: given a sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$, for what class of functions of exponential type does $L_n[f(a_n)] = 0$ $(n = 0, 1, 2, \cdots)$ imply $f(z) \equiv 0$? A somewhat more general problem asks, given a set of circles C_n, whether at least one of the functions $L_n[f(z)]$ is necessarily univalent in the corresponding C_n.[8]

Because of the representation (7) for the operators L_n, uniqueness theorems reduce to theorems of closure for sets of analytic functions. If every function analytic inside and on the contour C can be expanded in a uniformly convergent series of the functions $e^{a_n w}\lambda_n(w)$, then in particular the function e^{zw} can be so expanded,

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[7] Boas (unpublished).

[8] This is really a generalization of the L_n problem for $f'(z)$.

$$e^{zw} = \sum_{n=0}^{\infty} c_n(z)e^{a_n w}\lambda_n(w).$$

Substituting this expansion into (3) and integrating term by term, we obtain, if $L_n[f(a_n)]=0$,

$$f(z) = \sum_{n=0}^{\infty} c_n(z) \int_C e^{a_n w}\lambda_n(w)\phi(w)dw = \sum_{n=0}^{\infty} c_n(z)L_n[f(a_n)] = 0.$$

Similarly, the generalized uniqueness problem involving univalence leads to the closure problem for the set $\lambda_n(w)(e^{a_n w}-e^{b_n w})$, where a_n and b_n are arbitrary (unequal) points in the circles C_n.

4. **Uniqueness theorems.** The most elementary uniqueness theorem is that associated with $L_n=L^n$, $L[f(z)]=f'(z)$, $\{a_n\}=\{0\}$. Here the theorem states that a function of exponential type—or, of course, any analytic function—which vanishes with all its derivatives at a given point, vanishes identically. An immediate generalization is Whittaker's problem: if $f^{(n)}(a_n)=0$, $|a_n|\leq 1$, what is the largest number A such that if $f(z)$ is of type less than A it must vanish identically [32, p. 45]? Since the functions $z^n e^{a_n z}$ are known [5, pp. 481–482; 15] to form a closed set in $|z|<\log 2$, the general argument just outlined shows that[9] $A\geq\log 2$. Pólya has recently shown[10] that $\log 2 < A < 0.77 < \pi/4 = 0.785+$.[11] If the a_n are further restricted to be real, the corresponding constant A is actually $\pi/4$ [28]. If we make the restriction $a_{2n+1}=0$, $|a_{2n}|\leq 1$, then[12] the corresponding A exceeds 1.31.

The univalence theorem corresponding to Whittaker's problem is that one at least of the derivatives of $f(z)$ must be univalent in the unit circle (an infinite number, if $f(z)$ is not a polynomial), provided that $f(z)$ is of type less than A, where $A\geq\log 2$ [6, 18]; the best possible A is not known.

If $L_n[f(z)]=f^{(n)}(z)+b_n f(0)$, where the b_n are sufficiently small numbers, we obtain a theorem [5, p. 485] stating that the numbers $f^{(n)}(a_n)$ cannot all be very small unless $f(z)\equiv 0$. If $L_{2n}[f(z)]=L_{2n+1}[f(z)]=f^{(2n)}(z)$, $a_{2n}=0$, $a_{2n+1}=1$, we have the uniqueness theorem for the so-called two-point boundary problem: if every derivative of even order of a function of exponential type less than or equal to c vanishes both at 0 and at 1, then $f(z)$ is a sine polynomial of order at most c/π; if $c<\pi$, $f(z)\equiv 0$.[13] A similar uniqueness theorem holds for more general sequences $\{a_{2n}\}$, $\{a_{2n+1}\}$, near 0 and 1, respectively; the corresponding univalence result also is true.

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[9] First proved in another way by Takenaka [29].

[10] Unpublished result.

[11] Thus disproving the conjecture that $A=\pi/4$. The example showing that $A<\pi/4$ is $f(\pi z/4)$, where $f(z)=e^z+\omega^{-1}e^{\omega z}+e^{\omega^2 z}$ and $\omega^2+\omega+1=0$.

[12] The result $A\geq 0.78$ is given in [5, p. 486]; the result stated here can be obtained by modifying the argument on p. 482.

5. **Further uniqueness theorems.** When the function $\lambda(w)$ of (7) is univalent inside the contour C, we can establish the closure of the functions $\lambda^n(w)$ by making a conformal map which carries them into the functions z^n [14]. In this way it is found, for example, that the operators $L_n[f(a_n)]=\Delta^n f(0)$ have a uniqueness theorem for functions of type less than log 2; from this result it can be shown that a function of type less than log 2, taking integral values at the points $n=0, 1, 2, \cdots$, is necessarily a polynomial.[14] More generally, the operators $L_n[f(a_n)]=\Delta^n f(a_n)$ can be studied; for example, the critical type for the operators $\Delta^n f(a_n)$, $|a_n|\leqq 1$, is at least log 3/2.

If $\lambda(w)$ is not only univalent, but also has no zeros inside the contour C, the sequence of operators L^n can be imbedded in the family of operators L^t corresponding to the functions $[\lambda(w)]^t$, and uniqueness theorems obtained corresponding to various sequences L^{λ_n}. A great deal of attention has been paid to the operator $L[f(z)]=f(z+1)$, corresponding to the function $\lambda(w)=e^w$; results for this operator naturally can be transformed by conformal mapping so as to apply to other operators. With this operator the uniqueness problem can be formulated more simply as the problem of finding conditions under which $f(\lambda_n)=0$ implies $f(z)\equiv 0$.

The oldest result of this kind is Carlson's theorem: If $f(z)$ is of exponential type c, $c<\pi$,[15] then $f(n)=0$ $(n=0, 1, 2, \cdots)$ implies that $f(z)\equiv 0$.[16] A more delicate result, applying only to entire functions, states that if

$$(8)\qquad |f(z)| < \epsilon(|z|)e^{\pi|x|}(1+|y|)^{-2\delta},\quad 0<\delta<1/2,\quad \lim_{r\to\infty}\epsilon(r)=0,$$

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[13] [27]. Another proof in [7]. For generalizations, see [14].

[14] Pólya and Hardy; see [32, p. 55].

[15] Or, more generally, of (some) exponential type in a right half-plane and of type less than π along its boundary.

[16] See, for example, [30, p. 186]. A related result of Estermann (Dienes [13, p. 259]) is that if $f(z)$ is of type less than 1, and $f(z_n)=0$, $|z_n|=n$ $(n=0, 1, 2, \cdots)$, then $f(z)\equiv 0$.

and $f(\lambda_n)=0$ $(n=0,\ 1,\ 2,\cdots)$, where $|\lambda_n-n|<\delta$, then $f(z)\equiv 0$.[17] More general sequences $\{\lambda_n\}$ are allowable if we go back to the stronger requirement that $f(z)$ is of type c, $c<\pi$. For example, if $f(z)$ is of type less than π in the half-plane $x\geqq 0$, and if $f(\lambda_n)=0$ where the real numbers λ_n satisfy $\lim_{n\to\infty} n/\lambda_n=D\geqq 1$, and $|\lambda_n-\lambda_m|\geqq |n-m|\delta$, $\delta>0$, then $f(z)\equiv 0$.[18] Much more general sequences $\{\lambda_n\}$ can be used if the growth of $f(z)$ is restricted by a condition stating roughly that $f(z)$ is of order less than one along the real axis. More precisely,[19] if $f(z)$ is an entire function of exponential type c, such that

$$\lim_{R\to\infty}\int_1^R (1/x^2)\log|f(x)f(-x)|\,dx$$

exists and is finite, and

$$\limsup_{x\to\infty}\,(1/x)\log|f(\pm x)|\leqq 0,$$

then the zeros of $f(z)$ in $x\geqq 0$ and in $x\leqq 0$ have a density B at most equal to c/π. Consequently we cannot have $f(\lambda_n)=0$ for a sequence $\{\lambda_n\}$ whose density exceeds c/π.

Even more generally, the hypothesis that $f(\lambda_n)=0$ can be replaced by one stating that the numbers $f(\lambda_n)$ are very small;[20] for example, that $f(\lambda_n)=O(e^{-\delta\lambda_n})$, $\delta>0$.

Another interesting result is that if (8) is satisfied with $\delta=1$, then $f(z)\equiv 0$ if $f(z)$ vanishes at least once in every interval $(n,\ n+1)$ [31, p. 213]. This suggests that if we let the zeros coalesce in pairs, so that $f(z)$ and $f'(z)$ both vanish at $z=2n$, then $f(z)\equiv 0$ if $f(z)$ is of type less than π. More generally, we can consider the case where we have two operators L and M, $L[f(2n)]=M[f(2n)]=0$, and the functions λ and μ associated with L and M satisfy suitable conditions [4].

6. **Growth theorems.** Let L_t be a one-parameter family of operators, for example the powers of an operator L whose $\lambda(w)$ has no zeros. A uniqueness theorem states that, given sequences $\{a_n\}$ and $\{\lambda_n\}$, any function $f(z)$ of sufficiently small type, such that $L_{\lambda_n}[f(a_n)]=0$, vanishes identically. If $f(z)$ is identically zero, then of course so are $L_t[f(z)]$ and $M_t[f(z)]$ for every t, where M_t is another family of operators. We now suppose, not that the elements of the sequence $\{L_{\lambda_n}[f(a_n)]\}$ are zero, but their growth is restricted in some way;

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[17] [3, p. 158]. The case $\delta=0$ is due to Pólya [24] and Valiron [31, p. 204].

[18] This is an easy deduction from Carleman's theorem (for Carleman's theorem see, for example [30, p. 130]).

[19] See [17, especially pp. 13, 25].

[20] Levinson [17, p. 19]; Levinson gives a number of much more general results.

and we ask what can be said about the growth of a sequence $\{M_{\mu_n}[f(b_n)]\}$. The case most extensively studied has been that in which $a_n=b_n=0$; in this case, it is more convenient to think of the problem as that of determining the growth properties of $\{M^*[f(\mu_n)]\}$ from those of $\{L^*[f(\lambda_n)]\}$, where $L^*[f(z)]=L_z[f(0)]$ and M^* is defined similarly.

It is not always desirable to restrict $\{\lambda_n\}$ and $\{\mu_n\}$ to be countable sequences. If we replace both sequences by the set of all real numbers x, our original uniqueness problem becomes trivial; but we have the following non-trivial growth problem: given that $L[f(x)]$ is bounded for real x, what bound can be assigned to $M[f(x)]$, where L and M are given operators? The oldest case of the problem is S. Bernstein's problem of the bound for the derivative of a function, given a bound for the function; here L is the identity operator and $M[f(z)]=f'(z)$. Here we have the result that $|f'(x)|\leq cK$ if $|f(x)|\leq K(-\infty<x<\infty)$.[31] More generally, if M is an operator with associated function $\mu(w)$ analytic in $|w|<c$, then for every $f(z)$ of type $c'<c$ there is a number $A(c')$, independent of f, such that[32]

$$(9)\qquad |M[f(x)]|\leq A(c')\sup_{-\infty<x<\infty}|f(x)|.$$

Under suitable restrictions on $\mu(w)$ on the imaginary axis, it can be shown by use of the representation (5) that $A(c')$ may be replaced by a number $A(c)$ depending only on c. Still more generally we can obtain results of the form

$$|M[f(x)]|\leq A(c')\sup_{-\infty<x<\infty}|L[f(x)]|.$$

Such a result is no more general than (9) if L has an inverse, that is, if $\lambda(w)$ has no zeros; but may hold even when $\lambda(w)$ does have zeros if the zeros are off the imaginary axis.

7. **Growth theorems involving sequences.** The theorems to be discussed in this section deduce the growth properties of $M[f(x)]$ from those of $L[f(\lambda_n)]$. The necessary restrictions on the sequence $\{\lambda_n\}$ vary with the expression taken as measure of the rate of growth. We consider first functions bounded on the real axis. If $f(z)$ is of type c, $c<\pi$, and $|f(n)|\leq K$ $(n=0,\pm 1,\cdots)$, then[33] $|f(x)|\leq A(c)K$; the best possible $A(c)$ is[34] $O(-1/\log(\pi-c))$ as $c\to\pi$. This result (apart from

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[31] See [26, vol. 2, p. 35, section IV, problem 201].

[32] Civin [12], for a restricted class of functions of exponential type. The results can be extended to the general case by the method used in [3, pp. 150–152].

[33] Cartwright [11]. Other proofs by Pfluger [22], Macintyre [20], Boas [3].

[34] Boas and Schaeffer [8].

the evaluation of $A(k)$) extends to functions of exponential type in a half-plane,[25] and in that form carries over directly to other rates of growth, for example, $f(x)=O(x^p)$ $(p>0)$ or $f(x)=O(e^{Ax})$. However, results involving an exponential rate of growth for $f(z)$ on the real axis extend to more general sequences of points than do those involving merely boundedness for $f(x)$. For boundedness, the most general result so far obtained is that if $|\lambda_n-n|<A$ and $|\lambda_m-\lambda_n| \geqq \delta(m\neq n;\ m,\ n=0,1,2,\cdots)$, then $f(z)$, of type less than π in $x>0$ and bounded at $\{\lambda_n\}$, is bounded on the real axis.[26] Results of Levinson [17, p. 127] show that one cannot go much farther in this direction. On the other hand, results of the type

$$\limsup_{x\to\infty}\frac{\log|f(x)|}{x}=\limsup_{n\to\infty}\frac{\log|f(\lambda_n)|}{\lambda_n}$$

hold for sequences $\{\lambda_n\}$ satisfying such general conditions as[27]

$$|\lambda_n-\lambda_m|\geqq|n-m|\,d,\qquad d>0,\ n/\lambda_n\to D,$$

where the type of $f(z)$ is less than πD.

In these results quoted so far in this section, the operators L and M have both been the identity operator. It is clear that more general theorems, at least for the case of boundedness, can be deduced indirectly from the results of §6. For, from the boundedness of $f(\lambda_n)$ we can infer first that of $f(x)$ and then that of $L[f(x)]$. In this way we can show, for example, that if $|f(n)|\leqq K$ and f is of type $c'<\pi$, then $|f'(x)|\leqq c'A(c')K$, where $A(c')$ is the same as in (9). However, we may in some cases obtain sharper results by proceeding directly; thus Macintyre has pointed out [20, p. 6] that $|f(n)|\leqq K$ implies

$$|f''(x)+\pi^2 f(x)|\leqq AK, \tag{10}$$

where A is a constant, even when $f(z)$ is of type π. The reason for the difference between the operators $L_1[f]=f'$ and $L_2[f]=f''+\pi^2f$ is clearly shown by the difference between the corresponding functions $\lambda_1(w)=w$ and $\lambda_2(w)=w^2+\pi^2$. The second vanishes at $w=\pm i\pi$; it is generally true that an L whose λ has this property has the stronger boundedness property typified by (10).

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[25] Cartwright [11], Macintyre [20].

[26] Duffin and Schaeffer (unpublished). I am indebted to these authors for permission to include their result in this report.

For a theorem in which the boundedness of $f(z)$ on two sequences implies its boundedness on two lines (and hence that it is a constant) see Levinson [17, p. 122].

[27] For this, and much more general results, see Levinson [17, pp. 100 ff.]. Other theorems of this character are given by Junnila [16].

8. **Integrability.** Bernstein's theorem on derivatives states in effect that a function of exponential type, if bounded on a line, cannot oscillate too rapidly on the line. Another illustration of this fact is that if $f(z)$ is of type less than π and is bounded on the average near the integers, it is bounded on the real axis; more precisely, if

$$\sup_{-\infty<n<\infty}\int_{n-\delta}^{n+\epsilon}\psi(|f(t)|)dt < \infty,$$

where $\epsilon \geqq 0$, $\delta \geqq 0$, $\epsilon+\delta>0$, and $\psi(t)$ is non-decreasing and unbounded, then $f(x)$ is bounded [6, p. 163]. In particular, $f(x)$ is bounded (and hence approaches zero) on the real axis if it belongs to L^p for some positive p [23, p. 124].

A result of similar character is that

$$\sum_{-\infty}^{\infty}|f(x_n)|^p < \infty, \qquad p > 0, \tag{11}$$

with

$$|x_n - n| \leqq L < (2\pi^2)^{-1}$$

implies

$$\int_{-\infty}^{\infty}|f(x)|^p dx < \infty \tag{12}$$

if[28] $f(z)$ is of type less than π; conversely, if $f(z)$ is of any finite type and if the number of x_n's in any interval of unit length is bounded, then (12) implies (11).[29]

Still another indication of the restrictions imposed on the oscillation of $f(x)$ by its integrability properties is that if $f(x)$ belongs to L^p $(p>0)$, so does $f'(x)$ [23, p. 127]. This can be considered as a generalization of Bernstein's theorem mentioned in §6; Bernstein's theorem is the limiting case $p\rightarrow\infty$.

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DUKE UNIVERSITY

SOME NOTES ON AN EXPANSION THEOREM OF PALEY AND WIENER

R. J. DUFFIN AND J. J. EACHUS

Paley and Wiener[1] have formulated a criterion for a set of functions $\{g_n\}$ to be "near" a given orthonormal set $\{f_n\}$. The interest of this criterion is that it guarantees the set $\{g_n\}$ to have expansion properties similar to an orthonormal set.[2] In particular, they show that the set $\{g_n\}$ approximately satisfies Parseval's formula. In the first part of this paper we show that, conversely, if a set $\{g_n\}$ approximately satisfies Parseval's formula then there exists at least one orthonormal set which it is "near."

In the second part of the paper we consider sets which are on the borderline of being near a given orthonormal set.

The last part of this paper gives a simple formula for constructing sets near a given orthonormal set. As an application of this formula we obtain new properties of the so called non-harmonic Fourier series.

We shall handle these problems abstractly, using the notation of Hilbert space.[3] Subscript variables are assumed to range over all positive integers and $\sum$ shall mean a sum over all positive integers. By a finite sequence shall be meant a sequence with only a finite number of nonzero members. For application to the space L_2 the norm of a function $f(x)$ is defined in the usual way as $\|f\| = (\int_a^b |f(x)|^2 dx)^{1/2}$. A complete set which satisfies the Paley-Wiener criterion shall be termed strongly complete.

The principal novelty in the proof is the association of a linear transformation G with each set of elements $\{g_n\}$. Thus if $\{\psi_n\}$ is an orthonormal set we define $G\sum a_n\psi_n = \sum a_n g_n$ for every finite sequence of constants $\{a_n\}$. The norm of G is the limit superior of $\|Gx\|$ for elements x such that $\|x\| = 1$. With this definition of norm the aggregate of bounded linear transformations clearly forms a normed linear

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Presented to the Society, April 12, 1940 under the title *Converse of a closure theorem of Paley and Wiener*, and September 5, 1941 under the title *A Paley-Wiener type expansion theorem*; received by the editors October 8, 1941.

[1] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 19, 1934, p. 100.

[2] R. P. Boas, Jr., Journal of the London Mathematical Society, vol. 14 (1939), p. 242; Duke Mathematical Journal, vol. 6 (1940), p. 148; American Journal of Mathematics, vol. 63 (1941), p. 361.

[3] Because of the difficulty of finding adequate references to non-separable Hilbert space we confine ourselves to separable space. However, our theorems remain true for non-separable Hilbert space provided the range of subscript variable is redefined.

space. It is in this space that the concept of one sequence of functions being "near" another is made precise; thus two points are "near" if they are within unit distance of each other.

1. A converse theorem.

We prove the following theorem.

THEOREM A. *Suppose that the set of elements* $\{g_n\}$ *satisfies the relation*

$$(1) \qquad (1-\theta)\left(\sum |a_n|^2\right)^{1/2} \leqq \left\|\sum a_n g_n\right\| \leqq (1+\theta)\left(\sum |a_n|^2\right)^{1/2}$$

for every finite sequence of constants $\{a_n\}$ *and for some fixed number* θ; $0 \leqq \theta < 1$. *Then there exists an orthonormal set* $\{f_n\}$ *such that*

$$(2) \qquad \left\|\sum a_n(f_n - g_n)\right\| \leqq \theta\left(\sum |a_n|^2\right)^{1/2}.$$

PROOF. Relation (1) guarantees that the set $\{g_n\}$ is linearly independent; hence the closure of all linear combinations of the set $\{g_n\}$ is an infinite-dimensional manifold, a Hilbert space. Let $\{\psi_n\}$ be some complete orthonormal set of this Hilbert space and define the transformation G by $G\sum a_n\psi_n = \sum a_n g_n$. Thus if $y = \sum a_n\psi_n$ we may express relation (1) in the form

$$(3) \qquad (1-\theta)\|y\| \leqq \|Gy\| \leqq (1+\theta)\|y\|.$$

The following lemmas are well known:[4]

LEMMA 1. *A bounded linear transformation whose range and domain both determine Hilbert space may be factored in the form FP where F is a unitary transformation and P is a positive definite self-adjoint transformation.*

LEMMA 2. *If S is a self-adjoint transformation then*

$$\|Sx\|/\|x\|$$

and

$$|(Sx, x)|/(x, x)$$

have the same upper and lower bounds.

Clearly G satisfies the conditions of Lemma 1, so $G = FP$. From relation (3) it follows that

$$(1-\theta)\|y\| \leqq \|Py\| \leqq (1+\theta)\|y\|.$$

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[4] M. H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, American Mathematical Society Colloquium Publications, vol. 15, 1932; J. von Neumann, Annals of Mathematics, (2), vol. 33 (1932), p. 308; A. Wintner, Mathematische Annalen, vol. 37 (1933), p. 257.

Moreover, this inequality clearly remains valid if y is an arbitrary element; so by Lemma 2

$$(1-\theta)(y, y) \leqq (Py, y) \leqq (1+\theta)(y, y).$$

This may be written as

$$-\theta(y, y) \leqq (Py - y, y) \leqq \theta(y, y).$$

Again using Lemma 2, we have $\|y-Py\| \leqq \theta\|y\|$. Let us define $f_n = F\psi_n$. Because F is unitary it follows that $\{f_n\}$ is an orthonormal set. Moreover,

$$\begin{aligned} \left\|\sum a_n(f_n - g_n)\right\| &= \|Fy - Gy\| \\ &= \|F(y - Py)\| \\ &= \|y - Py\| \\ &\leqq \theta\|y\| = \theta\left(\sum |a_n|^2\right)^{1/2}. \end{aligned}$$

2. **The borderline case, $\theta = 1$.** We now establish this theorem.

THEOREM B. *The set of elements $\{g_n\}$ and the orthonormal set $\{f_n\}$ satisfy*

$$\left\|\sum a_n(f_n - g_n)\right\| < \left(\sum |a_n|^2\right)^{1/2} \tag{4}$$

for every sequence of coefficients $\{a_n\}$ such that

$$0 < \sum |a_n|^2 < \infty.$$

Then $\{g_n\}$ is complete if $\{f_n\}$ is complete.[5]

PROOF. If $\{g_n\}$ is not complete, there exists an element z such that $(g_n, z) = 0$ for all n. We can express z in the form $z = \sum a_n f_n$, where $0 < \sum |a_n|^2 < \infty$. Thus

$$\left\|\sum a_n(f_n - g_n)\right\|^2 = \|z\|^2 - 2\Re \sum a_n(g_n, z) + \left\|\sum a_n g_n\right\|^2 \geqq \|z\|^2.$$

This contradicts (4), and the theorem is proved.

The following counterexample shows that the completeness of the set $\{g_n\{$ does not imply that the orthonormal set $\{f_n\}$ is complete. Let $\{c_n\}$ be a sequence of constants such that $c_n > c_{n+1} > 1$. Let $\{\psi_n\}$

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[5] The referee has pointed out that a mean ergodic theorem may be used to prove the following generalization of Theorem B.

Let B be a reflexive Banach space and let $\{f_n\}$ be a basis in B. Suppose $\{g_n\}$ is a sequence in B such that $\|\sum a_n(f_n - g_n)\| < \|\sum a_n f_n\|$ for all $x = \sum a_n f_n$ in B. Then the elements $\{g_n\}$ span B.

We have been able to extend his theorem to spaces whose unit sphere has weak sequential compactness.

be a complete orthonormal set. Define $f_n=\psi_{n+1}$ and $g_n=\psi_{n+1}-\psi_n c_{n+1}/c_n$ for $n=1, 2, \cdots$. Thus

$$\left\|\sum a_n(f_n-g_n)\right\|^2=\sum\left|\frac{c_{n+1}}{c_n}a_n\right|^2<\sum|a_n|^2,$$

and so relation (4) is satisfied. Obviously the set $\{f_n\}$ is not complete; however $\sum|c_n|^2=\infty$, so the set $\{g_n\}$ is complete.

If the $<$ sign is replaced by the $\leqq$ sign in relation (4), Theorem B is certainly false because some or all of the elements $\{g_n\}$ could be zero. Nevertheless, we are able to prove an analogue of Theorem A.

THEOREM C. *Let $\{g_n\}$ be a complete set and $\{f_n\}$ an orthonormal set such that*

$$\left\|\sum a_n(f_n-g_n)\right\| \leqq \left(\sum |a_n|^2\right)^{1/2} \tag{5}$$

for every sequence of constants $\{a_n\}$ such that

$$0<\sum|a_n|^2<\infty. \tag{6}$$

Then

$$0<\left\|\sum a_n g_n\right\| \leqq 2\left(\sum|a_n|^2\right)^{1/2}. \tag{7}$$

Conversely, the truth of (7) *for a complete set $\{g_n\}$ and for every sequence of constants $\{a_n\}$ satisfying* (6) *implies the existence of a* (*complete*) *orthonormal set $\{f_n\}$ satisfying* (5).

PROOF. The novelty in the first part of this theorem is the appearance of the strict inequality on the left side of (7). Suppose then, on the contrary, that $\{a_n\}$ is a sequence for which $\sum a_n g_n=0$. Let $\sum a_n f_n=z$. Because $\{g_n\}$ is complete, there exists an element g_0 such that $(g_0, z)\neq 0$. Thus, according to (5), if λ is a constant:

$$\left\|\sum a_n(f_n-g_n)+\lambda(f_0-g_0)\right\| \leqq \left\|\sum a_n f_n+\lambda f_0\right\|,$$
$$\|z+\lambda f_0-\lambda g_0\| \leqq \|z+\lambda f_0\|,$$
$$-2\Re\lambda(g_0, z)+|\lambda|^2\{|g_0|^2-2\Re(g_0, f_0)\} \leqq 0.$$

This last inequality is clearly impossible for all values of λ. The remainder of the proof is omitted as it parallels the proof of Theorem A.

3. The method of separation of variables. We prove the following theorem.

THEOREM D. *Let $\{C_{nk}\}$, $n, k=1, 2, \cdots$, be a matrix of constants such that $|C_{nk}|\leqq c_k$. Let $\{T_k\}$ be a sequence of bounded linear trans-*

formations with corresponding bounds $\{t_k\}$. Let $\{f_n\}$ be a complete orthonormal set, and define

$$g_n = f_n + \sum_{k=1}^{\infty} C_{nk} T_k f_n.$$

Then if $\sum c_k t_k < 1$, the set $\{g_n\}$ is strongly complete in the sense of Paley and Wiener.

Proof. Let $\{a_n\}$ be an arbitrary finite sequence of constants. Then

$$\begin{aligned}
\left\| \sum a_n(f_n - g_n) \right\| &= \left\| \sum_n a_n \sum_k C_{nk} T_k f_n \right\| \\
&= \left\| \sum_k T_k \sum_n C_{nk} a_n f_n \right\| \\
&\leqq \sum_k \left\| T_k \sum_n C_{nk} a_n f_n \right\| \\
&\leqq \sum_k t_k \left\| \sum_n C_{nk} a_n f_n \right\| \\
&\leqq \sum_k t_k c_k \left\| \sum_n a_n f_n \right\|.
\end{aligned}$$

Thus the set $\{g_n\}$ satisfies the Paley-Wiener criterion (2) with $\theta = \sum t_k c_k < 1$.

A simple way to apply Theorem D depends upon the fact that multiplication by a bounded function is a bounded linear transformation of the space L_2. In particular we consider the sequence of functions $\{e^{i\lambda_n x}\}$, where $\{\lambda_n\}$, $n = 0, \pm 1, \pm 2, \cdots$, is a sequence of complex constants satisfying $|\lambda_n - n| \leqq L$ for some constant L. The interval under consideration is $-\pi \leqq x \leqq \pi$. We may write

$$e^{i\lambda_n x} = e^{inx} + \sum_{k=1}^{\infty} \frac{(i\lambda_n - in)^k}{k!} x^k e^{inx}.$$

Comparing with Theorem D, we have:

$$\begin{aligned}
C_{nk} &= (i\lambda_n - in)^k/k!, & c_k &= L^k/k!; \\
T_k &= x^k, & t_k &= \pi^k.
\end{aligned}$$

Clearly, if $L < \log 2/\pi$, the set $\{e^{i\lambda_n x}\}$ is strongly complete in the interval $(-\pi, \pi)$. (Actually the same is true in any interval of length 2π.)

The transformations $\{x^k\}$ never attain their least upper bound. This fact permits Theorem B to be employed in showing that the set $\{e^{i\lambda_n x}\}$ is complete even if $L = \log 2/\pi$. The proof parallels the proof of Theorem D.

The above results on the non-harmonic Fourier series are an extension of previous knowledge in two respects: In the first place, Paley and Wiener were forced to assume that $\{\lambda_n\}$ was a real sequence. Secondly, they[6] obtained the value $1/\pi^2 = .10+$ where we have $\log 2/\pi = .22+$. The best value for L is not known; however a theorem of Levinson[7] gives an upper limit of $1/4$.[8]

A second application of Theorem D is to furnish a proof of an analytic function expansion theorem of Boas.[9] In turn, Boas' theorem contains analytic function expansion theorems of Birkhoff, Walsh, Takenaka, G. S. Ketchum and others.

The operator in Theorem D, $\sum C_{nk} T_k$, has been assumed to be a discrete series; however the method of separation of variables is still available if we replace the series by an integral or Stieltjes integral. In particular, Cauchy's integral formula is of the right form.

UNIVERSITY OF ILLINOIS AND
PURDUE UNIVERSITY

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[6] Paley and Wiener, loc. cit., p. 113. A slightly better value than theirs has been obtained by Malin, thesis, Massachusetts Institute of Technology, 1934.

[7] N. Levinson, Annals of Mathematics, (2), vol. 37 (1936), p. 919; *Gap and Density Theorems*, American Mathematical Society Colloquium Publications, vol. 26, 1940, chap. IV.

[8] It is a curious parallelism that $\log 2/\pi$ and $1/4$ are in the same ratio as the limits of Takenaka and Schoenberg in a somewhat similar unsolved problem. For references see R. P. Boas, Jr., Proceedings of the National Academy of Sciences, vol. 26 (1940), p. 139; Transactions of this Society, vol. 48 (1940), p. 485.

[9] R. P. Boas, Jr., Transactions of this Society, vol. 48 (1940), p. 473, Theorem 3.1.

A GENERALIZATION OF THE POLAR REPRESENTATION OF NONSINGULAR MATRICES

JOHN WILLIAMSON

1. Introduction. If A is a square matrix with elements in the complex number field, then

$$A = PU, \tag{1}$$

where P is a positive definite hermitian matrix and U is a unitary matrix.[1] In this polar representation of the matrix A, as it is called, the two matrices P and U are unique. Since the matrix P is positive definite and nonsingular, it has the same signature as the identity matrix E while the unitary matrix U is a conjunctive automorph of E. From (1) we may deduce a somewhat similar representation of A in terms of a positive definite hermitian matrix and a conjunctive automorph, not of E, but of any nonsingular positive definite hermitian matrix.

Let H be a nonsingular hermitian matrix which is positive definite, so that there exists a nonsingular matrix Q satisfying

$$Q^{-1}H(Q^{-1})^* = E,$$

where Q^* is the conjugate transposed of Q. If $B=Q^{-1}AQ$ and $B=PU$ is the polar representation of B, then $A=QPQ^{-1}QUQ^{-1}=DR$, where $D=QPQ^{-1}$ and $R=QUQ^{-1}$. Since $DH=QPQ^{-1}QQ^*=QPQ^*$, DH is hermitian with the same signature as H. Further $RHR^*=QUQ^{-1}QQ^*\cdot(Q^{-1})^*U^*Q^*=QQ^*=H$. Hence we have this result as an analogue of the polar representation (1) of A.

RESULT (1). *If H is any nonsingular positive definite hermitian matrix and A is a nonsingular matrix, then*

$$A = DR, \tag{2}$$

where DH is a positive definite hermitian matrix and $RHR^*=H$.

If $H=H^{-1}$, $A=DR=DHHR$ and, since $HRH(HR)^*=H^3=H$, $A=P_1R_1$, where P_1 is a positive definite hermitian matrix and $R_1HR_1^*=H$. Therefore we have as a second analogue of (1) the following result.

Received by the editors October 13, 1941.

[1] L. Autonne, *Sur les groupes linéaires, réels et orthogonaux*, Bulletin de la Société Mathématique de France, vol. 30 (1902), pp. 121–134. A. Wintner and F. D. Murnaghan, *On a polar representation of non-singular matrices*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 676–678.

RESULT (2). *If H is a positive definite hermitian matrix, whose square is the identity matrix, and, if A is a nonsingular matrix, then*

$$A = P_1R_1, \tag{3}$$

where P_1 is a positive definite hermitian matrix and R_1 is a conjunctive automorph of H. The two matrices P_1 and R_1 are unique.

It is our intention here to determine what corresponds to results (1) and (2), if H is a nonsingular hermitian matrix but not necessarily definite.

2. Generalization in the complex field.

Let H be any nonsingular hermitian matrix and let A be a nonsingular matrix. If

$$A = DR, \tag{4}$$

where

$$DH = HD^*\tag{5}$$

and

$$RHR^* = H, \tag{6}$$

we shall call (4) *a polar representation of A with respect to H* or, for brevity an *H-representation* of A. If (4), (5) and (6) are satisfied, $AHA^*H^{-1} = DRHR^*D^*H^{-1} = DHD^*H^{-1} = D^2$. Further, if $AHA^*H^{-1} = D^2$ and $DH = HD^*$, then (6) is satisfied with $R = D^{-1}A$. For $D^{-1}AH(D^{-1}A)^* = D^{-1}AHA^*(D^{-1})^* = D^{-1}D^2H(D^{-1})^* = H$ by (5). Therefore we have proved this lemma.

LEMMA 1. *A necessary and sufficient condition that a matrix A have an H-representation is that there exist a matrix D such that $DH = HD^*$ and $AHA^*H^{-1} = D^2$.*

If A has the H-representation (4) and $QAQ^{-1} = A_1$, $QDQ^{-1} = D_1$, $QRQ^{-1} = R_1$ and $QHQ^* = H_1$, then $A_1 = D_1R_1$ where $D_1H_1 = H_1D_1^*$ and $R_1H_1R_1^* = H_1$. Moreover $H_1 = QHQ^*$ and $A_1H_1A_1^* = QAHA^*Q^*$, so that the two pencils of hermitian matrices $AHA^* - xH$ and $A_1H_1A_1^* - xH_1$ are conjunctive. Therefore, if A has an H-representation and if the two pencils $AHA^* - xH$ and $A_1H_1A_1^* - xH_1$ are conjunctive, the matrix A_1 has an H_1-representation. Accordingly we may suppose that the pencil $AHA^* - xH$ is in a canonical or normal form previously determined.[2] Since DH is hermitian, the elementary divisors of $D-xE$ are the same as those of the pencil of hermitian matrices $DH-xH$. Complex elementary divisors of such a pencil must therefore occur in conjugate pairs and in particular this is true of the pure imaginary elementary divisors. Hence, if $D-xE$ has the elementary divisor $(x-ib)^r$ repeated s times, where b is real, then $D-xE$ also has the elementary divisor $(x+ib)^r$ repeated s times and D^2-xE has the elementary divisor $(x+b^2)^r$ repeated $2s$ times. Since the elementary divisors of D^2-xE are the same as those of AHA^*-xH, the matrix A cannot have an H-representation if the pencil AHA^*-xH has an elementary divisor $(x+b^2)^r$, that is, a negative elementary divisor, repeated an odd number of times.

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[2] G. R. Trott, *On the canonical form of a nonsingular pencil of hermitian matrices*, American Journal of Mathematics, vol. 56 (1934), pp. 359–371. H. W. Turnbull, *Pencils of hermitian forms*, Proceedings of the London Mathematical Society (2), vol. 39 (1935), pp. 232–248.

We proceed to determine what further conditions, if any, must be satisfied in order that A may have an H-representation. In canonical form the matrices H and AHA^* are similarly partitioned diagonal block matrices of such a nature that it is only necessary to consider the three special cases in which the pencil AHA^*-xH has

(i) only the two elementary divisors $(x-a)^n$, $(x-\bar{a})^n$, $a\neq\bar{a}$,

(ii) the single elementary divisor $(x-b^2)^n$, b real,

(iii) only the two elementary divisors $(x+b^2)^n$, $(x+b^2)^n$, b real.

Case (i). In canonical form

$$H = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \quad \text{and} \quad AHA^*H^{-1} = \begin{pmatrix} a(E+U) & 0 \\ 0 & \bar{a}(E+U)' \end{pmatrix},$$

where U is the auxiliary unit matrix of order n. Let

$$G = (E+U)^{1/2} = E + U/2 - U^2/8 + \cdots + \alpha U^{n-1}, \tag{7}$$

$$2^{2n-4}(n-3)!(n-1)!\alpha = (-1)^n(2n-5)!.$$

Then $G' = (E+U')^{1/2}$. If $a=\rho^2e^{i\theta}$, $a^{1/2}=\delta\rho e^{i\theta/2}$ and $\bar{a}^{1/2}=\delta\rho e^{-i\theta/2}$, where $\delta=\pm 1$. Therefore, if $D=[a^{1/2}G, \bar{a}^{1/2}G']$, $DH=HD^*$ and $D^2=AHA^*H^{-1}$. The signature of DH is the same as that of H since the signature of both matrices is zero. Since δ may have either of the values 1 or -1, D is not unique.

Case (ii). In canonical form

$$H = \epsilon T \quad \text{and} \quad AHA^* = b^2(E+U)H,$$

where $\epsilon=\pm 1$ and T is the counter unit matrix, so that

$$T^2 = E \quad \text{and} \quad TU = U'T. \tag{8}$$

If $D=bG$, where G is defined by (7), $D^2=AHA^*H^{-1}$ and $DH=HD^*$ by (8). If n is even the signatures of DH and H are both zero and, if in DH, b is replaced by $-b$ the signature of the resulting matrix is also zero. If n is odd, the signature of H is ϵ and that of DH is $\pm\epsilon$ according as b is positive or negative. Therefore, if DH has the same signature as H and n is even, D is not unique but, if n is odd, D is unique.

Case (iii). If the matrix D exists the pencil $DH-xH$ has only the two elementary divisors $(x+ib)^n$, $(x-ib)^n$ and we may take $DH-xH$ in the canonical form

$$(9)\qquad H=\begin{pmatrix}0 & E\\ E & 0\end{pmatrix},\qquad D=\begin{pmatrix}ibG & 0\\ 0 & -ibG'\end{pmatrix},$$

where G is defined by (7). Hence

$$AHA^*H^{-1}=D^2=\begin{pmatrix}-b^2(E+U) & 0\\ 0 & -b^2(E+U')\end{pmatrix}$$

and

$$H=\begin{pmatrix}0 & E\\ E & 0\end{pmatrix},$$

while the matrix H may be transformed without disturbing D^2 into

$$\begin{pmatrix}T & 0\\ 0 & -T\end{pmatrix}.$$

For a matrix pencil AHA^*-xH with elementary divisors $(x+b^2)^n$, $(x+b^2)^n$ there are three distinct possible canonical forms in which H is one of the matrices

$$\begin{pmatrix}\epsilon_1 T & 0\\ 0 & \epsilon_2 T\end{pmatrix},$$

$\epsilon_1=1$, $\epsilon_2=-1$; $\epsilon_1=\epsilon_2=1$ or $\epsilon_1=\epsilon_2=-1$. If we call ϵ_1 and ϵ_2 the *indices* associated with the elementary divisor $(x+b^2)^n$, we see that A has an H-representation, if and only if one index is positive and the other is negative. Further, even when A does have an H-representation, D is not unique. Since the canonical form for the pencil AHA^*-xH is diagonal block, the following theorem follows immediately from Lemma 1.

THEOREM 1. *Let H be any nonsingular hermitian matrix and A be a nonsingular matrix. Then A has an H-representation $A=DR$, if and only if the negative elementary divisors of the pencil AHA^*-xH occur in pairs and exactly half of the indices associated with each negative elementary divisor are positive. The matrix, D, and therefore the matrix*

R, is unique, if and only if all elementary divisors of the pencil are positive and of odd order.

If $H^2=E$, $A=DHHR=SV$, where $S=DH$ and $V=HR$. Moreover $S=S^*$ and $VHV^*=H^3=H$. Hence we have as a corollary the following.

COROLLARY 1. *If $H^2=E$, the matrix A can be written in the form $A=SV$, where S is hermitian with the same signature as H and, $VHV^*=H$, if and only if the conditions of Theorem 1 are satisfied.*

The known theorem on the uniqueness of the polar components of the matrix A in (1) is a particular case of Theorem 1. For, if $H=E$, $AHA^*=AA^*$ and the elementary divisors of the pencil AA^*-xE are all positive and linear.

3. **Modified representation of any nonsingular matrix.** Even if the matrix A does not satisfy the conditions of Theorem 1, it is possible to find a somewhat different representation of A. We first reduce the pencil AHA^*-xH by conjunctive transformation to

$$\begin{pmatrix} A_1H_1A_1^* & 0 \\ 0 & A_2H_2A_2^* \end{pmatrix} - x\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

where no elementary divisor of $A_1H_1A_1-xH_1$ is negative and all elementary divisors of $A_2H_2A_2^*-xH_2$ are negative. Then $A_1=D_1R_1$ and, since all elementary divisors of $-A_2H_2A_2^*-xH_2$ are positive, there exists a matrix D_2, such that $D_2^2=-A_2H_2A_2^*H_2^{-1}$ and $D_2H_2=H_2D_2^*$. If $R_2=D_2^{-1}A_2$, then $R_2H_2R_2^*=D_2^{-1}A_2H_2A_2^*(D_2^*)^{-1}=D_2\cdot(-D_2^2H_2)(D_2^*)^{-1}=-H_2$. Therefore $A_2=D_2R_2$, where D_2H_2 is hermitian with the same signature as H_2 and $R_2H_2R_2^*=-H_2$. Let

$$V=\begin{pmatrix} E_1 & 0 \\ 0 & -E_2 \end{pmatrix}=[E_1,\ -E_2],$$

where E_i is the unit matrix of the same order as H_i. Then $V^2=E$. If $A=[A_1,\ A_2]$, $H=[H_1,\ H_2]$, $R=[R_1,\ R_2]$ and $D=[D_1,\ D_2]$, then $A=DR$ where DH is hermitian with the same signature as H and $RHR^*=HV$, so that the signature of HV is the same as that of H. Further from its form it is apparent that V is a polynomial in AHA^*H^{-1} and that $VH=HV=HV^*$, so that $VHV^*=H$. While we have used a special form of the pencil AHA^*-xH it follows that similar results are true when the pencil is not in this form. Accordingly we have proved this theorem.

THEOREM 2. *Let H be a nonsingular hermitian matrix and let A be* *a nonsingular matrix of the same order as* H. *Then there exists at least one matrix* V, *a polynomial in* AHA^*H^{-1}, *with the properties that* $V^2=E$, VH *is hermitian with the same signature as* H *and that no characteristic number of* $AHA^*(VH)^{-1}$ *is negative. The matrix* $A=DR$ *where* DH *is hermitian with the same signature as* H *and* $RHR^*=VH$. *If all the characteristic numbers of* AHA^*H^{-1} *are real, the matrix* V *is uniquely determined.*

4. **Representation in the real field.** If H is a real symmetric matrix and A a real nonsingular matrix our argument may be carried through in the field of real numbers with only a few alterations. In (i), if the pair of conjugate elementary divisors is $(x-c\pm id)^n$, we take $H=qT$, $AHA^*=p(E+U)qT$, where

$$p=\begin{pmatrix} c & d \\ -d & c \end{pmatrix} \quad \text{and} \quad q=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, if

$$\begin{pmatrix} r & s \\ -s & r \end{pmatrix}^2 = p,$$

$$\begin{pmatrix} r & s \\ -s & r \end{pmatrix} GH$$

is symmetric and

$$\left\{\begin{pmatrix} r & s \\ -s & r \end{pmatrix} G\right\}^2 = AHA^*H^{-1}.$$

Case (ii) is unaltered. In (iii) we replace the matrices D and H of (9) by

$$\begin{pmatrix} 0 & bG \\ -bG & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix},$$

respectively. It therefore follows that Theorems 1 and 2 are true in the field of real numbers, if hermitian is replaced by symmetric and conjugate transposed by transposed.

5. **Conditions for the commutativity of the polar components.** Let the matrix A have an H-representation $A=DR$. Then $A=RD_1$ where $D_1=R^{-1}DR$. The matrix D_1H is hermitian. Further $HA^*H^{-1}A =HD_1^*R^*H^{-1}RD_1=D_1HH^{-1}D_1=D_1^2$. Since $D^2=AHA^*H^{-1}$, it follows that, if A^* is commutative with $H^{-1}AH$, $D_1^2=D^2$ and that for a proper choice $D_1=D$. Conversely, if $D_1=D$, $H^{-1}AH$ is commutative with A^*. We have therefore proved this theorem.

THEOREM 3. *If A has an H-representation $A=DR$, A also has an H-representation $A=RD_1$. A necessary and sufficient condition that $D=D_1$ or that D be commutative with R is that A^* be commutative with $H^{-1}AH$.*

Therefore, if $A=DR$ and A is normal[3] with respect to H, $A=RD$ so that R and D are commutative. That the converse is not true may be shown as follows. Let

$$H=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\quad\text{and}\quad A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}.$$

Then

$$AHA^*H^{-1}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\1&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&1&0&0\\0&1&0&0\end{pmatrix}=\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}.$$

The characteristic numbers of AHA^*H^{-1} are all plus one and therefore A has an H-representation. In fact

$$D=\begin{pmatrix}1&1/2&0&0\\0&1&0&0\\0&0&1&0\\0&0&1/2&1\end{pmatrix}\quad\text{and}\quad R=\begin{pmatrix}1&-1/2&0&0\\0&1&0&0\\0&0&1&0\\0&0&1/2&1\end{pmatrix}.$$

Hence

$$H^{-1}AH=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

and is commutative with A^*. However, $H^{-1}AH$ is not a polynomial in A^* and is therefore not normal with respect to H.

If the elementary divisors of AHA^*-xH are all positive and linear and if $A=DR=RD$, it is comparatively easy to show that $H^{-1}AH$ is a polynomial in A^* and therefore that A is normal with respect to H.

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[3] John Williamson, *Matrices normal with respect to an hermitian matrix*, American Journal of Mathematics, vol. 60 (1938), pp. 355–373.

This seems to be the generalization of the classical result that a necessary and sufficient condition for the polar components of a matrix A to be commutative is that A be a normal matrix.

QUEENS COLLEGE

REMARKS ON REGULARITY OF METHODS OF SUMMATION

G. E. FORSYTHE AND A. C. SCHAEFFER

A doubly infinite matrix[1] (a_{mn}) $(m, n=1, 2, \cdots)$ is said to be *regular*, if for every sequence $x=\{x_n\}$ with limit x' the corresponding sums $y_m=\sum_{n=1}^{\infty}a_{mn}x_n$ exist for $m=1, 2, \cdots$, and if $\lim_{m\to\infty} y_m=x'$. An apparently more inclusive definition of regularity is that for each sequence x with limit x' the sums defining y_m shall exist for all $m\geqq m_0(x)$ and $\lim_{m\to\infty} y_m=x'$. Tamarkin[2] has shown that (a_{mn}) is regular in the latter sense if and only if there exists an m_1 *independent of* x such that the matrix (a_{mn}) $(m\geqq m_1, n\geqq 1)$ is regular in the former sense. Using point set theory in the Banach space (c), he proves a theorem[3] from which follows the result just mentioned. This note presents an elementary proof of that theorem and discusses some related topics.

THEOREM 1. *Suppose the doubly infinite matrix* (a_{mn}) *has the property that for each sequence* $x=\{x_n\}$ *with limit* 0 *there exists an* $m_0=m_0(x)$ *such that for all* $m\geqq m_0(x)$, $u_m=\lim\sup_{k\to\infty}\left|\sum_{n=1}^{k}a_{mn}x_n\right|<\infty$. *Then there exists an* m_1 *such that* $\sum_{n=1}^{\infty}|a_{mn}|<\infty$ *for all* $m\geqq m_1$.

If in addition $\lim_{m\to\infty} u_m=0$ for each sequence x with limit 0, it will follow[4] that there exists an N such that $\sum_{n=1}^{\infty}|a_{mn}|\leqq N<\infty$, for all $m\geqq m_1$.

To prove Theorem 1, suppose there were an infinite sequence $m_1<m_2<\cdots$ such that $\sum_{n=1}^{\infty}|a_{mn}|=\infty$ for $m\in\{m_\nu\}$. Let $x_1, \cdots, x_{k_1}$ be chosen with unit moduli and with amplitudes such that

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Presented to the Society, April 11, 1942 under the title *A remark on Toeplitz matrices*; received by the editors January 22, 1942.

[1] In this note a_{mn}, x_n and x' denote finite complex numbers.

[2] J. D. Tamarkin, *On the notion of regularity of methods of summation of infinite series*, this Bulletin, vol. 41 (1935), pp. 241–243.

[3] J. D. Tamarkin, loc. cit., p. 242, lines 1–6.

[4] See, for example, I. Schur, *Über lineare Transformationen in der Theorie der unendlichen Reihen*, Journal für die reine und angewandte Mathematik, vol. 151 (1921), pp. 79–111; p. 85, Theorem 4.

$$\sum_{n=1}^{k_1} a_{m_1 n} x_n = \sum_{n=1}^{k_1} | a_{m_1 n} x_n | > 1.$$

Let $x_{k_1+1}, \cdots, x_{k_2}$ be chosen with moduli 1/2 and with amplitudes such that

$$\left| \sum_{n=k_1+1}^{k_2} a_{m_2 n} x_n \right| > 2 + \sum_{n=1}^{k_1} | a_{m_2 n} x_n |.$$

Let $x_{k_2+1}, \cdots, x_{k_3}$ be chosen with moduli 1/3 and with amplitudes such that

$$\left| \sum_{n=k_2+1}^{k_3} a_{m_1 n} x_n \right| > 3 + \sum_{n=1}^{k_2} | a_{m_1 n} x_n |.$$

Writing $y_m(k) = \sum_{n=1}^{k} a_{mn} x_n$, the sequence $\{x_n\}$ and integers $k_1 < k_2 < \cdots$ are thus chosen successively so that $|y_{m_1}(k_1)| > 1$, $|y_{m_2}(k_2)| > 2$; $|y_{m_1}(k_3)| > 3$, $|y_{m_2}(k_4)| > 4$, $|y_{m_3}(k_5)| > 5$; $|y_{m_1}(k_6)| > 6$, $\cdots$; while $|x_n| = 1/r$, for $k_{r-1} < n \leqq k_r$. This is a sort of alternating or "sweeping-out" process. So defined, $\{x_n\}$ is a sequence with limit 0, but $\limsup_{k\to\infty} \left| \sum_{n=1}^{k} a_{mn} x_n \right| = \infty$, for $m \in \{m_\nu\}$. This contradiction completes the proof of Theorem 1.

The matrix (a_{mn}) is said to be ***null-preserving***, if for every sequence $x = \{x_n\}$ with limit 0 the corresponding sums defining y_m exist for $m = 1, 2, \cdots$ and if $\lim_{m\to\infty} y_m = 0$. An apparently more inclusive definition of null-preserving is that for each sequence x with limit 0 we have $u_m = \limsup_{k\to\infty} \left| \sum_{n=1}^{k} a_{mn} x_n \right| < \infty$ for all $m \geqq m_0(x)$ and $\lim_{m\to\infty} u_m = 0$. We remark that it is a consequence of Theorem 1 that (a_{mn}) is null-preserving in the latter sense if and only if there exists an m_1 such that the matrix (a_{mn}) $(m \geqq m_1, n \geqq 1)$ is null-preserving in the former sense.[5]

To consider a problem which is related to the above in the method of proof, let each element of a matrix (a_{mn}) be either $+1$ or -1. For $0 \leqq t \leqq 1$ and $n = 1, 2, \cdots$ let $\{\phi_n(t)\}$ be the Rademacher orthogonal functions,[6] and let $y_{mk}(t) = \sum_{n=1}^{k} a_{mn} \phi_n(t)$. Then it is well known[7] that for almost all t, for all $m = 1, 2, \cdots$ and for all $\epsilon > 0$, $\lim_{k\to\infty} k^{-1/2-\epsilon} y_{mk}(t) = 0$. It is clear that for a particular fixed m there is a t such that $\lim_{k\to\infty} k^{-1} y_{mk}(t) = 1$. The problem is to show that there is a t such that simultaneously for all $m=1, 2, \cdots$, $\limsup_{k\to\infty} k^{-1}y_{mk}(t)=1$. That there exists such a t can be shown by using the alternating process of Theorem 1.

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[5] For conditions that (a_{mn}) be null-preserving, see T. Kojima, *On generalised Toeplitz's theorems on limit and their applications*, Tôhoku Mathematical Journal, vol. 12 (1917), pp. 291–326; p. 300.

[6] A. Zygmund, *Trigonometrical Series*, Warsaw, 1935, p. 5.

[7] For references to this and more precise results, see A. Khintchine, *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Ergebnisse der Mathematik, Berlin, 1933, pp. 60–61.

Theorem 2 follows immediately from a theorem of Banach.[8]

THEOREM 2. *If E_m is a linear manifold satisfying Baire's condition*[9] *in a Banach space E $(m=1, 2, \cdots)$ and if $\lim_{m\to\infty} E_m=E$, then there exists an m_1 such that $E_{m_1}=E$.*

Theorem 2 furnishes a Banach space analogue and a proof of Theorem 1 which is related to Tamarkin's proof. To see this, let E be the Banach space (c_0) of sequences $x=\{x_n\}$ convergent to 0, with $\|x\|=\max_n |x_n|$, and with addition and multiplication by a (complex) scalar defined as usual. Let (a_{mn}) be as in Theorem 1. Let E_m be the subset of E for which $\limsup_{k\to\infty}\left|\sum_{n=1}^{k} a_{rn}x_n\right| < \infty$ for all $r\geqq m$. The hypotheses of Theorem 2 are satisfied, and from its conclusion it may be proved directly for an arbitrary $m\geqq m_1$ that $\sum_{n=1}^{\infty} |a_{mn}| < \infty$.

STANFORD UNIVERSITY

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[8] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 22, Theorem 2.

[9] See S. Banach, loc. cit., p. 17. By considering a Hamel base for E, G. W. Mackey has remarked to the authors that Theorem 2 is false if the words "satisfying Baire's condition" are omitted.

PROJECTIONS OF THE PRIME-POWER ABELIAN GROUP OF ORDER p^m AND TYPE $(m-1, 1)$

ROSS A. BEAUMONT

1. **Introduction.** A function f of the subgroups of the group G upon the subgroups of the group H is called a projectivity of G upon H ($f(G)=H$) if the following hold.

(1) For every subgroup S of $G, f(S)$ is a subgroup of H.

(2) If S' is a subgroup of H, then there exists a subgroup S of G such that $f(S)=S'$.

(3) If S and T are subgroups of G, $S\leqq T$ is a necessary and sufficient condition that $f(S)\leqq f(T)$.

The correspondence f is a (1-1) correspondence which preserves the partial ordering of the set of subgroups of the group G.

Further, a projectivity f is called index-preserving if $[T:S]=[f(T):f(S)]$ for subgroups S of cyclic subgroups T of G; and f is called strictly index-preserving if $[T:S]=[f(T):f(S)]$ for subgroups S of subgroups T of G.

If G is the direct product of cyclic groups of order p, p a prime number, R. Baer[1] has given necessary and sufficient conditions that a group H be a projection of G. In particular he has shown that if the projectivity of G upon H is index-preserving, then G and H are isomorphic. Thus in a study of the projections of the prime-power abelian group of order p^m and type $(m-1, 1)$, we need consider only the case $m>2$.

Rottlaender[2] investigated the case $m=3$ and found necessary and sufficient conditions for the existence of a strictly index-preserving projectivity of the prime-power abelian group G of order p^3 and type $(2, 1)$ upon a group H.

In this note, Baer's general results are used to find the necessary and sufficient conditions for the existence of a projectivity of the prime-power abelian group G of order p^m and type $(m-1, 1)$ upon a group H.

2. **The necessary conditions.** If G is an abelian group of the type under consideration, $G=\{u_1\}\times\{u_2\}$ where u_1 is of order p^{m-1}, $m>2$,

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Presented to the Society, April 11, 1942; received by the editors January 19, 1942.

[1] R. Baer, *The significance of the system of subgroups for the structure of the group*, American Journal of Mathematics, vol. 61 (1939), pp. 1–44. Hereafter this paper will be referred to as B.

[2] Ada Rottlaender, *Nachweis der Existenz nicht-isomorpher Gruppen von gleicher Situation der Untergruppen*, Mathematische Zeitschrift, vol. 28 (1928), pp. 641–653.

and u_2 is of order p. If $f(G)=H$, it follows from known results[3] that f is index-preserving, and hence $H=\{f(\{u_1\}), f(\{u_2\})\}$ where $f(\{u_1\})=\{u_1'\}$ is a cyclic group of order p^{m-1}, $f(\{u_2\})=\{u_2'\}$ is a cyclic group of order p. Thus $H=\{u_1', u_2'\}$ where u_1' and u_2' are independent generators of order p^{m-1} and p, respectively.

If K is any group, then K^p is the set of all pth powers of the elements of K, and K^p is a characteristic subset of K.

Lemma 1. *$f(G^p)=H^p$ is a characteristic subgroup of H.*

Proof. $G^p=\{u_1^p\}$ and $f(G^p)=f(\{u_1^p\})=f(\{u_1\})^p=\{u_1'^p\}$ since f is index-preserving. $\{u_1'^p\}\leqq H^p$. If x is in H^p, $x=y^p$ where y is in H. $f^{-1}(\{x\})=f^{-1}(\{y^p\})=f^{-1}(\{y\})^p=\{u_1^h u_2^k\}^p=\{u_1^{hp}\}$. $f(\{u_1^{hp}\})=f(\{u_1\})^{hp}=\{u_1'^{hp}\}=\{x\}$. Hence $x=u_1'^{shp}=(u_1'^p)^{sh}$ and x is in $\{u_1'^p\}$. Thus $H^p\leqq\{u_1'^p\}$ so that we have shown that $f(G^p)=\{u_1'^p\}=H^p$. Since H^p is a subgroup of H, it is a characteristic subgroup, and in particular normal.

Since $f(G)=H$ and $f(G^p)=H^p$, the index-preserving projectivity f of G upon H induces an index-preserving projectivity of G/G^p upon H/H^p. G/G^p is the direct product of two cyclic groups of order p and hence by the result[4] mentioned above, G/G^p and H/H^p are isomorphic. Thus H/H^p is abelian and H^p contains the commutator subgroup of H. This implies

$$u_2'^{-1}u_1'u_2' = u_1'^{1+jp}. \tag{1}$$

The following multiplication rules for elements in H follow from (1)

$$u_2'^{-1}u_1'^{k}u_2' = u_1'^{k(1+jp)}, \tag{2}$$

$$u_1'^{k}u_2'^{i} = u_2'^{i}u_1'^{k(1+jp)^i}, \tag{3}$$

$$\begin{aligned}(u_2'^{i}u_1'^{k})^{b} &= u_2'^{bi}u_1'^{k[1+(1+jp)^i+\cdots+(1+jp)^{(b-1)i}]}\\ &= u_2'^{bi}u_1'^{k[(1+jp)^{bi}-1]/[(1+jp)^i-1]}.\end{aligned} \tag{4}$$

It follows from (3) that the order of H is p^m so that we have proved the following theorem.

Theorem 1. *If G is the prime-power abelian group of order p^m, $m>2$, and type $(m-1, 1)$ and if f is a projectivity of G upon a group H, then H is a prime-power group of order p^m generated by independent generators u_1' of order p^{m-1} and u_2' of order p such that $u_2'^{-1}u_1'u_2'=u_1'^{1+jp}$.*

If $p=2$, we derive the additional necessary condition.

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[3] [B, Corollary 11.3].

[4] [B, Corollary 8.2].

LEMMA 2. *$j\equiv 0 \bmod 2$ so that $u_2'^{-1}u_1'u_2' = u_1'^{1+4w}$.*

PROOF. Since $f(\{u_1\}) = \{u_1'\}$, it follows from Baer's results[5] that there exists one and only one element u' in H such that $f(\{u_2\}) = \{u'\}$ and $f(\{u_2u_1\}) = \{u'u_1'\}$. Since $\{u'\} = \{u_2'\}$, $u' = u_2'^{\rho}$ where ρ is odd. $f(\{u_2u_1\}^2) = f(\{u_2u_1\})^2$ since f is index-preserving and we have

$$\begin{aligned} f(\{u_2u_1\}^2) &= f(\{u_1^2\}) = f(\{u_1\})^2 = \{u_1'^2\}, \\ f(\{u_2u_1\})^2 &= \{(u'u_1')^2\} = \{(u_2'^{\rho}u_1')^2\} \\ &= \{u_2'^{2\rho}u_1'^{[(1+2j)^{2\rho}-1]/[(1+2j)^{\rho}-1]}\} = \{u_1'^{(1+2j)^{\rho}+1}\}. \end{aligned}$$

Thus $u_1'^{(1+2j)^{\rho}+1} = u_1'^{2\gamma}$ where γ is odd and we have

$$\begin{aligned} (1+2j)^{\rho} + 1 &\equiv 2\gamma \bmod 2^{m-1}, \\ 2 + 2\rho j + [\rho(\rho-1)/2](2j)^2 + \cdots &\equiv 2\gamma \bmod 2^{m-1}, \\ 1 + \rho j + [\rho(\rho-1)/2]2j^2 + \cdots &\equiv \gamma \bmod 2^{m-2} \end{aligned}$$

and recall that $m>2$. Since γ is odd, the left member of the congruence is odd which implies ρj is even. Since ρ is odd, j is even, which completes the proof of the lemma.

3. **Construction of a projectivity for groups satisfying the necessary conditions.** If p is an odd prime and H is a group satisfying the necessary conditions, then H is either abelian or H is the unique non-abelian group $\{U_1, U_2\}$ where U_1 and U_2 are subject to the sole defining relations[6]

$$U_1^{p^{m-1}} = U_2^{p} = 1, \qquad U_2^{-1}U_1U_2 = U_1^{1+p^{m-2}}. \tag{5}$$

If H is abelian, then G and H are isomorphic and this isomorphism induces a projectivity of G upon H.

If H is non-abelian we will construct a projectivity of G upon H by establishing a correspondence between the cyclic subgroups and then extending this correspondence to a projectivity.

From (5) we find the following multiplication rule for elements of H

$$(U_2^{i}U_1^{k})^{h} = U_2^{ih}U_1^{k[h+ih(h-1)p^{m-2}/2]}. \tag{6}$$

Every element of G has the unique form $u_2^su_1^r$ where $0\leq s<p$, $0\leq r<p^{m-1}$. If $s>0$, there exists an integer x, uniquely determined mod p^{m-1}, such that $sx\equiv 1 \bmod p^{m-1}$, $(x, p)=1$. Then $u_1^r=(u_1^r)^{sx}=u_1^{rxs}=u_1^{ts}$ where $rx=t$. $u_2^s u_1^r=u_2^s u_1^{ts}$ and $\{u_2^s u_1^r\}=\{(u_2u_1^t)^s\}=\{u_2u_1^t\}$ where t is uniquely determined mod p^{m-1}. We define the following correspondence of the cyclic subgroups of G upon those of H

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[5] [B, (9.2), (d)].

[6] Carmichael, *Introduction to the Theory of Groups of Finite Order*, p. 132.

$$f(\{u_2u_1^t\}) = \{U_2U_1^t\}, \tag{7}$$

$$f(\{u_1^r\}) = \{U_1^r\}. \tag{8}$$

The correspondence f is a (1-1) correspondence of the set of cyclic subgroups of G upon a subset of the set of cyclic subgroups of H. To show that f is a (1-1) correspondence on the whole set of cyclic subgroups of H, it is only necessary to show that every cyclic subgroup of H has the form $\{U_2U_1^t\}$ or $\{U_1^r\}$ where t or r is uniquely determined mod p^{m-1}, respectively.

Every element of H has the unique form $U_2^sU_1^r$ where $0\leqq s<p$, $0\leqq r<p^{m-1}$. If $s>0$, there exists an integer x, uniquely determined mod p^{m-1}, such that $sx\equiv 1 \bmod p^{m-1}$, $(x, p)=1$. Then

$$\begin{aligned}\{U_2^sU_1^r\} &= \{(U_2^sU_1^r)^x\}\\ &= \{U_2^{sx}U_1^{r[x+(sx(x-1)/2)p^{m-2}]}\}\\ &= \{U_2U_1^{r[x+(sx(x-1)/2)p^{m-2}]}\}\end{aligned}$$

where $r[x+(sx(x-1)/2)p^{m-2}]$ is uniquely determined mod p^{m-1}.

The correspondence f preserves the indices of the cyclic subgroups since

$$f(\{u_2u_1^t\}^p) = f(\{u_1^{tp}\}) = \{U_1^{tp}\}$$

by (8),

$$f(\{u_2u_1^t\})^p = \{U_2U_1^t\}^p = \{U_1^{t[p+(p(p-1)/2)p^{m-2}]}\} = \{U_1^{tp}\},$$

by (7) and (6).

Since the only non-cyclic subgroups of H are those of the form $\{U_1^{p^\lambda}, U_2\}$, $0\leqq\lambda<m-1$, by extending f so that

$$f(\{u_1^{p^\lambda}, u_2\}) = \{U_1^{p^\lambda}, U_2\} \tag{9}$$

f becomes an index-preserving projectivity of G upon H.

Combining the above results with Theorem 1 we have this theorem.

THEOREM 2. *If G is the prime-power abelian group of odd order p^m, $m>2$, and type $(m-1, 1)$ then there exists a projectivity f of G upon a group H if, and only if, either H is isomorphic to G or H is the non-*

abelian group $\{U_1, U_2\}$ *where* U_1 *and* U_2 *are subject to the sole defining relations* (5).

If $p=2$ it follows from Lemma 2 that if $m=3$, then H is abelian and hence G and H are isomorphic groups. If $m>3$, it follows from Lemma 2 and from known results[7] that H is either abelian or the non-abelian group $\{U_1, U_2\}$ where U_1 and U_2 are subject to the sole defining relations

$$U_1^{2^{m-1}} = U_2^2 = 1, \qquad U_2U_1U_2 = U_1^{1+2^{m-2}} . \tag{10}$$

Since Baer has shown[8] that there exists an index-preserving projectivity of G upon this non-abelian group we have the following theorem.

THEOREM 3. *If G is the prime-power abelian group of order* 2^m, $m>2$, *and type* $(m-1, 1)$ *then*

(a) *if* $m=3$, *there exists a projectivity f of G upon a group H if and only if G and H are isomorphic groups*;

(b) *if* $m>3$, *there exists a projectivity f of G upon a group H if, and only if, either G and H are isomorphic groups or H is the non-abelian group* $\{U_1, U_2\}$ *where* U_1 *and* U_2 *are subject to the sole defining relations* (10).

UNIVERSITY OF WASHINGTON

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[7] Carmichael, loc. cit., p. 133.

[8] [B, p. 11].

SYMMETRIC DIFFERENTIAL EXPRESSIONS

HENRY J. RIBLET

1. Introduction. In a recent article,[1] it is shown that if

$$(1)\qquad f(x_1, \cdots, x_n) = \sum c_{\nu_{01},\cdots,\nu_{kn}} x_1^{\nu_{10}} \cdots (x_1^{(1)})^{\nu_{11}} \cdots (x_n^{(k)})^{\nu_{kn}},$$

where $x_i^{(k)}$ represents the kth derivative of x_i and $f(x_1, \cdots, x_n)$ is unchanged by all permutations of the variables, then

$$(2)\qquad f(x_1, \cdots, x_n) = \frac{1}{D^p} \sum C_{\lambda_{01},\cdots,\lambda_{kn}} a_1^{\lambda_{10}} \cdots (a_1^{(1)})^{\lambda_{11}} \cdots (a_n^{(k)})^{\lambda_{kn}}$$

where D is the discriminant of $x_1, \cdots, x_n$, and $a_1, \cdots, a_n$ are the elementary symmetric functions or E.S.F.'s. In applications to problems involving the differential equations satisfied by algebraic functions or the algebraic properties of the solutions of algebraic differential equations, it is desirable to have some method for passing from (1) to (2). The proof of the basic theorem, although constructive, gives a method that is prohibitively laborious and will often introduce unnecessary powers of D. It is the object of this paper to present an exhaustion procedure which greatly simplifies the work and obviates the latter danger.

2. Preliminaries. It is no restriction to limit ourselves to symmetric differential functions generated by a single term of (1). Term A of such a function will be said to be of higher *order* than term B if the first exponent ν_{ij} in A which differs from the corresponding exponent ν'_{ij} in B is the larger.

THEOREM 1. *The exponents of any power product of the derivatives of the E.S.F.'s are uniquely determined by the highest order term; and the exponents ν_{ij} of the highest order term satisfy the inequalities, where $j=1, \cdots, n-1$,*

$$(3)\qquad \nu_{0j} - \nu_{0j+1} \geqq \sum_{i=1}^{k} \nu_{ij+1}.$$

PROOF. It is clear that the highest order term in $a_i^{(j)}$ is $x_1 \cdots x_{i-1}x_i^{(j)}$ and that the highest order term in

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Presented to the Society, October 28, 1939 under the title *Symmetric differential expressions and applications*; received by the editors January 31, 1942.

[1] H. J. Riblet, *Algebraic differential fields*, American Journal of Mathematics, vol. 63 (1941), p. 341.

$$a_1^{\lambda_{01}} \cdots a_n^{\lambda_{0n}} (a_1^{(1)})^{\lambda_{11}} \cdots (a_n^{(k)})^{\lambda_{kn}}$$

is

$$x_1^{\nu_{01}} \cdots x_n^{\nu_{0n}} (x_1^{(1)})^{\nu_{11}} \cdots (x_n^{(k)})^{\nu_{kn}}$$

where we have $\nu_{ij} = \lambda_{ij}$ for i greater than zero and otherwise

$$\nu_{0l} = \lambda_{0l} + \sum_{i,j>l} \lambda_{ij}.$$

Now

$$\begin{aligned} \nu_{0l} - \nu_{0l+1} &= \lambda_{0l} + \sum_{i,j>l} \lambda_{ij} - \Big(\lambda_{0l+1} + \sum_{i,j>l+1} \lambda_{ij}\Big) \\ &= \lambda_{0l} + \sum_{i=1}^{k} \lambda_{il+1} = \lambda_{0l} + \sum_{i=1}^{k} \nu_{il+1} \end{aligned} \tag{4}$$

and since $\lambda_{0l} \geqq 0$ condition (3) is certainly satisfied. Moreover, if it is satisfied, the equations $\nu_{ij} = \lambda_{ij}$, for i greater than one and (4) otherwise determine all the λ's uniquely.

3. The procedure. We now consider the possibility of finding expression (2) by subtracting from $f(x_1, \cdots, x_n)$ a suitable multiple of that power product of the derivatives of the E.S.F.'s which its highest order term defines, by Theorem 1; and then proceeding in a similar manner with the resulting function.

THEOREM 2. *A necessary and sufficient condition that* (1) *be expressible as a rational integral function of the E.S.F.'s and their derivatives is that the highest order term of each function obtained by successively eliminating highest order terms satisfy condition* (3).

PROOF. From Theorem 1 it follows that condition (3) is certainly necessary. Sufficiency also follows from this theorem if we may be assured that there are only a finite number of terms lower than the original highest order term. This follows from the fact that in each term which occurs $\sum i\nu_{ij} = \sum i\lambda_{ij} = M$, a fixed quantity.

If the conditions of Theorem 2 are not satisfied, we will encounter a symmetric differential function f_2 whose highest order term does not satisfy condition (3). The highest order term of D is $x_1^{2n-2}x_2^{2n-4} \cdots x_n^2$. It is clear that multiplication of f_2 by a sufficiently high power p_2 of D will give a function whose highest order term satisfies condition (3). Thus we can continue as before hoping that the process will terminate.

THEOREM 3. *The exhaustion procedure terminates and when it does it gives the minimum power of D for which an expression of the form of* (2) *exists.*

PROOF. Let p be the smallest power of D for which an expression of the form (2) exists, and suppose that in the exhaustion process we have reached a point so that

$$f(x_1, \cdots, x_n) = f_1 + \frac{f_2}{D^{p_2}} + \frac{f_3}{D^{p_3}} + \cdots + \frac{f_r + f_r'}{D^{p_r}}, \tag{5}$$

where f_i denotes a rational integral function of the E.S.F.'s and their derivatives, $p_r \geq p$, and b_r, the highest order term of f_r', does not satisfy condition (3). If we replace $f(x_1, \cdots, x_n)$ by its equivalent as given by equation (2) and multiply both sides of the result by D^{p_r}, we obtain an expression for f_r' as a rational, integral function of the E.S.F.'s and their derivatives. This however contradicts Theorem 1 unless $b_r = 0$ and this implies that $f_r' \equiv 0$ which in turn proves the first assertion. Now if $p_r > p$, we can immediately prove that D divides f_r considered as a function of the E.S.F.'s and their derivatives. This completes the proof of the theorem.

As a corollary of Theorem 2 and as an example of the fact that the power of D occurring in the denominator of (2) may be less, for an extensive class of functions, than might be expected from the original construction, we prove this theorem:

THEOREM 4. *An integral rational symmetric differential expression which is linear in its differentiated terms may be expressed as an integral rational function of the E.S.F.'s and their first derivatives.*

PROOF. It is clear that the theorem is proved, if we can show its truth for the symmetric function generated by the term

$$\sum_{i=1}^{n} x_i^{a_i} x_j^{(1)}.$$

Now the highest order term for any such differential expression satisfies condition (3), for, if not, $a_{j-1} \geq a_j$ and simply by interchanging x_j and x_{j-1} we could obtain a term of higher order. Thus our result follows directly from Theorem 2.

HOFSTRA COLLEGE

NON-INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED WITH A LINEAR CONGRUENCE

M. L. VEST

1. **Introduction.** Vogt[1] has studied, using synthetic methods, the space transformations associated with a linear congruence and with a bundle of lines. In the present paper the non-involutorial transformations associated with these configurations are found analytically, and by an entirely different geometric process. It will be shown that the general transformation associated with a linear congruence so obtained differs from that of Vogt in one important respect, having one more fundamental curve in each of the projectively related spaces. The transformations in the planes containing the directrices of the congruence are also shown.

Given the directrices r and s of the congruence and two projective pencils of surfaces

$$\left| F_{m+n+1} \right| : \; r^m s^n g_{2m+2n+2mn+1}$$

and

$$\left| F'_{m'+n'+1} \right| : \; r^{m'} s^{n'} g_{2m'+2n'+2m'n'+1}.$$

Through a generic point $P(y)$ there passes a single F of $|F|$. The unique line t of the congruence through $P(y)$ meets the associated F' of $|F'|$ in one residual point $P'(x)$, the image of $P(y)$ under the transformation thus defined. The residual base curves of $|F|$ and $|F'|$, other than r and s, have been denoted by g and g', respectively. Through a point $O_{g'}$ on g' there is a unique line t' of the congruence, this line lying upon one surface of $|F'|$. The associated surface of $|F|$ meets t' in a point $\bar{P}$ which generates a curve $\bar{g}$. Similarly, beginning with a point O_g on g, a point $\bar{P}'$ generating a curve $\bar{g}'$ is found. It will be shown that r, s, g, g', $\bar{g}$, $\bar{g}'$ are fundamental curves of the transformation.

2. **Equations of the transformation.** Let us take the equations of the directrices r and s, respectively, as

$$(1) \qquad x_1 = x_2 = 0, \qquad x_3 = x_4 = 0,$$

and the pencils of surfaces

$$(2) \qquad \left| F_{m+n+1} \right| \equiv U - \lambda \bar{U} = 0, \quad \left| F'_{m'+n'+1} \right| \equiv U' - \lambda \bar{U}' = 0$$

Received by the editors January 24, 1942.

[1] Vogt, *Zentrale und windschiefe Raum-Verwandtschaften*, Jahresbericht der Schlesischen Gesellschaft für Vaterlanddische Kultur, class 84, 1906, pp. 8–16.

where

$$
(3)\qquad
\begin{aligned}
U_{m+n+1} &= \sum_{j,k=0}^{m,n} u_j v_k (c_{jk}x), &\qquad \bar{U}_{m+n+1} &= \sum_{j,k=0}^{m,n} \bar{u}_j \bar{v}_k (\bar{c}_{jk}x),\\
U'_{m'+n'+1} &= \sum_{j,k=0}^{m',n'} u'_j v'_k (c'_{jk}x), &\qquad \bar{U}'_{m'+n'+1} &= \sum_{j,k=0}^{m',n'} \bar{u}'_j \bar{v}'_k (\bar{c}'_{jk}x),\\
u_j &= \sum_{i=0}^{m} a_{ij} x_1^i x_2^{m-i}, &\qquad v_k &= \sum_{i=0}^{n} b_{ik} x_3^i x_4^{n-i},\\
\bar{u}_j &= \sum_{i=0}^{m} \bar{a}_{ij} x_1^i x_2^{m-i}, &\qquad \bar{v}_k &= \sum_{i=0}^{n} \bar{b}_{ik} x_3^i x_4^{n-i},\\
u'_j &= \sum_{i=0}^{m'} a'_{ij} x_1^i x_2^{m'-i}, &\qquad v'_k &= \sum_{i=0}^{n'} b'_{ik} x_3^i x_4^{n'-i},\\
\bar{u}'_j &= \sum_{i=0}^{m'} \bar{a}'_{ij} x_1^i x_2^{m'-i}, &\qquad \bar{v}'_k &= \sum_{i=0}^{n'} \bar{b}'_{ik} x_3^i x_4^{n'-i},
\end{aligned}
$$

$$(c_{jk}x) = c_{jk,1}x_1 + c_{jk,2}x_2 + c_{jk,3}x_3 + c_{jk,4}x_4,$$

and so on.

Through a generic point $P(y)$ there is an F of $|F|$ with parameter $\lambda = U(y)/\bar{U}(y)$ and to this corresponds the F' having equation

$$(4)\qquad U'(x)\bar{U}(y) - \bar{U}'(x)U(y) = 0.$$

The unique line t of the congruence passing through $P(y)$ intersects the surface (4) in the point P' having coordinates

$$
(5)\qquad \overset{-1}{T}_{m+m'+n+n'+2}:\
\left|
\begin{aligned}
\sigma x_1 &= Ry_1 + Ky_1 = Sy_1\\
\sigma x_2 &= Ry_2 + Ky_2 = Sy_2\\
\sigma x_3 &= Ry_3\\
\sigma x_4 &= Ry_4
\end{aligned}
\right.
$$

where

$$
(6)\qquad
\begin{aligned}
&K_{m+m'+n+n'+2} = U'\bar{U} - U\bar{U}', \quad R_{m+m'+n+n'+2} = U\bar{W}' - \bar{U}W',\\
&W'_{m'+n'+1} = \sum_{j,k=0}^{m',n'} u'_j v'_k (c'_{jk}z), \qquad \bar{W}'_{m'+n'+1} = \sum_{j,k=0}^{m',n'} \bar{u}'_j \bar{v}'_k (\bar{c}'_{jk}z),\\
&(c_{jk}z) = c_{jk,1}y_1 + c_{jk,2}y_2.
\end{aligned}
$$

Equations (5) are those of the inverse transformation.

In a similar manner the equations of the direct transformation are found to be

$$(7)\qquad T_{m+m'+n+n'+2}: \left| \begin{array}{l} \sigma y_1 = R'x_1 + K'x_1 = S'x_1 \\ \sigma y_2 = R'x_2 + K'x_2 = S'x_2 \\ \sigma y_3 = R'x_3 \\ \sigma y_4 = R'x_4 \end{array} \right.$$

where

$$(8)\qquad \begin{array}{ll} K'_{m+m'+n+n'+2} = U\overline{U}' - \overline{U}U' = -K, & R'_{m+m'+n+n'+2} = U'\overline{W} - \overline{U}'W, \\ W_{m+n+1} = \sum_{j,k=0}^{m,n} u_j v_k(c_{jk}x), & \overline{W}_{m+n+1} = \sum_{j,k=0}^{m,n} \bar{u}_j \bar{v}_k(\bar{c}_{jk}x). \end{array}$$

3. **Images of fundamental curves.** The transformations T^{-1} and T applied to an F' and F of $|F'|$ and $|F|$, respectively, give

$$(9)\qquad U' \sim (T^{-1})US^{m'}R^{n'}G, \qquad U \sim (T)U'S'^{m}R'^{n}G'$$

where

$$(10)\qquad \begin{array}{ll} G_{2m'+2n'+2} = U'\overline{W}' - \overline{U}'W', & G'_{2m+2n+2} = U\overline{W} - \overline{U}W, \\ S_{m+m'+n+n'+2} = U\overline{M}' - \overline{U}M', & S'_{m+m'+n+n'+2} = U'\overline{M} - \overline{U}'M, \\ M'_{m'+n'+1} = W' - U' = \sum_{j,k=0}^{m',n'} u'_j v'_k(c'_{jk}w), & \\ & M_{m+n+1} = W - U = \sum_{j,k=0}^{m,n} u_j v_k(c_{jk}w), \\ \overline{M}'_{m'+n'+1} = \overline{W}' - \overline{U}' = \sum_{j,k=0}^{m',n'} \bar{u}'_j \bar{v}'_k(\bar{c}'_{jk}w), & \\ & \overline{M}_{m+n+1} = \overline{W} - \overline{U} = \sum_{j,k=0}^{m,n} \bar{u}_j \bar{v}_k(\bar{c}_{jk}w), \\ (c_{jk}w) = c_{jk,3}x_3 + c_{jk,4}x_4. & \end{array}$$

Here U and U' are the corresponding surfaces of $|F|$ and $|F'|$ and $g' \sim (T^{-1})G$, $s' \sim (T^{-1})S$, $g \sim (T)G'$, $s \sim (T)S'$. Similarly,

$$(11)\qquad \begin{array}{ll} K' \sim (T^{-1})K'R^{n+n'}S^{m+m'}GG', & K \sim (T)KR'^{n+n'}S'^{m+m'}GG', \\ K' \sim (T)K'R'^{n+n'}S'^{m+m'}GG', & K \sim (T)KR^{n+n'}S^{m+m'}GG', \\ G' \sim (T^{-1})S^{2m+1}R^{2n+1}G', & G \sim (T)S'^{2m'+1}R'^{2n'+1}G, \\ G' \sim (T)S'^{2m+1}R'^{2n+1}G', & G \sim (T^{-1})S^{2m'+1}R^{2n'+1}G, \\ R' \sim (T^{-1})S^{m'+m+1}R^{n'+n}GG', & R \sim (T)S'^{m+m'+1}R'^{n+n'}GG', \\ R' \sim (T)S'^{m+m'+1}R'^{n+n'}[R'^2 - K'(W'\overline{W} - \overline{W}'W)], & \\ \qquad R \sim (T^{-1})S^{m+m'+1}R^{n+n'}[R^2 - K(W'\overline{W} - \overline{W}'W)], & \\ S' \sim (T^{-1})S^{m+m'}R^{n+n'+1}GG', & S \sim (T)S'^{m+m'}R'^{n+n'+1}GG', \\ S' \sim (T)S'^{m+m'}R'^{n+n'+1}[R'S' + K'(W'\overline{M} - \overline{W}'M)], & \\ \qquad S \sim (T^{-1})S^{m+m'}R^{n+n'+1}[RS + K(W\overline{M}' - \overline{W}M')]. & \end{array}$$

Through a point O_r on r there is a pencil of transversals through s. Each direction in this pencil determines an F' and the associated F cuts the line in one residual point. The locus of such points is a plane curve c which generates the surface R, image of r. The order of c, determined by the intersection of R and a homaloidal surface, is $n+n'+1$. Similarly, taking a point O_s on s, we find a plane curve k of order $m+m'+1$ generating the surface S, image of s.

Through a point $O_{g'}$ on g' there is a unique line t of the congruence, but every F of $|F|$ is associated with $O_{g'}$, hence $O_{g'}\sim(T^{-1})t$. The ruled surface G generated by t is the image of g' under T^{-1}. Furthermore, every point P' of the transversal determines the same F' and t meets the associated F in one point $\bar{P}$ so that $\bar{P}\sim(T)t$. The locus of points $\bar{P}$ is the curve $\bar{g}$ and $\bar{g}\sim(T)G$. The order of $\bar{g}$, determined by the intersection of two homaloidal surfaces, is $2m'n'+2mn'+2m'n+m+3m'+n+3n'+3$. In a similar manner we find a curve $\bar{g}'$ of order $2mn+2mn'+2m'n+3m+m'+3n+n'+3$ such that $\bar{g}'\sim(T^{-1})G'$.

We can now write the following correspondences:

$$
(12)\qquad
\begin{aligned}
r &\sim (T)R'\colon\ s^{m+m'+1}r^{n+n'}g'\bar{g}', & r &\sim (T^{-1})R\colon\ s^{m+m'+1}r^{n+n'}g\bar{g},\\
s &\sim (T)S'\colon\ s^{m+m'}r^{n+n'+1}g'\bar{g}', & s &\sim (T^{-1})S\colon\ s^{m+m'}r^{n+n'+1}g\bar{g},\\
g &\sim (T)G'\colon\ s^{2m+1}r^{2n+1}g\bar{g}', & g' &\sim (T^{-1})G\colon\ s^{2m'+1}r^{2n'+1}\bar{g}g',\\
\bar{g}' &\sim (T^{-1})G'\colon\ s^{2m+1}r^{2n+1}g\bar{g}', & \bar{g} &\sim (T)G\colon\ s^{2m'+1}r^{2n'+1}\bar{g}g'.
\end{aligned}
$$

4. **Invariant and homaloidal surfaces.** The eliminant of the parameter from $|F|$ and $|F'|$ is the pointwise invariant surface K. Generic planes subjected to the transformations give

$$\pi' \equiv (A'x) \sim (T^{-1})R(A'y) + K(A'z) = \phi_{m+m'+n+n'+3},$$
$$\pi \equiv (Ay) \sim (T)R'(Ax) + K'(Az) = \phi'_{m+m'+n+n'+3}$$

where the ϕ's are homaloidal surfaces of the transformation. Further,

$$\phi \sim (T)(A'x)S'^{\,m+m'+1}R'^{\,n+n'+1}GG',$$
$$\phi' \sim (T^{-1})(Ay)S^{m+m'+1}R^{n+n'+1}GG',$$

hence the homaloidal webs are

$$\infty^3|\phi|\colon\ s^{m+m'+1}r^{n+n'+1}g\bar{g},\qquad \infty^3|\phi'|\colon\ s^{m+m'+1}r^{n+n'+1}g'\bar{g}'.$$

The intersection of two homaloidal surfaces gives the homaloidal net

$$H \equiv [\phi\phi]\colon s^{m^2+m'^2+2mm'+2m+2m'+1}r^{n^2+2nn'+n'^2+2n+2n'+1}g\bar{g}b_{m+m'+n+n'+3},$$
$$H' \equiv [\phi'\phi']\colon s^{m^2+2mm'+m'^2+2m+2m'+1}r^{n^2+2nn'+n'^2+2n+2n'+1}g'\bar{g}'b'_{m+m'+n+n'+3}.$$

We now write the additional correspondences:

$$(13)\quad \begin{aligned} &\pi \sim (T)\phi':\ s^{m+m'+1}r^{n+n'+1}g'\bar{g}', \quad \pi \sim (T^{-1})\phi:\ s^{m+m'+1}r^{n+n'+1}g\bar{g},\\ &K \sim (T)K':\ s^{m+m'}r^{n+n'}gg'\bar{g}\bar{g}', \quad K' \sim (T^{-1})K;\ s^{m+m'}r^{n+n'}gg'\bar{g}\bar{g}'. \end{aligned}$$

The jacobian of the transformation is $J = RGG'S$.

5. **Tangency along r and s.** The projectivity $x_1 = y_1$, $x_2 = y_2$, $x_3 = ky_3$, $x_4 = y_3 + y_4$ is applied to the fundamental surfaces of the transformation and an examination of the coefficients of the highest powers of x_3 shows K and S to have

$$\begin{aligned} D_{m+m'} \equiv \sum_{j,k=0}^{m',n'} u'_j\Big(\sum_{i=0}^{n'} b_{ik}k^i\Big)&(kc'_{jk,3} + c'_{jk,4})\\ \cdot \sum_{j,k=0}^{m,n} \bar{u}_j\Big(\sum_{i=0}^{n} \bar{b}_{ik}k^i\Big)&(k\bar{c}'_{jk,3} + \bar{c}_{jk,4})\\ - \sum_{j,k=0}^{m,n} u_j\Big(\sum_{i=0}^{n} b_{ik}k^i\Big)&(kc_{jk,3} + c_{jk,4})\\ \cdot \sum_{j,k=0}^{m',n'} \bar{u}'_j\Big(\sum_{i=0}^{n'} \bar{b}'_{ik}k^i\Big)&(k\bar{c}'_{jk,3} + \bar{c}'_{jk,4}) = 0 \end{aligned}$$

as common tangent surface along r. D is also a common tangent surface to K' and S' along r. A similar process shows

$$\begin{aligned} E_{n+n'} \equiv \sum_{j,k=0}^{m',n'} v'_k\Big(\sum_{i=0}^{m'} a'_{ij}k^i\Big)&(kc'_{jk,1} + c'_{jk,2})\\ \cdot \sum_{j,k=0}^{m,n} \bar{v}_k\Big(\sum_{i=0}^{m} \bar{a}_{ij}k^i\Big)&(k\bar{c}_{jk,1} + \bar{c}_{jk,2})\\ - \sum_{j,k=0}^{m,n} v_k\Big(\sum_{i=0}^{m} a_{ij}k^i\Big)&(kc_{jk,1} + c_{jk,2})\\ \cdot \sum_{j,k=0}^{m',n'} \bar{v}'_k\Big(\sum_{i=0}^{m'} \bar{a}'_{ij}k^i\Big)&(k\bar{c}'_{jk,1} + \bar{c}'_{jk,2}) = 0 \end{aligned}$$

to be common tangent surface to K, K', R and R' along s.

6. **Intersection table.** Referring to (12), (13), and §5 we can now write the following intersection table.

$[R'S']$: $s^{m^2+m'^2+2mm'+m+m'}r^{n^2+n'^2+2nn'+n+n'}g'\bar{g}'$

$[R'G']$: $s^{2m^2+2mm'+3m+m'+1}r^{2n^2+2nn'+n'+n}\bar{g}'d_{1,1}\cdots d_{1,2mn+2n}$

$[R'\phi']$: $s^{m^2+m'^2+2mm'+2m+2m'+1}r^{n^2+n'^2+2nn'+n+n'}g'\bar{g}'c_{n+n'+1}$

$[R'K']$: $s^{m^2+m'^2+2mm'+m+m'}r^{n'^2+2nn'+n^2+(n+n')t}g'\bar{g}'$

$$[S'G']:\ s^{2m^2+2mm'+m+m'}r^{2n^2+2nn'+3n+n'+1}\bar{g}'e_{1,1}\cdots e_{1,2mn+2m}$$
$$[S'\phi']:\ s^{m^2+m'^2+2mm'+m+m'}r^{n^2+n'^2+2nn'+2n+2n'+1}g'\bar{g}'k_{m+m'+1}$$
$$[S'K']:\ s^{m^2+2mm'+m'^2+(m+m')}r^{n^2+n'^2+2nn'+n+n'}g'\bar{g}'$$
$$[G'\phi']:\ s^{2m^2+2m'm+3m+m'+1}r^{2n^2+2nn'+3n+n'+1}\bar{g}'l_{1,1}\cdots l_{1,2m+2mn+2n+1}$$
$$[G'K']:\ s^{2m^2+m+2mm'+m'}r^{2n^2+n+2nn'+n'}g\bar{g}'$$
$$[\phi'K']:\ s^{m^2+m'^2+2mm'+m+m'}r^{n^2+n'^2+2nn'+n+n'}g'\bar{g}'c_{m+m'+n+n'+2}.$$

7. The $T_{n+n'+2}$ in a plane through r. A plane $\pi\equiv x_1=\delta x_2$ cuts the surfaces of $|F|$ and $|F'|$ in residual pencils of curves

$$(14)\qquad |f_{n+1}|\equiv u-\mu\bar{u}=0,\qquad |f'_{n'+1}|\equiv u'-\mu\bar{u}'=0$$

where

$$u_{n+1}=\sum_{j,k=0}^{m,n}\psi_j\omega_k(\alpha_{jk}x),\qquad \bar{u}_{n+1}=\sum_{j,k=0}^{m,n}\bar{\psi}_j\bar{\omega}_k(\bar{\alpha}_{jk}x),$$
$$u'_{n'+1}=\sum_{j,k=0}^{m',n'}\psi'_j\omega'_k(\alpha'_{jk}x),\qquad \bar{u}'_{n'+1}=\sum_{j,k=0}^{m',n'}\bar{\psi}'_j\bar{\omega}'_k(\bar{\alpha}'_{jk}x),$$

$$(15)\qquad \begin{aligned}\psi_j&=\sum_{i=0}^{m}a_{ij}\delta^i,\quad \bar{\psi}_j=\sum_{i=0}^{m}\bar{a}_{ij}\delta^i,\quad \psi'_j=\sum_{i=0}^{m'}a'_{ij}\delta^i,\quad \bar{\psi}'_j=\sum_{i=0}^{m'}\bar{a}'_{ij}\delta^i,\\ \omega_k&=\sum_{i=0}^{n}b_{ik}x_3^ix_4^{n-i},\qquad \bar{\omega}_k=\sum_{i=0}^{n}\bar{b}_{ik}x_3^ix_4^{n-i},\\ \omega'_k&=\sum_{i=0}^{n'}b'_{ik}x_3^ix_4^{n'-i},\qquad \bar{\omega}'_k=\sum_{i=0}^{n'}\bar{b}'_{ik}x_3^ix_4^{n'-i},\end{aligned}$$

$$(\alpha_{jk}x)=(\alpha_{jk,1}\delta+\alpha_{jk,2})x_2+\alpha_{jk,3}x_3+\alpha_{jk,4}x_4.$$

The second directrix s cuts π in the point P: $(\delta, 1, 0, 0)$ which is an n-fold point on the curves of $|f|$ and an n'-fold point on the curves of $|f'|$. Through a generic point $P(y)$ of π passes an f of $|f|$ having parameter $\mu=u(y)/\bar{u}(y)$ and to this corresponds the f' of $|f'|$ having equation

$$(16)\qquad u'(x)\bar{u}(y)-\bar{u}'(x)u(y)=0.$$

The unique transversal t through $P(y)$ and P meets (16) in a point $P'(x)$, image of $P(y)$ under the transformation. The inverse transformation is thus

$$(17)\qquad \overset{-1}{T}_{n+n'+2}:\ x_2=\sigma y_2+\kappa,\qquad x_3=\sigma y_3,\qquad x_4=\sigma y_4,$$

where

$$\kappa_{n+n'+2} = u'\bar{u} - \bar{u}'u \qquad \sigma_{n+n'+1} = u\bar{w}' - \bar{u}w',$$

$$(18) \qquad w'_{n'} = \sum_{j,k=0}^{m',n'} \psi'_j\omega'_k(\alpha'_{jk}s), \qquad \bar{w}'_{n'} = \sum_{j,k=0}^{m',n'} \bar{\psi}'_j\bar{\omega}'_k(\bar{\alpha}'_{jk}s),$$

$$(\dot{\alpha}_{jk}s) = \alpha_{jk,1}\delta + \alpha_{jk,2}$$

while the direct transformation is

$$(19) \qquad T_{n+n'+2}\colon\ y_2 = \sigma'x_2 + \kappa', \qquad y_3 = \sigma'x_3, \qquad y_4 = \sigma'x_4$$

where

$$\kappa' = -\kappa, \qquad \sigma'_{n+n'+1} = u'\bar{w} - \bar{u}'w,$$

$$(20) \qquad w_n = \sum_{j,k=0}^{m,n} \psi_j\omega_k(\alpha_{jk}s), \qquad \bar{w}_n = \sum_{j,k=0}^{m,n} \bar{\psi}_j\bar{\omega}_k(\bar{\alpha}_{jk}s).$$

The curve κ is pointwise invariant under the transformations.

The application of the transformation and its inverse gives

$$(21) \qquad u'(x) \sim (T^{-1})\sigma^{n'}u\gamma, \qquad u(y) \sim (T)\sigma'^{n}u'\gamma',$$

where $\gamma_{2n'+1} = u'\bar{w}' - \bar{u}'w'$, $\gamma'_{2n+1} = u\bar{w} - \bar{u}w$, and, furthermore,

$$\begin{aligned}
&\kappa' \sim (T^{-1})\kappa'\sigma^{n+n'}\gamma\gamma', && \kappa \sim (T)\kappa\sigma'^{n+n'}\gamma\gamma',\\
&\kappa' \sim (T)\kappa'\sigma'^{n+n'}\gamma\gamma', && \kappa \sim (T^{-1})\kappa\sigma^{n+n'}\gamma\gamma',\\
&\gamma' \sim (T^{-1})\sigma^{2n+1}\gamma', && \gamma \sim (T)\sigma'^{2n'+1}\gamma,\\
&\gamma' \sim (T)\sigma'^{2n+1}\gamma', && \gamma \sim (T^{-1})\sigma^{2n'+1}\gamma,\\
&\sigma' \sim (T^{-1})\sigma^{n+n'}\gamma\gamma', && \sigma \sim (T)\sigma'^{n+n'}\gamma\gamma',\\
&\sigma' \sim (T)\sigma'^{n+n'}[\sigma'^2 + \kappa'(w'\bar{w} - w\bar{w}')], &&\\
& && \sigma \sim (T^{-1})\sigma^{n+n'}[\sigma^2 + \kappa(w'\bar{w} - w\bar{w}')].
\end{aligned}$$

The base points of the pencil $|f|$ are P and $2n+1$ other points $\Gamma_1, \Gamma_2, \cdots, \Gamma_{2n+1}$ while those of $|f'|$ are P and $\Gamma'_1, \Gamma'_2, \cdots, \Gamma'_{2n'+1}$. We designate the lines $P\Gamma_1, P\Gamma_2, \cdots, P\Gamma_{2n+1}$ as $\gamma'_1, \gamma'_2, \cdots, \gamma'_{2n+1}$ and $P\Gamma'_1, P\Gamma'_2, \cdots, P\Gamma'_{2n'+1}$ as $\gamma_1, \gamma_2, \cdots, \gamma_{2n'+1}$. The curves γ and γ' in (21) consist of the lines $\gamma_1, \gamma_2, \cdots, \gamma_{2n'+1}$ and $\gamma'_1, \gamma'_2, \cdots, \gamma'_{2n+1}$, respectively. It is easily seen that $\Gamma_i \sim (T)\gamma'_i$, $\Gamma'_i \sim (T^{-1})\gamma_i$, $\gamma'_i \sim (T^{-1})\Gamma_i$, $\gamma_i \sim (T)\Gamma'_i$.

Moreover, there is a particular f of $|f|$ consisting of the lines γ'_1 and an n-ic through $\Gamma_2, \Gamma_3, \cdots, \Gamma_{2n+1}, P$, the latter being an $(n-1)$-fold point on the n-ic. The n-ic intersects γ'_1 in another point $\bar{P}_1$. Then $\bar{P}_1 \sim (T)\bar{\Gamma}'_1$ and $\bar{\Gamma}'_1 \sim (T^{-1})\gamma'_1$. Similarly, we can find points $\bar{\Gamma}'_2, \cdots, \bar{\Gamma}'_{2n+1}, \bar{\Gamma}_1, \cdots, \bar{\Gamma}_{2n'+1}$ such that $\bar{\Gamma}'_i \sim (T^{-1})\gamma'_i$ and $\bar{\Gamma}_i \sim (T)\gamma_i$.

We can now write the following correspondences:

$$P \sim (T)\rho': P^{n+n'}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}',$$

$$\Gamma_i \sim (T)\gamma_i': P\Gamma_i\bar{\Gamma}_i',$$

$$\bar{\Gamma}_i' \sim (T^{-1})\gamma_i': P\Gamma_i\bar{\Gamma}_i',$$

$$\kappa \sim (T)\kappa': P^{n+n'}\Gamma_1 \cdots \Gamma_{2n+1}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1 \cdots \bar{\Gamma}_{2n'+1}\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'.$$

When the transformations are applied to a generic line we get

$$\lambda' = (Ax) \sim (T^{-1})\sigma(Ay) + \kappa A_2 = \phi_{n+n'+2},$$

$$\lambda = (Ay) \sim (T)\sigma'(Ax) + \kappa' A_2 = \phi_{n+n'+2}',$$

$$\phi \sim (T)(Ay)\sigma'^{n+n'+1}\gamma\gamma', \qquad \phi' \sim (T^{-1})(Ax)\sigma^{n+n'+1}\gamma\gamma'$$

and the holaloidal nets are therefore

$$\infty^2 |\phi| : P^{n+n'+1}\Gamma_1 \cdots \Gamma_{2n+1}\bar{\Gamma}_1 \cdots \bar{\Gamma}_{2n'+1},$$

$$\infty^2 |\phi'| : P^{n+n'+1}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'.$$

The translation $x_2 = x_2' + 1$, $x_3 = x_3'$, $x_4 = x_4'$ is applied to the fundamental curves of the transformation and σ' and κ' are found to have the curve

$$l_{n+n'} = \sum_{j,k=0}^{m',n'} \psi'\omega'(\alpha_{jk,1}'\delta + \alpha_{jk,2}') \cdot \sum_{j,k=0}^{m,n} \bar{\psi}\bar{\omega}(\bar{\alpha}_{jk,1}\delta + \bar{\alpha}_{jk,2})$$

$$- \sum_{j,k=0}^{m',n'} \bar{\psi}'\bar{\omega}'(\bar{\alpha}_{jk,1}'\delta + \bar{\alpha}_{jk,2}') \cdot \sum_{j,k=0}^{m,n} \psi\omega(\alpha_{jk,1}\delta + \alpha_{jk,2}) = 0$$

as common tangent at the point P.

The jacobian is $\sigma^2\gamma\gamma'$.

The intersection table is as follows

$$[\phi'\phi']: P^{n^2+2nn'+n'^2+2n+2n'+1}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'F$$

(the homaloidal web)

$$[\sigma'\gamma_i']: P^{n+n'}\bar{\Gamma}_i', \qquad [\gamma_i'\kappa']: P^{n+n'}\Gamma_i\bar{\Gamma}_i', \qquad [\gamma_i'\phi']: P^{n+n'+1}\bar{\Gamma}_i'$$

$$[\sigma'\kappa']: P^{n^2+2nn'+n'^2+(n+n')t}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'$$

$$[\sigma'\phi']: P^{n^2+2nn'+n'^2+n+n'}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'$$

$$[\kappa'\phi']: P^{n^2+2nn'+n'^2+n+n'}\Gamma_1' \cdots \Gamma_{2n'+1}'\bar{\Gamma}_1' \cdots \bar{\Gamma}_{2n+1}'.$$

As the plane π generates the pencil on r the $T_{n+n'+3}$ generates the space $T_{m+m'+n+n'+2}$ whose equations may be obtained from (17) and (19) by replacing u, $\bar{u}$, u', $\bar{u}'$, w, $\bar{w}$, w', $\bar{w}'$, (αx) and δ by U, $\overline{U}$, U', $\overline{U}'$, W, $\overline{W}$, W', $\overline{W}'$, (cx) and x_1/x_2, respectively.

When the surfaces are cut by a plane $\pi' \equiv x_3 = \epsilon x_4$ through s a similar plane transformation of order $m+m'+2$ is found.

8. **The transformation associated with a special linear congruence.** If we take the directrices r and s of the congruence as

$$x_1 = x_4 = 0, \qquad x_2 = x_4 = 0, \tag{22}$$

respectively, and pencils of surfaces F and F' as in (2) and (3) where $u_j = \sum_{i=0}^{m} a_{ij} x_1^i x_4^{m-i}$, $v_k = \sum_{i=0}^{n} b_{ik} x_2^i x_4^{n-i}$ and similarly for $\bar{u}_j$, u_j', and so on, the equations of the inverse transformation obtained by the present technique are

$$\text{(23)} \quad \overset{-1}{T}_{m+m'+n+n'+2}\colon\ x_1 = Ry_1, \quad x_2 = Ry_2, \quad x_3 = Ry_3 + K, \quad x_4 = Ry_4$$

where $K_{m+m'+n+n'+2} = U'\overline{U} - U\overline{U}'$, $R_{m+m'+n+n'+1} = U\overline{W}' - \overline{U}W'$,

$$W'_{m'+n'} = \sum_{j,k=0}^{m',n'} u_j' v_k' (c'_{jk,3}), \qquad \overline{W}'_{m'+n'} = \sum_{j,k=0}^{m',n'} \bar{u}_j' \bar{v}_k' (\bar{c}'_{jk,3}).$$

Such transformations are well known so that no discussion will be given.

9. **Discussion.** Vogt, in his general skew transformation, finds three fundamental curves and their images in each space. These he designates as g_1, g_2, l_1, l_2, R_1, R_2 having images γ_2, γ_1, λ_2, λ_1, ρ, ρ, respectively. Of these g_1, g_2, l_1, l_2, γ_2, γ_1, λ_2, λ_1, correspond to r, r, s, s, R', R, S', S of the present paper. However, R_1, R_2, ρ, ρ, are replaced by g, $\bar{g}$; g', $\bar{g}'$; G', G; G, G', respectively. The sums of the orders of g and $\bar{g}$, g' and $\bar{g}'$, G and G' are equal to the orders of R_1, R_2, and ρ, respectively, as given by Vogt.

The properties of the transformation in §8 are the same as those of the "central" transformation which he describes.

Quadratic transformations associated with a linear congruence may of course be gotten from those of the present paper by taking $|F|$ and $|F'|$ as pencils of planes, that is, making $m = n = 0$, in the plane transformations of §7 and the space transformation of §8.

WEST VIRGINIA UNIVERSITY

THE MEASURE OF THE CRITICAL VALUES OF DIFFERENTIABLE MAPS

ARTHUR SARD

1. Introduction. Consider the *map*

$$y^j = f^j(x^1, x^2, \cdots, x^m), \qquad j = 1, 2, \cdots, n, \tag{1.1}$$

of a region R of euclidean m-space into part of euclidean n-space. Suppose that each function f^j $(j=1, \cdots, n)$ is of class C^q in R $(q \geqq 1)$.[1] A *critical point* of the map (1.1) is a point in R at which the matrix of first derivatives $\mathfrak{M}=\|f^j_i\|$ $(i=1, \cdots, m; j=1, \cdots, n)$ is of less than maximum rank. The *rank* of a critical point x is the rank of $\mathfrak{M}$ at x. A *critical value* is the image under (1.1) of a critical point. If $n=1$, these definitions are the usual definitions of critical point and value of a continuously differentiable function.

We prove the following result: *If $m \leqq n$, the set of critical values of the map* (1.1) *is of m-dimensional measure*[2] *zero* without further hypothesis on q; *if $m > n$, the set of critical values of the map* (1.1) *is of n-dimensional measure zero providing that* $q \geqq m-n+1$. Using an example due to Hassler Whitney[3] we show that the hypothesis on q cannot be weakened. We prove also that the critical values of (1.1) corresponding to critical points of rank zero constitute a set of (m/q)-dimensional measure zero.

The idea of considering the measure of the set of critical values of one function or of several functions is due to Marston Morse.

The first result stated above reduces, if $n=1$, to the known theorem: The critical values of a function of m variables of class C^m constitute a set of linear measure zero. A. P. Morse[4] has given a proof of this theorem for all m. In the present paper we make use of one of A. P. Morse's results.

Presented to the Society, January 1, 1941 under the title *The measure of the critical values of differentiable maps of euclidean spaces*; received by the editors February 9, 1942.

[1] A function is of class C^q if all its partial derivatives of order q exist and are continuous.

[2] In the sense of Hausdorff-Saks. The definition is given in §2.

[3] H. Whitney, *A function not constant on a connected set of critical points*, Duke Mathematical Journal, vol. 1 (1935), pp. 514–517.

[4] A. P. Morse, *The behaviour of a function on its critical set*, Annals of Mathematics, (2), vol. 40 (1939), pp. 62–70. Proofs for the cases $m=1, 2, 3$ had previously been given by M. Morse and for the cases $m=4, 5, 6$ by M. Morse and the author in unpublished papers.

In a study of functional dependence,[5] A. B. Brown has shown that the set of critical values of (1.1) corresponding to a closed bounded set of critical points is nowhere dense providing that q satisfies certain conditions. The theorems of the present paper imply this particular result.

2. s-dimensional measure.

Given a set A and positive quantities s and α. Denote the diameter of A by $\delta(A)$. Let $\{A_1, A_2, \cdots\}$ be a covering of A by sets of diameter less than α; and let $L_s(A;\alpha)$ be the greatest lower bound of the sums $\sum_k [\delta(A_k)]^s$ for all such coverings. The *s-dimensional outer measure* of A is

$$L_s(A) = c_s \lim_{\alpha \to 0} L_s(A;\alpha), \qquad c_s = \pi^{s/2}/2^s\Gamma[(s+2)/2], \tag{2.1}$$

where c_s is, for integral s, the s-dimensional volume of a sphere of unit diameter in s-space.[6]

We shall say that A is an *s-null* set if $L_s(A)=0$. The value of c_s in (2.1) is immaterial to this paper as we are concerned with the nullity of sets. An s-null set is a fortiori an $(s+p)$-null set (p positive). $L_s(A)$ is a regular Carathéodory measure. In n-space $L_n(A)=|A|$, where $|A|$ is the outer Lebesgue measure of A.[7]

3. Critical points of positive rank: change of variables.

Consider a critical point x_0 of rank $r>0$. Suppose (without real loss of generality) that the determinant $\Delta=|f^d_c|$ $(c, d=1, \cdots, r)$ is not zero at x_0. Consider the change of variables from x to u defined by the equations

$$u^d = f^d(x), \quad u^{r+g} = x^{r+g}, \qquad d = 1, \cdots, r;\ g = 1, \cdots, m-r. \tag{3.1}$$

Let u_0 be the image of x_0 under (3.1); and let J be the Jacobian matrix of (3.1). Since $|J|=\Delta\neq 0$ at x_0, the inverse of (3.1)

$$x^i = \phi^i(u), \qquad i = 1, \cdots, m,$$

exists and is of class C^q near u_0. In terms of the new variables u the map (1.1) is

$$\begin{cases} y^d = u^d = F^d(u), & d = 1, \cdots, r, \qquad (3.2)\\ y^{r+h} = f^{r+h}[\phi(u)] = F^{r+h}(u), & h = 1, \cdots, n-r. \qquad (3.3)\end{cases}$$

Let $\mathfrak{M}'$ be the functional matrix $\|F^j_i\|$ $(i=1, \cdots, m; j=1, \cdots, n)$.

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[5] A. B. Brown, *Functional dependence*, Transactions of this Society, vol. 38 (1935), pp. 379–394.

[6] F. Hausdorff, *Dimension und äusseres Mass*, Mathematische Annalen, vol. 79 (1919), pp. 157–179. S. Saks, *Theory of the Integral*, Warsaw, 1937, pp. 53–54.

[7] We make use of this fact for the case $|A|=0$ only.

Then $\mathfrak{M}=\mathfrak{M}'J$ and therefore $\mathfrak{M}$ and $\mathfrak{M}'$ have the same rank since $|J|\neq 0$. Hence *we may consider* (3.2, 3.3) *near* u_0 *instead of* (1.1) *near* x_0 *without changing either critical values or ranks of critical points.*

4. **Case I:** $m\leqq n$. We prove Theorem 4.1.

THEOREM 4.1. *The critical values of the map* (1.1) *constitute an* m*-null set if* $m\leqq n$.

PROOF. We consider first the set A of critical points of rank zero. We shall show that there is a neighborhood N of each point of A such that $f(NA)$, the image of NA under (1.1), is an m-null set. As A can be covered by denumerably many such neighborhoods N it will follow that $f(A)$ is an m-null set.

Let N be an open m-cube[8] whose closure is in R. Suppose that x_1 is a point of NA and x_2 is a point of N. Let $y_e=f(x_e)$ $(e=1, 2)$. Then by the mean value theorem,

$$y_2^j-y_1^j=\sum_{i=1}^{m}\zeta_i^j(x_2^i-x_1^i),\qquad j=1,\cdots,n, \tag{4.2}$$

where $\zeta_i^j\to 0$ as $x_2\to x_1$.

Let $C(\gamma)$ be a closed m-cube of side γ. It follows from (4.2) applied twice and the triangle inequality that if $C(\gamma)$ contains a point of A, then $\delta\{f[NC(\gamma)]\}\leqq 2m^2\zeta\gamma$ and $\delta\{f[NC(\gamma)]\}^m\leqq(2m^2\zeta)^m|C(\gamma)|$, where ζ is the least upper bound of the functions $|\zeta_i^j|$ for all x_1 in NA, x_2 in N such that $|x_2^i-x_1^i|\leqq\gamma$ $(i=1,\cdots,m;\ j=1,\cdots,n)$. Then $\zeta\to 0$ with γ.

Therefore given $\alpha>0$ and $\epsilon>0$, there exists a $G>0$ such that if $C(\gamma)$ contains a point of A, $\gamma<G$ implies that

$$\delta\{f[NC(\gamma)]\}<\alpha,\qquad \delta\{f[NC(\gamma)]\}^m<\frac{\epsilon}{|N|+1}|C(\gamma)|. \tag{4.3}$$

Consider the set of all cubes $C(\gamma)$ centered at points of NA and such that $\gamma<G$. By a theorem related to Vitali's covering theorem,[9] there exists a sequence $\{C_1, C_2,\cdots\}$ of cubes of this set which covers all of NA and is such that

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[8] By a cube we mean a cube with sides parallel to the axes.

[9] Hans Rademacher, *Eineindeutige Abbildung und Messbarkeit*, Monatshefte für Mathematik und Physik, vol. 27 (1916), Theorem II, p. 190, with "circles centered at P" replaced by "squares centered at P." It is not necessary to use the theorem related to Vitali's covering theorem. One may instead consider a network of cubes; or one may give a proof like A. P. Morse's proof of his Theorem 4.3 (loc. cit., pp. 68–69). This remark applies also to our later applications of the theorem related to Vitali's covering theorem. The author is indebted to A. P. Morse for his having suggested the applicability of Vitali's covering theorem.

$$\sum_k |C_k| \leqq |N| + 1. \tag{4.4}$$

The sequence $\{f(NC_k)\}$ covers $f(NA)$. Let δ_k be the diameter of $f(NC_k)$. Then (4.3) and (4.4) imply that $\delta_k < \alpha$ ($k = 1, 2, \cdots$) and

$$\sum_k \delta_k^m < \frac{\epsilon}{|N| + 1} \sum_k |C_k| \leqq \epsilon.$$

Hence $f(NA)$ is an m-null set. This completes the first part of the proof.

We consider now the set B of critical points of rank r, where r is any positive integer less than m. To prove that $f(B)$ is an m-null set we shall show that there is a neighborhood N of each point of B such that $f(NB)$ is an m-null set.

Consider a point x_0 of B. Introduce the change of variables of §3 and consider the map (3.2, 3.3) in a closed cube K centered at u_0. Let D be the set of critical points of (3.2, 3.3) of rank r in K.

Suppose that u_1 is a point of D and u_2 is a point of K. Let $y_e = F(u_e)$ ($e = 1, 2$). Then by the mean value theorem,

$$\begin{aligned} & y_2^d - y_1^d = u_2^d - u_1^d, && d = 1, \cdots, r, \\ & y_2^{r+h} - y_1^{r+h} = \sum_{c=1}^{r} F_c^{r+h}(u_1)(u_2^c - u_1^c) + \sum_{i=1}^{m} \zeta_i^{r+h}(u_2^i - u_1^i), && \\ & && h = 1, \cdots, n - r, \end{aligned} \tag{4.5}$$

where $\zeta_i^{r+h} \to 0$ as $u_2 \to u_1$.

Let $C(\gamma)$ be a closed cube of side γ in the space $(u^1, \cdots, u^m)$; and let $\Pi(\gamma)$ be the projection of $C(\gamma)$ on the space $(u^1, \cdots, u^r)$. Let p be a positive integer. Divide $\Pi(\gamma)$ by bisections into 2^{rp} congruent closed r-cubes each of side $\gamma/2^p$. Each such subcube is the projection of a *strip* of $C(\gamma)$ in which $u^1, \cdots, u^r$ may change by at most $\gamma/2^p$ and $u^{r+1}, \cdots, u^m$ may change by at most γ.

Consider any strip S that contains a point of D. Let u_1 and u_2 be any two points of S and let $y_e = F(u_e)$ ($e = 1, 2$). Then (4.5) applied twice implies that

$$\begin{aligned} & |y_2^d - y_1^d| \leqq \gamma/2^p, && d = 1, \cdots, r, \\ & |y_2^{r+h} - y_1^{r+h}| \leqq 2rU\gamma/2^p + 2m\zeta\gamma, && h = 1, \cdots, n - r, \end{aligned} \tag{4.6}$$

where U is the least upper bound of the functions $|F_c^{r+h}(u)|$ for all u in K and ζ is the least upper bound of the functions $|\zeta_i^{r+h}|$ for all u_1 in D, u_2 in K such that $|u_2^i - u_1^i| \leqq \gamma$ ($c = 1, \cdots, r$; $h = 1, \cdots, n - r$; $i = 1, \cdots, m$). Then $\zeta \to 0$ with γ.

Let δ be the diameter of $F(S)$. The triangle inequality and (4.6) imply that

$$\delta \leqq (V/2^p + W\zeta)\gamma, \tag{4.7}$$

where $V = r + (n-r)2rU$ and $W = (n-r)2m$. V and W are constants, and $V \geqq r \geqq 1$.

Let α and ϵ be given, $\alpha > 0$, $1 > \epsilon > 0$. Put

$$\eta = \epsilon/(|K| + 1). \tag{4.8}$$

There exists a number G,

$$0 < G < \alpha/(V + 1), \tag{4.9}$$

such that $\gamma < G$ implies that

$$W\zeta < \eta^{1/(m-r)}(4V)^{-r/(m-r)}/2 < 1. \tag{4.10}$$

Let p be the positive integer determined by the relation

$$2V(4V)^{r/(m-r)}\eta^{-1/(m-r)} < 2^p \leqq 4V(4V)^{r/(m-r)}\eta^{-1/(m-r)}. \tag{4.11}$$

(Note that the first member of (4.11) is greater than 1, as it should be.)

Then $\gamma < G$ implies that $\delta < \alpha$ by (4.7), (4.9) and (4.10), since $2^p > 1$; and $\gamma < G$ also implies that

$$\delta < \eta^{1/(m-r)}(4V)^{-r/(m-r)}\gamma \tag{4.12}$$

by (4.7), the first half of (4.11) and (4.10). Now consider all strips S containing points of D. There are at most 2^{rp} such strips; for these

$$\sum \delta^m < 2^{rp}\eta^{m/(m-r)}(4V)^{-mr/(m-r)}\gamma^m \leqq \eta\gamma^m = \eta\,|C(\gamma)|, \tag{4.13}$$

by (4.12) and the second half of (4.11). Thus $\gamma < G$ implies that there are sets covering $F[DC(\gamma)]$ each of whose diameters is less than α and the sum of the mth powers of whose diameters is less than $\eta|C(\gamma)|$.

Consider the set of all cubes $C(\gamma)$ centered at points of D and such that $\gamma < G$. As before, a sequence $\{C_k\}$ of these cubes covers D and is such that

$$\sum_k |C_k| \leqq |K| + 1. \tag{4.14}$$

Then the covering of $F(D)$ consisting of all the coverings of the sets $F(DC_k)$ is a covering by sets each of whose diameters is less than α and the sum of the mth powers of whose diameters is less than $\eta\sum_k |C_k| \leqq \epsilon$, by (4.13), (4.8) and (4.14). Hence $F(D)$ is an m-null set.

Let N be the inverse image under (3.1) of the interior of K. Then N

is a neighborhood of x_0. Further $f(NB)$ is contained in $F(D)$ and is therefore an m-null set. This completes the proof of Theorem 4.1.

5. **A. P. Morse's theorem.** We state the following theorem, due to A. P. Morse.[10]

THEOREM 5.1. *Given a positive integer q and a set A in the space of the variables x. There exists a sequence $A_0, A_1, A_2, \cdots$ of sets with the following properties:* (i) $A=\sum_{k=0}^{\infty} A_k$, (ii) *$A_0$ is denumerable,* (iii) A_k $(k=1, 2, \cdots)$ *is bounded,* (iv) *if $g(x)$ is any function of class C^q whose critical set includes A and if x_1 and x_2 are points of A_k, then*

$$\lim_{x_2 \to x_1} \frac{g(x_2)-g(x_1)}{|x_2-x_1|^q}=0, \qquad k=1, 2, \cdots .$$

6. **Critical points of rank zero.** We prove Theorem 6.1.

THEOREM 6.1. *Let A be the set of critical points of rank zero of the map* (1.1). *Then $f(A)$ is an s-null set if $s \geqq m/q$.*

PROOF. Decompose A into the subsets of Theorem 6.1. Then $f(A_0)$ is denumerable and hence is an s-null set. We shall prove that $f(A_k)$ is an s-null set $(k=1, 2, \cdots)$. It will follow that $f(A)$ is an s-null set.

Consider a non-empty A_k (k fixed and positive). For simplicity put $B=A_k$. Given $\alpha>0$ and $\epsilon>0$. Given a point x of B. Let $C(\gamma)$ be the closed cube of side γ *centered at x*. By continuity and Theorem 5.1 there is a positive $G_x<1$ such that $\gamma<G_x$ implies that

$$\delta\{f[C(\gamma)]\}<\alpha \tag{6.2}$$

and

$$|f^j(x_1)-f^j(x)|<\frac{\epsilon^{1/s}2^{q-1}}{m^q n(|B|+1)^{1/s}}|x_1-x|^q, \quad j=1, \cdots, n, \tag{6.3}$$

whenever x_1 is in $BC(\gamma)$. Now by the triangle inequality $|x_1-x| \leqq m\gamma/2$ if x_1 is in $BC(\gamma)$. Hence (6.3) twice and the triangle inequality imply that

$$\delta\{f[BC(\gamma)]\}<2\frac{\epsilon^{1/s}2^{q-1}}{m^q(|B|+1)^{1/s}}\left(\frac{m\gamma}{2}\right)^q=\frac{\epsilon^{1s}/\gamma^q}{(|B|+1)^{1/s}},$$

and therefore

$$\delta\{f[BC(\gamma)]\}^s<\epsilon\gamma^{qs}/(|B|+1), \tag{6.4}$$

providing that $\gamma<G_x$.

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[10] Loc. cit., Theorem 4.2. By denumerable we mean denumerably infinite or finite. The sets A_k $(k=1, 2, \cdots)$ are condensed; however we do not make use of this property. The symbol $|x_2-x_1|$ denotes the distance between x_1 and x_2.

Consider the set of all cubes $C(\gamma)$, $\gamma < G_n$, centered at points x of B. As before, a sequence $\{C_k\}$ of these cubes covers B and is such that

$$\sum_k |C_k| \leqq |B| + 1. \tag{6.5}$$

Let γ_k be the side of C_k $(k=1, 2, \cdots)$.

The sequence $\{f(BC_k)\}$ covers $f(B)$. Further $\delta[f(BC_k)] < \alpha$ by (6.2); and

$$\sum_k \delta[f(BC_k)]^s < \epsilon \sum_k \gamma_k^m/(|B| + 1) \leqq \epsilon \sum_k |C_k|/(|B| + 1) \leqq \epsilon$$

by (6.4) and (6.5), since $\gamma_k \leqq 1$ by construction, $qs \geqq m$ by hypothesis and $|C_k| = \gamma_k^m$ $(k=1, 2, \cdots)$. Hence $f(B)$ is an s-null set.

7. **Case** II: $m > n$. We prove Theorems 7.1 and 7.2.

THEOREM 7.1. *Let A be the set of critical points of rank r of the map* (1.1). *Then $f(A)$ is an n-null set if $q \geqq (m-r)/(n-r)$, $m > n$.*

PROOF. If $r=0$ the theorem reduces to Theorem 6.1 with $s=n$.

Suppose that $0 < r < n$. To prove that $f(A)$ is an n-null set we shall show that there is a neighborhood N of each point of A such that $f(NA)$ is an n-null set.

Consider a point x_0 of A. Introduce the change of variables of §3 and consider the map (3.2, 3.3) in a closed neighborhood $\overline{N}$ of u_0. Regard $u^1, \cdots, u^r$ as parameters for each permissible set of values of which (3.3) defines a map of the $(m-r)$-space $(u^{r+1}, \cdots, u^m)$ into the $(n-r)$-space $(y^{r+1}, \cdots, y^n)$. For each $(u^1, \cdots, u^r)$ let $\mathfrak{M}^*$ be the functional matrix of (3.3). Then $\mathfrak{M}'$ is of rank r if and only if $\mathfrak{M}^*$ is of rank zero.

Thus if $(u^1, \cdots, u^m)$ is a critical point of (3.2, 3.3) of rank r, then $(u^{r+1}, \cdots, u^m)$ is a critical point of (3.3) of rank zero for the values $u^1, \cdots, u^r$ of the parameters. But for each set of values of the parameters, the critical points of (3.3) of rank zero map into an $(n-r)$-null set, by Theorem 6.1 and hypothesis.

Let B be the set of critical points of (3.2, 3.3) of rank r in $\overline{N}$. The cross-section of $F(B)$ for each $(y^1, \cdots, y^r)$ is thus an $(n-r)$-null set. Further $F(B)$ is closed and therefore measurable. Hence $F(B)$ is an n-null set by the theorem of Fubini.

THEOREM 7.2. *If $m > n$ the critical values of the map* (1.1) *constitute an n-null set providing that $q \geqq m-n+1$.*

PROOF. Apply Theorem 7.1 with $r=0, 1, \cdots, n-1$. For these values of r, $m-n+1 \geqq (m-r)/(n-r)$.

8. The hypothesis on q in Theorem 7.2 cannot be weakened. Let $W(x^1, \cdots, x^t)$ be a function of t variables, $x^1, \cdots, x^t$, of class C^{t-1}, which takes on every value from 0 to 1 on a Jordan arc of critical points ($t \geqq 2$). H. Whitney has constructed such a function.[11] Consider the map

$$(8.1) \qquad y^1 = W(x^1, \cdots, x^{m-n+1}), \qquad y^a = x^{m-n+a}, \qquad a = 2, 3, \cdots, n; \; m > n,\text{[12]}$$

in the unit cube. The map (8.1) is of class C^{m-n}, but the critical values of (8.1) do not constitute an n-null set. Indeed the set of critical values of (8.1) is the unit cube and is thus of n-dimensional measure 1.

Queens College

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[11] Loc. cit.

[12] If $n=1$, (8.1) is to consist of the equation for y^1 only.

THE RADICAL OF A NON-ASSOCIATIVE ALGEBRA

A. A. ALBERT

1. **Introduction.** An algebra $\mathfrak{A}$ is said to be nilpotent of index r if every product of r quantities of $\mathfrak{A}$ is zero, and is said to be a zero algebra if it is nilpotent of index two. It is said to be simple if it is not a zero algebra and its only nonzero (two-sided) ideal is itself, and is said to be semi-simple if it is a direct sum of simple algebras.

The radical of an associative algebra $\mathfrak{A}$ is a nilpotent ideal $\mathfrak{N}$ of $\mathfrak{A}$ which is maximal in the strong sense in that it contains[1] all nilpotent ideals of $\mathfrak{A}$. No such ideal exists in an arbitrary non-associative algebra, and so the radical of such an algebra has never[2] been defined. However the property that $\mathfrak{A}-\mathfrak{N}$ be semi-simple is really the vital one and we shall define the concept of radical here by proving this theorem.

THEOREM 1. *Every algebra $\mathfrak{A}$ which is homomorphic to a semi-simple algebra has an ideal $\mathfrak{N}$, which we shall call the* **radical** *of $\mathfrak{A}$, such that $\mathfrak{A}-\mathfrak{N}$ is semi-simple, $\mathfrak{N}$ is contained in every ideal $\mathfrak{B}$ of $\mathfrak{A}$ for which $\mathfrak{A}-\mathfrak{B}$ is semi-simple.*

The hypothesis that $\mathfrak{A}$ shall be homomorphic to a semi-simple algebra is equivalent to the property that there shall exist an ideal $\mathfrak{B}$ in $\mathfrak{A}$ such that $\mathfrak{A}-\mathfrak{B}$ shall be semi-simple. It is a necessary assumption even in the associative case, since $\mathfrak{A}$ may be nilpotent and then $\mathfrak{A}=\mathfrak{N}$, every $\mathfrak{A}-\mathfrak{B}$ is nilpotent. Moreover it is satisfied by every algebra $\mathfrak{A}$ with a unity quantity. We shall, nevertheless, carry our study a step farther in that we shall define explicitly a certain proper ideal $\mathfrak{N}$ for every algebra $\mathfrak{A}$ such that either $\mathfrak{N}$ is the radical of $\mathfrak{A}$ in the sense above or $\mathfrak{A}$ is not homomorphic to a semi-simple algebra. In the latter case $\mathfrak{A}-\mathfrak{N}$ is a zero algebra.

Our results will be consequences of the remarkable fact[3] that the major structural properties of any non-associative algebra $\mathfrak{A}$ over $\mathfrak{F}$ are determined by almost the same properties of a certain related associative algebra $T(\mathfrak{A})$. We define the right multiplications R_x and the

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Presented to the Society April 18, 1942; received by the editors February 11, 1942.

[1] For these results see my *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939, chap. 2.

[2] For the case of alternative algebras see M. Zorn, *Alternative rings and related questions* I: *Existence of the radical*, Annals of Mathematics, (2), vol. 42 (1941), pp. 676–686.

[3] Cf. my *Non-associative algebras* I: *Fundamental concepts and isotopy*, Annals of Mathematics, (2), vol. 43 (1942), pp. 685–708. See also N. Jacobson, *A note on non-associative algebras*, Duke Mathematical Journal, vol. 3 (1937), pp. 544–548.

left multiplications L_x of $\mathfrak{A}$ for every x of $\mathfrak{A}$ to be the respective linear transformations

$$a \to a \cdot x = aR_x, \qquad a \to x \cdot a = aL_x, \qquad a \text{ in } \mathfrak{A},$$

on $\mathfrak{A}$, and let the transformation algebra $T(\mathfrak{A})$ of $\mathfrak{A}$ be the polynomial ring over $\mathfrak{F}$ generated by the R_x, the L_x, and the identity transformation I. If $\mathfrak{S}$ is any set of linear transformations S on $\mathfrak{A}$ we define $\mathfrak{A}\mathfrak{S}$ to be the linear subspace of $\mathfrak{A}$ spanned by the images aS of every a of $\mathfrak{A}$. Then if $\mathfrak{H}$ is the radical of $T(\mathfrak{A})$ the set $\mathfrak{A}\mathfrak{H}$ is a proper ideal of $\mathfrak{A}$ which is zero if and only if $\mathfrak{H}=0$. When $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is a zero algebra the algebra $T(\mathfrak{A})-\mathfrak{H}$ is a field of order one and we shall prove these theorems.

THEOREM 2. *An algebra $\mathfrak{A}$ is homomorphic to a semi-simple algebra if and only if $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is not a zero algebra.*

THEOREM 3. *If $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is a zero algebra and $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ the algebra $\mathfrak{A}-\mathfrak{B}$ is a zero algebra if and only if $\mathfrak{B}$ contains $\mathfrak{A}\mathfrak{H}$.*

THEOREM 4. *Let $\mathfrak{A}$ be homomorphic to a semi-simple algebra. Then either $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is semi-simple and $\mathfrak{A}\mathfrak{H}$ is the radical of $\mathfrak{A}$ or $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is the direct sum of a semi-simple algebra and a zero algebra $\mathfrak{N}_0=\mathfrak{N}-\mathfrak{A}\mathfrak{H}$ such that $\mathfrak{N}$ is the radical of $\mathfrak{A}$.*

We shall close our discussion with a study of algebras with a unity quantity and the radicals of isotopes with unity quantities. Moreover we shall exhibit an algebra with a unity quantity and a radical which is a field.

2. **A fundamental lemma.** Let $\mathfrak{B}$ be a linear subspace of an algebra $\mathfrak{A}$ of order n over $\mathfrak{F}$ and m be the order of $\mathfrak{B}$ so that there exists an idempotent E of rank m in the algebra $(\mathfrak{F})_n$ of all linear transformations on $\mathfrak{A}$ such that

$$\mathfrak{B} = \mathfrak{A}E.$$

Then $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ if and only if

$$ET(\mathfrak{A}) = ET(\mathfrak{A})E. \tag{1}$$

Since $T(\mathfrak{A})$ contains the identity transformation I it follows that

$$\mathfrak{B}T(\mathfrak{A}) = \mathfrak{B}. \tag{2}$$

We let $\mathfrak{S}$ be the intersection

$$T(\mathfrak{A})E \cap T(\mathfrak{A}), \tag{3}$$

so that $\mathfrak{S}$ consists of all $S=SE$ in $T(\mathfrak{A})$. Then $\mathfrak{S}T(\mathfrak{A})$ is contained in $T(\mathfrak{A})ET(\mathfrak{A})=T(\mathfrak{A})ET(\mathfrak{A})E$ by (1), $\mathfrak{S}T(\mathfrak{A})$ is contained in $\mathfrak{S}$. Also $T(\mathfrak{A})T(\mathfrak{A})E=T(\mathfrak{A})E$, $T(\mathfrak{A})\mathfrak{S}=\mathfrak{S}$ is an ideal[4] of $T(\mathfrak{A})$. Then $T(\mathfrak{A})-\mathfrak{S}$ is defined and we may prove the fundamental lemma.

LEMMA 1. *The algebra* $T(\mathfrak{A}-\mathfrak{B})$ *is equivalent to* $T(\mathfrak{A})-\mathfrak{S}$.

For let a be any quantity of $\mathfrak{A}$ and $\{a\}=a+\mathfrak{B}$ be the corresponding coset in the decomposition of the additive group $\mathfrak{A}$ relative to $\mathfrak{B}$. If T is in $T(\mathfrak{A})$ the set $\mathfrak{B}T$ is contained in $\mathfrak{B}$, the coset $\{aT\}=aT+\mathfrak{B}$ is independent of a. Then the correspondence

$$a\rightarrow\{aT\}=\{a\}T_0 \tag{4}$$

is a transformation T_0 of $\mathfrak{A}-\mathfrak{B}$ uniquely determined for every T of $T(\mathfrak{A})$. Moreover T_0 is a linear transformation. But then we have determined a mapping

$$T\rightarrow T_0 \tag{5}$$

of $T(\mathfrak{A})$ on a set $\mathfrak{T}_0$ of linear transformations T_0 on $\mathfrak{A}-\mathfrak{B}$. It is clear from (4) that

$$\{a(T_1\alpha+T_2\beta)\}=\{aT_1\}\alpha+\{aT_2\}\beta=\{a\}(T_{10}\alpha+T_{20}\beta), \tag{6}$$

$$\{a(T_1T_2)\}=\{(aT_1)T_2\}=\{aT_1\}T_{20}=\{a\}T_{10}T_{20} \tag{7}$$

for every α and β of $\mathfrak{F}$, T_1 and T_2 of $T(\mathfrak{A})$. Then (5) determines a homomorphism of $T(\mathfrak{A})$ on $\mathfrak{T}_0$.

The general right multiplication $R_{\{x\}}$ of $\mathfrak{A}-\mathfrak{B}$ is the transformation $\{a\}\rightarrow\{a\}\cdot\{x\}$, and this is the transformation $(R_x)_0$ given by (5). For

$$\{a\}\ \{x\}=\{a\cdot x\}=\{aR_x\}. \tag{8}$$

Similarly $L_{\{x\}}=(L_x)_0$, $\mathfrak{T}_0$ contains $T(\mathfrak{A}-\mathfrak{B})$. If $u_1,\cdots,u_n$ are a basis of $\mathfrak{A}$ over $\mathfrak{F}$ and $S_i=R_{u_i}$, $T_i=L_{u_i}$ every transformation of $T(\mathfrak{A})$ is a polynomial $T=\phi(I,S_1,\cdots,S_n,T_1,\cdots,T_n)$, $T_0=\phi(I,S_{10},\cdots,S_{n0},T_{10},\cdots,T_{n0})$ is in $T(\mathfrak{A}-\mathfrak{B})$, $T(\mathfrak{A}-\mathfrak{B})=T_0$. If $T_0=0$ we have $\{aT\}=0$ for every a, aT is in $\mathfrak{B}$ for every a of $\mathfrak{A}$, $aT=aTE$, $T=TE$ is in $\mathfrak{S}$. Thus the algebra $T(\mathfrak{A})$ is homomorphic under (5) to $T(\mathfrak{A}-\mathfrak{B})$ such that $\mathfrak{S}$ is the ideal of all transformations T of $T(\mathfrak{A})$ such that $T_0=0$. Then $T(\mathfrak{A})-\mathfrak{S}$ is equivalent to $T(\mathfrak{A}-\mathfrak{B})$. This proves our lemma.

3. **Algebras with a semi-simple transformation algebra.** A quantity

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[4] This proof is so brief that I repeat it rather than refer to the proof in the article quoted in Footnote 3.

$z \neq 0$ of an algebra A is called an absolute divisor of zero if $z \cdot a = a \cdot z = 0$ for every a of A. Suppose first that $\mathfrak{A}$ contains no absolute divisor of zero and that $T(\mathfrak{A})$ is semi-simple. Then $T(\mathfrak{A}) = \mathfrak{T}_1 \oplus \cdots \oplus \mathfrak{T}_r$ for simple algebras $\mathfrak{T}_i$ and $\mathfrak{T}_i = T(\mathfrak{A})E_i$ where E_i is the unity quantity of $\mathfrak{T}_i$, E_i is a nonzero idempotent of the center of $T(\mathfrak{A})$. We let $\mathfrak{A}_i = \mathfrak{A}E_i$ and have (1) for each E_i, $\mathfrak{A}_i$ is an ideal of $\mathfrak{A}$. Now $\mathfrak{A}_i = \mathfrak{A}_i E_i$ and it follows that the $\mathfrak{A}_i$ are supplementary ideals $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$. Write $\mathfrak{A} = \mathfrak{A}_i \oplus \mathfrak{B}_i$ and have $T(\mathfrak{A}) = \mathfrak{T}_i \oplus \mathfrak{S}_i$, $\mathfrak{T}_i = T(\mathfrak{A})E_i$, $\mathfrak{S}_i = T(\mathfrak{A})E_{i0}$ where $E_{i0} = I - E_i$. Then $\mathfrak{S}_i$ is the algebra $\mathfrak{S}$ of Lemma 1 for $\mathfrak{B} = \mathfrak{B}_i$, $T(\mathfrak{A} - \mathfrak{B}) \simeq T(\mathfrak{A}) - \mathfrak{S}_i$. Clearly $\mathfrak{A} - \mathfrak{B}_i = \mathfrak{A}_i$, $T(\mathfrak{A}) - \mathfrak{S}_i = \mathfrak{T}_i$, $T(\mathfrak{A}_i) = \mathfrak{T}_i$ is simple. But $\mathfrak{A}_i$ has no absolute divisor of zero and then is known[5] to be simple when $\mathfrak{T}_i$ is simple. Thus we have shown that if $T(\mathfrak{A})$ is semi-simple and $\mathfrak{A}$ has no absolute divisors of zero it is semi-simple.

If $\mathfrak{B}$ is a linear subset of $\mathfrak{A}$ we have $\mathfrak{B}\mathfrak{A} \subset \mathfrak{B}T(\mathfrak{A})$, $\mathfrak{A}\mathfrak{B} \subset \mathfrak{B}T(\mathfrak{A})$. Then if $\mathfrak{H}$ is any right ideal of $T(\mathfrak{A})$ we have $(\mathfrak{A}\mathfrak{H})\mathfrak{A} \subset (\mathfrak{A}\mathfrak{H})T(\mathfrak{A}) \subset \mathfrak{A}\mathfrak{H}$, $\mathfrak{A}(\mathfrak{A}\mathfrak{H}) \subset (\mathfrak{A}\mathfrak{H})T(\mathfrak{A}) \subset \mathfrak{A}\mathfrak{H}$. Hence $\mathfrak{A}\mathfrak{H}$ is an ideal of $\mathfrak{A}$. If $\mathfrak{H} \neq 0$ is a nilpotent ideal of $T(\mathfrak{A})$ we cannot have $\mathfrak{A}\mathfrak{H} = 0$. Also $\mathfrak{A}\mathfrak{H} \neq \mathfrak{A}$ since otherwise $\mathfrak{H}^t = 0$ would imply that $\mathfrak{A} = 0$. Let then $\mathfrak{A}$ be semi-simple, $\mathfrak{H}$ be the radical of $T(\mathfrak{A})$. We write $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ for simple algebras $\mathfrak{A}_i$ and may choose pairwise orthogonal idempotents E_i of $(\mathfrak{F})_n$ such that $\mathfrak{A}_i = \mathfrak{A}E_i$, $E_1 + \cdots + E_r = I$. Then $\mathfrak{A}_i$ is an ideal of $\mathfrak{A}$, $E_i T(\mathfrak{A}) = E_i T(\mathfrak{A}) E_i$, $E_i \mathfrak{H} = E_i \mathfrak{H} E_i = \mathfrak{H}_i$ is clearly a nilpotent ideal of $E_i T(\mathfrak{A})$. But it follows that $\mathfrak{A}\mathfrak{H} = \mathfrak{A}_1\mathfrak{H}_1 \oplus \cdots \oplus \mathfrak{A}_1\mathfrak{H}_r$ for ideals $\mathfrak{A}_i\mathfrak{H}_i$ of $\mathfrak{A}_i$. This is impossible unless each $\mathfrak{H}_i = 0$ since each $\mathfrak{A}_i$ is simple. Thus if $\mathfrak{A}$ is semi-simple so is $T(\mathfrak{A})$.

Suppose finally that $\mathfrak{A}$ does have absolute divisors of zero and let $\mathfrak{N}$ be the set of all absolute divisors of zero of $\mathfrak{A}$. Then clearly $\mathfrak{N}$ is an ideal of $\mathfrak{A}$ which is a zero algebra, $\mathfrak{N} = \mathfrak{N}E$ for an idempotent E of $(\mathfrak{F})_n$. If $\mathfrak{A} = \mathfrak{N}$ we have $T(\mathfrak{A}) = I\mathfrak{F}$ and $T(\mathfrak{A})$ is semi-simple. Otherwise $I - E = E_0$ is an idempotent of $(\mathfrak{F})_n$, $\mathfrak{A} = \mathfrak{A}(E + E_0)$ is the direct sum $\mathfrak{A} = \mathfrak{G} \oplus \mathfrak{N}$ where $\mathfrak{G} = \mathfrak{A}E_0$ contains no absolute divisors of zero. If a and x are in $\mathfrak{A}$ we write $x = g + h$ with g in $\mathfrak{G}$ and h in $\mathfrak{N}$, $a \cdot x = a \cdot g = aR_g$, $R_x = R_g$, $(a \cdot x)E_0$ is in $\mathfrak{G}$, $R_x E_0 = R_x$ for every x of $\mathfrak{A}$. Similarly every L_x is in $T(\mathfrak{A})E_0$ and it is clear that $T(\mathfrak{G})$ is equivalent to $T(\mathfrak{A})E_0$. The algebra $\mathfrak{S}$ of Lemma 1 defined for $\mathfrak{B} = \mathfrak{G}$ is $T(\mathfrak{A})E_0$ and is an ideal of $T(\mathfrak{A})$, $T(\mathfrak{A}) = T(\mathfrak{A})E_0 + I\mathfrak{F}$. The algebra $\mathfrak{S}$ of Lemma 1 defined for $\mathfrak{B} = \mathfrak{N}$ is $T(\mathfrak{A})E$ and is EF since $E_0E = 0$. Then $T(\mathfrak{A}) = T(\mathfrak{A})E_0 \oplus EF$, $T(\mathfrak{A})E_0$ is semi-simple when $T(\mathfrak{A})$ is semi-simple, $T(\mathfrak{G})$ is semi-simple and so is $\mathfrak{G}$. Conversely, if $\mathfrak{G}$ is semi-simple so is $T(\mathfrak{A})E_0$ and so is $T(\mathfrak{A})$. We have proved this lemma.

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[5] In the paper referred to in Footnote 3, N. Jacobson defined $T(\mathfrak{A})$ to be generated by the right and left multiplications of $\mathfrak{A}$ and with I *omitted*. He then proved our result. We require the more general statement including the case where $\mathfrak{A}$ may be a zero algebra and so refer to Lemma 10 of my own paper of that reference.

LEMMA 2. *The transformation algebra $T(\mathfrak{A})$ is semi-simple if and only if $\mathfrak{A}$ is either semi-simple, a zero algebra, or a direct sum of a semi-simple algebra and a zero algebra.*

4. The radical of an algebra. If $\mathfrak{A}$ is an associative semi-simple algebra and $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ we have $\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{C}$, where $\mathfrak{C}$ is an ideal of $\mathfrak{A}$ equivalent to $\mathfrak{A}-\mathfrak{B}$ and is semi-simple. Let $\mathfrak{A}$ now be an associative algebra with radical $\mathfrak{N}\neq 0$ and $\mathfrak{B}$ be an ideal of $\mathfrak{A}$. If $\mathfrak{B}$ contains $\mathfrak{N}$ the algebra $\mathfrak{A}-\mathfrak{B}$ is equivalent to $(\mathfrak{A}-\mathfrak{N})-(\mathfrak{B}-\mathfrak{N})$ and is semi-simple by the argument above. Conversely, let $\mathfrak{B}$ be an ideal of $\mathfrak{A}$ such that $\mathfrak{A}-\mathfrak{B}$ is semi-simple, $\mathfrak{A}-\mathfrak{B}$ contains no properly nilpotent classes. Then every properly nilpotent quantity of $\mathfrak{A}$ must define the zero class of $\mathfrak{A}-\mathfrak{B}$, $\mathfrak{B}$ contains all properly nilpotent quantities of $\mathfrak{A}$, $\mathfrak{B}$ contains $\mathfrak{N}$. This proves Theorem 1 in the associative case.

We now let $\mathfrak{B}$ be any ideal of an arbitrary algebra $\mathfrak{A}$, $\mathfrak{H}$ be the radical of $T(\mathfrak{A})$ so that $\mathfrak{A}\mathfrak{H}$ is a proper ideal of $\mathfrak{A}$. If $\mathfrak{A}-\mathfrak{B}$ is semi-simple so is $T(\mathfrak{A}-\mathfrak{B})$ by Lemma 2, and so is $T(\mathfrak{A})-\mathfrak{S}$ by Lemma 1. But by the result just proved $\mathfrak{S}$ contains $\mathfrak{H}$. However $\mathfrak{B}=\mathfrak{A}E$, $\mathfrak{S}=T(\mathfrak{A})E=\mathfrak{S}E$, $\mathfrak{H}=\mathfrak{H}E$, $\mathfrak{A}\mathfrak{H}=\mathfrak{A}\mathfrak{H}E$ is contained in $\mathfrak{A}\mathfrak{S}=\mathfrak{A}\mathfrak{S}E$ and hence in $\mathfrak{B}$. Then we have proved that the radical $\mathfrak{N}$ of $\mathfrak{A}$ contains $\mathfrak{A}\mathfrak{H}$, $\mathfrak{A}-\mathfrak{B}$ is equivalent to $(\mathfrak{A}-\mathfrak{A}\mathfrak{H})-(\mathfrak{B}-\mathfrak{A}\mathfrak{H})$. Hence $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ cannot be a zero algebra. If $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is semi-simple our definition implies that $\mathfrak{A}\mathfrak{H}=\mathfrak{N}$. Otherwise $\mathfrak{A}_0=\mathfrak{A}-\mathfrak{A}\mathfrak{H}=\mathfrak{G}\oplus\mathfrak{N}_0$ where $\mathfrak{G}$ is semi-simple and $\mathfrak{N}_0$ is a zero algebra, $\mathfrak{B}-\mathfrak{A}\mathfrak{H}$ is an ideal $\mathfrak{B}_0$ of $\mathfrak{A}_0$ such that $\mathfrak{A}_0-\mathfrak{B}_0$ is semi-simple. If there is a quantity of $\mathfrak{N}_0$ not in $\mathfrak{B}_0$ the corresponding class of $\mathfrak{A}_0-\mathfrak{B}_0$ is an absolute divisor of zero of that algebra, contrary to our hypothesis. Hence $\mathfrak{B}_0$ contains $\mathfrak{N}_0$, $\mathfrak{B}$ contains the algebra $\mathfrak{N}$ of all the quantities in the classes of $\mathfrak{N}_0$, $\mathfrak{N}$ is the radical of $\mathfrak{A}$. This proves Theorem 1 and Theorem 4.

If $\mathfrak{A}$ is homomorphic to an algebra $\mathfrak{A}_0$ and $\mathfrak{B}$ is the set of all quantities of $\mathfrak{A}$ mapped into zero by the given homomorphism then $\mathfrak{A}-\mathfrak{B}$ is equivalent to $\mathfrak{A}_0$. If $\mathfrak{A}-\mathfrak{B}$ is semi-simple we have seen that $\mathfrak{B}$ contains $\mathfrak{A}\mathfrak{H}$, $\mathfrak{A}-\mathfrak{B}$ is equivalent to $(\mathfrak{A}-\mathfrak{A}\mathfrak{H})-(\mathfrak{B}-\mathfrak{A}\mathfrak{H})$, $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ cannot be a zero algebra. This proves Theorem 2. We have also seen that if $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is a zero algebra and $\mathfrak{A}-\mathfrak{B}$ is a zero algebra then $T(\mathfrak{A}-\mathfrak{B})=T(\mathfrak{A})-\mathfrak{S}$ for an ideal $\mathfrak{S}$ of $T(\mathfrak{A})$. But then by Lemma 2 $T(\mathfrak{A})-\mathfrak{S}$ is semi-simple, $\mathfrak{S}$ contains $\mathfrak{H}$, $\mathfrak{A}\mathfrak{S}=\mathfrak{A}\mathfrak{S}E$ contains $\mathfrak{A}\mathfrak{H}$ and is contained in $\mathfrak{A}E=\mathfrak{B}$. This proves Theorem 3.

5. The radicals of isotopic algebras. Let $\mathfrak{A}$ and $\mathfrak{A}_1$ be algebras of the same order so that we may regard them as having quantities in the same linear space. Then $\mathfrak{A}$ and $\mathfrak{A}_1$ are principal isotopes if multiplication in $\mathfrak{A}_1$ is given by $[a, x]=aR_x^{(1)}$ for $R_x^{(1)}=PR_{xQ}$, P and Q nonsingular linear transformations on $\mathfrak{A}$. If $\mathfrak{A}$ and $\mathfrak{A}_1$ have unity quantities the transformations P and Q are in $T(\mathfrak{A})$ and $T(\mathfrak{A})=T(\mathfrak{A}_1)$.

Let $\mathfrak{B}$ be an ideal of $\mathfrak{A}$, $\mathfrak{B}=\mathfrak{A}E$, $ET(\mathfrak{A})=ET(\mathfrak{A})E$. Then $EP=EPE$, $EQ=EQE$ and $EPP^{-1}=E=(EP)(EP^{-1})$, $P_1=EP$ is a nonsingular quantity of $ET(\mathfrak{A})$. Similarly $Q_1=EQ$ is a nonsingular quantity of $ET(\mathfrak{A})$. Write $\mathfrak{B}_1=\mathfrak{A}_1E$, so that since $ET(\mathfrak{A}_1)=ET(\mathfrak{A}_1)E$ the space $\mathfrak{B}_1$ is an ideal[6] of $\mathfrak{A}_1$. Then if b and y are in $\mathfrak{B}_1$ we have $b=bE$, $y=yE$,

$$[b, y] = bEPR_{yEQ} = bP_1R_{yQ_1}. \tag{9}$$

It follows that $\mathfrak{B}$ and $\mathfrak{B}_1$ are principal isotopes with isotopy given by

$$R_y^{(1)} = P_1R_{yQ_1}. \tag{10}$$

Every isotope $\mathfrak{A}_1$ of $\mathfrak{A}$ is equivalent to a principal isotope and we have proved the first part of this theorem.

THEOREM 5. *Let $\mathfrak{A}$ and $\mathfrak{A}_1$ be isotopic algebras with unity quantities. Then every ideal $\mathfrak{B}$ of $\mathfrak{A}$ is an isotope of an ideal $\mathfrak{B}_1$ of $\mathfrak{A}_1$ such that the difference algebras $\mathfrak{A}-\mathfrak{B}$ and $\mathfrak{A}_1-\mathfrak{B}_1$ are isotopes.*

We now observe that the homomorphism (5) of $T(\mathfrak{A})$ on $T(\mathfrak{A}-\mathfrak{B})$ carries every nonsingular P of $T(\mathfrak{A})$ into a nonsingular P_0 of $T(\mathfrak{A}-\mathfrak{B})$. Then if we define

$$(R_{\{x\}})^{(1)} = P_0R_{\{x\}Q_0}, \tag{11}$$

the algebra with multiplication defined by

$$[\{a\}, \{x\}] = \{a\}(R_{\{x\}})^{(1)} \tag{12}$$

is a principal isotope of $\mathfrak{A}-\mathfrak{B}$. But the difference algebra $\mathfrak{A}_1-\mathfrak{B}_1$ has multiplication defined by $[\{a\}, \{x\}]=\{[a, x]\}=\{aR_x^{(1)}\}=\{aPR_{xQ}\}$ $=\{aP\}R_{\{xQ\}}=\{a\}(R_{\{x\}})^{(1)}$ since $\{aP\}=\{a\}P_0$, $\{xQ\}=\{x\}Q_0$. This proves our theorem.

We should observe that while P_0 and Q_0 are in $T(\mathfrak{A}-\mathfrak{B})$ the transformations P_1 and Q_1 defining the isotopy of B and B_1 need not be in $T(\mathfrak{B})$. This is an evident consequence of the fact that if $\mathfrak{A}$ has a unity quantity so does $\mathfrak{A}-\mathfrak{B}$, but certainly $\mathfrak{B}$ need not have a unity quantity. Observe also that if $\mathfrak{A}$ of order n does not have a unity quantity and we pass to an algebra $\underline{\mathfrak{A}}$ of order $n+1$ with a unity quantity the algebra $\mathfrak{A}$ will be an ideal of $\underline{\mathfrak{A}}$. The results above then become of particular importance in the study of isotopes of algebras *without* unity quantities.

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[6] It follows from this that if $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ the same linear space is an ideal $\mathfrak{B}_1$ of $\mathfrak{A}_1$. However, we wish to prove the stronger result that $\mathfrak{B}$ and $\mathfrak{B}_1$ are isotopic.

We conclude our general results by proving the following theorem

THEOREM 6. *Let* $\mathfrak{A}$ *be an algebra with a unity quantity,* $\mathfrak{H}$ *be the radical of* $T(\mathfrak{A})$. *Then* $\mathfrak{A}\mathfrak{H}$ *is the radical of* $\mathfrak{A}$ *and is isotopic to the radical* $\mathfrak{A}_1\mathfrak{H}_1$ *of any isotope* $\mathfrak{A}_1$ *of* $\mathfrak{A}$ *with a unity quantity. Moreover the semi-simple algebras* $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ *and* $\mathfrak{A}_1-\mathfrak{A}_1\mathfrak{H}$ *are isotopic.*

For every homomorph $\mathfrak{A}-\mathfrak{B}$ of an algebra $\mathfrak{A}$ with a unity quantity has a unity quantity and cannot be a direct sum of a zero algebra and another algebra. Thus Theorem 4 implies that $\mathfrak{A}\mathfrak{H}$ is the radical of $\mathfrak{A}$. Our result follows from Theorem 5.

6. **An algebra whose radical is a field.** Let $\mathfrak{A}$ be an algebra with a basis e, u, v over $\mathfrak{F}$ so that every quantity of $\mathfrak{A}$ is uniquely expressible in the form $a=\alpha e+\beta u+\gamma v$ for α, β, γ in $\mathfrak{F}$. We let e be the unity quantity of $\mathfrak{A}$ and complete the definition of $\mathfrak{A}$ with the relations

$$u^2 = e, \qquad uv = v, \qquad v^2 = v, \qquad vu = 0.$$

Let $\mathfrak{B}$ be a nonzero ideal of $\mathfrak{A}$ and $a\neq 0$ be in $\mathfrak{B}$ so that the corresponding α, β, γ are not all zero. Then $au=\alpha u+\beta e$, $(au)u=\alpha e+\beta u$, $a-(au)u=\gamma v$, $v[(au)u]=\alpha v$, $v(au)=\beta v$ are all in $\mathfrak{B}$, $\mathfrak{B}$ contains the algebra $\mathfrak{N}$ of order one over $\mathfrak{F}$ spanned by v. Now $(\alpha e+\beta u+\gamma v)v=(\alpha+\beta+\gamma)v$, $v(\alpha e+\beta u+\gamma v)=(\alpha+\gamma)v$, $\mathfrak{N}$ is an ideal of $\mathfrak{A}$. If $\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{C}$ for ideals $\mathfrak{B}$ and $\mathfrak{C}$ we have proved that both $\mathfrak{B}$ and $\mathfrak{C}$ would contain $\mathfrak{N}$. This is impossible. Also $\mathfrak{N}$ is a nonzero proper ideal of $\mathfrak{A}$ and $\mathfrak{A}$ cannot be simple. It follows that $\mathfrak{A}$ is not semi-simple. But $\mathfrak{A}=\mathfrak{N}+\mathfrak{S}$ where $\mathfrak{S}$ is the semi-simple associative algebra spanned by e and u, $\mathfrak{A}-\mathfrak{N}=\mathfrak{S}$, $\mathfrak{A}-\mathfrak{N}$ is semi-simple. Then $\mathfrak{N}$ is the radical of $\mathfrak{A}$ according to our definition and is a field of order one over $\mathfrak{F}$.

THE UNIVERSITY OF CHICAGO

AN ARITHMETICAL IDENTITY FOR THE FORM $ab-c^2$

WALTER H. GAGE

1. **Introduction.** The number of solutions in positive integers of the equation $n=xy+yz+zx$, n a positive integer, has been investigated Liouville,[1] Bell,[2] and Mordell.[3] Mordell, who was the first to obtain complete results, gave a strictly arithmetical treatment, while Bell made use of formulae which he obtained by paraphrasing theta-function identities. Although the latter considered only the case of n prime, his methods were extended later to the general case.[4]

Making use of other formulae derived by the method of paraphrase, Bell[5] has also solved the problem of representations in the forms $xy+yz+2zx$, $xy+2yz+2zx$. As he has pointed out, a feature of the method is the handling of the two forms simultaneously.

In this paper we derive by elementary methods a simple identity which on specialization not only yields complete results for representations of n in the forms

$$xy + yz + zx, \qquad xy + 2yz + 2zx, \qquad xy + yz + 2zx,$$

but as in Bell's paper,[5] handles the latter two forms at the same time.

2. **Fundamental identity.** Let $f(a, b, c)$ be a function, uniform and finite for all integer triples (a, b, c), but otherwise (so far) completely arbitrary. If the summation sign refers to the sum over all those integer solutions (a, b, c) of $n=ab-c^2$ subject to the restrictions indicated under it, we then have

$$\sum_{a,\,b>c>0} f(a, b, c) = \sum_{a>b>c>0} f(a, b, c) + \sum_{b>a>c>0} f(a, b, c) + \sum_{a=b>c>0} f(a, b, c). \tag{1}$$

Imposing on $f(a, b, c)$ the condition

$$f(a, b, c) = f(b, a, c) \tag{2}$$

Received by the editors February 13, 1942.

[1] Journal de Mathématiques, (2), vol. 7 (1862), p. 44.

[2] E. T. Bell, *Class numbers and the form $yz+zx+xy$*, Tôhoku Mathematical Journal, vol. 19 (1921), pp. 105–116.

[3] L. J. Mordell, *On the number of solutions in positive integers of the equation $yz+zx+xy=n$*, American Journal of Mathematics, vol. 45 (1923), pp. 1–4.

[4] W. H. Gage, *Representations in the form $xy+yz+zx$*, American Journal of Mathematics, vol. 51 (1929), pp. 345–348.

[5] E. T. Bell, *Numbers of representations of integers in a certain triad of ternary quadratic forms*, Transactions of this Society, vol. 32 (1930), pp. 1–5.

we may write

$$
\begin{aligned}
\sum_{a,b>c>0} f(a,b,c) &= 2\sum_{a>b>c>0} f(a,b,c) + \sum_{a=b>c>0} f(a,b,c) \\
&= 2\Bigg[\sum_{a>b>2c>0} f(a,b,c) + \sum_{a>b>c>0,\, b<2c} f(a,b,c) \\
&\qquad + \sum_{a>b=2c>0} f(a,b,c)\Bigg] + \sum_{a=b>c>0} f(a,b,c).
\end{aligned}
\tag{3}
$$

Note that $n=ab-c^2=a(a+b-2c)-(a-c)^2$, so that if we replace (a, b, c) by $(a+b-2c, a, a-c)$ in the second summation of the right member of (3) the conditions $a>b>c>0$, $b<2c$ become $a, b>2c>0$. Hence, if $f(a, b, c)$ satisfies (2), we have

$$
\begin{aligned}
\sum_{a,b>c>0} f(a,b,c) &= 2\sum_{a>b>2c>0} f(a,b,c) + 2\sum_{a,b>2c>0} f(a+b-2c, a, a-c) \\
&\quad + 2\sum_{d\equiv\delta\,(\mathrm{mod}\,2),\, d>3\delta>0} f\left(\frac{d+\delta}{2}, 2\delta, \delta\right) \\
&\quad + \sum_{d\equiv\delta\,(\mathrm{mod}\,2),\, d>\delta>0} f\left(\frac{d+\delta}{2}, \frac{d+\delta}{2}, \frac{d-\delta}{2}\right)
\end{aligned}
\tag{4}
$$

where the last two summations refer to all integer solutions (d, δ) of $n=d\delta$ subject to the given conditions.

3. **Specialization.** In (4) let $f(a, b, c)=1$, and in the left member replace (a, b, c) by $(x+z, y+z, z)$. Then, if N denotes the number of integer representations of n in the form stated after it, we get

$$
\begin{aligned}
N(n &= xy+yz+zx;\ x, y, z>0) \\
&= 6N(n=ab-c^2;\ a>b>2c>0) \\
&\quad + N(n=d\delta;\ d\equiv\delta\ (\mathrm{mod}\ 2),\ d>\delta>0) \\
&\quad + 2N(n=d\delta;\ d\equiv\delta\ (\mathrm{mod}\ 2),\ d>3\delta>0) \\
&\quad + 2N(n=d\delta;\ d\equiv\delta(\mathrm{mod}\ 2),\ 3\delta>d>\delta>0).
\end{aligned}
\tag{5}
$$

Let $G(n)$ denote the number of classes of binary quadratic forms of determinant $-n$, subject to all the conventions of H. J. S. Smith's *Report on the Theory of Numbers*.[4]

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[4] *Mathematical Papers*, vol. 1. See also Dickson's *History of the Theory of Numbers*, vol. 3, pp. 108–109.

$$\begin{aligned} G(n) = {} & N(n = ab - c^2;\ a > b > 2|c| > 0) \\ & + N(n = ab - c^2;\ a > b > 0,\ c = 0) \\ & + N(n = ab - c^2;\ a = b > 2c > 0) \\ & + N(n = ab - c^2;\ a > b = 2c > 0) \\ & + (1/2)N(n = ab - c^2;\ a = b > 0,\ c = 0) \\ & + (1/3)N(n = ab - c^2;\ a = b = 2c > 0). \end{aligned}$$

Hence we have

$$\begin{aligned} & 2N(n = ab - c^2;\ a > b > 2c > 0) \\ & \quad = G(n) - N(n = d\delta;\ d > \delta > 0) - (1/2)N(n = d^2;\ d > 0) \\ (6) & \qquad - N(n = d\delta;\ d \equiv \delta \pmod 2),\ d > 3\delta > 0) \\ & \qquad - (1/3)N(n = 3d^2;\ d > 0) \\ & \qquad - N(n = d\delta;\ d \equiv \delta \pmod 2),\ 3\delta > d > \delta > 0). \end{aligned}$$

From (5) it therefore follows that

$$(7) \quad N(n = xy + yz + zx;\ x,\ y,\ z > 0) = 3\left[G(n) - (1/2)\zeta(n)\right],$$

where $\zeta(n)$ is the number of divisors of n.

On choosing $f(a, b, c) = 1$ if a, b are not both even, $f(a, b, c) = 0$ if $a \equiv b \equiv 0 \pmod 2$, we likewise obtain the result

$$\begin{aligned} & N(n = 2xy + 2yz + 4zx;\ x,\ y,\ z > 0,\ y \equiv 1 \pmod 2)) \\ (8) & \qquad + N(n = xy + 2yz + 2zx;\ x, y, z > 0,\ x \equiv y \equiv 1 \pmod 2)) \\ & \quad = F(n) - (1/2)\zeta'(n) - (1/2)\zeta'(n/2), \end{aligned}$$

where $F(n)$ is the number of uneven classes of binary quadratic forms of determinant $-n$, $\zeta'(n)$ is the number of odd divisors of n, and $\zeta'(n/2) = 0$ for n odd.

If, in (8), we replace n by $2^{\alpha+1}m$, $2m$, m, respectively, $m \equiv 1 \pmod 2$, we get Bell's theorems[5] 1, 2, 3 from which the complete results for $xy+yz+2zx$, $xy+2yz+2zx$ follow.

THE UNIVERSITY OF BRITISH COLUMBIA

DUAL GEODESICS ON A SURFACE

C. E. SPRINGER

Introduction. Union curves and dual union curves have been defined and studied in projective space by Sperry.[1] It is well known that the union curves of the congruence of normals to a metric surface are the geodesics on the surface. The principal aim of this note is to obtain the differential equation of the *dual* geodesics on a metric analytic surface in ordinary space.

The notation of Eisenhart[2] will be employed for the most part. However, $\Gamma^{\alpha}_{\beta\gamma}$ will be used here as the Christoffel symbol of the second kind. Greek indices will take the range 1, 2, and Latin indices the range 1, 2, 3.

1. **Ray-point corresponding to a point of a curve on the surface.** The tangent planes to the surface S $(x^i = x^i(u^1, u^2))$ at the point $P(x^i)$ and at two "successive" points of the curve C $(u^\alpha = u^\alpha(s))$ on S are given by

$$
\begin{aligned}
&(\xi^i - x^i)X^i = 0,\\
(1)\qquad &(\xi^i - x^i)\frac{\partial X^i}{\partial u^\alpha}u'^\alpha = 0,\\
&(\xi^i - x^i)\left(\frac{\partial^2 X^i}{\partial u^\alpha \partial u^\beta}u'^\alpha u'^\beta + \frac{\partial X^i}{\partial u^\alpha}u''^\alpha\right) = \frac{\partial X^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta}u'^\alpha u'^\beta,
\end{aligned}
$$

where the primes indicate differentiation with respect to s.

The ray-point[3] R of the curve C corresponding to the point P is the point of intersection of the three planes (1). The coordinates of R are given by

$$
(2)\qquad S(\xi^i - x^i) = \delta^{jk}_{\sigma\tau}X^\sigma \frac{\partial X^l}{\partial u^\alpha}\frac{\partial x^l}{\partial u^\beta}\frac{\partial X^\tau}{\partial u^\gamma}u'^\alpha u'^\beta u''^\gamma,
$$

where i, j, k take the values 1, 2, 3 cyclically, $\delta^{\rho\mu}_{\sigma\tau}$ is a Kronecker delta, and S is defined by

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Presented to the Society, November 28, 1942; received by the editors of the American Journal of Mathematics February 19, 1942; later transferred to this Bulletin.

[1] Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, American Journal of Mathematics, vol. 40 (1928), p. 213.

[2] Eisenhart, *Differential Geometry*, Princeton University Press, 1940.

[3] Lane, *Projective Differential Geometry of Curves and Surfaces*, The University of Chicago Press, 1932.

$$S = \delta^{123}_{ijk} X^i \frac{\partial X^j}{\partial u^\gamma} u'^\gamma \left(\frac{\partial^2 X^k}{\partial u^\alpha \partial u^\beta} u'^\alpha u'^\beta + \frac{\partial X^k}{\partial u^\alpha} u''^\alpha \right). \tag{3}$$

The tangents to the parametric curves at P, and the normal to the surface at P constitute a local system of reference. In order to deduce the local coordinates of the point R from equations (2), we employ the formulas from Eisenhart,[4]

$$\frac{\partial X^k}{\partial u^\alpha} = - d_{\alpha\gamma} g^{\gamma\sigma} \frac{\partial x^k}{\partial u^\sigma}, \tag{4}$$

$$\frac{\partial^2 X^k}{\partial u^\alpha \partial u^\beta} = \frac{\partial X^k}{\partial u^\gamma} \Gamma^\gamma_{\alpha\beta} - k_{\alpha\beta} X^k, \tag{5}$$

$$\Gamma^\pi_{\alpha\beta} = d^{\sigma\pi} \frac{\partial d_{\alpha\sigma}}{\partial u^\beta} - d^{\sigma\pi} d_{\alpha\gamma} \Gamma^\gamma_{\sigma\beta}; \tag{6}$$

and we use the fact that

$$\delta^{123}_{ijk} X^i \frac{\partial x^j}{\partial u^1} \frac{\partial x^k}{\partial u^2} = g^{1/2} \tag{7}$$

to obtain for the first part of the expression for S

$$\begin{aligned} &\delta^{123}_{ijk} X^i \frac{\partial X^j}{\partial u^\gamma} \frac{\partial^2 X^k}{\partial u^\alpha \partial u^\beta} u'^\alpha u'^\beta u'^\gamma \\ &\qquad = g^{1/2} (g^{\tau 1} g^{\epsilon 2} - g^{\tau 2} g^{\epsilon 1}) d_{\gamma\tau} d_{\pi\epsilon} \Gamma^\pi_{\alpha\beta} u'^\alpha u'^\beta u'^\gamma . \end{aligned} \tag{8}$$

On summing out τ and ϵ, and on writing $|g^{\alpha\beta}| = g^{-1}$, the right member of (8) reduces to

$$g^{-1/2} \delta^{\epsilon\tau}_{12} d_{\gamma\epsilon} d_{\pi\tau} \Gamma^\pi_{\alpha\beta} u'^\alpha u'^\beta u'^\gamma . \tag{9}$$

In consequence of (4), we have for the second part of the expression for S

$$\delta^{123}_{ijk} X^i \frac{\partial X^j}{\partial u^\alpha} \frac{\partial X^k}{\partial u^\beta} u'^\alpha u''^\beta = \delta^{123}_{ijk} X^i \frac{\partial x^j}{\partial u^\sigma} \frac{\partial x^k}{\partial u^\tau} d_{\alpha\gamma} d_{\beta\delta} g^{\gamma\sigma} g^{\delta\tau} u'^\alpha u''^\beta . \tag{10}$$

On summing σ and τ, and on making use of (7), the right member of (10) takes the form

$$g^{1/2} (g^{\gamma 1} g^{\delta 2} - g^{\gamma 2} g^{\delta 1}) d_{\alpha\gamma} d_{\beta\delta} u'^\alpha u''^\beta, \tag{11}$$

which, in turn, reduces to

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[4] Eisenhart, loc. cit., p. 217, p. 257.

$$g^{-1/2}\delta_{12}^{\sigma\tau}d_{\alpha\sigma}d_{\beta\tau}u'^{\alpha}u''^{\beta}. \tag{12}$$

Because of (9) and (12), equation (3) takes the form

$$S \equiv g^{-1/2}\delta_{12}^{\sigma\tau}(d_{\gamma\sigma}d_{\epsilon\tau}\Gamma_{\alpha\beta}^{\epsilon}u'^{\alpha}u'^{\beta}u'^{\gamma} + d_{\alpha\sigma}d_{\beta\tau}u'^{\alpha}u''^{\beta}). \tag{13}$$

In order to express the right member of (2) as a linear combination of the partial derivatives $\partial x^i/\partial u^\alpha$, we use the definition

$$\frac{\partial X^l}{\partial u^\alpha}\frac{\partial x^l}{\partial u^\beta} = -d_{\alpha\beta}, \tag{14}$$

and also the formulas from Eisenhart[5]

$$\begin{aligned} \delta_{\sigma\nu}^{jk}X^{\sigma}\frac{\partial X^{\nu}}{\partial u^1} &= ek^{1/2}h^{2\alpha}\frac{\partial X^i}{\partial u^\alpha}, \\ \delta_{\sigma\nu}^{jk}X^{\sigma}\frac{\partial X^{\nu}}{\partial u^2} &= -ek^{1/2}h^{1\alpha}\frac{\partial X^i}{\partial u^\alpha}, \end{aligned} \tag{15}$$

where i, j, k take the values 1, 2, 3 cyclically, and where e is $+1$ or -1 according as the Gaussian curvature is positive or negative. In consequence of (14), (15), and (4), the right member of (2) assumes the form

$$ek^{1/2}d_{\alpha\beta}u'^{\alpha}u'^{\beta}(u'^1h^{2\sigma} - u'^2h^{1\sigma})d_{\sigma\tau}g^{\tau\delta}\frac{\partial x^i}{\partial u^\delta}. \tag{16}$$

If we multiply the equation[6]

$$h_{\alpha\beta} = d_{\alpha\gamma}d_{\beta\delta}g^{\gamma\delta} \tag{17}$$

by $d^{\alpha\tau}$ and sum on α, and then multiply the resulting equation by $h^{\beta\sigma}$ and sum on β, we obtain

$$d^{\sigma\tau} = h^{\beta\sigma}d_{\beta\delta}g^{\tau\delta}. \tag{18}$$

Because of (18), the expression (16) may be written in the form

$$ek^{1/2}d_{\alpha\beta}u'^{\alpha}u'^{\beta}\delta_{\gamma\sigma}^{12}u'^{\gamma}d^{\sigma\delta}\frac{\partial x^i}{\partial u^\delta}, \tag{19}$$

which is the expression desired for the right member of equation (2). Use of (13) and (19) in (2) yields the coordinates ξ^i of the ray-point R of the curve C ($u^\alpha = u^\alpha(s)$) at the point $P(x^i)$ in the form

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[5] Eisenhart, loc. cit., p. 259.

[6] Eisenhart, loc. cit., p. 253.

$$(20)\quad \xi^i = x^i + e(hg)^{1/2} d_{\alpha\beta}u'^{\alpha}u'^{\beta}\,\frac{\delta^{12}_{\gamma\sigma}u'^{\gamma}d^{\sigma\delta}}{\delta^{\sigma\tau}_{12}(d_{\gamma\sigma}d_{\mu\tau}\Gamma^{\mu}_{\alpha\beta}u'^{\alpha}u'^{\beta}u'^{\gamma}+d_{\alpha\sigma}d_{\beta\tau}u'^{\alpha}u''^{\beta})}\cdot\frac{\partial x^i}{\partial u^\delta}.$$

The local coordinates η^δ of the point R are given by

$$(21)\qquad \frac{\eta^\delta}{(g_{\delta\delta})^{1/2}} = d\,d_{\alpha\beta}u'^{\alpha}u'^{\beta}\,\frac{\delta^{12}_{\gamma\sigma}u'^{\gamma}d^{\sigma\delta}}{\delta^{\sigma\tau}_{12}(d_{\gamma\sigma}d_{\mu\tau}\Gamma^{\mu}_{\alpha\beta}u'^{\alpha}u'^{\beta}u'^{\gamma} + d_{\alpha\sigma}d_{\beta\tau}u'^{\alpha}u''^{\beta})},$$

where δ is not summed, and where $d = |d_{\alpha\beta}|$ has the value $e(hg)^{1/2}$.

2. **Dual geodesics.** We consider next the line l_2 lying in the tangent plane to the surface at P which is in G. M. Green's "relation R"[7] to the normal l_1 to the surface at P. It can readily be shown that the equation of the line l_2 in local coordinates is

$$(22)\qquad (g_{22})^{1/2}\Gamma^{2}_{12}\xi^1 + (g_{11})^{1/2}\Gamma^{1}_{12}\xi^2 + (g_{11}g_{22})^{1/2} = 0.$$

A geodesic curve on the surface has the property that its osculating plane at every point P contains the normal line l_1 to the surface at P. We may define a *dual geodesic* to be a curve on the surface which enjoys the property that its ray-point R corresponding to every point P lies on the line l_2 given by (22). If we require that the coordinates (21) satisfy the equation (22), we find, after some reduction, that the curve C is a solution of the differential equation

$$(23)\quad d\delta^{12}_{\alpha\beta}u'^{\alpha}u''^{\beta} + [d_{\alpha\beta}(d_{\gamma 1}\Gamma^{1}_{12} - d_{\gamma 2}\Gamma^{2}_{12}) + \delta^{\sigma\tau}_{12}d_{\gamma\sigma}d_{\mu\tau}\Gamma^{\mu}_{\alpha\beta}]u'^{\alpha}u'^{\beta}u'^{\gamma} = 0,$$

where $\Gamma^{\mu}_{\alpha\beta}$ is given by (6).

If we write $u^1 = u$, $u^2 = v$, $d_{11} = D$, $d_{12} = d_{21} = D'$, $d_{22} = D''$, $v' = dv/du$, $v'' = d^2v/du^2$, and make use of (6), equation (23) takes the form

$$(24)\qquad (DD'' - D'^2)v'' = p + qv' + rv'^2 + sv'^3,$$

where

$$\begin{aligned}
p &= D'D_u - DD'_u + (2\Gamma^{2}_{12} - \Gamma^{1}_{11})DD' - \Gamma^{2}_{11}D'^2,\\
q &= D''D_u - DD''_u + D'D_v - DD'_v + (DD'' + D'^2)(2\Gamma^{2}_{12} - \Gamma^{1}_{11})\\
&\quad + D^2\Gamma^{1}_{22} + DD'(\Gamma^{2}_{22} - 2\Gamma^{1}_{12}) - 2D'D''\Gamma^{2}_{11},\\
(25)\quad r &= D''D_v - DD''_v + D''D'_u - D'D''_u\\
&\quad + (DD'' + D'^2)(\Gamma^{2}_{22} - 2\Gamma^{1}_{12}) - D''^2\Gamma^{2}_{11}\\
&\quad + D'D''(2\Gamma^{2}_{12} - \Gamma^{1}_{11}) + 2DD'\Gamma^{1}_{22},\\
s &= D''D'_v - D'D''_v + (\Gamma^{2}_{22} - 2\Gamma^{1}_{12})D'D'' + \Gamma^{1}_{22}D'^2.
\end{aligned}$$

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[7] Lane, loc. cit., p. 81.

If the asymptotic lines are taken as parametric, we have $D = D'' = 0$, and equation (24) takes the form

$$v'' = \Gamma^{2}_{11} + (\Gamma^{1}_{11} - 2\Gamma^{2}_{12})v' + (2\Gamma^{1}_{12} - \Gamma^{2}_{22})v'^{2} - \Gamma^{1}_{22}v'^{3}. \tag{26}$$

The geodesics on a surface satisfy the differential equation[8]

$$v'' = -\Gamma^{2}_{11} + (\Gamma^{1}_{11} - 2\Gamma^{2}_{12})v' + (2\Gamma^{1}_{12} - \Gamma^{2}_{22})v'^{2} + \Gamma^{1}_{22}v'^{3}. \tag{27}$$

From (26) it is seen that if the asymptotic curve given by $v = \text{const.}$ is a dual geodesic, then $\Gamma^{2}_{11} = 0$, which, by (27), is the condition that the curve be a geodesic. But if a curve is both an asymptotic and a geodesic it is a straight line. Hence, *if an asymptotic line is a dual geodesic it is a straight line.*

Equation (26) is independent of D'. Hence, we conclude that *isometric surfaces have the same equations of dual geodesics.*

It can be shown that when the asymptotics are parametric the directions of Segrè[9] at the point P on the surface are given by

$$\Gamma^{2}_{11}du^{3} - \Gamma^{1}_{22}dv^{3} = 0. \tag{28}$$

Comparison of equations (26) and (27) shows that the *directions of Segrè may be characterized as the directions in which the geodesics and dual geodesics coincide.*

3. **Ray-points of geodesics.** Each curve through the point P on the surface has a ray-point R. We shall now find the locus of the ray-points for all geodesics through P. Along a geodesic curve we have

$$u''^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}u'^{\beta}u'^{\gamma} = 0. \tag{29}$$

In consequence of (29) and the definition of $\Gamma^{\kappa}_{\alpha\beta}$ from (6), the coordinates (η^1, η^2) of the ray-point R of a geodesic curve $u^{\alpha} = u^{\alpha}(s)$ through P are found from (21) to be given by

$$\begin{aligned} T\eta^1 &= -(g_{11})^{1/2}(u'^1)^2(u'^2), \\ T\eta^2 &= (g_{22})^{1/2}(u'^1)(u'^2)^2, \end{aligned} \tag{30}$$

where

$$T = \Gamma^{2}_{11}(u'^1)^3 + \Gamma^{2}_{12}(u'^1)^2(u'^2) - \Gamma^{1}_{12}(u'^1)(u'^2)^2 - \Gamma^{1}_{22}(u'^2)^3. \tag{31}$$

Elimination of u'^1 and u'^2 between the equations (30) yields a cubic equation which may be written in the form

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[8] Eisenhart, *Differential Geometry*, Ginn and Co., 1909, p. 205.

[9] Lane, loc. cit., p. 77.

$$\Gamma^2_{11}\left(\frac{\eta^1}{(g_{11})^{1/2}}\right)^3+\Gamma^1_{22}\left(\frac{\eta^2}{(g_{22})^{1/2}}\right)^3 - \frac{\eta^1\eta^2}{(g_{11}g_{22})^{1/2}}\left(\frac{\Gamma^2_{12}}{(g_{11})^{1/2}}\eta^1+\frac{\Gamma^1_{12}}{(g_{22})^{1/2}}\eta^2+1\right)=0. \tag{32}$$

From the form of equation (32) it can be seen that the locus has a double point at P, with the asymptotics at P as double point tangents. Furthermore, the curve has three points of inflection which lie on the line represented by equation (22), some one of the inflections lying on each of the lines through P given by the equation

$$\Gamma^2_{11}\left(\frac{\eta^1}{(g_{11})^{1/2}}\right)^3+\Gamma^1_{22}\left(\frac{\eta^2}{(g_{22})^{1/2}}\right)^3=0. \tag{33}$$

On comparing equations (28) and (33), we recognize that the latter gives the directions of Darboux[10] at P. Hence, we have the following theorem. *The locus of the ray-points of the geodesics through a point P on a surface is a cubic curve lying in the tangent plane to the surface at P. The asymptotics are the double point tangents to the curve at P, and the three points of inflection of the curve are given by the intersections of the tangents of Darboux at P with the line which is in Green's Relation R with the normal to the surface at the point P.* Lane[11] gives a similar theorem for projective space.

UNIVERSITY OF OKLAHOMA

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[10] Lane, loc. cit., p. 76.

[11] Lane, loc. cit., p. 98.

GENERATORS OF PERMUTATION GROUPS SIMPLY ISOMORPHIC WITH $LF(2, p^n)$

F. A. LEWIS

It is well known that the group $LF(2, p^n)$ of linear fractional transformations of determinant unity in the $GF[p^n]$ can be represented as a permutation group G of degree p^n+1. The purpose of this note is to show that the generators of G follow from a slight extension of an argument used in a recent paper.[1]

We obtain a representation of the abstract group L simply isomorphic with the special linear homogeneous group $SLH(2, p^n)$ by means of the cosets K and KTS_λ, where λ ranges over the p^n marks of the field $u_0(=0)$, $u_1, \cdots, u_m$, $(m=p^n-1)$. Let $k_\infty=K$ and $k_{u_i}=KTS_{u_i}$ for $i=0, 1, \cdots, m$.

If ρ is any mark, $KS_\rho=K$ and $KTS_\lambda \cdot S_\rho=KTS_{\lambda+\rho}$, so that to S_ρ there corresponds the permutation

$$s_\rho = \begin{pmatrix} k_\infty & k_0 & k_{u_1} & \cdots & k_{u_m} \\ k_\infty & k_\rho & k_{u_1+\rho} & \cdots & k_{u_m+\rho} \end{pmatrix}. \tag{1}$$

If $\lambda\neq 0$, $KTS_\lambda T=KTS_{-\lambda^{-1}}$. Further, $KTS_0T=K$, so that to T there corresponds the permutation

$$t = (k_0k_\infty \cdot k_{u_1}k_{-u_1^{-1}} \cdots k_{u_m}k_{-u_m^{-1}}). \tag{2}$$

Hence L has a $(d, 1)$ isomorphism with (s_ρ, t), where d is the order of a subgroup of K which is invariant in L. The quotient group (s_ρ, t) is simply isomorphic[2] with $LF(2, p^n)$ and is of order $p^n(p^{2n}-1)/d$, where $d=2$ or 1 according as $p>2$ or $p=2$.

THEOREM. *A permutation group simply isomorphic with the group $LF(2, p^n)$ of linear fractional transformations of determinant unity in the $GF[p^n]$ is generated by* (1) *and* (2), *where ρ ranges over an independent set of additive generators of the field.*

COROLLARY.[3] *A permutation group simply isomorphic with the group $LF(2, p)$ is generated by $(k_0k_1k_2 \cdots k_{p-1})$ and $(k_0k_\infty \cdot k_1k_{i_1} \cdot k_2k_{i_2} \cdots)$, where $ji_j \equiv -1$* (mod p).

UNIVERSITY OF ALABAMA

Presented to the Society, December 29, 1939 and April 4, 1942; received by the editors February 20, 1942.

[1] *A note on the special linear homogeneous group* $SLH(2, p^n)$, this Bulletin, vol. 47 (1941), pp. 629–632. The notation and results of this paper are assumed above.

[2] L. E. Dickson, *Linear Groups with an Exposition of the Galois Field Theory*, pp. 87–88.

[3] Compare with $x'=x+1$ and $x'=-1/x$, which generate $LF(2, p)$.

ON AN INEQUALITY OF SEIDEL AND WALSH

LYNN H. LOOMIS

Introduction. In a recent paper[1] Seidel and Walsh introduced the following concepts.

Let R be a Riemann surface (configuration) lying over the w-plane, and let C_p be a simply-connected region of R having the following properties:

(a) C_p contains precisely p points (counted according to branch-point multiplicity) lying over some point of the w-plane.

(b) C_p lies over the circle $|w-w_0|<r$, and the boundary of C_p lies over the circumference $|w-w_0|=r$.

It follows that C_p contains precisely p points lying over every point of $|w-w_0|<r$, and in particular, p points $\tilde{w}_i$ lying over w_0. Seidel and Walsh name such a region a *p-sheeted circle with centers $\tilde{w}_i$ and radius r.* Given a point $\tilde{w}_0$ of R, let r_p be the radius of the largest p-sheeted circle in R with center $\tilde{w}_0$; if none exists, let $r_p=0$. We then define the *radius of p-valence of R at $\tilde{w}_0$, $D_p(\tilde{w}_0)$*, as the maximum of the r_n for $n\leqq p$.

Let $w=f(z)=a_1z+\cdots+a_pz^p+a_{p+1}z^{p+1}+\cdots$ be analytic in the unit circle $|z|<1$ with $|f(z)|<M$, and let the Riemann surface R be the image of $|z|<1$ under $w=f(z)$. Let $\tilde{w}_0$ be the image of $z=0$; $\tilde{w}_0$ lies over $w=0$. Seidel and Walsh establish the following relation between the first p coefficients of $f(z)$ and the radius of p-valence, $D_p(\tilde{w}_0)$, of R at $\tilde{w}_0$.

There exist two constants, λ_p depending only on p, and Λ_p depending on p and M, such that

$$\lambda_p D_p(\tilde{w}_0) \leqq \sum_{n=1}^{p} |a_n| \leqq \Lambda_p[D_p(\tilde{w}_0)]^{2^{-p}}. \tag{1}$$

Seidel and Walsh find for Λ_p the value

$$\Lambda_p = 24pM^r, \qquad r = 1-2^{-p}.$$

In this note we prove the following two statements concerning the inequalities (1).

A. *The exponent 2^{-p} may be replaced by $1/(p+1)$ and this exponent is the best possible (for $D_p\to 0$).*

Received by the editors February 24, 1942.

[1] W. Seidel and J. L. Walsh, *On the derivatives of functions analytic in the unit circle and their radii of univalence and of p-valence*, Transactions of this Society, vol 52 (1942), pp. 129–216.

We show, in fact, that there is a constant K_p depending only on p such that

$$\sum_{n=1}^{p} |a_n| \leqq K_p M^{p/(p+1)} [D_p(w_0)]^{1/(p+1)}, \tag{2}$$

and that there is a class of functions actually attaining the bound M for which

$$\sum_{n=1}^{p} |a_n| \geqq M^{p/(p+1)} [D_p(w_0)]^{1/(p+1)}. \tag{3}$$

B. *The coefficient* λ_p *may be replaced by* 1, *and this value of the constant is the best possible.*

All proofs are based on the theorem of Rouché. We shall always suppose, without explicit statement, that z is confined to the unit circle $|z| < 1$.

Proof of (2). To prove (2) we assume $a_p \neq 0$ and let

$$P(z) = \sum_{n=1}^{p} a_n z^n = a_p \prod_{n=1}^{p} (z - b_n), \tag{4}$$

from which we have

$$|P(z)| \leqq \sum_{n=1}^{p} |a_n| = C. \tag{5}$$

We observe that $|a_n| \leqq M$, so that $|P(z)| < pM$, and therefore by the Schwarz lemma,

$$|f(z) - P(z)| < (p+1)M|z|^{p+1}. \tag{6}$$

Now let

$$\begin{aligned} P_1(r) &= |a_p| \prod_{n=1}^{p} (r - |b_n|) = c_1 r + \cdots + c_p r^p, \\ C' &= \sum_{n=1}^{p} |c_n| = |a_p| \prod_{n=1}^{p} (1 + |b_n|). \end{aligned} \tag{7}$$

Then by (4) and (7), $|P(re^{i\theta})| \geqq |P_1(r)|$. Also, since c_n is the same polynomial in the $|b_j|$ as a_n is in the b_j, and since only plus signs (or only minus signs) occur, we have $|c_n| \geqq |a_n|$, and $C' \geqq C$. Together with (7) this gives

$$|P(re^{i\theta})| \geqq C \left| \prod_{n=1}^{p} \frac{r - |b_n|}{1 + |b_n|} \right|. \tag{8}$$

Note that if $|b_i| > 1$ we only strengthen the inequality (8) if we replace $|b_i|$ by 1 in the quotients occurring there. But then (8) implies that

$$|P(re^{i\theta})| \geqq C2^{-p}\prod_{n=1}^{p}|r - b_n'|, \tag{9}$$

where $b_n' = |b_n|$ if $|b_n| \leqq 1$, and $b_n' = 1$ otherwise. Now choose any positive real number $x_0 < 1$. In the interval $(0, x_0)$ there is at least one point r_1 such that for $i \leqq p$,

$$|r_1 - b_i'| \geqq x_0/2^p,$$

and therefore by (9),

$$|P(r_1e^{i\theta})| \geqq Cx_0^p(4p)^{-p}, \tag{10}$$

and by (6) and (10)

$$|P(r_1e^{i\theta})| - |f(r_1e^{i\theta}) - P(r_1e^{i\theta})| \geqq Cx_0^p(4p)^{-p} - x_0^{p+1}(p+1)M.$$

This expression is maximum for $x_0 = Cp/(4p)^p(p+1)^2M$, its maximum value being C^{p+1}/M^pK^{p+1}, where $K = (p+1)((p+1)/p)^{p/(p+1)}(4p)^p$.

The polynomial $P(z)$ has n zeros in $|z| \leqq r_1$ with $n \leqq p$. Therefore by Rouché's theorem, $f(z)$ assumes in $|z| \leqq r_1$ every value in $|w| \leqq C^{p+1}/K^{p+1}M^p$ precisely n times. Therefore $D_p(0) \geqq C^{p+1}/K^{p+1}M^p$; that is,

$$\sum_{n=1}^{p}|a_n| \leqq KM^{p/(p+1)}[D_p(0)]^{1/(p+1)}$$

as was to be proved.

In the above proof we have assumed that $a_p \neq 0$. If $a_p = 0$ the proof is valid for some $n \leqq p$, and p can be reintroduced in the later stages, say in equation (8), where the product can be taken over p factors if some of the $|b_n|$ are allowed to vanish.

Proof of (3). We prove (3) by considering the class of functions

$$f(z) = Mz^p\frac{a - Mz}{M - az} \tag{11}$$

$$= az^p\frac{1 - (Mz/a)}{1 - (az/M)} \tag{12}$$

where $a < M$. These functions have been studied by the author in a previous note.[3] From (11) it is clear that $|f(z)| \leqq M$ in $|z| \leqq 1$, and that the bound is attained for $z=1$. It is also evident from (11) that the minimum of $|f(z)|$ on $|z| = r < a/M$ occurs at $z=r$. Thus $|f(z)| \geqq f(r)$ for $|z| = r < a/M$. By Rouché's theorem $f(z)$ covers the circle $|w| < f(r)$ precisely p times in $|z| < r$, since $f(z) = 0$ has p roots in this circle. By Rolle's theorem $f'(z)$ vanishes for some r_0 between $r=0$ and $r=a/M$. Then every value of $|w| \leqq f(r_0)$ except $w=f(r_0)$ is assumed precisely p times in $|z| \leqq r_0$, and $w=f(r_0)$ is assumed $p+1$ times. Therefore $D_p(0) = f(r_0)$. But by (12)

$$D_p(0) = f(r_0) < a(r_0)^p < a(a/M)^p = a^{p+1}/M^p. \tag{13}$$

Also by (12), the first term in the expansion of $f(z)$ is az^p so that the sum of the moduli of the first p coefficients is a. Thus by (13)

$$\sum_{n=1}^{p} | a_n | = a > M^{p/(p+1)} [D_p(0)]^{1/(p+1)}$$

which is the desired inequality (3).

The coefficient λ_p. We have left to prove that λ_p can be taken as 1; in other words, that

$$D_p(0) \leqq \sum_{n=1}^{p} | a_n | = C.$$

As before, let $P(z) = a_1 z + \cdots + a_p z^p$, and suppose that $D_p(0) > C$. Then $|f(z)| = C$ defines an analytic Jordan curve Γ in $|z| < 1$ and Γ contains the origin in its interior. On Γ we have $|P(z)| < C = |f(z)|$. Therefore by Rouché's theorem $f(z) - P(z)$ and $f(z)$ have the same number of zeros interior to Γ. But $f(z)$ has at most p zeros inside Γ since the image of the interior of Γ under $f(z)$ is an n-sheeted circle with $n \leqq p$. And $f(z) - P(z) \equiv a_{p+1} z^{p+1} + \cdots$ has at least $p+1$ zeros interior to Γ. By this contradiction we see that $D_p(0) \leqq C$ as we wished to prove.

This result is trivially the best possible, for if $f(z) \equiv az^p$, then $D_p(0) = a = C = M$.

HARVARD UNIVERSITY

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[3] L. H. Loomis, *The radius and modulus of n-valence for analytic functions whose first n−1 derivatives vanish at a point*, this Bulletin, vol. 46 (1940), pp. 406–501.

Two problems were considered in this note. One was the problem of the present paper for the class of functions whose first non-vanishing coefficient is a_p. The second was the problem of determining for this restricted class of functions the largest circle $|z| < r$ (the radius depending on p, M and $|a_p|$) in which all the functions are at most p-valent. The present note does not treat the corresponding problem for the more general class of functions considered here.

THE CAUCHY THEOREM FOR FUNCTIONS ON CLOSED SETS

PHILIP T. MAKER

The object of this paper is to extend the theorem of Cauchy to functions of a complex variable defined on any bounded closed set, E, by determining conditions on $f(z)$ in order that for certain coverings of E, C_n, and an extension of $f(z)$, $f^*(z)$, $\lim_{n\to\infty} \int_{c_n} f^*(z)dz=0$. It was suggested partly by the notion of a general monogenic function due to Trjitzinsky[1] and partly by the measure theory methods of Menchoff[2] and others, which succeed so well in lightening the restrictions on the real and imaginary parts of a complex function in order that $f(z)$ be regular.

Throughout this paper we shall consider only rectangles with sides parallel to the real and imaginary axes. A C-covering of a plane set F, denoted by C, will be a set of closed rectangles, possibly abutting, but nonoverlapping, which contain F. c will denote the boundary of C. The covering C_n is to be composed of rectangles R_{mn} so that $C_n=\sum_m R_{mn}$ $(m, n=1, 2, \cdots)$.

1. **The extension, $f^*(z)$.** If $u(P)$ is a positive continuous function defined on the closed and bounded set F in the plane, we shall let[3] $u^*(P)=\max_{Q\in F} u(Q)\{2-d(P, Q)/d(P, F)\}$ for P not in F, and $u^*(P)=u(P)$ for P in F, where $d(P, Q)$ denotes the distance from P to Q and $d(P, F)$ the distance from the set F to P. In general, if $u(P)$ is continuous, since $u(P)=(u(P)+|u(P)|)/2-(|u(P)|-u(P))/2$, that is, since $u(P)$ is the difference of two continuous positive functions, $u^*(P)$ will denote the extension of $u(P)$ obtained by extending as before these parts. If $f(z)$ $(=u(x, y)+iv(x, y))$ is defined on a bounded closed set and continuous, $f^*(z)$ will denote $u^*(x,y)+iv^*(x,y)$.

LEMMA 1. *If $u(P)$ is defined on a bounded closed set F and $|u(Q)-u(P)|<M(P)d(P, Q)$ where $M(P)$ is a finite function of P defined on F, then $|u^*(P)-u^*(Q)|<20\ M(P)\ d(P, Q)$, for P in F and Q arbitrary.*

Presented to the Society, December 29, 1941; received by the editors February 25, 1942.

[1] W. J. Trjitzinsky, *Théorie des Fonctions d'une Variable Complexe Définies sur des Ensembles Généraux*, Annales Scientifique de L'École Normale Supérieure, Paris, 1938, p. 120.

[2] D. Menchoff, *Les Conditions de Monogénéité*, Actualités Scientifiques et Industrielles, no. 329, Paris, 1936.

[3] S. Bochner, *Fourier Lectures*, 1936–1937, Princeton, p. 62.

PROOF. Consider first the case for which $u(P) \geqq 0$. If Q is any point not in D, if Q_0 is a point in F for which $d(Q, Q_0) = d(Q, F)$, and if, of the points S satisfying the inequality $d(S, Q) \leqq 2d(Q, Q_0)$, R is the point where the maximum of u is attained, then $u^*(Q_0) \leqq u^*(Q) \leqq u^*(R)$. Hence

$$\begin{aligned} |u^*(Q) - u^*(P)| &\leqq |u^*(Q) - u^*(R)| + |u^*(R) - u^*(P)| \\ &\leqq |u^*(Q_0) - u^*(P)| + 2|u^*(R) - u^*(P)| \\ &< M(P)d(Q_0, P) + 2M(P)d(R, P). \end{aligned}$$

It is easily verified that $d(Q_0, P) \leqq 2d(Q, P)$ and $d(R, P) \leqq 4d(Q, P)$, so that $|u^*(Q) - u^*(P)| < 10M(P)\, d(P, Q)$ and the lemma is proved for case of $u(P)$ positive. In the general case, $u(P) = (|u| + u)/2 - (|u| - u)/2 = g(P) - h(P)$, where g and h are positive functions, and satisfy the conditions of the lemma, so that for P in F and Q arbitrary $|g^*(Q) - g^*(P)|$ and $|h^*(Q) - h^*(P)|$ are each less than $10M(P)\, d(P, Q)$. Hence for $u^* = g^* - h^*$, it readily follows that $|u^*(Q) - u^*(P)| < 20M(P)\, d(P, Q)$, and the proof of the lemma is complete.

2. **Bounded derivatives.** We shall use the following fundamental lemma:[4]

LEMMA 2. *Let $w(x, y)$ be a real, continuous function defined in the square S, the sides of which are parallel to the coordinate axes, and let F be a closed set in S and such that*

$$\begin{aligned} |w(x + h, y) - w(x, y)| &\leqq M|h|, \\ |w(x, y + k) - w(x, y)| &\leqq M|k| \end{aligned}$$

for all points (x, y) in F and for all points $(x+h, y)$, $(x, y+k)$ of the square S, where M is a constant. Finally let R be the least rectangle with sides parallel to the axes containing F.[5]

Under these conditions[6] *the following inequalities hold:*

$$\left| \int_{x_1}^{x_2} [w(x, y_2) - w(x, y_1)]dx - \iint_F \frac{\partial w}{\partial y}\, dxdy \right| \leqq 5Mm(S - F),$$

$$\left| \int_{y_1}^{y_2} [w(x_2, y) - w(x_1, y)]dy - \iint_F \frac{\partial w}{\partial x}\, dxdy \right| \leqq 5Mm(S - F)$$

where (x_1, y_1), (x_2, y_1), (x_2, y_2) and (x_1, y_2), $(x_1 \leqq x_2,\ y_1 \leqq y_2)$ are the

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[4] For the proof of this lemma, cf. loc. cit., p. 10.

[5] The "least rectangle" may be only a segment or a point.

[6] The conditions on w imply the existence of the partial derivatives a.e. in F.

corners of the rectangle R and $m(S-F)$ is the measure of the set $S-F$, which is composed of points of S not in F.

THEOREM 1. *Let $f(z)$ $(=u(x, y)+iv(x, y))$ be defined on the bounded closed set E, and let R be a rectangle with sides parallel to the axes containing at least one point of E on each side. If (letting $F=E\cdot R$)*

(1) *for all z and $z+h$ in F, and a constant B,*

$$\left|\frac{f(z+h)-f(z)}{h}\right|<B,$$

(2) *the Cauchy-Riemann equations hold a.e. (almost everywhere) in F, where the partial derivatives of u and v exist,*[7] *then $\left|\int_r f^*(z)dz\right| <400Bm(S-F)$ where r is the boundary of R, and S is a square of least area containing R.*

PROOF. If $h=k+il$, condition (1) implies

$$\left|\frac{u(x+k, y+l)-u(x, y)}{h}\right|<B,$$

and a similar condition on $v(x, y)$, for every point z, and $z+h$, in F. According to Lemma 1,

$$\left|\frac{u^*(x+k, y+l)-u^*(x, y)}{h}\right|<20B,$$

for each point z in F, and $z+h$ in R. Hence by Lemma 2,

$$\left|\int_{x_1}^{x_2}[u^*(x, y_2)-u^*(x, y_1)]dx-\int\int_F \frac{\partial u^*}{\partial y}\,dxdy\right|<100Bm(S-F),$$

(x_1, y_1) and (x_2, y_2) being corners of R. Similar inequalities for $u^*(x, y)$ with respect to y, and $v^*(x, y)$ with respect to x and y also hold. But

$$\int f^*(z)dz=\int_r u^*dx-v^*dy+i\int_r v^*dx+u^*dy$$

and

$$\int_r u^*dx=-\int_{x_1}^{x_2}[u^*(x, y_2)-u^*(x, y_1)]dx.$$

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[7] This is in no way a further restriction on E, for almost all points of any measurable plane set are points of linear density for it both in the direction of the x-axis and in that of the y-axis.

The condition that the limits, $\lim_{h\to 0}(f(z+h)-f(z))/h$, as $z+h$ approaches z through points of E along either of two curves having non-collinear tangents at z, should be equal, is equivalent to the condition (2) in the presence of (1). For (1) and (2) imply that $f^*(z)$ is monogenic a.e. in E (Menchoff, loc. cit., Theorem 2, p. 27 and Theorem 5, p. 23).

Let $\epsilon' = \int_{x_1}^{x_2} [u^*(x, y_2) - u^*(x, y_1)]\, dx - \int\int_F (\partial u^*/\partial y) dy dx$. Then $\int_r u^* dx = -\int\int_F (\partial u^*/\partial y) dy dx - \epsilon'$. Taking into account similar reasoning for the other parts of $\int_r f^*(z) dz$ we have

$$\int_r f^*(z) dz = - \int\int_F \left(\frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x} \right) dx dy + i \int\int_F \left(\frac{\partial u^*}{\partial x} - \frac{\partial v^*}{\partial y} \right) dx dy + \epsilon$$

where $|\epsilon| < 400Bm(S-F)$. Since the Cauchy-Riemann equations hold a.e. in F, $\int_r f^*(z)dz = \epsilon$, and the proof of the theorem is complete.

COROLLARY. *Let $f(z)$ be defined on the bounded closed set E with a bounded derivative there. Let $S = \sum_m S_m$ be a C-covering of E by squares with R_m the least rectangle within S_m containing $S_m \cdot E$. Then there is a constant B for which $\sum_m |\int_{r_m} f^*(z)dz| < 400Bm(S-E)$, and if $S \to E$, $C = \sum_m R_m \to E$, and* $\lim_{C \to E} \int_c f^*(z)dz = 0$.

3. **Derivatives finite, except for a denumerable set.** We prove this theorem:

THEOREM 2. *If $f(z)$ is defined and continuous on the bounded closed set E, and if, except for a denumerable number of points,* $\limsup_{h \to 0} |(f(z+h) - f(z))/h| < \infty$, *and the Cauchy-Riemann equations hold a.e. where the partial derivatives of u and v exist, then there is a sequence of C-coverings, $\{C_n\}$, for which* $\lim_{n \to \infty} \sum_m |\int_{r_{mn}} f^*(z)dz| = 0$.

PROOF. Define $I(C) = \sum_m |\int_{r_m} f^*(z)dz|$. If for every point z of E there is a neighborhood $N(z)$ such that for every closed subset of E in N, there is a sequence of coverings $\{C_n\}$ for which $\lim_{n \to \infty} I(C_n) = 0$, then by the Heine-Borel theorem there exists a sequence of coverings of E with the property mentioned in the theorem. The proof will be complete therefore, if we show that there is such a neighborhood for each point of E. Let P be those points of E such that in every neighborhood of z there is a closed subset of E for which there is no sequence of C-coverings, $\{C_n\}$, for which $\lim_{n \to \infty} I(C_n) = 0$. We shall assume that P is not empty and show that this leads to an absurdity.

Let P_m $(m = 1, 2, \cdots)$ be the points of P for which each of the absolute values,

$$|u^*(x+k, y) - u^*(x, y)|, \quad |v^*(x+k, y) - v^*(x, y)|,$$
$$|u^*(x, y+k) - u^*(x, y)|, \quad |v^*(x, y+k) - v^*(x, y)|$$

is less than or equal to $m|k|$ for $|k| \leq 1/m$, k a real number. Since

u^* and v^* are continuous and P is closed, P_m is closed. Since at each point of E, except for a denumerable set H, the partial derivates are finite, $P=\sum_m P_m+P\cdot H$. By Baire's[3] theorem, there is an isolated point of P in H, or for some integer N there is a point z_0 in P, the center of a square S which contains only points of P which are in P_N. The former alternative is quickly dismissed as impossible; we proceed on the basis of the latter, and let F be any closed subset of $E\cdot S$. Subdivide the sides of S into n equal parts, $n>2N$, and obtain the squares S_j $(j=1, 2, \cdots, n^2)$. ϵ being given, choose n so great that the squares $\overline{S}_j$ which contain points of $F\cdot P$ satisfy the inequality, $m(\sum_j\overline{S}_j-P\cdot S)<\epsilon/800N$. If $\overline{R}_j$ is the least rectangle containing $P\cdot\overline{S}_j$, and $\overline{C}$ is the covering $\sum_j\overline{R}_j$, by Theorem 1, $I(\overline{C})<400\, N\sum_j m(\overline{S}_j-P\ \overline{S}_j)<\epsilon/2$. Since $I(R)$ is a continuous function of r, $\overline{C}$ may be extended by the addition of more small rectangles, so that, if C' is the new covering, $I(C')$ remains less than $\epsilon/2$, but so that the points of $F\cdot P$ are inner points of the covering. The part of F not already covered (denote it by G) is such that its closure contains only points z of F for which there is some neighborhood $N(z)$ with the property that every closed subset of F in N can be C-covered, say by $C_n(z)$ $(n=1, 2, \cdots)$ and $\lim_{n\to\infty} I(C_n(z))=0$. Let $S(z)$ be a square with z as center entirely within $N(z)$. Of these squares a finite number, k, cover G, and within each of these is a covering, $C(z)$, of G for which $I(C(z))<\epsilon/2k$. Hence G is C-covered by a covering C for which $I(C)<\epsilon/2$. F is therefore C-covered by $C+C'$ for which $I(C+C')<\epsilon$, so that z_0 cannot belong to P, contrary to assumption. This completes the proof of Theorem 2.

COROLLARY. *If $f(z)$, defined on the bounded closed set E and continuous there, has a derivative at each point except at most a denumerable set, there is a sequence of C-coverings of E with E as their limit for which* $\lim_{n\to\infty}\int_{C_n} f^*(z)dz=0$.

DUKE UNIVERSITY

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[3] S. Saks, *The Theory of the Integral*, New York, 1937, p. 54.

ON VALUE REGIONS OF CONTINUED FRACTIONS

WALTER LEIGHTON AND W. J. THRON

The results of this paper are contained in the following theorem.

THEOREM 1. *If the elements* $a_1, a_2, \cdots$ *of the continued fraction*

$$(1) \qquad 1 + \frac{a_1}{1} + \frac{a_2}{1} + \cdots$$

lie in the parabola

$$(2) \qquad \rho \leqq \frac{2d(1-d)}{1-\cos\theta}, \qquad 1/2 \leqq d < 1,$$

then the approximants of the continued fraction (1) *lie in the hyperbolic region*

$$(3) \qquad R > \frac{2d(1-d)}{1-2d+\cos\phi}, \qquad -\beta < \phi < \beta,$$

where $\beta = \operatorname{arc\,cos}(2d-1)$. *If* z *is any value on the boundary of the region* (3), *there exists one and only one continued fraction of the form* (1), *with elements in the parabola* (2), *converging to* z, *namely:*[1]

$$(4) \qquad 1 + \frac{a}{1} + \frac{\bar{a}}{1} + \frac{a}{1} + \cdots$$

where $a = (z-1)\bar{z}$ *is a value on the boundary of the parabola* (2).

For the case $d = 1/2$, Scott and Wall [3] determined the value region of the approximants and Paydon [1] established the uniqueness property of (4) for that case.

A convergence criterion due to Scott and Wall [2] insures the convergence of the continued fraction (1) if in addition to the conditions of Theorem 1 it is required that the series $\sum |b_n|$ diverges, where $b_1 = 1/a_1$, $b_{n+1} = 1/a_{n+1}b_n$. The value of such a continued fraction must lie in the closure of the region (3). Finally it follows from Theorem 1 that all values in the region (3) are assumed by a continued fraction of the form (4). The following result is now seen to be a consequence of Theorem 1.

Presented to the Society, April 11, 1942; received by the editors December 7, 1941, and, in revised form, March 27, 1942.

[1] Here and elsewhere in the paper a bar over a number means the complex conjugate of the number.

THEOREM 2. *If the elements* a_n $(n \geqq 1)$ *of the continued fraction* (1) *lie in the parabola* (2) *and if in addition* $\sum |b_n|$ *diverges, where* $b_1 = 1/a_1$, $b_{n+1} = 1/a_{n+1}b_n$, *the continued fraction* (1) *converges, and its value lies in the closure of the hyperbolic region* (3). *Further, every value in* (3) *is taken on by at least one convergent continued fraction* (1) *with elements in* (2).

We now prove the first part of Theorem 1, namely, that the approximants of (1) lie in the region (3) if the elements of the continued fraction lie in the parabola (2). We first note, that if z lies in (3) then $w = re^{i\psi} = z - 1$ satisfies the relation

$$\begin{aligned} r &< \frac{-2d(1-d)}{1-2d+\cos\psi}, \qquad && \beta < \psi < 2\pi - \beta; \\ r &< \infty, && -\beta < \psi < \beta. \end{aligned} \tag{5}$$

The proof is by induction. $A_1/B_1 - 1 = a_1$ satisfies relation (5), if a_1 lies in the parabola (2), as

$$\frac{2d(1-d)}{1-\cos\phi} \leqq \frac{-2d(1-d)}{1-2d+\cos\phi}$$

for $\beta < \phi < 2\pi - \beta$. Hence A_1/B_1 lies in the region (4). To complete the proof it is sufficient to show that $w = z - 1 = a/z'$ lies in the region defined by relation (5) if a and z' are any arbitrary complex numbers in the regions (2) and (3), respectively. To this end we prove the following lemma.

LEMMA. *The number*

$$w = \frac{a}{z} = \frac{\rho e^{i\phi}}{Re^{i\theta}} = \frac{\rho}{R} e^{i(\phi-\theta)}$$

satisfies relation (5), *if* a *and* z *are arbitrary complex numbers in the regions defined by relations* (2) *and* (3), *respectively.*

We shall choose a and z in such a way, that they maximize $|w|$ for any arbitrary fixed $\arg w = \alpha$, $\beta < \alpha < 2\pi - \beta$. It is clear that the numbers a and z must then be chosen on the boundaries of the regions over which they are allowed to vary, and we then have

$$|w| = \frac{\rho}{R} = \frac{1-2d+\cos\theta}{1-\cos\phi}, \qquad \phi - \theta = \alpha.$$

It follows from elementary considerations that if $|w|$ is to be a maximum, θ must satisfy the relation

$$\sin\theta + \sin\alpha + (1-2d)\sin(\alpha+\theta) = 0.$$

This is equivalent to

$$2\sin\frac{\alpha+\theta}{2}\left[\cos\frac{\alpha-\theta}{2} + (1-2d)\cos\frac{\alpha+\theta}{2}\right] = 0.$$

The solution $\theta = -\alpha$ is excluded since the ranges of α and $-\theta$ have no values in common. It remains to find the value of $|w|$ when

$$\cos\frac{\alpha-\theta}{2} + (1-2d)\cos\frac{\alpha+\theta}{2} = 0. \tag{6}$$

We shall show that (6) is equivalent to the two relations $|w| = -2d(1-d)/(1-2d+\cos\alpha)$ and $z = 1+\bar{w}$. If these two relations are satisfied the sine law leads to

$$\frac{\sin(\pi-\alpha)}{\sin(-\theta)} = \frac{\sin\alpha}{-\sin\theta} = -\frac{1-2d+\cos\alpha}{1-2d+\cos\theta}.$$

This gives

$$(1-2d)(\sin\alpha - \sin\theta) + \sin(\alpha-\theta) = 0,$$

or

$$2\sin\frac{\alpha-\theta}{2}\left[\cos\frac{\alpha-\theta}{2} + (1-2d)\cos\frac{\alpha+\theta}{2}\right] = 0.$$

Since θ cannot equal α, the only solution of this equation is that of equation (6). Thus the lemma is proved.

This completes the proof of the first part of Theorem 1. It remains to show that any z on the boundary of (3) is uniquely expressible as a continued fraction of the form (4).

From the proof of the lemma it follows that a_1 must lie on the boundary of (2) for z to lie on the boundary of (3). It was further shown that

$$z = 1 + \frac{a_1}{1} + \frac{a_2}{1} + \cdots$$

lies on the boundary only if

$$z = 1 + \frac{a_2}{1} + \frac{a_3}{1} + \cdots;$$

but then z lies on the boundary and hence we must have

$$z = 1 + \frac{a_{2n-1}}{\bar{z}}, \qquad \bar{z} = 1 + \frac{a_{2n}}{z}, \qquad n \geqq 1.$$

This gives

$$a_{2n-1} = (z - 1)\bar{z},$$
$$a_{2n} = (\bar{z} - 1)z = \bar{a}_{2n-1},$$

and it is easily seen that all a_n lie on the boundary of the parabola. The theorem is now completely proved.

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THE RICE INSTITUTE

A TABLE OF COEFFICIENTS FOR NUMERICAL DIFFERENTIATION

ARNOLD N. LOWAN, HERBERT E. SALZER AND ABRAHAM HILLMAN[1]

The following table lists the coefficients $A_{m,s}$ for $m=1, 2, \cdots, 20$ and $s=m, \cdots, 20$ in Markoff's formula for the mth derivative in terms of advancing differences, namely

$$\omega^m f^{(m)}(x) = \sum_{s=m}^{n-1} (-1)^{m+s} A_{m,s}\Delta^s f(x) + (-1)^{m+n}\omega^n A_{m,n} f^{(n)}(\xi).$$

In this formula ω is the tabular interval and

$$A_{m,s} = (-1)^{m+s} m B_{s-m}^{(s)}/s(s-m)!$$

and $B_{s-m}^{(s)}$ is the $(s-m)$th Bernoulli number of the sth order.

Presented to the Society, April 4, 1942 under the title *Coefficients of differences in the expansion of derivatives in terms of advancing differences*; received by the editors March 7, 1942.

[1] The results reported here were obtained in the course of the work done by the Mathematical Tables Project, Work Projects Administration, New York City.

If a function has been tabulated to sufficiently great accuracy and for some suitable interval of the argument along the real axis, the accompanying table may be used to generate the values of the derivatives which in turn may be employed to generate the values of the function in the complex plane within a region where the function is analytic.

The coefficients were computed from the recurrence formula

$$sA_{m,s} = (s-1)A_{m,s-1} + mA_{m-1,s-1}$$

and checked by independent calculations using the identity

$$x(x+1)(x+2)\cdots(x+s-1) \equiv s!\sum_{j=1}^{s} A_{j,s}x^j/j!.$$

From the identity

$$(x+x^2/2+x^3/3+\cdots)^m \equiv A_{m,m}x^m + A_{m,m+1}x^{m+1} + \cdots$$

it was discovered that a prime p is not effectively present in the denominator of an $A_{m,s}$ for which $s<m+p-1$. The cancellation of prime factors in accordance with this rule was a further check on the accuracy of the work.

The Markoff formula is used at the beginning and end of a table where advancing differences are the only types available. For a full discussion see L. M. Milne-Thomson, *The Calculus of Finite Differences*, chap. 7, pp. 157–159. According to Milne-Thomson the relative simplicity of the remainder term is another advantage over central difference formulae.

Comparison of the Markoff coefficients with central difference coefficients shows the latter to be much smaller and obviously more convenient for obtaining the derivatives of a polynomial sufficiently far away from the ends of a table. However for many important functions in applied mathematics such as Bessel, error, and gamma functions, use of the Markoff formula for a polynomial approximation of some fixed degree might yield a smaller total error due to the particular form of its remainder term.

The first few coefficients of the various formulae may be found in H. T. Davis, *Table of the Higher Mathematical Functions*, vol. 1, pp. 73–77; Whittaker and Robinson, *Calculus of Observations*, pp. 62–65, and in an article by W. S. Bickley *Numerical differentiation near the limits of a difference table*, Philosophical Magazine, (7), vol. 33 (1942), pp. 12–14. (This article lists coefficients of the first 12 derivatives up to those of the 12th difference.)

COEFFICIENTS $A_{m,s}$ IN MARKOFF'S EXPANSION

$m \diagdown s$	1	2	3.	4	5	6	7	8	9	10	11	12	13	14
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$	$\frac{1}{12}$	$\frac{1}{13}$	$\frac{1}{14}$
2		1	1	$\frac{11}{12}$	$\frac{5}{6}$	$\frac{137}{180}$	$\frac{7}{10}$	$\frac{363}{560}$	$\frac{761}{1260}$	$\frac{7129}{12600}$	$\frac{671}{1260}$	$\frac{83711}{166320}$	$\frac{6617}{13860}$	$\frac{1145993}{2522520}$
3			1	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{15}{8}$	$\frac{29}{15}$	$\frac{469}{240}$	$\frac{29531}{15120}$	$\frac{1303}{672}$	$\frac{16103}{8400}$	$\frac{190553}{100800}$	$\frac{128977}{69300}$	$\frac{9061}{4950}$
4				1	2	$\frac{17}{6}$	$\frac{7}{2}$	$\frac{967}{240}$	$\frac{89}{20}$	$\frac{4523}{945}$	$\frac{7645}{1512}$	$\frac{341747}{64800}$	$\frac{412009}{75600}$	$\frac{9301169}{1663200}$
5					1	$\frac{5}{2}$	$\frac{25}{6}$	$\frac{35}{6}$	$\frac{1069}{144}$	$\frac{285}{32}$	$\frac{31063}{3024}$	$\frac{139381}{12096}$	$\frac{1148963}{90720}$	$\frac{355277}{25920}$
6						1	3	$\frac{23}{4}$	9	$\frac{3013}{240}$	$\frac{781}{48}$	$\frac{242537}{12096}$	$\frac{48035}{2016}$	$\frac{1666393}{60480}$
7							1	$\frac{7}{2}$	$\frac{91}{12}$	$\frac{105}{8}$	$\frac{4781}{240}$	$\frac{13321}{480}$	$\frac{314617}{8640}$	$\frac{790153}{17280}$
8								1	4	$\frac{29}{3}$	$\frac{55}{3}$	$\frac{10831}{360}$	$\frac{897}{20}$	$\frac{944311}{15120}$
9									1	$\frac{9}{2}$	12	$\frac{99}{4}$	$\frac{1747}{40}$	$\frac{5551}{80}$
10										1	5	$\frac{175}{12}$	$\frac{65}{2}$	$\frac{491}{8}$
11											1	$\frac{11}{2}$	$\frac{209}{12}$	$\frac{1001}{24}$
12												1	6	$\frac{41}{2}$
13													1	$\frac{13}{2}$
14														1

MATHEMATICAL TABLES PROJECT
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$\omega^m D^m f(x) \sim \sum_{s=m}^{\infty} (-1)^{m+s} A_{m,s} \Delta^s f(x)$

15	16	17	18	19	20	s / m
$\frac{1}{15}$	$\frac{1}{16}$	$\frac{1}{17}$	$\frac{1}{18}$	$\frac{1}{19}$	$\frac{1}{20}$	1
$\frac{1171733}{2702700}$	$\frac{1195757}{2882880}$	$\frac{143327}{360360}$	$\frac{42142223}{110270160}$	$\frac{751279}{2042040}$	$\frac{275295799}{775975200}$	2
$\frac{30946717}{17199000}$	$\frac{39646461}{22422400}$	$\frac{58433327}{33633600}$	$\frac{344499373}{201801600}$	$\frac{784809203}{467812800}$	$\frac{169704792667}{102918816000}$	3
$\frac{406841}{71280}$	$\frac{35118025721}{6054048000}$	$\frac{4446371981}{756756000}$	$\frac{80847323107}{13621608000}$	$\frac{2263547729}{378378000}$	$\frac{32262100943}{5360355000}$	4
$\frac{21939781}{1496880}$	$\frac{2065639}{133056}$	$\frac{2195261857}{134534400}$	$\frac{371446039969}{21794572800}$	$\frac{27566944753}{1556755200}$	$\frac{31938836201}{1743565824}$	5
$\frac{22463}{720}$	$\frac{277382447}{7983360}$	$\frac{38101097}{997920}$	$\frac{1356664151597}{32691859200}$	$\frac{162356544377}{3632428800}$	$\frac{694142313941}{14529715200}$	6
$\frac{899683}{16200}$	$\frac{2271089}{34560}$	$\frac{86853967}{1140480}$	$\frac{13195009}{152064}$	$\frac{227663026369}{2335132800}$	$\frac{2022480780283}{18681062400}$	7
$\frac{35717}{432}$	$\frac{54576553}{518400}$	$\frac{8424673}{64800}$	$\frac{334947281}{2138400}$	$\frac{9764119}{52800}$	$\frac{5013017410969}{23351328000}$	8
$\frac{515261}{5040}$	$\frac{23915}{168}$	$\frac{76492463}{403200}$	$\frac{21878439}{89600}$	$\frac{4065163957}{13305600}$	$\frac{3975325483}{10644480}$	9
$\frac{2485}{24}$	$\frac{324509}{2016}$	$\frac{59279}{252}$	$\frac{79243781}{241920}$	$\frac{11795941}{26880}$	$\frac{6063698587}{10644480}$	10
$\frac{30217}{360}$	$\frac{1199}{8}$	$\frac{494351}{2016}$	$\frac{1513391}{4032}$	$\frac{18843187}{34560}$	$\frac{367394203}{483840}$	11
$\frac{105}{2}$	$\frac{26921}{240}$	$\frac{6341}{30}$	$\frac{5490071}{15120}$	$\frac{976163}{1680}$	$\frac{354467473}{403200}$	12
$\frac{143}{6}$	65	$\frac{35269}{240}$	$\frac{46631}{160}$	$\frac{3965533}{7560}$	$\frac{10596053}{12096}$	13
7	$\frac{329}{12}$	$\frac{238}{3}$	$\frac{136241}{720}$	$\frac{31521}{80}$	$\frac{6406481}{8640}$	14

WORK PROJECTS ADMINISTRATION
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COEFFICIENTS $A_{m,s}$ IN MARKOFF'S EXPANSION
$\omega^m D^m f(x) \sim \sum_{s=m}^{\infty} (-1)^{m+s} A_{m,s} \Delta^s f(x)$

s m	15	16	17	18	19	20
15	1	$\frac{15}{2}$	$\frac{125}{4}$	$\frac{765}{8}$	$\frac{11519}{48}$	$\frac{50255}{96}$
16		1	8	$\frac{106}{3}$	114	$\frac{18017}{60}$
17			1	$\frac{17}{2}$	$\frac{119}{3}$	$\frac{1615}{12}$
18				1	9	$\frac{117}{4}$
19					1	$\frac{19}{2}$
20						1

NEW YORK CITY

ON MAJORANTS OF SUBHARMONIC AND ANALYTIC FUNCTIONS

FRANTIŠEK WOLF

This paper represents a different approach to a whole group of problems connected with majorants of subharmonic functions. The same method has been used previously in order to prove a generalization of the Phragmén-Lindelöf theorem.[1] It seems that the best approach is to prove first Lemma 4, and then the most important results are easily deducible. Corollary 6 is a generalization of a result of N. Levinson.[2] His theorem has made me realize the importance of these results.

LEMMA 1. *If* (i) $0 < f(x) \leq 1$ *and* (ii) $\int_a^b \log f(x)\cdot dx$ *is finite, then*

$$\int_a^b \log\left|\int_\xi^x f(y)dy\right| dx \tag{1}$$

is a continuous function of ξ *in* (a, b).

We first suppose that $f(x)$ is non-decreasing and that $(0, 1) = (a, b)$. We get

$$\int_0^x f(y)dy > \int_{x/2}^x f(y)dy \geq (x/2)f(x/2).$$

Hence

$$\begin{aligned} \int_0^1 \log\left(\int_0^x f(y)dy\right)dx &> \int_0^1 \log\,(x/2)dx + \int_0^1 \log f(x/2)dx \\ &> 2\int_0^1 \log x\cdot dx + 2\int_0^1 \log f(x)dx \\ &= -2 + 2\int_0^1 \log f(x)\cdot dx. \end{aligned} \tag{2}$$

If $f(x)$ is replaced by $f(a+(b-a)x)$, we obtain

Presented to the Society, November 22, 1941 under the title *On majorants of analytic functions*; received by the editors March 2, 1942.

[1] Cf. the end of this paper and Journal of the London Mathematical Society, vol. 14 (1939), p. 208.

[2] *Gap and Density Theorems*, American Mathematical Society Colloquium Publications, vol. 26, 1940, p. 127, Theorem 43.

$$\int_a^b \log\left(\int_a^x f(y)dy\right)dx \geqq -2(b-a) + (b-a)\log(b-a) \tag{3}$$
$$+ 2\int_a^b \log f(x)\cdot dx.$$

Next, if $f(x)$ is general, we form the rearranged non-decreasing function $\bar{f}(x)$ for which

$$\text{meas}\, \underset{x}{E}\, [f(x) \leqq y] = \text{meas}\, \underset{x}{E}\, [\bar{f}(x) \leqq y]$$

for all y. We know that[3]

$$\int_0^1 \log f(x)\ dx = \int_0^1 \log \bar{f}(x)\cdot dx,$$

and that

$$\left|\int_{x_0}^x f(y)\ dy\right| \geqq \int_0^{|x-x_0|} \bar{f}(y)\cdot dy.$$

Hence, if $x_0 \subset (0, 1)$,

$$\int_0^1 \log\left|\int_{x_0}^x f(y)\cdot dy\right|\cdot dx$$
$$\geqq \int_0^1 \log\left(\int_0^{|x-x_0|} \bar{f}(y)\cdot dy\right)\cdot dx$$
$$= \int_0^{x_0} \log\left(\int_0^{x_0-x} \bar{f}(y)\cdot dy\right)\cdot dx + \int_{x_0}^1 \log\left(\int_0^{x-x_0} \bar{f}(y)\ dy\right)dx$$
$$= \int_0^{x_0} \log\left(\int_0^x \bar{f}(y)\cdot dy\right) dx + \int_0^{1-x_0} \log\left(\int_0^x \bar{f}(y)dy\right)dx$$
$$\geqq 2\int_0^1 \log\left(\int_0^x \bar{f}(y)\cdot dy\right)\cdot dx.$$

This, with (2) gives

$$\int_0^1 \log\left(\int_{x_0}^x f(y)\cdot dy\right)\cdot dx \geqq -4 + 4\int_0^1 \log \bar{f}(x)\cdot dx$$
$$= -4 + 4\int_0^1 \log f(x)\cdot dx.$$

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[3] Hardy, Littlewood, Pólya, *Inequalities*, Cambridge, 1934, p. 276.

In a similar fashion we get, for $x_0 \subset (a, b)$,

$$\begin{aligned}\int_a^b \log\left|\int_{x_0}^x f(y)dy\right|dx \geqq &-4(b-a)+2(b-a)\log(b-a)\\ &+4\int_a^b \log f(x)\cdot dx.\end{aligned}\tag{4}$$

If $\epsilon > 0$ is given, we can find a $\delta > 0$ such that

$$8\delta - 4\delta\log\delta - 4\int_{x_0-\delta}^{x_0+\delta}\log f(x)\cdot dx < \epsilon/3.$$

Let us denote by J_1 the common part of (a, b) and $(x_0-\delta, x_0+\delta)$, and by J_2 the rest of (a, b). Then, by (3), for $\xi \subset J_1$

$$\begin{aligned}0 > \int_{J_1}\log\left|\int_\xi^x f(y)\ dy\right|\ dx \geqq &-8\delta + 4\delta\log\delta\\ &+4\int_{x_0-\delta}^{x_0+\delta}\log f(x)\cdot dx > -\epsilon/3.\end{aligned}$$

Further

$$g_2(\xi) = \int_{J_2}\log\left|\int_\xi^x f(y)\cdot dy\right|\cdot dx$$

is a continuous function for $\xi = x_0$ and, hence, there is a $\delta_1 < \delta$ such that

$$|g_2(\xi) - g_2(x_0)| \leqq \epsilon/3 \qquad \text{for } |\xi - x_0| < \delta_1.$$

Hence, if we call the integral (1) $g(\xi)$, then

$$|g(\xi) - g(x_0)| < \epsilon \qquad \text{for } |\xi - x_0| < \delta_1.$$

This shows that (1) is a continuous function of ξ at an arbitrary point $x_0 \subset (a, b)$.

LEMMA 2. *Given a non-negative $\psi(x) \subset L$, there is a domain D, bounded by two continuous curves*

$$C_1 \equiv y = g_1(x), \qquad C_2 \equiv y = g_2(x)$$

and two straight lines $x = x_0 - a$, $x = x_0 + a$, such that, if $x + iy = f(re^{i\theta})$ represents D, conformally on the unit circle $r < 1$, then

$$\left|\frac{dx}{d\theta}\right|_{r=1} \geqq \text{const. } \exp\,[\psi(x)]\tag{5}$$

on C_1 and C_2.

Further the rectangle

$$R_0\colon\ [\,|y-y_0| < a \log 3/2,\ |x-x_0| < a\,] \tag{6}$$

is interior to D and

$$|y-y_0| \leqq c(a), \qquad |x-x_0| \leqq a, \qquad \textit{with} \lim_{a\to 0} c(a) = 0,$$

contains D.

There is no loss of generality in supposing $x_0 = y_0 = 0$. We define D by constructing its conformal representation on the unit circle. First we define the boundary function of the harmonic function $x(r, \theta)$ so that it satisfies the above condition for the derivative. We define it as the inverse function of

$$\theta(x) = b\int_{-a}^{x} e^{-\psi(t)}dt \tag{7}$$

and, for later convenience, we take b such that

$$b\int_{-a}^{a} e^{-\psi(t)}dt = \theta_0 < 2 \operatorname{arc\,cos} 3/4 < \pi/3. \tag{8}$$

Then the inverse function $x(1, \theta)$, defined in $(0, \theta_0)$ satisfies condition (5) and is continuous and decreasing. Further, we define $x(1, \theta) = a$, for $\theta \subset (\theta_0, \pi)$; $x(1, \pi+\theta) = x(1, \theta_0-\theta)$ for $\theta \subset (0, \theta_0)$, and finally $x(1, \theta) = -a$ for $\theta \subset (\pi+\theta_0, 2\pi)$. In the interval $(\pi, \pi+\theta_0)$ condition (5) is again satisfied. Now, the boundary function is completely defined and

$$x(r, \theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{x(1, \phi)d\phi}{1-2r\cos(\theta-\phi)+r^2}.$$

The conjugate harmonic function is

$$y(r, \theta) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\, x(1, \phi)d\phi.$$

By partial integration

$$y(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|1-re^{i(\theta-\phi)}\right| dx(1, \phi).$$

Since x is constant in some intervals, we get, using the above definition of $x(1, \phi)$,

$$y(r,\theta)=\frac{1}{2\pi}\int_0^{\theta_0}\Big[\log\left|1-re^{i(\theta-\phi)}\right| \tag{9}$$
$$-\log\left|1+re^{i(\theta-\theta_0+\phi)}\right|\Big]dx(1,\phi).$$

Now, the required conformal representation is given by $x(r,\theta)+iy(r,\theta)$. The boundary curve of D in the parametric form is $x(1,\theta)+iy(1,\theta)$, and we shall investigate it.

If $\theta\subset(0,\theta_0)$, then $x(1,\theta)$ runs from $-a$ to $+a$ and, since $\left|1-e^{i(\theta-\phi)}\right|<1$ for $\theta,\phi\in(0,\theta_0)\subset(0,\pi/3)$ (cf. (8)),

$$\begin{aligned}y(1,\theta) &\leqq -\frac{1}{2\pi}\int_0^{\theta_0}\log\left|1+e^{i(\theta-\theta_0+\phi)}\right|dx\\ &\leqq -\frac{1}{2\pi}\log\left|1+e^{i\theta_0}\right|\int_{-a}^{a}dx\\ &= -(a/\pi)\log 2\cos\theta_0/2 < -(a/\pi)\log 3/2.\end{aligned}$$

The first term I_1 on the right in (9) is more difficult to dispose of. We have

$$\begin{aligned}I_1 &= \frac{1}{2\pi}\int_0^{\theta_0}\log\left|2\sin\frac{\theta-\phi}{2}\right|\cdot dx(1,\phi)\\ &= \frac{1}{2\pi}\int_0^{\theta_0}\log\frac{2\sin(\theta-\phi)/2}{\theta-\phi}\,dx(1,\phi)\\ &\quad+\frac{1}{2\pi}\int_0^{\theta_0}\log|\theta-\phi|\cdot dx(1,\phi)=I_1'+I_1'',\end{aligned}$$

Here, I_1' is evidently a continuous function of θ and

$$0>I_1>-\frac{a}{\pi}\log\frac{2\sin\theta_0/2}{\theta_0}\cdot$$

By means of (7), we change variables in I_1'' and obtain

$$I_1''=\frac{1}{2\pi}\int_{-a}^{a}\log\left|b\int_{\xi}^{x(\theta)}e^{-\psi(\alpha)}\cdot d\alpha\right|\cdot d\xi.$$

By Lemma 2, this is a continuous function of x and therefore also of θ and from (3) we get

$$0>I_1''>-\frac{2a}{\pi}+\frac{a\log 2a}{\pi}+\frac{1}{\pi}\int_{-a}^{a}\log\left|be^{-\psi(x)}\right|dx.$$

If we combine the last inequalities and introduce the abbreviation $c(a)$, we get

$$-\frac{a}{\pi}\log 3/2 > y(1,\theta) > -c(a), \qquad \theta \subset (0,\theta_0),$$

with

$$\lim_{a\to 0} c(a) = 0.$$

From (9) it is easy to deduce that

$$y(r,\theta+\pi) = -y(r,\theta_0-\theta),$$

and we obtain

$$(a/\pi)\log 3/2 < y(1,\theta) \leqq c(a), \text{ for } \theta \subset (\pi, \pi+\theta_0).$$

Using the same notation we shall prove the following lemma.

LEMMA 3. *If $\psi(x_0-a)$, $\psi(x_0+a)$ are finite, then every subharmonic function $\alpha(x, y)$, defined in D, which satisfies*

$$\alpha(x,y) \leqq e^{\psi(x)}, \qquad |x| < a, \tag{10}$$

has an upper bound in $R[|y| < a\cdot\log 3/2,\ |x| < a]$ which depends only on $\psi(x)$.

If we represent D conformally on the unit circle C, we get from $\alpha(x, y)$ a subharmonic function $\alpha(r, \theta)$ defined in C. There, it must be less than any harmonic function with boundary values not less than those of $\alpha(r, \theta)$. By (10), these are not greater than $\exp\{\psi(x(1,\theta))\}$, where $x(1, \theta)$ has been defined in the preceding lemma. Such a harmonic function is the Poisson integral of $\exp\{\psi(x(1,\theta))\}$. We have to show that this Poisson integral is not identically equal to ∞. A sufficient condition is

$$\int_0^{2\pi} \exp\{\psi(x(1,\theta))\}\cdot d\theta < \infty.$$

In view of the definition of $x(1, \theta)$, this integral is obviously less than

$$\left|\int_{-a}^{a} e^{\psi(x)}\frac{d\theta}{dx}dx\right| + \int_{\theta_0}^{\pi} e^{\psi(a)}d\theta + \left|\int_{a}^{-a} e^{\psi(x)}\frac{d\theta}{dx}dx\right| + \int_{\pi+\theta_0}^{2\pi} e^{\psi(-a)}d\theta,$$

and using (5), also less than

$$2b\int_{-a}^{a} e^{\psi(x)}e^{-\psi(x)}dx + \pi(e^{\psi(a)} + e^{\psi(-a)}) \leqq 4ab + \pi(e^{\psi(a)} + e^{\psi(-a)}) < \infty.$$

Hence $\alpha(x, y)$ is, inside of D, less than a finite harmonic function depending only on $\psi(x)$. In the domain D, completely interior to R, it is therefore bounded from above by a constant depending only on $\psi(x)$.

THEOREM. *If* (i) $\alpha(x, y)$ *is subharmonic in* $R[|x| < c,\ |y| < d]$ *and*

(ii) $\alpha(x, y) < e^{\psi(x)}$, $\psi(x) \subset L$, $x \subset (-c, c)$, $\psi(-c) < \infty$, $\psi(c) < \infty$,

then for any δ, *such that* $0 < \delta < d$, *there is an upper bound* C *for* $\alpha(x, y)$ *in* $D_0[|x| < c,\ |y| < d-\delta]$ *dependent only on* δ *and* $\psi(x)$, *but independent of the particular* $\alpha(x, y)$.

In view of the Lemma 3, it is sufficient to show that there is an a and a finite number of domains D_k satisfying the conditions of this lemma, and, such that D_k contains (cf. (6)) $R_k[|x-x_k| < a,\ |y-y_k| \leq c(a)]$, is contained in R, and that $\sum R_k$ covers completely D_0. Since $\alpha(x, y)$ has a finite upper bound in every R_k, it must be so also in D_0. And the upper bound will depend only on $\psi(x)$ and D_0.

We determine $a > 0$ such that $c(a) - a \log 3/2 < \delta/2$. If we define $D^*[|x| < c,\ |y| < d - \delta/2]$, then $D_0 \subset D^* \subset R$, and, if $R_k \subset D^*$, then the corresponding $D_k \subset R$ (cf. (3.2)). Since R_k can be any rectangle, of the above size, in D^* and such that $\psi(x_k \pm a)$ is finite, it is evident that we can find a finite number of them covering completely D_0. The theorem is proved.

COROLLARY: *Let* $f(z)$ *be a function, analytic in* $R[|x| < c,\ |y| < d]$, *such that*

$$|f(z)| \leq M(x).$$

If $\int_{-c}^{c} \log^+ \log^+ M(x)dx < \infty$, *then for every domain* D_0 *completely interior to* R, *there exists a* ϕ *depending only on* D_0 *and* $M(x)$, *such that*

$$|f(z)| \leq \phi \qquad \textit{for } z \subset D_0.$$

There is no loss of generality to suppose $M(x) > e$. and then the sign $+$ can be omitted over the log signs. If $f(z)$ is analytic, then $\log|f(z)|$ is subharmonic and the result follows from the preceding theorem.

Nils Sjöberg has proved the following theorem:[4]

Let $M(\theta)$ *be given and* $0 < \epsilon < 1$. *In order that the class of subharmonic functions, defined in* $|z| < 1$, *which satisfy in* $1 - \epsilon \leq |z| < 1$

$$|\mu(re^{i\theta})| \leq M(\theta),$$

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[4] Comptes Rendus du Congrès des Mathématiques à Helsinfors, 1938.

should be bounded from above in every circle $|z| \leqq r_0 < 1$, *it is sufficient that*

$$\int_{-\pi}^{\pi} \log^+ M(\theta)\, d\theta < \infty.$$

This theorem can easily be deduced from our result. By $\zeta = \log z$, we straighten out the concentric circles, and $1-\epsilon \leqq |z| < 1$ corresponds to $\log(1-\epsilon) \leqq \Re(\zeta) < 0$. In this strip $\mu(\zeta) \leqq M(\theta)$, $\theta = \Im(\zeta)$. Hence, if $M(\theta_0) < \infty$, then, by our result, the class of subharmonic functions $\mu(\zeta)$ will be bounded in $\theta_0 \leqq \arg z = \theta \leqq \theta_0 + 2\pi$ and $\log(1-\epsilon/2) \leqq \Re(\zeta) \leqq \log(1-\epsilon/4)$. Hence the same is true of $\mu(z)$ in $1-\epsilon/2 \leqq |z| \leqq 1-\epsilon/4$. But the class of subharmonic functions must have the same upper bound in $|z| \leqq 1-\epsilon/4$. The theorem is proved.

Similarly we can prove a generalization of the Phragmén-Lindelöf theorem:[5]

If: (i) $f(z)$ *is analytic in* $\Im(z) > 0$, (ii) *its boundary values on* $\Im(z) = 0$ *are in absolute value less than* 1, (iii) *there are two sequences* $r_k \to \infty$ *and* $\epsilon_k \to 0$ *such that*

$$|f(z)| \leqq \exp\left\{\epsilon_k r_k e^{\psi(\theta)}\right\}, \qquad \psi(\theta) \subset L,$$

for $r_k(1-\delta) < |z| < r_k$, *then*

$$|f(z)| \leqq 1 \qquad \text{for } \Im(z) > 0.$$

From condition (iii) it follows that

$$\log |f(r_k z)| / \epsilon_k r_k \leqq e^{\psi(\theta)},$$

for $1-\delta < |z| < 1$. By condition (ii), we can suppose $\psi(0) = \psi(\pi) = 0$. In the same way as in the preceding we deduce the existence of a ϕ, such that

$$\log |f(r_k z)| / \epsilon_k r_k < \phi \qquad \text{for } 1 - \delta/2 < |z| < 1 - \delta/4,$$

or

$$|f(z)| \leqq \exp\left\{\epsilon_k \phi r_k\right\}, \qquad (1 - \delta/2) r_k < |z| < (1 - \delta/4) r_k.$$

Now we use the Phragmén-Lindelöf theorem in its classical form[6] to deduce the desired result.

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[5] Journal of the London Mathematical Society, vol. 14 (1939), p. 208.

[6] R. Nevanlinna, *Eindeutige analytische Funktionen*, Berlin, 1936, p. 43.

HOMOGENEOUS AND NONHOMOGENEOUS DIOPHANTINE EQUATIONS

ANTHONY A. AUCOIN

In an earlier paper[1] we considered homogeneous polynomials f and g whose degrees are relatively prime, and solved the Diophantine equation $f(x)=g(y)$. These results are generalized in the present paper. The solutions are given in terms of arbitrary parameters and if the parameters are integral the solutions are also.

We begin our discussion with the hypothesis that the functions $f(x)=f(x_{11}, \cdots, x_{1m}, \cdots, x_{n1}, \cdots, x_{nm})$, $g(x)=g(x_{11}, \cdots, x_{1m}, \cdots, x_{n1}, \cdots, x_{nm})$ are polynomials with integral coefficients, homogeneous in each set of variables $x_{k1}x_{k2} \cdots x_{km}$; f being of degree $\alpha_k \geqq 0$, g being of degree $\beta_k \geqq 0$ in the sets $x_{k1}x_{k2} \cdots x_{km}$ and $d_k = \alpha_k - \beta_k$. We suppose further that integers λ_i, $\mu_i \geqq 0$ exist such that[2]

$$\sum_{i=1}^{n} d_i\lambda_i = -\sum_{i=1}^{n} d_i\mu_i = 1. \tag{1}$$

THEOREM 1. *The Diophantine equation*

$$f(x) = g(x) \tag{2}$$

has integral solutions, and every solution for which the members of (2) *do not vanish, is equivalent* (*in a sense to be defined*) *to one of the solutions given by*

$$x_{kj} = \alpha_{kj}[g(\alpha)]^{\lambda_k}[f(\alpha)]^{\mu_k}, \qquad k = 1, \cdots, n;\ j = 1, \cdots, m, \tag{3}$$

where the α_{kj} are arbitrary integers.

PROOF. Let $x_{kj} = \alpha_{kj}s^{\lambda_k}t^{\mu_k}$. Then by (1), (2) becomes[3] $sf(\alpha)=tg(\alpha)$, which is satisfied identically in the α_{kj} if $s=g(\alpha)$, $t=f(\alpha)$. Hence (3) is a solution of (2).

We now define the concept of equivalent solutions. Suppose $x_{kj}=\rho_{kj}$ is a solution of (2). If there are no integers $b>1$, ρ'_{kj} such that $\rho_{kj}=\rho'_{kj}b^{\sigma_k}$, where the σ_k are positive integers such that $\sum_{k=1}^{n}\alpha_k\sigma_k$

Received by the editors March 2, 1942.

[1] A. A. Aucoin and W. V. Parker, *Diophantine equations whose members are homogeneous*, this Bulletin, vol. 45 (1939), pp. 330–333.

[2] We need to postulate only the existence of λ_i since it may be shown that for n odd $\mu_i=\sum_{j=1}^{n}d_j-d_i-2\sum_{k=1}^{(n-1)/2}d_{i+k}-\lambda_i$ where $d_{n+p}=d_p$, and for n even $\mu_i=\sum_{j=1}^{n}d_j-d_i-2\sum_{k=1}^{(n/2)-1}d_{i+k}-2d'_{n/2+i}-\lambda_i$ where $d'_k=d_k$ for $k\leqq n$, $d'_k=0$ for $k>n$ and $d_{n+p}=d_p$. We must, however, assume that $\mu_i\geqq 0$.

[3] It will be shown later that $s, t\neq 0$.

$=\sum_{k=1}^{n}\beta_k\sigma_k$, then $x_{kj}=\rho_{kj}$ is defined to be a primitive solution of (2). If $x_{kj}=\rho_{kj}$ is a primitive solution of (2), then $x_{kj}=\rho_{kj}b^{\sigma_k}$ (derived from the primitive solution), where b is a nonzero integer,[4] σ_k are positive integers such that $\sum_{k=1}^{n}\alpha_k\sigma_k=\sum_{k=1}^{n}\beta_k\sigma_k$, is also a solution. Two solutions are said to be equivalent if they may be derived from the same primitive solution.

Suppose now that $x_{kj}=\rho_{kj}$ is a solution of (2). Then $f(\rho)=g(\rho)$. If we choose $\alpha_{kj}=\rho_{kj}$, (3) becomes $x_{kj}=\rho_{kj}[f(\rho)]^{\lambda_k+\mu_k}$ which is equivalent to the given solution $x_{kj}=\rho_{kj}$ provided $f(\rho)\neq 0$, since from (1), $\sum_{k=1}^{n}\alpha_k(\lambda_k+\mu_k)=\sum_{k=1}^{n}\beta_k(\lambda_k+\mu_k)$. If $f(\rho)\neq 0$, then s, $t\neq 0$.

As our next topic we suppose that for the functions considered above, f is of degree $-\alpha_p$ and g is of degree $-\beta_p$ in the set $x_{p1}\,x_{p2}\cdots x_{pm}$, where α_p, β_p are positive integers.[5] Here we let $d_p=\beta_p-\alpha_p$.

THEOREM 2. *The Diophantine equation*

$$f(x) = g(x) \tag{4}$$

has solutions, and every solution, for which the members of (4) *do not vanish, is equivalent to a solution given by*

$$x_{kj} = \alpha_{kj}[A(\alpha)B(\alpha)]^{\lambda_k+\mu_k}[g(\alpha)]^{\lambda_k}[f(\alpha)]^{\mu_k}, \tag{5}$$

where $A(x)=\prod_{j=1}^{m}x_{pj}^{\alpha_p}$, $B(x)=\prod_{j=1}^{m}x_{pj}^{\beta_p}$, *the* α_{kj} *being arbitrary integers.*

PROOF. If we multiply (4) by $A(x)B(x)$ we have

$$A(x)B(x)f(x) = A(x)B(x)g(x), \tag{6}$$

each member of which is a polynomial. If we let

$$x_{kj} = \alpha_{kj}s^{\lambda_k}t^{\mu_k}, \tag{7}$$

equation (6) becomes[6] $sA(\alpha)B(\alpha)f(\alpha)=tA(\alpha)B(\alpha)g(\alpha)$, which is identically satisfied in the α_{kj} if $s=A(\alpha)B(\alpha)g(\alpha)$, $t=A(\alpha)B(\alpha)f(\alpha)$. Hence (5) is a solution of (4).

It is evident that more than one of the corresponding sets of variables may be of negative degree.

Suppose that $x_{kj}=\rho_{kj}$ is a given solution of (4). Then $f(\rho)=g(\rho)$. If we choose $\alpha_{kj}=\rho_{kj}$, (5) becomes

$$x_{kj}=\rho_{kj}[A(\rho)B(\rho)f(\rho)]^{\lambda_k+\mu_k}$$

which is equivalent to the given solution $x_{kj}=\rho_{kj}$ provided $f(\rho)\neq 0$, since from (1)

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[4] If $b=0$, the solution is trivial.

[5] The λ_i, and hence the μ_i, here are different from those of Theorem 1.

[6] It will be shown later that s, $t\neq 0$.

$$\sum_{k=1}^{n} \alpha_k(\lambda_k + \mu_k) - 2\alpha_p(\lambda_p + \mu_p) = \sum_{k=1}^{n} \beta_k(\lambda_k + \mu_k) - 2\beta_p(\lambda_p + \mu_p).$$

If $f(\rho) \neq 0$ then s, $t \neq 0$. We now pass on to our next result.

Suppose that f is of degree $-\alpha_p$ in the set $x_{p1} x_{p2} \cdots x_{pm}$ and g is of degree $-\beta_q$ in the set $x_{q1}x_{q2} \cdots x_{qm}$ where α_p, β_q are positive integers. Here[7] $d_p = -(\alpha_p + \beta_p)$, $d_q = \alpha_q + \beta_q$.

THEOREM 3. *The Diophantine equation* $f(x) = g(x)$ *has solutions, and every solution for which the members do not vanish, is equivalent to a solution given by* $x_{kj} = \alpha_{kj}[C(\alpha)D(\alpha)]^{\lambda_k+\mu_k}[g(\alpha)]^{\lambda_k}[f(\alpha)]^{\mu_k}$ *where* $C(x) = \prod_{j=1}^{m} x_{pj}^{\alpha_p}$, $D(x) = \prod_{j=1}^{m} x_{qj}^{\beta_q}$, *the* α_{kj} *being arbitrary integers.*

The proof is similar to that of Theorem 2.

We can now extend the above methods of solution to nonhomogeneous equations. The nature of this extension is contained in the following theorem.

THEOREM 4. *The Diophantine equation*[8]

$$f(x) = g(x) \tag{8}$$

where $f(x) = \sum_{k=1}^{m} a_k \prod_{i=1}^{n} x_i^{\alpha_{ki}}$, $g(x) = \sum_{k=1}^{p} b_k \prod_{j=1}^{q} x_j^{\beta_{kj}}$, a_k, b_k *are integers,* α_{ki}, β_{kj} *positive integers, has solutions if there exist positive integers* u_i, v_j, M, N *such that*

$$\begin{aligned}
&\sum_{i=1}^{n} \alpha_{ki} u_i = M + 1, && k = 1, \cdots, m,\\
&\sum_{j=1}^{q} \beta_{kj} u_j = M, && k = 1, \cdots, p,\\
&\sum_{i=1}^{n} \alpha_{ki} v_i = N, && h = 1, \cdots, m,\\
&\sum_{j=1}^{q} \beta_{kj} v_j = N + 1, && k = 1, \cdots, p,
\end{aligned} \tag{9}$$

and every solution for which the members of (8) *do not vanish, is equivalent to a solution given by*

$$x_j = \alpha_j [g(\alpha)]^{u_j} [f(\alpha)]^{v_j}, \tag{10}$$

where the α_j *are arbitrary integers.*

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[7] As in the previous theorem the λ_i and μ_i will be different from those of Theorem 1.

[8] A. A. Aucoin and W. V. Parker, op. cit., p. 331. Theorem 4, which generalizes Theorem 2 of the reference may be obtained from that theorem directly by setting $y_j = x_i$.

PROOF. Let $x_j=\alpha_j s^{u_j}t^{v_j}$. Then (8) becomes, by (9), $s^{M+1}t^Nf(\alpha) = s^Mt^{N+1}g(\alpha)$, which is satisfied identically in the α_j if $s=g(\alpha)$, $t=f(\alpha)$. Hence (10) is a solution of (8).

When the conditions of (9) are satisfied there also exist non-negative integers R, u_j' such that

$$(11)\quad \sum_{i=1}^{n}\alpha_{hi}u_i' = R,\ \sum_{j=1}^{q}\beta_{kj}u_j' = R,\quad h = 1,\cdots,m;\ k = 1,\cdots,p.$$

If $x_j=\rho_j$ is a solution of (8) and there are no integers $d>1$, u_j', ρ_j' such that $\rho_j=\rho_j'd^{u_j'}$ where u_j' satisfies (11), then $x_j=\rho_j$ is defined to be a primitive solution of (8). If $x_j=\rho_j$ is a primitive solution of (8), then $x_j = \rho_jd^{u_j'}$ (derived from the primitive solution) where d ($\neq 0$) is an integer and u_j' are non-negative integers satisfying (11), is also a solution.

Suppose $x_j=\rho_j$ is a solution of (8). Then $f(\rho)=g(\rho)$. If we choose $\alpha_j=\rho_j$, (10) becomes $x_j=\rho_j[f(\rho)]^{u_j+v_j}$ which is equivalent to the solution $x_j=\rho_j$ since from (9)

$$\sum_{i=1}^{n}\alpha_{hi}(u_i+v_i) = \sum_{j=1}^{q}\beta_{kj}(u_j+v_j) = M+N+1.$$

In Theorems 2 and 3 it is conceivable that the method applies to cases other than those in which an entire set of variables appear in the denominator. The cases are too numerous to consider. It is likewise apparent that the method of Theorem 4 applies when some of the exponents are negative. As there is no typical case the procedure will be illustrated by a particular example.

Consider the equation

$$(12)\qquad ax^3u^5/v^7 + by^4v^4/u^2 = cx^5/w$$

which may be written as

$$(13)\qquad ax^3u^7w + by^4v^{11}w = cx^5u^2v^7.$$

If we let $x=\alpha s^{p_1}t^{q_1}$, $y=\beta s^{p_2}t^{q_2}$, $u=\lambda s^{p_3}t^{q_3}$, $v=\mu s^{p_4}t^{q_4}$, $w=\nu s^{p_5}t^{q_5}$ and substitute in (13) we have

$$s^{3p_1+7p_3+p_5}t^{3q_1+7q_3+q_5}a\alpha^3\lambda^7\nu + s^{4p_2+11p_4+p_5}t^{4q_2+11q_4+q_5}b\beta^4\mu^{11}\nu$$
$$= s^{5p_1+2p_3+7p_4}t^{5q_1+2q_3+7q_4}c\alpha^5\lambda^2\mu^7.$$

We require an integral solution of the equations

$$\begin{aligned} 3p_1 + 7p_3 + p_5 &= m, & 3q_1 + 7q_3 + q_5 &= n+1,\\ 4p_2 + 11p_4 + p_5 &= m, & 4q_2 + 11q_4 + q_5 &= n+1, \end{aligned}$$

$$5p_1 + \ 2p_3 + 7p_4 = m + 1, \qquad 5q_1 + \ 2q_3 + 7q_4 = n.$$

A solution of these equations is $p_1=4$, $p_2=3$, $p_3=11$, $p_4=7$, $p_5=1$, $q_1=4$, $q_2=3$, $q_3=11$, $q_4=7$, $q_5=3$. Hence a solution of (12) is $x=\alpha s^4t^4$, $y=\beta s^3t^3$, $u=\lambda s^{11}t^{11}$, $v=\mu s^7t^7$, $w=\nu st^3$ where $s=a\alpha^3\lambda^7\nu+b\beta^4\mu^{11}\nu$, $t=c\alpha^5\lambda^2\mu^7$.

If $x=x'$, $y=y'$, $u=u'$, $v=v'$, $w=w'$ is a given solution of (12) and the choice $\alpha=x'$, $\beta=y'$, $\lambda=u'$, $\mu=v'$, $\nu=w'$ is made then $s=t$ and the solution becomes $x=x't^8$, $y=y't^6$, $u=u't^{22}$, $v=v't^{14}$, $w=w't^4$ which is equivalent to the given solution provided $t\neq 0$.

United States Naval Academy and
University of Houston

VECTOR ANALOGUES OF MORERA'S THEOREM

E. F. BECKENBACH

Let the vector

$$X = X(x_1, x_2, x_3) = X(x) = X_1i + X_2j + X_3k$$

be defined and continuous in the domain (non-null connected open set) D. Consider the mean-value vector

$$X^{(\rho)}(x) = \frac{1}{|V_\rho|}\int_{V_\rho} X(x+\xi)dV, \tag{1}$$

where V_ρ denotes the sphere

$$\xi_1^2 + \xi_2^2 + \xi_3^2 < \rho^2,$$

and $|V_\rho|$ its volume,

$$|V_\rho| = 4\pi\rho^3/3.$$

The vector (1) can be defined thus for only a part D_ρ of D, but this is of no consequence since ρ is arbitrarily small.

Since $X(x)$ is continuous, it follows that $X^{(\rho)}(x)$ has continuous partial derivatives of the first order; these are given by

$$\frac{\partial}{\partial x_\nu} X^{(\rho)}(x) = \frac{1}{|V_\rho|}\int_{S_\rho} X(x+\rho\alpha)\alpha_\nu d\sigma, \tag{2}$$

where S_ρ denotes the surface of V_ρ and α_1, α_2, α_3 are the components of the unit vector along the outer normal to S_ρ.

Presented to the Society, September 5, 1941 under the title *Vector formulations of Morera's theorem*; received by the editors March 3, 1942.

Mean-value functions are particularly useful in giving sufficiency proofs involving lessening of differentiability conditions. We shall use mean-value vectors to establish two generalizations of Morera's theorem to vector functions.

A vector $X(x)$ having continuous partial derivatives of the first order in the finite domain D has been called[1] a ***Newtonian vector*** provided it is both irrotational and solenoidal; that is, provided its curl and its divergence both vanish identically in D:

$$\begin{aligned}\text{curl } X \equiv \nabla \times X \equiv \left(\frac{\partial X_3}{\partial x_2} - \frac{\partial X_2}{\partial x_3}\right)i + \left(\frac{\partial X_1}{\partial x_3} - \frac{\partial X_3}{\partial x_1}\right)j \\ + \left(\frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2}\right)k = 0;\end{aligned} \tag{3}$$

$$\text{div } X \equiv \nabla \cdot X \equiv \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3} = 0. \tag{4}$$

It follows readily[2] that every Newtonian vector is a harmonic vector.

For a vector of components $X_1(x_1, x_2)$, $X_2(x_1, x_2)$, equations analogous to (3) and (4) are the Cauchy-Riemann differential equations

$$\frac{\partial X_2}{\partial x_1} = \frac{\partial X_1}{\partial x_2}, \qquad \frac{\partial X_2}{\partial x_2} = -\frac{\partial X_1}{\partial x_1},$$

for the function

$$f(z) \equiv X_2(x_1, x_2) + iX_1(x_1, x_2), \qquad z \equiv x_1 + ix_2.$$

This fact is taken as justification for considering Newtonian vectors as a generalization of analytic functions of a complex variable.

Let the vector $X(x)$ have continuous partial derivatives of the first order in D. By Stokes' theorem, $X(x)$ is irrotational if and only if for each reducible[3] closed curve C in D,

$$\int_C X \cdot dR = 0, \tag{5}$$

where R is the vector from the origin to a moving point on C. By

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[1] D. G. Fulton and G. Y. Rainich, *Generalizations to higher dimensions of the Cauchy integral formula*, American Journal of Mathematics, vol. 54 (1932), pp. 235–241.

[2] See, for instance, H. B. Phillips, *Vector Analysis*, New York, 1933, pp. 149, 153.

[3] A closed curve (or surface) in D is reducible provided it is rectifiable (or of finite area) and can be shrunk continuously in D to a point of D. The forms of Stokes' theorem and Green's theorem referred to in this paragraph may be found in H. B. Phillips, loc. cit., pp. 62, 68, 72.

Green's theorem, $X(x)$ is irrotational if and only if for each reducible closed surface S in D,

$$\int_S n \times X d\sigma = 0, \tag{6}$$

where n is the unit vector along the outer normal to S; and again by Green's theorem, $X(x)$ is solenoidal if and only if for each reducible closed surface S in D,

$$\int_S n \cdot X d\sigma = 0. \tag{7}$$

It follows that Newtonian vectors are characterized by (5) and (7), and also by (6) and (7); but (cf. Morera's theorem) *the assumption that $X(x)$ has continuous partial derivatives of the first order now is redundant*,[4] as we shall show. We shall further lessen the sufficiency conditions by replacing the integral conditions (5), (6) and (7) by weaker local conditions which are implied by (5), (6) and (7), respectively.

We shall denote by $S(x, r)$ the sphere with center (x) and radius r, and by $C_1(x, r)$, $C_2(x, r)$, $C_3(x, r)$ the circles with center (x) and radius r in planes perpendicular to the x_1, x_2, x_3 axes, respectively. Let

$$C(x, r) \equiv C_1(x, r)i + C_2(x, r)j + C_3(x, r)k,$$

and let $o(r^t)$ denote a function (not always the same function) of r such that

$$\lim_{r \to 0} \frac{o(r^t)}{r^t} = 0.$$

THEOREM 1. *Let the vector $X(x)$ be continuous in the finite domain D. If*

$$\int_{C(x,r)} X \cdot dR = o(r^2) \tag{8}$$

and

$$\int_{S(x,r)} n \cdot X d\sigma = o(r^3) \tag{9}$$

uniformly in each sphere in D, then $X(x)$ is a Newtonian vector.

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[4] See H. B. Phillips, loc. cit., pp. 177, 178, for a proof of the result that if the continuous vector function $X(x)$ satisfies (5) and (7), then $X(x)$ is a Newtonian vector.

PROOF. Since (8) and (9) hold uniformly in each sphere in D, we have

$$\frac{1}{|V_\rho|}\int_{V_\rho}\left(\int_{C(x,r)} X\, dR\right)dV = o(r^2)$$

and

$$\frac{1}{|V_\rho|}\int_{V_\rho}\left(\int_{S(x,r)} n\cdot X d\sigma\right)dV = o(r^3),$$

in D_ρ. Interchanging the order of integration, we obtain

$$\int_{C(x,r)} X^{(\rho)}\cdot dR = o(r^2) \tag{10}$$

and

$$\int_{S(x,r)} n\cdot X^{(\rho)} d\sigma = o(r^3). \tag{11}$$

Since $X^{(\rho)}(x)$ has continuous partial derivatives of the first order in D_ρ, about any point $(x^{(0)})$ in D_ρ we have the finite Taylor development

$$X^{(\rho)}(x) = X^{(\rho)}(x^{(0)}) + \frac{\partial X^{(\rho)}}{\partial x_1}(x_1 - x_1^{(0)}) + \frac{\partial X^{(\rho)}}{\partial x_2}(x_2 - x_2^{(0)}) + \frac{\partial X^{(\rho)}}{\partial x_3}(x_3 - x_3^{(0)}) + o(r),$$

where the partial derivatives are evaluated at $(x^{(0)})$. Computations yield

$$\int_{C(x^{(0)},r)} X^{(\rho)}\cdot dR = \pi r^2 \operatorname{curl} X^{(\rho)}(x^{(0)}) + o(r^2) \tag{12}$$

and

$$\int_{S(x^{(0)},r)} n\cdot X^{(\rho)} d\sigma = (4/3)\pi r^3 \operatorname{div} X^{(\rho)}(x^{(0)}) + o(r^3). \tag{13}$$

Now (10)–(13) give

$$\operatorname{curl} X^{(\rho)}(x) = 0$$

and

$$\operatorname{div} X^{(\rho)}(x) = 0$$

throughout D_ρ; that is, $X^{(\rho)}(x)$ is a Newtonian vector. Hence $X(x)$, the uniform limit of $X^{(\rho)}(x)$ as $\rho \to 0$, is a Newtonian vector.

THEOREM 2. *Let the vector* $X(x)$ *be continuous in the finite domain* D. *If*

$$\int_{S(x,r)} n \times X d\sigma = o(r^3)$$

and

$$\int_{S(x,r)} n \cdot X d\sigma = o(r^3)$$

uniformly in each sphere in D, *then* $X(x)$ *is a Newtonian vector.*

PROOF. The proof is analogous to that of Theorem 1, with

$$\int_{S(x^{(0)},r)} n \times X^{(\rho)} d\sigma = (4/3)\pi r^3 \operatorname{curl} X^{(\rho)}(x^{(0)}) + o(r^3)$$

in place of (12).

THE UNIVERSITY OF MICHIGAN

ON THE INTERIORITY OF REAL FUNCTIONS[1]

G. T. WHYBURN

If X and Y are metric spaces, a continuous transformation $f(X)=Y$ is said to be interior at a point x of X provided that if U is any open subset of X containing x, $f(x)$ is interior to $f(U)$ in Y. In 1928 Kuratowski[2] proved a theorem essentially to the effect that if X is compact, the set G of all points y in Y such that f is interior at all points of $f^{-1}(y)$ is a G_δ-set dense in Y. We first establish the following related result.

THEOREM. *If $f(x)$ is a real-valued continuous function defined on a locally connected separable metric space M (therefore transforming M into a set Y of real numbers), there exists a countable subset C of Y such that f is interior at every point of $M-f^{-1}(C)$.*

PROOF. For each $y \varepsilon Y$ let Y_1 and Y_2 be the sets of all numbers in Y which are less than y and greater than y, respectively. Let $M_1(y)=f^{-1}(Y_1)$, $M_2(y)=f^{-1}(Y_2)$. Now there must exist[3] a countable subset C of Y such that if y is any point of $Y-C$,

$$f^{-1}(y) \subset \overline{M_1(y)}\ \overline{M_2(y)}.$$

For if not, there would exist an uncountable subset K of Y such that for any $y \varepsilon K$ there is a point $p_y \varepsilon f^{-1}(y)$ which fails to be a limit point either of $M_1(y)$ or of $M_2(y)$. Clearly we may suppose that for an uncountable subset of K_1 of K, $p_y \cdot \overline{M_1(y)}=0$. Now since M is separable and metric, there exists a countable sequence $R_1, R_2, \cdots$ of open sets in M such that if U is any open set in M and $p \varepsilon U$, there is an m such that $p \varepsilon R_m \subset U$. Hence, for each $y \varepsilon K_1$ there exists an integer m_y such that $p_y \subset R_{m_y}$ and $R_{m_y} \cdot M_1(y)=0$. But if y, $y' \varepsilon K_1$, $y'<y$, we have $p_{y'} \subset M_1(y)$ so that $m_y \neq m_{y'}$. This is absurd, since K_1 was uncountable. Hence we have established the existence of the set C as asserted above.

We shall show that f is interior at any point x of $M-f^{-1}(C)$. To this end let U be any open set in M containing x and let $y=f(x)$. Since M is locally connected, there exists a connected open subset V

Presented to the Society, September 8, 1939; received by the editors March 4, 1942.

[1] This paper, in slightly different form, was sent to the editors of Fundamenta Mathematicae in June, 1939.

[2] See Fundamentae Mathematicae, vol. 11 (1926), p. 176.

[3] This could be established also with the aid of a lemma of Zarankiewicz. See Fundamenta Mathematicae, vol. 12 (1928), p. 119.

of M with $x \subset V \subset U$. Now V contains points x_1 and x_2 of $M_1(y)$ and $M_2(y)$, respectively. Since $f(V)$ is connected and contains $f(x_1)$ and $f(x_2)$, it therefore contains the interval $[f(x_1), f(x_2)]$. Hence $f(x)$, which is interior to this interval, is interior to $f(V)$ and hence surely to $f(U)$.

Examples are easily constructed to show that our conclusion is invalid in the absence of either the local connectedness condition on M or the linearity restriction on the image space.

However, we can extend the theorem considerably. First we note the following corollary. (Note: All sets referred to in this paper are assumed separable and metric.)

COROLLARY. *Let* $f(M) = Y = \sum_{i=1}^{\infty} L_i$ *be continuous where* M *is locally connected, each* L_i *is open in* Y *and is homeomorphic with a subset of a line and* $L_i \cdot L_j = 0$ *for* $i \neq j$. *There exists a countable subset* C *of* L *such that* f *is interior at every point of* $M - f^{-1}(C)$.

For under the conditions on the L_i it results at once that Y is homeomorphic with a subset of a line and hence with a set of real numbers. (For if L_i' is a subset of the open interval $(i-1, i)$ homeomorphic with L_i, $i = 1, 2, \cdots$, clearly Y is homeomorphic with $\sum L_i'$.) Thus f is effectively a real continuous function on M and the corollary follows from the preceding theorem.

THEOREM. *Let* $f(M) = K$ *be continuous where* M *is locally connected and let* D *be the closure of the set of all points of order greater than* 2 *of* K. *There exists a countable subset* C *of* $K - D$ *such that* f *is interior at every point of* $M - f^{-1}(D + C)$.

PROOF. By continuity of f, $M - f^{-1}(D) = M'$ is open in M and hence is locally connected. Consider the mapping $f(M') = K - D = K'$. Since every point of K' is of order at most 2, K' is locally connected. Hence the components $R_1, R_2, \cdots$ of K' are open (thus countable). Further, each R_i is homeomorphic with a subset of a line, since it is a connected set all of whose points are of order at most 2. Thus by the above corollary there exists a countable subset C of K' such that f is interior at all points of $M' - f^{-1}(C)$. Obviously this yields our theorem.

COROLLARY. *If* D *is countable, there exists a countable subset* C *of* K *such that* f *is interior at all points of* $M - f^{-1}(C)$.

COROLLARY. *Any continuous mapping* $f(M) = K$ *of a locally connected set* M *into a graph* K *is interior at all points of* $M - f^{-1}(C)$, *where* C *is some countable subset of* K.

If X and Y are topological spaces, X_1 is a subset of X and $f(X_1) \subset Y$ and $\phi(X) \subset Y$ are single-valued continuous transformations, $\phi(x)$ is said to be an extension of f to X provided $f(X) = \phi(x)$ for $x \epsilon X_1$. Extension theorems in various forms for continuous transformations are well known.[4] However, not much consideration seems to have been given to the important problem of carrying over to an extension ϕ of f properties which f may enjoy in addition to continuity. In this note a beginning in this direction is made with the following theorem.

THEOREM. *Let M be a subcontinuum of a cyclic locally connected compact continuum L and let $f(M) = (0,1)$ be continuous and such that for any y, $0 \leq y \leq 1$, $L - f^{-1}(y)$ is connected. There exists an extension $\phi(L) = (0, 1)$ of f to L which is interior at every point of $L - M$ and at every point of M where f is interior.*[5] *(Thus if f is interior on M, ϕ is interior on L.)*

PROOF. Let us decompose L upper semi-continuously[6] into the subsets $[f^{-1}(y)]$, $0 \leq y \leq 1$, of M and individual points of $L - M$. Call L' the hyperspace of this decomposition and let $h(L) = L'$ be the associated transformation. Clearly $h(M)$ is a simple arc $a'b'$ joining $a' = hf^{-1}(0)$ and $b' = hf^{-1}(1)$. By a theorem of the author,[7] there exists a non-alternating interior retracting transformation $g(x)$ retracting the cyclic chain $C(a,'b')$ in L' into $a'b'$. But since no one of the sets $f^{-1}(y)$ disconnects L and L is cyclic, it follows that L' is cyclic and hence $L' = C(a', b')$. Thus $g(x)$ retracts L' into $a'b'$.

Let us define

$$\phi(x) = fh^{-1}gh(x), \qquad x \epsilon L.$$

Then ϕ has the required properties. For if $x \epsilon M$, we have $h(x) \epsilon a'b'$ so that $gh(x) = h(x)$; whence $h^{-1}gh(x) = h^{-1}h(x) = f^{-1}f(x)$ so that $fh^{-1}gh(x) = f(x)$. This proves $\phi(x) = f(x)$ on M. To show that ϕ is interior at any point x where f is interior or at any point of $L - M$, let $x \epsilon U$ where U is open in L. If $x \epsilon M$, then since f is interior at x, $h[U \cdot M]$ contains an open subset V of $a'b'$ about $h(x)$. Hence $g(V) = V$

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[4] See, for example, F. Hausdorff, Fundamenta Mathematicae, vol. 30 (1938), p. 40, and C. Kuratowski, ibid., p. 48.

[5] It is supposed that L is imbedded in a metric space and (0, 1) denotes the interval $0 \leq y \leq 1$ of the real numbers. A continuum is *cyclic* if it has no cut point.

[6] See R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, 1932, chap. 5; note also references to P. Alexandroff and to C. Kuratowski.

[7] See Whyburn, *Non-alternating interior retracting transformations*, Annals of Mathematics, (2), vol. 40 (1939), pp. 914–921.

is open in $a'b'$ and $h^{-1}g(V)\supset x$ and is open in M. Since f is interior at x, $fh^{-1}g(V)$ has $f(x)$ as an interior point; and since this set surely is contained in $fh^{-1}gh(U)=\phi(U)$, it follows that $\phi(x)$ is interior to $\phi(U)$. If $x\epsilon L-M$, we may suppose $U\subset L-M$. Then h is topological on U, so that $h(U)$ is open in L'. Thus $gh(U)$ is open in $a'b'$, since g is interior on L'. Hence $h^{-1}gh(U)$ is open in M and $fh^{-1}gh(U)=\phi(U)$ has $\phi(x)$ as an interior point, since f is interior at x.

It is clear that essentially the same argument suffices to establish the following somewhat more general extension theorem:

Let M be a subcontinuum of a locally connected continuum L of the form $L=C(a, b)$ where a, $b\epsilon M$. Let $f(M)=(0, 1)$ be such that $f(a)=0$, $f(b)=1$ and for any y with $0<y<1$, every component of $L-f^{-1}(y)$ contains either a or b. There exists an extension $\phi(L)=(0, 1)$ of f to L which is interior at every point of $L-M$ and also at every point of M where f is interior.

UNIVERSITY OF VIRGINIA

THE MODULAR SPACE DETERMINED BY A POSITIVE FUNCTION

R. W. BARNARD AND H. H. GOLDSTINE

At the suggestion of T. H. Hildebrandt the authors undertook to determine the nature of the space of modular functions of E. H. Moore when the range $\mathfrak{P}$ is taken to be the infinite interval $-\infty < x < +\infty$ and the base matrix ϵ to be of the form

$$(1) \qquad \epsilon(x, y) = \int_{-\infty}^{+\infty} e^{i(x-y)t} dV(t),$$

where V is a monotonically increasing bounded function. This form of ϵ is suggested by the work of Bochner on positive functions.[1] In this note we determine the form of functions modular as to ϵ and of the J-integral.

To avoid, at first, convergence questions we turn our attention to functions ϕ finite as to ϵ, that is, functions of the form

$$(2) \qquad \phi(x) = \sum_{j=1}^{n} \epsilon(x, y_j)a_j = \int_{-\infty}^{+\infty} e^{ixt}\lambda(t)dV(t),$$

where

$$(3) \qquad \lambda(t) = \sum_{j=1}^{n} a_j e^{-iy_j t}.$$

In the formulas (2) and (3) the a_j are arbitrary constants and the y_j are points on the interval $(-\infty, +\infty)$. It is known from standard results in the theory of modular and finite functions[2] that every function ϕ finite as to ϵ is modular and that

$$(4) \qquad \begin{aligned} N\phi &= J\bar{\phi}\phi = \sum_{j,k=1}^{n} \bar{a}_j\epsilon(x_j, x_k)a_k, \\ J\bar{\phi}_1\phi_2 &= [N(\phi_1 + \phi_2) - N(\phi_1 - \phi_2) - iN(\phi_1 + i\phi_2) \\ &\qquad + iN(\phi_1 - i\phi_2)]/4. \end{aligned}$$

Calculating the values of $N\phi$ and $J\bar{\phi}_1\phi_2$, we see that

$$J\bar{\phi}_1\phi_2 = \int_{-\infty}^{+\infty} \bar{\lambda}_1\lambda_2 dV, \qquad N\phi = \int_{-\infty}^{+\infty} |\lambda|^2 dV.$$

Received by the editors March 7, 1942.

[1] S. Bochner, *Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse*, Mathematische Annalen, vol. 108 (1933), pp. 378–410.

[2] E. H. Moore, *General Analysis*, Part II, Philadelphia, 1939, pp. 94 ff.

To determine the form of an arbitrary modular function μ we consider a sequence

$$\phi_n(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda_n dV$$

of functions finite as to ϵ, converging mode 2 to a modular function[3] μ. Since ϕ_n converges strongly, it follows that

$$\lim_{m,n} N(\phi_m - \phi_n) = \lim_{m,n} \int_{-\infty}^{+\infty} |\lambda_m - \lambda_n|^2 dV = 0,$$

and hence there exists a measurable function λ such that λ^2 is integrable with respect to V and[4]

$$\lim_{n} \int_{-\infty}^{+\infty} |\lambda_n - \lambda|^2 dV = 0.$$

With the help of Schwarz' inequality one sees that

$$\mu(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda(t) dV, \qquad N\mu = \int_{-\infty}^{+\infty} |\lambda|^2 dV.$$

THEOREM 1. *To each modular function μ there corresponds a measurable function λ such that λ^2 is integrable with respect to V and*

$$\mu(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda(t) dV(t) \quad \textit{and} \quad N\mu = \int_{-\infty}^{+\infty} |\lambda|^2 dV. \tag{5}$$

Conversely, if λ is measurable and λ^2 integrable with respect to V, then the first of the formulas (5) *defines a modular function μ for which the second of these formulas is valid. If μ_1, μ_2 are two modular functions, then*

$$J\bar{\mu}_1\mu_2 = \int_{-\infty}^{+\infty} \bar{\lambda}_1\lambda_2 dV \tag{6}$$

where λ_1, λ_2 are the square integrable functions associated with μ_1, μ_2.

It remains to prove only the latter part of the theorem. To do this let $\xi(x) = \int_{-\infty}^{+\infty} e^{ixt}\lambda dV$, where λ is any measurable function such that λ^2 is integrable with respect to V, and let x_j, a_j $(j = 1, 2, \cdots, n)$ be constants such that

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[3] Ibid., p. 116.

[4] E. W. Hobson, *The Theory of Functions of a Real Variable*, 2d edition, 1926, vol. I, p. 246.

$$\sum_{j,k=1}^{n} \bar{a}_j \epsilon(x_j, x_k) a_k = \int_{-\infty}^{+\infty} \left| \sum_{j=1}^{n} a_j e^{-ix_j t} \right|^2 dV \leqq 1.$$

It then follows with the help of Schwarz' inequality that

$$\left| \sum_{j=1}^{n} \bar{a}_j \xi(x_j) \right|^2 = \left| \int_{-\infty}^{+\infty} \left(\sum_{j=1}^{n} \bar{a}_j e^{ix_j t} \right) \lambda dV \right|^2 \leqq \int_{-\infty}^{+\infty} |\lambda|^2 dV,$$

and hence ξ is modular.[5] The formula (6) follows at once from the second equation (5) and equation (4).

Finally, we seek conditions that the matrix ϵ should be proper. These are contained in the following result:

THEOREM 2. *The base matrix ϵ is proper if the measure function V is such that every set E whose complement has zero measure has a finite limit point.*

It is clear that

$$0 = \sum_{j,k=1}^{n} \bar{a}_j \epsilon(x_j, x_k) a_k = \int_{-\infty}^{+\infty} \left| \sum_{j=1}^{n} a_j e^{-ix_j t} \right|^2 dV$$

implies the vanishing of the analytic function

$$\sum_{j=1}^{n} a_j e^{-ix_j t} \tag{7}$$

for almost all t. If the constants a_j were not all zero, the expression (7) would have a non-finite number of zeros in a bounded interval, which is false, and hence $a_1, \cdots, a_n$ are all zero and ϵ is a proper matrix.

THE UNIVERSITY OF CHICAGO AND
THE UNIVERSITY OF MICHIGAN

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[5] Ibid., p. 84.

A CORRECTION TO "ON THE ITERATION OF LINEAR HOMOGENEOUS TRANSFORMATIONS"

ARNOLD DRESDEN

The theorem proved in the note entitled *On the iteration of linear homogeneous transformations*, published in this Bulletin, vol. 48 (1942), pp. 577–579, was proved by Rufus Oldenburger, in a paper on *Infinite powers of matrices and characteristic roots*, which appeared in the Duke Mathematical Journal, vol. 6 (1940), pp. 357–361. My apologies are due to Professor Oldenburger.

SWARTHMORE COLLEGE

Received by the editors August 30, 1942.

AN ADDITION TO "GENERALIZATIONS OF THE BERNOULLI POLYNOMIALS AND NUMBERS AND CORRESPONDING SUMMATION FORMULAS"

TOMLINSON FORT

In the third paragraph of my paper *Generalizations of the Bernoulli polynomials and numbers and corresponding summation formulas*, published in this Bulletin, vol. 48 (1942), pp. 567–574, no explicit statement is made of the fact that it is assumed that P and Q as operators on x obey the laws: $PF(x+c)\,]_{c=0}=PF(x)$; $QF(x+c)\,]_{c=0}=QF(x)$.

Also the statement $f_n(0)=0$, should read $f_n(0)=0$, $n>0$.

LEHIGH UNIVERSITY

Received by the editors November 4, 1942.

ERRATA, VOLUME 48

M. R. Hestenes, *The problem of Bolza in the calculus of variations*, 57–75.

p. 63, first display line. Instead of $g(t)$ read $g(a)$.

p. 72, line 32. Instead of by E_0^* the set $\cdots$ read by $\mathcal{E}_0^*$ the set $\cdots$.

Isaac Opatowski, *Confluent hypergeometric functions and Markoff chains*, abstract 48-5-179.

p. 365, line 2. Instead of $k_t=0$ read $k_0=0$.

A. N. Lowan, N. Davids and A. Levenson, *Table of the zeros of the Legendre polynomials of order* 1–16 *and the weight coefficients for Gauss' mechanical quadrature formula*, pp. 739–743.

p. 742, last display line. Instead of x_k^3, in the denominator, read x_k+3.

INDEX OF VOLUME 48

INDEX OF ABSTRACTS[1]

Subjects:

Algebra and Theory of Numbers, 29, 210, 355, 518, 665, 825.
Analysis, 35, 212, 361, 519, 666, 827.
Applied Mathematics, 44, 216, 369, 522, 669, 831.
Geometry, 48, 216, 371, 522, 670, 833.
Logic and Foundations, 49, 219, 373, 834.
Statistics and Probability, 52, 220, 374, 523, 673, 834.
Topology, 52, 220, 374, 523, 675, 835.

Authors:

Adams, C. R., 805 (666); Albert, A. A , 7 (29), 336 (210), 502 (355); Albert, G. E., 806 (666); Allendoerfer, C. B., 192 (49); Arnold, B. H., 493 (355); Aroian, L. A., 805 (673); Artin, E., 502 (355); Avann, S. P., 9 (29), 9 (29), 495 (356), 496 (356).

Baer, R., 193 (29), 503 (356), 806 (518); Barker, C. B., 9 (35), Barnett, I. A., 192 (30); Basoco, M. A., (827); Beaumont, R. A., 496 (210); Beckenbach, E. F., 4 (47; 705), 5 (47; 862), 192 (36), 193 (36); Begle, E. G., 503 (374); Bell, P. O., 336 (216); Bergman, S., 191 (36), 336 (216), 492 (370), 493 (369), 806 (669), 806 (670); Berkeley, E. C., 193 (49); Bernstein, D. L., 501 (361); Berry, G. D. W., 193 (50), Bers, L , 4 (47; 860), 492 (362), (831); Birkhoff, G., (831); Blanch, G., 337 (213); Blumenthal, L. M., 7 (48); Bochner, S., 191 (37), 193 (30); Bourgin, D. G., 4 (47; 711), Brauer, R., 493 (356); Bradshaw, J. W , 192 (37); Brown, A. B., 4 (47; 860); Brown, H. K., 806 (522); Bruck, R. H., 7 (47; 856), 191 (30), 193 (30), 195 (34); Busemann, H., 502 (371); Bushey, J. H., 492 (374), 493 (374), 493 (374).

Cairns, S. S., 4 (47; 711), 4 (47, 711); Cameron, R. H., (666); Campbell, A. D., 4 (47; 856); Carlitz, L., 193 (35), 493 (357), 806 (825), 806 (825); Church, A., 193 (50); Civin, P., 336 (220), 492 (376); Clarkson, J. A., 805 (666); Clippinger, R. F., 503 (357); Coburn, N., 9 (48), 336 (217), 336 (217), (832); Cohen, I. S , 193 (31), Cohen, L. W., 806 (828); Comenetz, G., 4 (47, 863); Court, N. A., 193 (48); Coxeter, H. S. M., 806 (670); Curtiss, J. H., 4 (47, 710).

Davids, N., 5 (47; 704); Day, M. M., 503 (375); DeCicco, J. J., 4 (47; 863), 336 (217), 336 (218), 336 (218), (670), 806 (671), 807 (672), (833); Dilworth, R. P., 9 (31), 9 (31), 191 (31), 493 (357), 805 (825); Dines, L. L., 193 (210), Doob, J. L., 193 (37); Douglas, J., 336 (218); Dressel, F. G., 493 (362).

Eilenberg, S., 193 (54), 805 (836); Engelbrecht, A. E., 7 (44); Erdos, P., 805 (666); Ettlinger, H. J., 193 (37); Everett, C. J., 807 (522), (826); Ewing, G M., 501 (364).

Feller, W., 807 (828); Fialkow, A., 493 (372); Ficken, F. A., 493 (357); Flexner, W. W., 192 (52); Foote, W. R., 806 (832); Forsythe, G. E., 495 (212); Fort, T., 192 (47; 857); Frame, J. S., 493 (357).

Garabedian, H. L., 336 (212); Gaskell, R. E., 193 (45); Geiringer, H. P., 492 (370); Gelbart, A., 193 (37), 336 (212), 493 (371), (831); Gere, B. H., 807 (671); Goodman, N., 193 (50); Gordon, R. D., 9 (52); Gottschalk, W. H., 807 (523); Greenberg, H. J., 336 (212); Greville, T. N. E., 805 (834); Gumbel, E. J., 4 (47; 865), 805 (673).

Hall, D. W., 194 (53), 806 (675); Hall, M., 191 (31); Halmos, P. R., 493 (523);

---

[1] Numbers within parentheses refer to the page numbers for abstracts, and those outside parentheses refer to the pages for titles as listed in reports of meetings.

Harrold, O. G., 807 (835); Heins, A. E., 192 (45); Hellinger, E. D., 502 (362); Hildebrand, F. B., 194 (45); Hille, E., 9 (38), 495 (519), 806 (828); Hillman, A., 337 (213), 494 (363), (828); Hinrichsen, J. J. L., 7 (45); Holl, D. L., 6 (46); Horenstein, W., 5 (47; 861), 337 (213); Hughes, H. K., 502 (364); Hurewicz, W., 493 (519); Hutchinson, L. C., 493 (372); Hyers, D. H., 502 (368).

Isaacs, R. P., (832).

Jacobson, N., 493 (518); James, R. C., 8 (53), 9 (53), 495 (375); James, R. D., 502 (358); John, F., 501 (362); Johnson, R. E., 502 (358); Jonah, H. F. S., 502 (358); Jones, B. W., 4 (32); Jones, F. B., 194 (220), 806 (836).

Kac, M., 192 (38); 807 (667), 807 (673); Kales, M. L., 503 (363); Kaplan, S., 503 (375); Kaplansky, I., 194 (32), 337 (220), (826), (826); Karlin, M., 493 (363); Karush; W., 501 (519); Kasner, E., 336 (218), 337 (219), (671), (672), 806 (672), 807 (672), (833); Kelley, J. L., 192 (221); Kempner, A. J., 807 (826); Kershner, R. B., 192 (38); Kincaid, W. M., 807 (675); Kleene, S. C., 193 (51), 493 (373); Kogbetliantz, E., (829); Korn, A., 4 (47; 863); Krabill, D. M., 807 (829).

Laden, H. N., 194 (38); Lehmer, D. H., 8 (47; 857), 807 (665); Lehner, J., 194 (35), 493 (358); Leighton, W., 496 (213), 503 (213), 807 (520); Levi, H., 5 (47; 857); Levenson, A., 5 (47; 704); Levit, R. J., 8 (47; 857); Lewis, F. A., 493 (359); Lorch, E. R., 336 (47; 860); Lorch, L., 807 (829); Lowan, A. N., 5 (47; 704); 5 (47; 861), 337 (213), 337 (213), 494 (363), (520); Lubben, R. G., 807 (675)

McEwen, G. F., 9 (52); McShane, E. J., 194 (39), 194 (39); MacDuffee, C. C., 191 (32); Mackey, G. W., 8 (47; 861); MacLane, S., 192 (32), 193 (54), 805 (836); Maker, P. T., 191 (39); Mandelbrojt, S., 807 (520), (829); Mann, H. B., 336 (211), 805 (674), (834); Marth, E., 807 (667); Martin, W. T., 191 (37), (666); Mendel, C. W., 192 (30); Menger, K., 807 (833), 807 (833); Mersman, W. A., 495 (371), 496 (371); Michal, A. D., 496 (372); Mickle, E. J., 191 (39); Milgram, A. N., 503 (375); Miller, H. C., 192 (54); Miller, G. T., 502 (364); Millsaps, K., 9 (54), 496 (359); Mitchell, J. M., 807 (667); Mittleman, D., 337 (219), 337 (372); Montgomery, D., 806 (676), 807 (676), 808 (676); Moore, R. L., 808 (836), 808 (836); Morkovin, V., 806 (670); Morse, A. P., 805 (666); Morse, M., 501 (364), 808 (521); Munro, G. C., 7 (39); Munroe, M. E., 804 (521); Myers, S. B., 194 (39).

Nesbitt, C. J., 193 (34); Nielsen, K. L., 336 (214), 808 (668); Niven, I., 502 (359).

Oboukhoff, N. M., 194 (40); O'Connor, R. E., 5 (47; 858); 5 (47; 859), 5 (47; 859); Oldenburger, R., 493 (360), 494 (359), 502 (359); Opatowski, I., 7 (46), 502 (364); Owens, O. G., 9 (40).

Pall, G., 805 (827), 808 (827); Pate, R. S., 7 (32); Paydon, J. F., 194 (40); Pepper, P. M., 502 (373); Perlis, S., 194 (33); Peters, J. W., 192 (49); Phillips, R. S., 193 (30); Pinney, E. J., 9 (41), 496 (365), 496 (365); Pollard, H., 5 (47; 700), 336 (214), 494 (365), 494 (365); Polya, G., 503 (366); Poritsky, H., 806 (832); Prager, W., 192 (46); Prenowitz, W., 4 (47; 864).

Rademacher, H. A., 494 (366); Rado, T., 501 (366); Rainich, G. Y., 193 (41); Ramsey, B. P., 808 (668); Reade, M., 5 (47; 862), 192 (36), 194 (41); Reichelderfer, P. V., 501 (366); Reid, W. T., 806 (829); Reves, G. E., 191 (42); Reynolds, C. N., 808 (836); Ritt, J. F., 5 (47; 862), (827); Roberts, J. H., 492 (376); Robinson, C. V., 7 (47; 864); Robinson, R. M., 9 (42), (830); Rochford, M. D. P., 808 (521), 808 (521); Roever, W. H., 194 (219), 502 (367); Rommel, J. D., 502 (522); Rosenbloom, P. C., 805 (830), Ross, A. E., 502 (518); Rosser, B., 193 (51).

Sadowsky, M. A., 503 (360); Salem, R., 336 (214), 805 (522), 808 (668); Salzer, H. E., 494 (363), (520), (828); Samelson, H., 493 (523), 806 (676), 807 (676); Satterthwaite, F. E., 805 (674); Schaeffer, A. C., 495 (212), (215); Schafer, R. D., 502 (360);

Schatten, R., 4 (47; 862); Scheffé, H., 805 (523), 808 (674); Scherk, P., 502 (355); Schilling, O. F. G., 192 (32), 194 (32), 194 (33); Schwartz, H. M., 194 (42), 503 (367); Schweitzer, A. R., 5 (47; 709), 194 (51), 337 (211), 503 (219), 503 (220), 808 (834); Schwerdtfeger, H., 808 (519); Shanks, M. E., 7 (54); Shiffman, M., 808 (830); Shohat, J. A., (831); Silverman, L. L., 336 (215); Slade, J. J., 806 (832); Smiley, D. M., 808 (665); Smiley, M. F., 191 (33), 194 (211), 494 (360), 494 (367), 808 (367); Snapper, E., 4 (47; 858), (827); Snyder, W. S., 501 (368); Sorgenfrey, R. H., 194 (55); Spencer, D. C., 805 (522); Spencer, D. E., 805 (672); Struik, S. R. R., 805 (674), 805 (834), 808 (833); Swain, R. L., 503 (376); Swingle, P. M., 9 (55); Synge, J. L. 192 (47); Szász, O., 191 (42), 192 (42), 336 (215), 492 (368), 805 (668); Szego, G., 9 (43), 806 (828).

Theilheimer, F., 192 (47); Thorne, C. J., 805 (668); Thrall, R. M., 193 (34), 503 (360); Thron, W. J., 496 (213), 503 (213), 807 (520); Thurston, H. S., 808 (665); Titt, E. W., 194 (43), Tomlinson, T. W., 502 (372); Tompkins, C. B., 502 (373); Topel, B. J., 502 (373); Transue, W. R., 191 (33), 192 (43), 494 (360); Tripp, R. H., 7 (47); Trjitzinsky, W. J., 194 (43); Tuan, H. F., 493 (356).

Ulam, S. M., 195 (44), 501 (361), 502 (368); Ulrich, F. E., 807 (520).

Valentine, F. A., 495 (368); Vanderslice, J. L., 192 (49); Villars, D. S., 805 (675); Vinograde, B., 502 (361).

Wade, T. L., 7 (47; 856), 195 (34); Wagner, R. W., 502 (361); Wald, A., 805 (674), 805 (835); Wall, H. S., 5 (47; 863), 194 (40), 336 (212), 502 (362); Wallace, A. D., 195 (221), 808 (524), (837); Wallman, H., 193 (216); Walsh, J. L., 494 (368); Ward, M., 8 (34); Wasow, W., 336 (215); van Waters, M., 805 (674); Webster, M. S., 503 (369); Weil, A., 192 (49); Weinstein, A., 193 (47), 337 (216); Weyl, H., 492 (369), 808 (669); White, P. A., 195 (55); Whitehead, G. W., 192 (55); Whitman, P. M., 192 (34); Whitney, H., 806 (676), 808 (676), 809 (669), 809 (669), 809 (669), 809 (677); Whyburn, G. T., 494 (376), 494 (376), 494 (376), 809 (837); Wilcox, L. R., 195 (35); Wilkins, J. E., 502 (373); Wolf, F., 9 (44), 501 (369); Wolfowitz, J., 805 (835).

Youngs, J. W. T., 7 (47; 715), 503 (377), 806 (837).

Zippin, L., 808 (676); Zupnick, D., 807 (671).

INDEX OF REPORTS OF MEETINGS AND MISCELLANEOUS ARTICLES

Ayres, W. L. Reports of meetings of the American Mathematical Society: *November meeting in Manhattan*, 6; *April meeting in Chicago*, 496.

Birkhoff, G., MacLane, S., and Ore, O. *Conference on Algebra and Twenty-Third Colloquium*, 169.

Field, P. *James Waterman Glover—In Memoriam*, 199.

Goldstine, H. H., and Graves, L. M. *Conference on the Theory of Integration*, 175.

Graves, L. M. See Goldstine, H. H.

Hedrick, E. R. *A Proposed Journal of Applied Mathematics*, 791.

Hollcroft, T. R. Reports of meetings of the American Mathematical Society: *October meeting in New York*, 3; *Annual meeting of Society*, 183; *February meeting in New York*, 335; *April meeting in New York*, 491; *Summer meeting in Poughkeepsie*, 800.

Kline, J. R. See Morse, M.

MacLane, S. See Birkhoff, G.

Michal, A. D. Reports of meetings of the American Mathematical Society: *April meeting in Berkeley*, 495.

Morse, M. *Letter from Marston Morse*, 1.

Morse, M., and Kline, J. R. *Occupational Classification of Mathematicians*, 792.

Murphy, R. D., and Richardson, R. G. D. *Robert Henderson—In Memoriam*, 504.
Occupational Bulletin No. 10, 797; *Occupational Bulletin No.* 23, 795.
Ore, O. See Birkhoff, G.
Putnam, T. M. Reports of meetings of the American Mathematical Society: *November meeting in Pasadena*, 8.
Richardson, R. G. D. See Murphy, R. D.
Selective Service, 795, 797

INDEX OF BOOK REVIEWS

Ayres, W. L. See Wilder, R. L.
Bateman, H. See Carslaw, H. S.
Bell, E. T. See Courant, R.
Bergman, S. *Sur les Fonctions Orthogonales de Plusieurs Variables Complexes, avec les Applications à la Théorie des Fonctions Analytiques.* A. Gelbart, 15.
Biernacki, M. *Les Fonctions Multivalentes.* (Actualités Scientifiques et Industrielles, no. 657.) W. Seidel, 640.
Biot, M. A. See von Kármán, T.
Birkhoff, G., and MacLane, S. *A Survey of Modern Algebra.* R. M. Thrall, 342.
Bourgin, D. G. See Conway, A. W.
Brauer, R. See Hancock, H.
Carslaw, H. S., and Jaeger, J. C. *Operational Methods in Applied Mathematics.* H. Bateman, 510.
Churchill, R. V. See *Research—A National Resource.*
Conway, A. W., and McConnell, A. J., edited by. *The Mathematical Papers of Sir William Rowan Hamilton.* Vol. II. *Dynamics.* (Cunningham Memoir, no. 14.) D. G. Bourgin, 813.
Coulson, C. A. *Waves. A Mathematical Account of the Common Types of Wave Motion.* S. A. Schelkunoff, 202.
Courant, R., and Robbins, H. *What is Mathematics?* E. T. Bell, 810.
Curry, H. B. See Tarski, A.
Davis, H. T. *The Analysis of Economic Time Series.* (Cowles Commission for Research in Economics, Monograph, no. 6.) G. Tintner, 338.
Eisenhart, L. P. *An Introduction to Differential Geometry with Use of the Tensor Calculus.* G. A. Hedlund, 18.
Enzyklopadie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band I, Teil 1, Heft 2; Band I_1, Teil 1, Heft 4; Band I_1, Heft 5. O. Ore, 653.
Gelbart, A. See Bergman, S.
Halmos, P. R. See Murray, F. J.
Hancock, H. *Development of the Minkowski Geometry of Numbers.* R. Brauer, 651.
Hazeltine, A. See von Kármán, T.
Hedlund, G. A. See Eisenhart, L. P.
Hurewicz, W., and Wallman, H. *Dimension Theory.* (Princeton Mathematical Series, no. 4.) P. A. Smith, 641.
Jaeger, J. C. See Carslaw, H. S.
James, R. D. See Lehmer, D. H.
von Kármán, T., and Biot, M. A. *Mathematical Methods in Engineering.* A. Hazeltine, 646.
Lehmer, D. H. *Guide to Tables in the Theory of Numbers.* R. D. James, 201.
McConnell, A. J. See Conway, A. W.
MacLane, S. See Birkhoff, G.

Murray, F. J. *An Introduction to Linear Transformations in Hilbert Space.* (Annals of Mathematics Studies, no. 4.) P. R. Halmos, 204.
—— See Widder, D. V.
National Research Council. See *Research—A National Resource.*
Neyman, J. See Wald, A.
Opatowski, I. See *Tables of the Moments of Inertia and Section Modulus of Ordinary Angles, Channels and Bulb Angles with Certain Plate Combinations.*
Ore, O. See *Enzyklopadie.*
Poritsky, H. See Seth, B. R.
Quine, W. V. *Mathematical Logic.* B. Rosser, 21.
Research—A National Resource. II. *Industrial Research.* (Report of the National Research Council.) R. V. Churchill, 338.
Robbins, H. See Courant, R.
Rosser, B. See Quine, W. V.
Schelkunoff, S. A. See Coulson, C. A.
Schilpp, P. A., edited by. *The Philosophy of Alfred North Whitehead.* (The Library of Living Philosophers, vol 3.) A. R. Turquette, 506.
Seidel, W. See Biernacki, M.
Seth, B. R. *Two-dimensional Potential Problems Connected with Rectilinear Boundaries.* H. Poritsky, 510
Skolem, T. *Diophantische Gleichungen.* (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 5, Heft 4.) H. S. Vandiver, 10.
Smith, P. A. See Hurewicz, W.
Snyder, V. See *Tables of Probability Functions.*
Tables of the Moments of Inertia and Section Modulus of Ordinary Angles, Channels and Bulb Angles with Certain Plate Combinations. I. Opatowski, 813.
Tables of Probability Functions. Vol. 1, V. Snyder, 201.
Tarski, A. *Introduction to Logic and to the Methodology of Deductive Sciences.* H. B. Curry, 507.
Thrall, R. M. See Birkhoff, G.
Tintner, G. See Davis, H. T.
Turquette, A. R. See Schilpp, P. A.
Vandiver, H. S. See Skolem, T.
Wald, A. *On the Principles of Statistical Inference.* (Notre Dame Mathematical Lectures, no. 1.) J. Neyman, 639.
Wallman, H. See Hurewicz, W.
Walker, R. J. See Wilder, R. L.
Widder, D. V. *The Laplace Transform.* (Princeton Mathematical Series, no. 6.) F. J. Murray, 642.
Wilder, R. L., and Ayres, W. L , edited by *Lectures in Topology.* (The University of Michigan Conference of 1940.) R. J. Walker, 203.

INDEX OF ADDRESSES

Bewley, L. V. *Traveling waves on electric power systems*, 527.
Boas, R. P. *Entire functions of exponential type*, 839.
Carlitz, L. *Some topics in the arithmetic of polynomials*, 679.
Hestenes, M. R. *The problem of Bolza in the calculus of variations*, 57.
Murray, F. J. *The analysis of linear transformations*, 76.
Rademacher, H. *Trends in research: The analytic number theory*, 379.
Sokolnikoff, I. S. *Some new methods of solution of two-dimensional problems in elasticity*, 539.

Stoker, J. J. *Mathematical problems connected with the bending and buckling of elastic plates*, 247.
Whyburn, W. M. *Differential equations with general boundary conditions*, 692.
Wright, S. *Statistical genetics and evolution*, 223.
Zariski, O. *Normal varieties and birational correspondences*, 402.

INDEX OF RESEARCH PAPERS

Albert, A. A. *The radical of a non-associative algebra*, 891.
Aucoin, A. A. *Homogeneous and nonhomogeneous Diophantine equations*, 933.
Ayres, W. L. *A new proof of the cyclic connectivity theorem*, 627.
Baer, R. *Inverses and zero-divisors*, 630.
Barnard, R. W., and Goldstine, H. H. *The modular space determined by a positive function*, 946.
Bartels, R. C. F., and Churchill, R. V. *Resolution of boundary problems by the use of a generalized convolution*, 276.
Beaumont, R. A. *Projections of the prime-power abelian group of order* p^m *and type* $(m-1, 1)$, 866.
Beckenbach, E. F. *Vector analogues of Morera's theorem*, 937.
Bewley, L. V. See Index of Addresses.
Blanch, G. See Lowan, A. N.
Boas, R. P. *Generalized Laplace integrals*, 286.
—— See Index of Addresses.
Botts, T. *On convex sets in linear normed spaces*, 150.
Brauer, R. *On the nilpotency of the radical of a ring*, 752.
Bruck, R. H. *Generalized Fischer groups and algebras*, 618.
Carlitz, L. See Index of Addresses.
Carruth, P. W. *Arithmetic of ordinals with applications to the theory of ordered abelian groups*, 262.
Chevalley, C. *On the composition of fields*, 482.
Churchill, R. V. *Expansions in series of non-orthogonal functions*, 143.
—— See Bartels, R. C. F.
Clark, C. E. *On 3-dimensional manifolds*, 437.
Coleman, R. *Conformal geometry of one-parameter families of curves*, 94.
Court, N. A. *On the theory of the tetrahedron*, 583.
Davids, N. See Lowan, A. N.
Daum, J. A. *The basic analogue of Kummer's theorem*, 711.
DeCicco, J. See Kasner, E.
Dimsdale, B. *Approximation of continuous functions by means of lacunary polynomials*, 608.
Dines, L. L. *On the mapping of* n *quadratic forms*, 467.
Dresden, A. *On the iteration of linear homogeneous transformations*, 577.
—— *A correction to "On the iteration of linear homogeneous transformations,"* 949.
Duffin, R. J., and Eachus, J. J. *Some notes on an expansion theorem of Paley and Wiener*, 850.
Eachus, J. J. See Duffin, R. J.
Everett, C. J. *Vector spaces over rings*, 312.
Feld, J. M. *Whirl-similitudes, euclidean kinematics, and non-euclidean geometry*, 783.
Fialkow, A. *Correction to "Totally geodesic Einstein spaces,"* 167.
Forsythe, G. E., and Schaeffer, A. C. *Remarks on regularity of methods of summation*, 863.

Fort, T. *Generalizations of the Bernoulli polynomials and numbers and corresponding summation formulas*, 567.
—— *An addition to "Generalizations of the Bernoulli polynomials and numbers and corresponding summation formulas,"* 949.
Fox, R. H. *A characterization of absolute neighborhood retracts*, 271.
Gage, W. H. *An arithmetical identity for the form* $ab-c^2$, 898.
Garabedian, H. L. *Hausdorff methods of summation which include all of the Cesàro methods*, 124.
Goldstine, H. H. See Barnard, R. W.
Greenberg, H. J., and Wall, H. S. *Hausdorff means included between* (C,0) *and* (C,1), 774.
Griffiths, L. W. *A note on representation by polygonal numbers*, 122.
Harrold, O. G. *A mapping characterization of Peano spaces*, 561.
Hestenes, M. R. See Index of Addresses.
Hill, J. D. *Some theorems on subseries*, 103.
Hillman, A. See Lowan, A. N.
Horenstein, W. See Lowan, A. N.
Hua, L. K. *On the least primitive root of a prime*, 726.
—— *On the least solution of Pell's equation*, 731.
Ingram, W. H. *Forced oscillations of continuous dynamical systems*, 153.
Jackson, D. *Generalization of a theorem of Korous on the bounds of orthonormal polynomials*, 602.
Jeffery, R. L. *Perron integrals*, 714.
Jones, B. W. *An extension of a theorem of Witt*, 133.
Jones, F. B. *Connected and disconnected plane sets and the functional equation* $f(x)+f(y)=f(x+y)$, 115.
—— *Measure and other properties of a Hamel basis*, 472.
Kaplansky, I., and Schilling, O. F. G. *Some remarks on relatively complete fields*, 744.
Kasner, E., and DeCicco, J. *Pseudo-conformal geometry: functions of two complex variables*, 317.
Kober, H. *A note on Hilbert's operator*, 421.
Leighton, W., and Thron, W. J. *On value regions of continued fractions*, 917.
Levenson, A. See Lowan, A. N.
Lewis, F. A. *Generators of permutation groups simply isomorphic with* $LF(2,p^n)$, 907.
Loomis, L. H. *On an inequality of Seidel and Walsh*, 908.
Lowan, A. N., Blanch, G., and Horenstein, W. *On the inversion of the q-series associated with Jacobian elliptic functions*, 737.
Lowan, A. N., Davids, N., and Levenson, A. *Table of the zeros of the Legendre polynomials of order* 1–16 *and the weight coefficients for Gauss' mechanical quadrature formula*, 739.
Lowan, A. N., Salzer, H. E., and Hillman, A. *A table of coefficients for numerical differentiation*, 920.
McShane, E. J. *On Perron integration*, 718.
MacLane, S., and Schilling, O. F. G. *A formula for the direct product of crossed product algebras*, 108.
Maker, P. T. *The Cauchy theorem for functions on closed sets*, 912.
Montgomery, D., and Zippin, L. *A theorem on Lie groups*, 448.
Murray, F. J. See Index of Addresses.
Nichols, G. D. *A sufficient condition for Cesàro summability*, 580.

Patterson, J. O. *A note on the diophantine problem of finding four biquadrates whose sum is a biquadrate*, 736.
Peebles, G. H. *On equivalence of certain types of series of orthonormal functions*, 556.
Perlin, I. E. *Sufficient conditions that polynomials in several variables be positive*, 458.
Perlis, S. *A characterization of the radical of an algebra*, 128.
Rademacher, H. See Index of Addresses.
Riblet, H. J. *Factorization of differential ideals*, 575.
—— *Symmetric differential expressions*, 871.
Rosenthal, A. *On differentiation of integrals and approximate continuity*, 414.
Salzer, H. E. See Lowan, A. N.
Sanders, S. T. *A linear transformation whose variables and coefficients are sets of points*, 440.
Sard, A. *The measure of the critical values of differentiable maps*, 883.
Schaeffer, A. C. See Forsythe, G. E.
Schilling, O. F. G. See Kaplansky, I.; MacLane, S.
Schoenberg, I. J. *On local convexity in Hilbert space*, 432.
Scott, W. *A note on the lower semi-continuity of double integrals in the parametric form*, 763.
Shah, S. M. *On integral functions of integral or zero order*, 329.
Shohat, J. *Note on closure for othogonal polynomials*, 488.
Simmons, H. A. *Classes of maximum numbers associated with two symmetric equations*, 295.
Smith, P. A. *Everywhere dense subgroups of Lie groups*, 309.
Snyder, W. S. *A remark on the cardinal of limit spaces*, 121.
Sokolnikoff, I. S. See Index of Addresses.
Sprague, A. H. *A differential geometry problem using tensor analysis*, 747.
Springer, C. E. *Dual geodesics on a surface*, 901.
Stoker, J. J. See Index of Addresses.
Szász, O. *On the logarithmic means of rearranged partial sums of a Fourier series*, 705.
Thron, W. J. See Leighton, W.
Vogt, M. L. *A non-involutorial space transformation associated with a* $Q_{1,n}$ *congruence*, 767.
—— *Non-involutorial space transformations associated with a linear congruence*, 874.
Wade, T. L. *Euclidean concomitants of the ternary cubic*, 589.
Wall, H. S. *The behavior of certain Stieltjes continued fractions near the singular line*, 427.
—— See Greenberg, H. J.
Walsh, J. L. *Note on the coefficients of overconvergent power series*, 163.
Weisner, L. *Roots of certain classes of polynomials*, 283.
Whyburn, G. T. *On the interiority of real functions*, 942.
Whyburn, W. M. See Index of Addresses.
Wilcox, L. R. *A note on complementation in lattices*, 453.
Williamson, J. *A generalization of the polar representation of nonsingular matrices*, 856.
Wintner, A. *On a family of Fourier transforms*, 304.
—— *On an elementary analogue of the Riemann-Mangoldt formula*, 759.
Wolf, F. *On majorants of subharmonic and analytic functions*, 925.
Wright, S. See Index of Addresses.
Zariski, O. See Index of Addresses.
Zippin, L. See Montgomery, D.

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